NOTE ON PACKING PATTERNS IN COLORED PERMUTATIONS

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ABSTRACT. Packing patterns in permutations concerns finding the permutation with the maximum number of a prescribed pattern. In 2002, Albert, Atkinson, Handley, Holton and Stromquist showed that there always exists a layered permutation containing the maximum number of a layered pattern among all permutations of length $n$. Consequently the packing density for all but two (up to equivalence) patterns up to length 4 can be obtained. In this note we consider the analogous question for colored patterns and permutations. By introducing the concept of “colored blocks” we characterize the optimal permutations with the maximum number of a given colored pattern when it contains at most three colored blocks. As examples we apply this characterization to find the optimal permutations of various colored patterns and subsequently obtain their corresponding packing densities.

1. INTRODUCTION

Given a permutation $\pi$ of length $n$ a pattern $\sigma$ is said to be contained in $\pi$ or $\sigma$ occurs in $\pi$ if a subsequence of $\pi$ is order isomorphic to $\sigma$. For instance, the permutation $\pi = 51342$ contains two $\sigma = 321$ patterns as the subsequences 532 and 542. The occurrence of a pattern in a permutation has been vigorously studied over the past decades. However, most of the existing work involves studying permutations which avoid a particular pattern; i.e., pattern avoidance.

A symmetric problem to pattern avoidance is the study of which permutation contains the most instances of a given pattern; i.e., pattern packing. The central problem of pattern packing is maximizing the function $p(\pi, \sigma)$ among all permutations of given length, where $p(\pi, \sigma)$ is the number of occurrences of a pattern $\sigma$ in the permutation $\pi$. This question was first considered in the unpublished work of Stromquist [4] and in the PhD thesis of Price [2] where much of the foundational work on pattern packing can be found. In what follows we call a permutation $\tilde{\pi}$ optimal if

$$p(\tilde{\pi}, \sigma) \geq p(\pi, \sigma)$$

for any permutation $\pi$ of the same length. A useful way to compare the “packability” of two different patterns is by comparing their packing densities, defined as

$$\delta(\sigma) = \lim_{|\pi| \to \infty} \frac{p(\tilde{\pi}, \sigma)}{|\pi|/|\sigma|}.$$

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Definition 1. A permutation $\pi$ of length $n$ is called layered if it can be partitioned into segments $\pi_1, \pi_2, \ldots, \pi_r$ called layers such that each layer is decreasing and all elements in $\pi_i$ are less than all elements in $\pi_{i+1}$ for $1 \leq i \leq r - 1$.

For example, the permutation $\pi = 32154876$ contains three layers $\pi_1 = 321$, $\pi_2 = 54$, $\pi_3 = 876$. Because $\pi_1 < \pi_2 < \pi_3$ (notating all elements of one layer being less than all elements of another) we have a layered permutation. A simple representation of a layered permutation with lattice points is shown in Figure 1.

![Figure 1. A layered permutation 32154876](image)

Building upon previous work, Albert et al. [1] considered packing densities of layered patterns. One important result used there is from [4].

Theorem 1.1 ([4]). Given a layered pattern $\sigma$ there exists an optimal permutation $\tilde{\pi}$ that is also layered.

Since all but two symmetry classes of permutations of length 4 are layered permutations (the exceptions being represented by 1342 and 2413), most of the packing densities have been found for patterns up to length four while the others have been conjectured.

We next introduce the concept of colored permutations.

Definition 2. An $m$-colored permutation $\chi$ of length $n$ is a permutation of length $n$ in which each element is assigned one of $m$ distinct colors.

For example, let $\chi = 3a2a5b1a4b$ be a two-colored permutation where 3, 2 and 1 have color $a$ while 5 and 4 have color $b$. Analogously to the case of non-colored patterns the colored pattern $\phi = 2a1a3b$ occurs in $\chi$ as the subsequences $3a1a4b$ and $3a2a4b$.

Colored permutations are similar to permutations of a multi-set. The question of pattern avoidance on multi-sets has been studied in past years (see, for instance, [3]). In this note we focus on pattern packing in colored permutations.

We first define colored blocks, which are central to our study. Colored blocks are analogous to layers in non-colored permutations.

Definition 3. In a colored permutation $\chi$, a colored block is a maximal monochromatic segment $\chi_i^{(a)}$ in which each element has color $a$ and every element not in $\chi_i^{(a)}$ is either larger than or smaller than all elements in $\chi_i^{(a)}$.

Remark 1. Note that every entry in a colored permutation is in exactly one of its colored blocks.

In other words, a colored block is a monochromatic segment of elements with consecutive numerical values. For instance, the permutation $\chi = 3a1a2a6a5b4b$ contains three colored
blocks, $\chi_1^{(a)} = 3a_12a_2$, $\chi_2^{(a)} = 6a$, and $\chi_3^{(b)} = 5b_4a_b$. A graphical representation of colored blocks is shown in Figure 2. An important note is that colored blocks are both numerically and chromatically disjoint.

![Figure 2. A Colored Permutation $3a_12a_26a5b4b$](image)

In the rest of this note we provide some observations on the optimal colored permutations for colored patterns which contain either two or three colored blocks. For convenience we will reuse the notation $p(\chi, \phi)$ to represent the number of occurrences of the colored pattern $\phi$ in the colored permutation $\chi$. Colored blocks will often be denoted simply by their color and/or location, i.e. $\chi_1^{(a)} = A_1$, $\chi_2^{(b)} = B_2$, $\chi_3^{(a)} = A_3$, etc. Similarly for colored patterns $\phi_1^{(a)} = \alpha_1$, $\phi_2^{(b)} = \beta_2$, etc. The collection of all colored blocks in $\chi$ of color $a$ ($b$) will be denoted by $\chi_A$ ($\chi_B$).

2. PATTERN PACKING WITH TWO OR THREE COLORED BLOCKS

Note that a single-colored (or non-colored) permutation has exactly one colored block (namely the permutation itself). In what follows we assume the permutations/patterns under consideration to be at least two-colored.

2.1. Patterns with two colored blocks. Since the number of colored blocks is at least the number of colors, we may assume that a colored pattern $\phi$ with two colored blocks $\phi_1^{(r)} = \rho$ and $\phi_2^{(b)} = \beta$ to have exactly two colors and is of the form $\rho \beta$. Furthermore, we may assume without loss of generality that all elements in $\rho$ are less than all elements of $\beta$ (similarly to layers denoted $\alpha < \beta$). For the remainder of this section color $r$ will be referred to as “red” and $b$ will be referred to as “blue”.

**Theorem 2.1.** For a pattern $\phi$ with two blocks of the form $\rho \beta$ with $\rho < \beta$, there is an optimal length-$n$ permutation $\tilde{\chi}$ of the form $RB$ with $R < B$.

**Remark 2.** The proof below follows the simple idea that sliding all of the red entries to the left and all of the blue entries to the right leaves every instance of a $\phi$-pattern intact.

**Proof.** Let $\chi$ be an optimal permutation of length $n$ with colored blocks $\chi_1\chi_2\ldots\chi_k$. First we claim that $\chi_1 = R_1$ is red. If $\chi_1$ were blue,

$$p(\chi,\phi) = p(\chi_1,\rho) \cdot p(\chi_{>1},\beta) + p(\chi_{>1},\phi)$$

where $\chi_{>i_0}$ ($\chi_{<i_0}$) is the collection of all colored blocks in $\chi$ after (before) $\chi_{i_0}$. Clearly the first term in the sum is zero. By recoloring $\chi_1$ red, this term is replaced with

$$p(R_1,\rho) \cdot p(\chi_{>1},\beta) \geq 0,$$
and thus \( p(\chi, \phi) \) will only increase. Similarly, we may assume \( \chi_k = B_k \) is blue. Along the same lines we claim there is no blue block immediately preceding a red block. Otherwise, let \( \chi_j = B_j \) and \( \chi_{j+1} = R_{j+1} \) in \( \chi \), we have

\[
p(\chi, \phi) = P(\chi_{<j}, \rho) \cdot p(B_j, \beta) + P(\chi_{>j+1}, \beta) \cdot p(R_{j+1}, \rho) + p(\chi_{<j} \chi_{>j+1}, \phi).
\]

Let \( \chi' \) be obtained from \( \chi \) by switching \( \chi_j \) and \( \chi_{j+1} \), we have

\[
p(\chi', \phi) = p(\chi, \phi) + p(R_{j+1}, \rho) \cdot p(B_j, \beta).
\]

Thus \( p(\chi, \phi) \) may only increase.

Consequently, we now have an optimal permutation \( \chi \) of the form \( \chi = \chi_R \chi_B \). Because any \( \phi = \rho \beta \) pattern occurring in \( \chi \) must consist of a \( \rho \) pattern from \( \chi_R \) and a \( \beta \) pattern from \( \chi_B \), we have

\[
p(\chi, \phi) \leq p(\chi_R, \rho) \cdot p(\chi_B, \beta)
\]

with equality if \( \chi_R < \chi_B \). Hence there is an optimal permutation \( \tilde{\chi} = RB \) with \( R < B \).

For example, to pack the pattern \( \chi = 2, r_1, 3, b_4, b_7 \) an optimal permutation of length \( n \) consists of a decreasing sequence of the elements \( [\lfloor \frac{n}{2} \rfloor] \ldots 1 \) colored red followed by an increasing sequence of the elements \( ([\frac{n}{2}] + 1) \ldots n \) colored blue. More detailed applications will be discussed in Section 3.

### 2.2. Patterns with three colored blocks.

In this subsection we consider patterns with three colored blocks through several different cases. Some arguments are similar to those in the previous subsection and we omit some details.

First consider the case when the pattern has three distinct colors. Assume without loss of generality \( \phi_1^{(r)} = \rho \), \( \phi_2^{(b)} = \beta \), and \( \phi_3^{(g)} = \gamma \) (with colors red, blue and green) and thus the colored pattern has the form \( \phi = \rho \beta \gamma \).

**Theorem 2.2.** Given a pattern \( \phi \) with three colored blocks of distinct colors of the form \( \rho \beta \gamma \), there is an optimal permutation \( \tilde{\chi} \) (of length \( n \)) of the form RBG with the same numerical ordering as \( \rho \beta \gamma \).

**Remark 3.** For instance, given a pattern \( \phi \) of the form \( \rho \beta \gamma \) with \( \rho < \gamma < \beta \), there is an optimal permutation of the form RBG such that \( R < G < B \).

**Proof.** Following the same arguments as Theorem 2.1, it is easy to show that the optimal permutation is of the form

\[
R_1 \ldots R_i B_{i+1} \ldots B_j G_{j+1} \ldots G_k = \chi_R \chi_B \chi_G.
\]

Then

\[
p(\chi, \phi) \leq p(\chi_R, \rho) \cdot p(\chi_B, \beta) \cdot p(\chi_G, \gamma)
\]

with equality if any elements \( a \in \chi_R \), \( b \in \chi_B \) and \( c \in \chi_G \) assume the same numerical ordering as \( \rho \), \( \beta \) and \( \gamma \).
Suppose now that a pattern $\phi$ has three colored blocks with only two colors. Assume without loss of generality that there is one red block and two blue blocks. First consider the case when the two blue blocks are adjacent, i.e., $\phi = \rho \beta_1 \beta_2$. This case is representative of all patterns with two adjacent blue blocks since reversing the permutation/pattern turns all $\rho \beta_1 \beta_2$ patterns into $\beta_1 \beta_2 \rho$ patterns.

**Theorem 2.3.** For a pattern $\phi$ with three colored blocks of the form $\rho \beta_1 \beta_2$, there is an optimal permutation $\tilde{\chi}$ (of length $n$) that is also of the form $RB_1 B_2$ and the numerical ordering of the colored blocks in $\tilde{\chi}$ is the same as that of the colored blocks in $\phi$.

**Proof.** First note that since the two blue blocks are numerically disjoint, no element from $\beta_1$ may be (numerically) adjacent to an element in $\beta_2$. That is, either $\beta_1 < \rho < \beta_2$ or $\beta_2 < \rho < \beta_1$. Without loss of generality we will assume the former.

Once again arguments from Theorem 2.1 yield that all red blocks in an optimal permutation $\chi$ can be placed before any blue blocks. That is, an optimal permutation is of the form

$$\chi = R_1 \ldots R_i B_{i+1} \ldots B_k = \chi_R \chi_B.$$

Note that any $\rho \beta_1 \beta_2$ pattern is a result of a $\rho$ pattern in $\chi_R$ and a $\beta_1 \beta_2$ pattern in $\chi_B$. For any particular pattern $\rho$ in $\chi_R$, let $\chi_{B<\rho}$ be the set of all blue blocks less than $\rho$ and $\chi_{B>\rho}$ be the set of all blue blocks greater than $\rho$. Since $\rho \beta_1 \beta_2$ patterns are only formed using $\beta_1$ patterns from $\chi_{B<\rho}$ and $\beta_2$ patterns from $\chi_{B>\rho}$, the contribution from this $\rho$ pattern to $p(\chi, \phi)$ is at most

$$p(\chi_{B<\rho}, \beta_1) \cdot p(\chi_{B>\rho}, \beta_2).$$

This can be achieved (regardless of the choice of $\rho$) by putting the blue blocks in increasing order. Under this assumption, let $\chi_{B<j}$ ($\chi_{B>j}$) denote the collection of blue blocks before (after) $B_j$ in $\chi_B$ and $\tilde{j}_0$ be such that

$$p(\chi_{B<j_{\tilde{j}_0+1}}, \beta_1) \cdot p(\chi_{B>j_{\tilde{j}_0}}, \beta_2) \geq p(\chi_{B<j_{\tilde{j}_0+1}}, \beta_1) \cdot p(\chi_{B>j_{\tilde{j}_0}}, \beta_2)$$

for any $i + 1 \leq j \leq k - 1$, we now have

$$p(\chi, \phi) \leq p(\chi_R, \rho) \cdot p(\chi_{B<j_{\tilde{j}_0+1}}, \beta_1) \cdot p(\chi_{B>j_{\tilde{j}_0}}, \beta_2).$$

Equality holds if

$$\chi_{B<j_{\tilde{j}_0+1}} < \chi_R < \chi_{B>j_{\tilde{j}_0}}.$$

Consequently each of $R = \chi_R$, $B_1 = \chi_{B<j_{\tilde{j}_0+1}}$ and $B_2 = \chi_{B>j_{\tilde{j}_0}}$ is a single block and the optimal permutation is of the form $RB_1 B_2$ with $B_1 < R < B_2$. $\blacksquare$

Lastly we consider the case when the pattern is of the form $\phi = \beta_1 \rho \beta_2$.

**Theorem 2.4.** For a pattern $\phi$ with three colored blocks of the form $\beta_1 \rho \beta_2$, there is an optimal permutation $\tilde{\chi}$ that is of the form $B_1 RB_2$ with same numerical ordering as those in $\phi$.

**Proof.** First we may assume (following the same argument as before), that in an optimal permutation $\chi$, the first and last blocks are blue, i.e.,

$$\chi = \chi_1^{(b)} \ldots \chi_i^{(c)} \ldots \chi_k^{(b)}$$

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where \( c_i \in \{r, b\} \). Consider any \( \rho \) pattern formed in the sequence of the \( s \) red blocks \( \chi_R := \chi^{(r)}_{j_1} \chi^{(r)}_{j_2} \cdots \chi^{(r)}_{j_s} \). A \( \beta_1 \rho \beta_2 \) pattern can only be formed by a \( \beta_1 \) pattern in the sequence of blue blocks before \( \chi^{(r)}_{j_1} \) and a \( \beta_2 \) pattern in the sequence of blue blocks after \( \chi^{(r)}_{j_s} \). Thus the number of \( \phi \) patterns formed from this particular \( \rho \) pattern is at most

\[
p(\chi_{B_{<j_1}}, \beta_1) \cdot p(\chi_{B_{>j_s}}, \beta_2)
\]

where \( \chi_{B_{<j_1}} \) denotes the sequence of blue blocks before block \( j_1 \) and \( \chi_{B_{>j_s}} \) denotes the sequence of blue blocks after block \( j_s \).

Let \( j_0 \) (not necessarily unique) be a value such that \( c_{j_0} = r \) and

\[
p(\chi_{B_{<j_0}}, \beta_1) \cdot p(\chi_{B_{>j_0}}, \beta_2) \geq p(\chi_{B_{<j_1}}, \beta_1) \cdot p(\chi_{B_{>j_s}}, \beta_2)
\]

for all \( j \), then

\[
p(\chi, \beta_1 \rho \beta_2) \leq p(\chi_{B_{<j_0}}, \beta_1) \cdot p(\chi_R, \rho) \cdot p(\chi_{B_{>j_0}}, \beta_2)
\]

with equality if all red blocks are located in between the first blue block immediately preceding and following \( \chi^{(r)}_{j_0} \). Consequently \( \chi \) is an optimal permutation of the form

\[
\chi^{(b)}_1 \cdots \chi^{(b)}_\ell \chi^{(r)}_{\ell+1} \cdots \chi^{(r)}_{m} \chi^{(b)}_{m+1} \cdots \chi^{(b)}_k = \chi_{B_1} \chi_R \chi_{B_2}
\]

with colored blocks \( \chi_{B_1} := \chi^{(b)}_1 \cdots \chi^{(b)}_\ell, \chi_R := \chi^{(r)}_{\ell+1} \cdots \chi^{(r)}_{m} \), and \( \chi_{B_2} := \chi^{(b)}_{m+1} \cdots \chi^{(b)}_k \).

Through arguments similar to those of Theorem 2.3 one can see that the numerical ordering of \( \chi_{B_1} \chi_R \chi_{B_2} \) is the same as \( \beta_1 \rho \beta_2 \).

3. APPLICATIONS TO SPECIFIC PATTERNS

In this section, we apply our findings to some specific colored patterns and obtain their corresponding packing densities. For convenience we let \( p(n, \phi) \) denote the specific number of occurrences of \( \phi \) in an optimal permutation of length \( n \).

3.1. Patterns of Length 2. For non-colored patterns of length 2, the packing density is trivially equal to one.

In the colored case, Theorem 2.1 implies that the optimal permutation of length \( n \) of the colored pattern \( 1_r 2_b \) (or equivalently \( 2_r 1_b \)) is of the form \( RB \) with \( R < B \). Then

\[
p(n, 1_r 2_b) = p(R, 1) \cdot p(B, 1) = |R| \cdot |B|.
\]

Given \( |R| + |B| = n \), it is easy to see that

\[
p(n, 1_r 2_b) = \left\lfloor \frac{n^2}{4} \right\rfloor = \frac{2n^2 - 1 + (-1)^n}{8}.
\]

Therefore the packing density of all length two colored patterns (in which two distinct colors occur) is given by

\[
\delta(1_r 2_b) = \lim_{n \to \infty} \frac{p(n, 1_r 2_b)}{\binom{n}{2}} = \frac{1}{2}.
\]
3.2. **Patterns of Length 3.** For non-colored patterns of length 3, the packing densities for the decreasing and increasing patterns are trivial. The layered pattern 132 has packing density \(2\sqrt{3} - 3\) as established in [4].

3.2.1. *The pattern \(2, 1_3 3_b\) (and equivalents).* Theorem 2.3 implies that the optimal permutation of length \(n\) of the colored pattern \(2, 1_3 3_b\) is of the form \(RB_1 B_2\) with \(B_1 < R < B_2\), then

\[
p(n, 2, 1_3 3_b) = p(R, 1) \cdot p(B_2, 1) = |R| \cdot |B_1| \cdot |B_2|.
\]

With \(|R| + |B_1| + |B_2| = n\), it is easily shown that

\[
p(n, 2, 1_3 3_b) = \begin{cases} 
\frac{n^3}{27}, & \text{if } n \equiv 0 \mod 3 \\
\frac{(n-1)^2(n+2)}{27}, & \text{if } n \equiv 1 \mod 3 \\
\frac{(n+1)^2(n-2)}{27}, & \text{if } n \equiv 2 \mod 3
\end{cases}
\]

Consequently the packing density of \(2, 1_3 3_b\) (and equivalent patterns) is

\[
\delta(2, 1_3 3_b) = \lim_{n \to \infty} \frac{p(n, 2, 1_3 3_b)}{\frac{n^3}{6}} = \frac{2}{9}.
\]

3.2.2. *The pattern \(1_3 3_b 2_b\) (and equivalents).* Theorem 2.1 implies that the optimal permutation of length \(n\) of the colored pattern \(1_3 3_b 2_b\) is of the form \(RB\) in which \(R < B\), then

\[
p(n, 1_3 3_b 2_b) = p(R, 1) \cdot p(B, 21) = |R| \left(\frac{|B|}{2}\right).
\]

Let \(|B| = k\), then

\[
p(n, 1_3 3_b 2_b) = \max_{1 \leq k \leq n} \left\{ \binom{n-k}{2} \right\},
\]

achieved when \(k \sim \frac{2n}{3}\). Consequently \(\delta(1_3 3_b 2_b) = \frac{4}{9}\).

3.3. **Longer colored patterns.** Patterns of length over three may also be studied so long as they contain no more than three colored blocks and each colored block is equivalent to a non-colored pattern with known packing density.

3.3.1. *Pattern \(1_b 3_r 4, 2_b\).* Theorem 2.4 implies that the optimal colored permutation \(\tilde{\chi}\) is of the form \(B_1 RB_2\) with \(R_1 < B_2 < R\), then

\[
p(n, 1_b 3_r 4, 2_b) = p(B_1, 1) \cdot p(R, 12) \cdot p(B_2, 1).
\]

For convenience, let \(|B_1| = x\), \(|R| = y\), and \(|B_2| = z\). Then \(x + y + z = n\) and thus for any fixed \(y\)

\[
p(B_1, 1) \cdot p(B_2, 1) = x \cdot z \leq \frac{2(n-y)^2 - 1 + (-1)^{n-y}}{8}
\]

with equality when \(|x - z| \leq 1\). Consequently

\[
p(n, 1_b 3_r 4, 2_b) = \max_{2 \leq y \leq n-2} \left\{ \frac{2(n-y)^2 - 1 + (-1)^{n-y}}{8} \cdot \binom{y}{2} \right\},
\]

achieved when \(y \sim \frac{n}{2}\). Hence \(\delta(1_b 3_r 4, 2_b) = \frac{3}{16}\).

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3.3.2. Pattern $3_b 2_b 4, 6_r 5_r 1_b$. Theorem 2.3 implies that the optimal permutation $\tilde{\chi}$ is of the form $B_1 R B_2$ with $B_2 < B_1 < R$. Hence
\[
p(n, 3_b 2_b 4, 6_r 5_r 1_b) = p(B_1, 21) \cdot p(R, 132) \cdot p(B_2, 1).
\]
Letting $|B_1| = x$, $|R| = y$, $|B_2| = z$ and fixing $y$ again, we have $x + z = n - y$ and
\[
p(B_1, 21) \cdot p(B_2, 1) \leq \left\lfloor \frac{x}{2} \right\rfloor \cdot z.
\]
This expression is maximized when $x \sim \frac{2(n-y)}{3}$ and $z \sim \frac{n-y}{3}$. From [4] we have $p(y, 132) \sim (2\sqrt{3} - 3) \frac{y^3}{6}$, hence
\[
p(n, 3_b 2_b 4, 6_r 5_r 1_b)
\sim \max_{3 \leq y \leq n-3} \left\{ \frac{2(n-y)}{6} \left( \frac{2(n-y) - 3}{6} \right) \cdot \left( \frac{2(n-y)}{3} \right) \cdot (2\sqrt{3} - 3) \frac{y^3}{6} \right\},
\]
achieved when $y \sim \frac{n}{2}$. Thus $\delta(3_b 2_b 4, 6_r 5_r 1_b) = \frac{5}{9}(2\sqrt{3} - 3)$.

4. Concluding Remarks

In this note we considered the question of packing colored patterns into colored permutations. It is worth noting that, with our characterizations of the optimal permutations, the colored version of the pattern packing question is in some sense easier than the non-colored version and encompasses a wider range of patterns. For instance, the optimal permutation for the colored pattern $6_r 1_r 3_r 2_r 5_b 4_b$ can indeed be characterized since it contains only three colored blocks and each block is a layered pattern. However, the non-colored pattern $613254$ is not layered and its optimal permutation is much more difficult to characterize.

It is natural to conjecture that the following holds in general, which we post as a question.

**Question 4.1.** Is it true that the optimal colored permutation with respect to a given colored pattern always shares the same number and arrangement of the colored blocks as those of the pattern?

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**References**


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