

Problem Solving Seminar

AMS Grad Student Chapter, University of Rochester

30th September, 2019

Problem 1 (+¹, Putnam 2018). Let n be a positive integer, and let

$$f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$$

Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Problem 2 (++) . Consider an rectangle whose interior has been divided into smaller rectangles whose sides are parallel to the original rectangle. Further, suppose that for each of the smaller rectangles, at least one of their sides is an integer.

Prove that original rectangle has an integral side.

Problem 3 (−) . There is a library with a set of n books B and a set of $n + 1$ members S .

For $X \subseteq S$, define the set of books $B(X)$ read by X as follows:

$$B(X) = \{b \in B : \exists x \in X, x \text{ has read } b\}$$

Show that there exist disjoint sets $M, N \subseteq S$ such that

$$B(M) = B(N)$$

Problem 4 (+, IMO 2019). The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right.

He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H , then he turns over the k th coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \rightarrow HHT \rightarrow HTT \rightarrow TTT$, which stops after three operations.

- Show that, for each initial configuration, Harry stops after a finite number of operations.
- For each initial configuration C , let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C .

Problem 5 (+, Generalization of one of Naomi's extra credits). Let V be a vector space be a vector space over the field \mathbb{F} , and let $\{V_\alpha\}_{\alpha \in \Omega}$ be a collection of proper subspaces of V indexed by an ordinal Ω . Further, suppose that

$$V = \bigcup_{\alpha \in \Omega} V_\alpha$$

Show that $|\mathbb{F}| \leq |\Omega|$.

(In Naomi's homework set, Ω is a finite ordinal, and she states the problem as "show that a vector space over an infinite field cannot be decomposed as a finite union of its proper subspaces")

Problem 6 (+++). Does there exist a group G , and elements $x, y \in G$ such that:

¹+ indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, − indicates easiness.

i) G is torsion. That is, for any $g \in G$, $g^n = e$ for some $n > 0$.

ii) $|\langle x, y \rangle| = \infty$

What if the first condition is strengthened to be that G has finite exponent? That is, what if the same n can be chosen so that $g^n = e$ for every $g \in G$? (Note that this is not necessarily the same as $n = o(g)$).