COMBINATORIAL ANALYSIS OF INTEGER POWER PRODUCT EXPANSIONS

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Abstract. Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ be a formal power series with complex coefficients. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of nonzero integers. The Integer Power Product Expansion of $f(x)$, denoted $\mathbb{Z}$PPE, is $\prod_{k=1}^{\infty}(1 + w_k x^k)^{r_k}$. Integer Power Product Expansions enumerate partitions of multi-sets. The coefficients $\{w_k\}_{k=1}^{\infty}$ themselves possess interesting algebraic structure. This algebraic structure provides a lower bound for the radius of convergence of the $\mathbb{Z}$PPE and provides an asymptotic bound for the weights associated with the multi-sets.

1. Introduction

In the field of enumerative combinatorics, it iswell known that

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}(1 - x^n)^{-1}, \tag{1}$$

where $p(n)$ is the number of partitions of n [1]. Equally well known is the generating function for $p_d(n)$, the number of partitions of n with distinct parts [1]

$$1 + \sum_{n=1}^{\infty} p_d(n)x^n = \prod_{n=1}^{\infty}(1 + x^n). \tag{2}$$

Equation (2) is a special case of the *Generalized Power Product Expansion*, GPPE. The GPPE of a formal power series $1 + \sum_{n=1}^{\infty} a_n x^n$ is

$$1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty}(1 + g_n x^n)^{r_n}, \tag{3}$$

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where $\{r_n\}_{n=1}^{\infty}$ is a set of nonzero complex numbers. If $r_n = 1$ and $g_n = 1$, Equation (3) becomes Equation (2). Similarly, Equation (1) is a special case of the *Generalized Inverse Power Product Expansion,* GIPPE. The GIPPE of a formal power series $1 + \sum_{n=1}^{\infty} a_n x^n$ is

$$1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 - h_n x^n)^{-r_n}, \tag{4}$$

where $\{r_n\}_{n=1}^{\infty}$ is a set of nonzero complex numbers. Equation (1) is Equation (4) with $r_n = 1$ and $h_n = 1$. The analytic and algebraic properties of the GPPE and the GIPPE were extensively studied in [5, 6, 4]. Since Equations (1) and (2) are generating functions associated with partitions, it is only natural to define a single class of product expansions that incorporate both as special examples. Define the *Integer Power Product Expansion*, $\mathbb{Z}$PPE, of the formal power series $1 + \sum_{n=1}^{\infty} a_n x^n$ to be

$$1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 + w_n x^n)^{r_n}, \tag{5}$$

whenever $\{r_n\}_{n=1}^{\infty}$ is a set of nonzero *integers.* Then Equation (2) is Equation (5) with $r_n = 1$ and $w_n = 1$, while Equation (1) is Equation (5) with $r_n = -1$ and $w_n = -1$.

The purpose of this paper is to study, in a self-contained manner, the combinatorial, algebraic, and analytic properties of the $\mathbb{Z}$PPE. Section 2 discusses, in detail, the role of integer power product in the field of enumerative combinatorics. In particular, we show how integer power products enumerate partitions of multi-sets. We also discuss how the $\mathbb{Z}$PPE factors the formal power series associated with the number of compositions. Section 3 derives the algebraic properties of w_n in terms of $\{a_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$. The most important property, known as the Structure Property, writes w_n as a polynomial in $\{a_i\}_{i=1}^{n}$, whose coefficients are rational expressions of the form $\frac{p(r_1,r_2,..,r_n)}{q(r_1,r_2,..,r_n)}$. We exploit the Structure Property in Section 4 when determining a lower bound for the radius of convergence of $\prod_{n=1}^{\infty}(1 + w_n x^n)^{r_n}$. Section 4 also contains an asymptotic approximation for the integer power product expansion associated with $1 - \sum_{n=1}^{\infty} s^n x^n$ where $s = \sup_{n \geq 1} |a_n|^{\frac{1}{n}}$, namely the majorizing product expansion.

2. Combinatorial Interpretations of Integer Power Product Expansions

Given a formal power series $1 + \sum_{n=1}^{\infty} a_n x^n$ or an analytic function $f(x)$ with $f(0) = 1$ which has a Taylor series representation $1 + \sum_{n=1}^{\infty} a_n x^n$, we define the *Integer Power Product Expansion,* denoted $\mathbb{Z}$PPE, as

$$f(x) = \prod_{n=1}^{\infty} (1 + w_n x^n)^{r_n}, \tag{6}$$

where $\{w_n\}_{n=1}^{\infty}$ is a set of nonzero complex numbers and $\{r_n\}_{n=1}^{\infty}$ is a set of nonzero *integers*. We say $(1+w_nx^n)^{r_n}$ is an *elementary factor* of the $\mathbb{Z}$PPE. If $r_n \geq 1$, an elementary factor has the form $(1+w_nx^n)^{r_n} = (1+g_nx^n)^{r_n}$, while for $r_n \leq -1$, an elementary factor has the form $(1+w_nx^n)^{r_n} = (1-h_nx^n)^{-|r_n|}$. If $r_n = 1$ for all n, Equation (6) becomes the Power Product Expansion $f(x) = \prod_{n=1}^{\infty}(1+g_nx^n)$, while if $r_n = -1$ for all n, Equation (6) becomes the Inverse Power Product Expansion $f(x) = \prod_{n=1}^{\infty}(1-h_nx^n)^{-1}$.

Given a fixed set of nonzero integers $\{r_n\}_{n=1}^{\infty}$, there is a one-to-one correspondence between the set of formal power series and the set of $\mathbb{Z}$PPE's. To discover this correspondence, expand each elementary factor of Equation (6) in terms of Newton's Binomial Theorem and then compare the coefficient of x^n. In particular we find that

$$1+\sum_{n=1}^{\infty} a_nx^n = \sum_{k_1=0}^{\infty}\binom{r_1}{k_1}(w_1x)^{k_1}\sum_{k_2=0}^{\infty}\binom{r_2}{k_2}\left(w_2x^2\right)^{k_2}\sum_{k_3=0}^{\infty}\binom{r_3}{k_3}\left(w_3x^3\right)^{k_3}\cdots .$$

Hence,

$$a_n = \binom{r_n}{1}w_n + \sum_{\substack{l\cdot v=n\\ l_j<n}}\binom{r_{l_1}}{v_1}\cdots\binom{r_{l_\theta}}{v_\theta}w_{l_1}^{v_1}\cdots w_{l_\theta}^{v_\theta}, \tag{7}$$

where $l = [l_1, l_2, \cdots, l_\theta]$ and $v = [v_1, v_2, \cdots, v_\theta]$. Equation (7) implies that

$$w_n = \frac{1}{r_n}\left[a_n - \sum_{\substack{l\cdot v=n\\ l_j<n}}\binom{r_{l_1}}{v_1}\cdots\binom{r_{l_\theta}}{v_\theta}w_{l_1}^{v_1}\cdots w_{l_\theta}^{v_\theta}\right]. \tag{8}$$

We formalize the above discussion in the following proposition which is a statement about a bijection between the sequence of the coefficients in a given power series and the sequence of coefficients in its $\mathbb{Z}$PPE expansion.

Proposition 1:*Let $\{r_n\}_{n=1}^{\infty}$ denote a sequence of nonzero integers. Let $w_k \in \mathbb{C}, k = 1, 2, \ldots$, be an infinite sequence. Let the symbol $\prod_{k=1}^{\infty}(1+w_kx^k)^{r_k}$ stand for the infinite product*

$$\prod_{k=1}^{\infty}(1+w_kx^k)^{r_k} := (1+w_1x)^{r_1}(1+w_2x^2)^{r_2}\cdots(1+w_kx^k)^{r_k}\cdots \quad . \tag{9}$$

Then there exists a unique sequence $a_n \in \mathbb{C}, n = 1, 2, \ldots$, such that in the sense of power series the following holds

$$1+\sum_{n=1}^{\infty} a_nx^n = \prod_{k=1}^{\infty}(1+w_kx^k)^{r_k}. \tag{10}$$

Conversely, let $a_n \in \mathbb{C}, n = 1, 2, \ldots$, be an infinite sequence. Then there exists a unique sequence of elements $w_k \in \mathbb{C}, k = 1, 2, \ldots$, such that the identity (10) holds. Moreover, the *elements w_k have the representation provided by Equation (8).*

The one-to-one correspondence of Proposition 1 has many combinatorial interpretations. Let n be a positive integer. A *partition* of n is a sum of k positive integers i_k such that $n = i_1 + i_2 + \cdots + i_k$. Each i_l for $1 \leq l \leq k$ is called a *part* of the partition [1]. Without loss of generality assume $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$. Given $n = i_1 + i_2 + \cdots + i_k$, we associate each part i_k with the monomial x^{i_k}. Then each summand of $\sum_{j=0}^{\infty}(x^{i_k})^j = 1 + x^{i_k} + x^{2i_k} + x^{3i_k} + \cdots$ represents the part i_k occurring j times, and the product $\prod_{i=1}^{\infty}\sum_{j=0}^{\infty}(x^i)^j = \prod_{i=1}^{\infty}(1-x^i)^{-1}$ becomes

$$\prod_{i=1}^{\infty}(1-x^i)^{-1} = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots = \sum_{n=0}^{\infty} p(n)x^n, \tag{11}$$

where $p(n)$ is the number of partitions of n. Equation (11) is Equation (6) with $r_n = -1$ and $w_n = -1$ for all n. To obtain a combinatorial interpretation for Equation (6) with $r_n = 1$ and $w_n = 1$ we observe that

$$\prod_{i=1}^{\infty}(1+x^i) = (1+x)(1+x^2)(1+x^3)\cdots = \sum_{n=0}^{\infty} p_d(n)x^n, \tag{12}$$

where $p_d(n)$ counts the partitions of n composed of distinct parts [1], where a partition of n has distinct parts if $n = i_1 + i_2 + \cdots + i_k$ and $i_l = i_p$ if and only if $l = p$.

Equations (11) and (12) may be combined as follows. Let $\{r_k\}_{k=1}^{\infty}$ be a set of integers such that for each k, $r_k = 1$ or $r_k = -1$. Furthermore require that $w_k = r_k$. Equation (6) becomes

$$\prod_{i=1}^{\infty}(1+r_i x^i)^{r_i} = \sum_{n=0}^{\infty} p_H(n)x^n, \tag{13}$$

where $p_H(n)$ is the number of partitions of n composed of unlimited number of copies of the part x^k if $r_k = -1$, and at most one copy of the part x^k if $r_k = 1$. For example suppose that $r_i = -1$ if i is *odd* and $r_i = 1$ if i is *even*. Equation (13) becomes

$$(1-x)^{-1}(1+x^2)(1-x^3)^{-1}(1+x^4)(1-x^5)^{-1}(1+x^6)\cdots = \sum_{n=0}^{\infty} p_H(n)x^n.$$

In Equation (13) we required that $r_k = \pm 1$. Let us remove this restriction and just assume $\{r_k\}_{k=1}^{\infty}$ is an arbitrary set of integers. Define

$$\operatorname{sg}(r_n) = \begin{cases} 1, & r_n \geq 1, \\ -1, & r_n \leq -1. \\ 0, & r_n = 0. \end{cases} \tag{14}$$

Is there a combinatorial interpretation for $\prod_{i=1}^{\infty}(1+\mathrm{sg}(r_i)x^i)^{r_i}$? To answer this question we need the notion of a multi-set. Let $\{r_k\}_{k=1}^{\infty}$ be a set of nonnegative integers. Define the associated multi-set as $1^{r_1}2^{r_2}\ldots k^{r_k}\ldots$, where k^{r_k} denotes r_k distinct copies of the integer k. If $r_k = 0$, there are no copies of k in the multi-set. Given $\{r_k\}_{k=0}^{\infty}$, a set of positive integers, we form the generating function

$$\prod_{i=1}^{\infty}(1+x^i)^{r_i} = (1+x)^{r_1}(1+x^2)^{r_2}\ldots(1+x^k)^{r_k}\cdots = \sum_{n=0}^{\infty}\hat{p}_d(n)x^n, \tag{15}$$

where $\hat{p}_d(n)$ counts the partitions of $n$composed of *distinct* parts of the multi-set $1^{r_1}2^{r_2}\ldots i^{r_i}\ldots$. To clarify what is meant by *distinct* parts when working in the context of multi-sets, it helps to introduce the notion of color. Each of the r_i copies of i is assigned a unique color from a set of r_i colors. Differently colored i's are considered distinct from each other. Thus $\hat{p}_d(n)$ counts the partitions of n over the multi-set $1^{r_1}2^{r_2}\ldots k^{r_k}\ldots$ which have distinct colored parts. As a case in point, take the multi-set $1^2 2^4 3^3 4^5$, and represent it as
$\{1_R, 1_B, 2_R, 2_B, 2_O, 2_Y, 3_R, 3_B, 3_O, 4_R, 4_B, 4_O, 4_Y, 4_G\}$ where the color of the digit is denoted by the subscript and R = Red, B = Blue, O = Orange, Y = Yellow, and G = Green. The generating function for this multi-set is $\prod_{n=1}^{4}(1+x^n)^{r_n} = (1+x)^2(1+x^2)^4(1+x^3)^3(1+x^4)^5$ where exponent of x denotes the part while the exponent of each elementary factor denotes the number of colors available for the associated part.

Equation (15) is the multi-set generalization of Equation (12). There is also a multi-set generalization of Equation (11). Assume r_n is a positive integer. Equation (11) generalizes as

$$\prod_{i=1}^{\infty}(1-x^i)^{r_i} = (1-x)^{r_1}(1-x^2)^{r_2}(1-x^3)^{r_3}\cdots = \sum_{n=0}^{\infty}\hat{p}(n)x^n, \tag{16}$$

where $\hat{p}(n)$ is the number of partitions of n associated with the colored multi-set which contains an unlimited number of repetitions of each integer k in r_k colors. In other words, the multi-set is $S_1^{r_1}S_2^{r_2}\ldots S_i^{r_i}\ldots$, where $S_i = \{i, i+i, i+i+i, \ldots\}$. The factor $(1-x^i)^{r_i} = (1+x^i+x^{2i}+x^{3i}+\ldots)^{r_i}$ corresponds to $\{i, i+i, i+i+i, \ldots\}$ replicated in r_i colors. As an example of Equation (16), let $r_1 = 2$, $r_2 = 1$ and $r_3 = 3$. The associated generating function is $(1-x)^{-2}(1-x^2)^{-1}(1-x^3)^{-3}$, and the multi-set contains two copies of $\{1, 1+1, 1+1+1, \ldots\}$, one in Red and one in Blue; one copy of $\{2, 2+2, 2+2+2, \ldots\}$ in Red; and three copies of $\{3, 3+3, 3+3+3, \ldots\}$ in Red, Blue, and Orange.

We combine Equations (15) and (16) as

$$\prod_{i=1}^{\infty}(1+\mathrm{sg}(r_i)x^i)^{r_i} = \sum_{n=0}^{\infty} \hat{p}_H(n)x^n, \tag{17}$$

where $\hat{p}_H(n)$ is the number of partitions composed from $|r_i|$ copies of M_i, where $M_i = \{i, i+i, i+i+i, \cdots\}$ if $\mathrm{sg}(r_i) = -1$, and $M_i = \{i\}$ if $\mathrm{sg}(r_i) = 1$. As a specific example of Equation (17), let $r_1 = -1$, $r_2 = 2$, and $r_3 = -2$. Then $M_1 = \{1, 1+1, 1+1+1, \cdots\}$ occurs in Red, $M_2 = \{2\}$ occurs in Red and Blue, while $M_3 = \{3, 3+3, 3+3+3, \cdots\}$ occurs in Red and Blue, and the associated generating function is $(1-x)^{-1}(1+x^2)^2(1-x^3)^2$.

Equation (17) is the multi-set generalization of Equation (13). To further generalize Equation (17) we multiply each part i of the multi-set with the weight w_i to form

$$\prod_{i=1}^{\infty}(1+\mathrm{sg}(r_i)w_i x^i)^{r_i} = \sum_{n=0}^{\infty} \hat{p}_H(\bar{w}, n)x^n, \tag{18}$$

where $\hat{p}_H(\bar{w}, n)$ is a polynomial in $\{w_i\}_{i=0}^{\infty}$ such that each $\bar{w}$ is the sum of monomials $w_1^{\alpha_1} w_2^{\alpha_2} \ldots w_m^{\alpha_m}$, where $\sum_{i=1}^{m} i\alpha_1 = n$ and α_m counts the number of times colored part m appears in the partition. If $\mathrm{sg}(r_i) = -1$, there are $|r_i|$ colored copies of the weighted multi-set $M_i = \{w_i i, w_i i + w_i i, w_i i + w_i i + w_i i \ldots\} = \{kw_i i\}_{k=1}^{\infty}$, and each $kw_i i$ is associated with the monomial $w_i^k (x^i)^k = w_i^k x^{ik}$. If $\mathrm{sg}(r_i) = 1$, there are r_i colored copies of the weighted multi-set $M_i = \{w_i i\}$, where $w_i i$ is associated with the monomial $w_i x^i$. In the case of the previous example with $r_1 = -1$, $r_2 = 2$, and $r_3 = -2$, we now have one copy of the weighted multi-set $\{w_{1,}, w_1 + w_1, w_1 + w_1 + w_1, \cdots\}$, two copies of the multi-set $\{2w_2\}$, and two copies of the multi-set, and the generating function is $(1-w_1 x)^{-1}(1+w_2 x^2)^2(1-w_3 x^3)^2$.

The combinatorial interpretations of Equations (11) through (18) originated from the product side of Equation (6). To develop a combinatorial interpretation from the sum side of Equation (6), define $f(x) = 1 - \sum_{n=1}^{\infty} a_n x^n$ where $\{a_n\}_{n=1}^{\infty}$ is a set of *positive* integers. Equation (6) implies that

$$1 - \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty}(1+w_n x^n)^{r_n} \tag{19}$$

Take Equation (19) and form the reciprocal.

$$\frac{1}{1-\sum_{n=1}^{\infty} a_n x^n} = \frac{1}{\prod_{n=1}^{\infty}(1+w_n x^n)^{r_n}} = \prod_{n=1}^{\infty}(1+w_n x^n)^{-r_n}. \tag{20}$$

Equation (20) shows that the reciprocal of $1 - \sum_{n=1}^{\infty} a_n x^n$ is also a $\mathbb{Z}$PPE. Expand the left side of Equation (20) as

$$\begin{aligned} \frac{1}{1-\sum_{n=1}^{\infty} a_n x^n} &= 1 + \sum_{n=1}^{\infty} a_n x^n + \left[\sum_{n=1}^{\infty} a_n x^n\right]^2 + \left[\sum_{n=1}^{\infty} a_n x^n\right]^3 + \cdots + \left[\sum_{n=1}^{\infty} a_n x^n\right]^k + \dots \\ &= 1 + \sum_{n=1}^{\infty} C(n,1)x^n + \sum_{n=2}^{\infty} C(n,2)x^n + \cdots + \sum_{n=k}^{\infty} C(n,k)x^n + \dots \\ &= 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} C(n,k)\right] x^n, \end{aligned}$$

where $C(n,k)$ is a polynomial representation of the compositions of n with exactly k parts such that the part i is represented by a_i and the $+$ is replace by $*$. In other words, $C(n,k)$ is composed of monomials $ca_{i_1}a_{i_2}\dots a_{i_k}$ such that $i_1 + i_2 + \dots i_k$ is a partition of n. Recall that a *composition* of a positive integer n with k parts is a sum $i_1 + i_2 + \dots i_k = n$ where each part i_j is a positive integer with $1 \leq i_j \leq n$. The difference between a partition of n with k parts and a composition of n with k parts is that a composition distinguishes between the order of the parts in the summation [**?**, 2]. Our combinatorial interpretation of $C(n,k)$ is verified via a standard induction argument on k.

Since

$$\frac{1}{1-\sum_{n=1}^{\infty} a_n x^n} = 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} C(n,k)\right] x^n := 1 + \sum_{n=1}^{\infty} C_n x^n, \tag{21}$$

we may interpret C_n to be the sum of all non-trivial polynomial representations of the compositions of n with k parts, i.e. C_n is a polynomial representation of the compositions of $n$where C_n is constructed by taking the set of compositions of n, replacing i with a_i, replacing $+$ with $*$, and summing the monomials. If $a_n = 1$, $C(n,k)$ is the number of compositions of n with k parts, while C_n is the total number of compositions of n. In particular, we find that

$$\begin{aligned} \frac{1}{1-\sum_{n=1}^{\infty} x^n} &= \frac{1}{1-\left(\frac{x}{1-x}\right)} = 1 + \frac{x}{1-x} + \left(\frac{x}{1-x}\right)^2 + \cdots + \left(\frac{x}{1-x}\right)^k + \dots \\ &= 1 + \sum_{n=1}^{\infty} x^n + \left[\sum_{n=1}^{\infty} x^n\right]^2 + \left[\sum_{n=1}^{\infty} x^n\right]^3 + \cdots + \left[\sum_{n=1}^{\infty} x^n\right]^k + \dots \quad . \end{aligned} \tag{22}$$

Define $[\sum_{n=1}^{\infty} x^n]^k = \sum_{l=k}^{\infty} \hat{C}(l,k)x^l$ whenever $k \geq 1$. Clearly $\hat{C}(l,1) = 1$ and a standard induction argument on k shows that $\hat{C}(l,k) = \binom{l-1}{k-1}$. Equation (22) then becomes

$$\begin{aligned}
\frac{1}{1-\sum_{n=1}^{\infty} x^n} &= 1 + \sum_{n=1}^{\infty} x^n + \left[\sum_{n=1}^{\infty} x^n\right]^2 + \left[\sum_{n=1}^{\infty} x^n\right]^3 + \cdots + \left[\sum_{n=1}^{\infty} x^n\right]^k + \dots \\
&= 1 + \sum_{n=1}^{\infty} \hat{C}(n,1)x^n + \sum_{n=2}^{\infty} \hat{C}(n,2)x^n + \cdots + \sum_{n=k}^{\infty} \hat{C}(n,k) + \dots \\
&= 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \hat{C}(n,k)\right] x^n = 1 + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \binom{n-1}{k-1}\right] x^n \\
&= 1 + \sum_{n=1}^{\infty} 2^{n-1}x^n.
\end{aligned}$$

Our calculations have proven of the fact that number of compositions of n is 2^{n-1}, and the number of compositions of n with k parts is $\binom{n-1}{k-1}$. See Example I.6, Page 44 of [2] or Theorem 3.3 of [8]. But more importantly, by combining our observations with Equation (20), we see that the $\mathbb{Z}$PPE $\prod_{n=1}^{\infty}(1+w_n x^n)^{-r_n}$ provides a way of factoring the series $1+\sum_{n=1}^{\infty} C_n x^n$, where C_n is the polynomial representation of the compositions of n.

3. Algebraic Formulas for coefficients of Integer Power Product Expansions

In this section all calculations are done in the context of formal power series and formal power products. For a fixed set of nonzero integers $\{r_n\}_{n=1}^{\infty}$, there are three ways to describe the coefficients of the $\mathbb{Z}$PPE in terms of the coefficients of a given power series. First is Equation (8). An alternative formula for $\{w_n\}_{n=1}^{\infty}$ is found by computing the log of Equation (6). Since $\log(1+w_n x^n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(w_n x^n)^k}{k}$, we observe that

$$\log \prod_{n=1}^{\infty}(1+w_n x^n)^{r_n} = \sum_{n=1}^{\infty} r_n \log(1+w_n x^n) = \sum_{n=1}^{\infty} r_n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(w_n x^n)^k}{k}. \tag{23}$$

Represent $\log f(x) = \sum_{k=1}^{\infty} D_k x^k$. Comparing the coefficient of x^s in this expansion of $\log f(x)$ with the coefficient of x^s provided by the expansion in Equation (23) implies that

$$D_s = \frac{1}{s} \sum_{n:n|s}^{\infty} (-1)^{\frac{s}{n}-1} n r_n w_n^{\frac{s}{n}}. \tag{24}$$

Solving Equation (24) for w_s gives us

$$w_s = \frac{D_s - \frac{1}{s}\sum_{\substack{n|s \\ n\neq s}} (-1)^{\frac{s}{n}-1} n r_n w_n^{\frac{s}{n}}}{r_s}. \tag{25}$$

Although Equations (8) and (25) are useful for explicitly calculating w_n, neither of these formulas reveal the structure property of w_n crucial for determining a lower bound on the radius of convergence of the $\mathbb{Z}$PPE. Take Equation (6), define $a_n = C_{1,n}$, and rewrite it as

$$1 + \sum_{n=1}^{\infty} C_{1,n} x^n = (1 + w_1 x)^{r_1} [1 + \sum_{n=2}^{\infty} C_{2,n} x^n],$$

where $1 + \sum_{n=2}^{\infty} C_{2,n} x^n = \prod_{n=2}^{\infty} (1 + w_n x^n)^{r_n}$. Next write

$$1 + \sum_{n=2}^{\infty} C_{2,n} x^n = (1 + w_2 x^2)^{r_2} [1 + \sum_{n=3}^{\infty} C_{3,n} x^n],$$

where $1 + \sum_{n=3}^{\infty} C_{3,n} x^n = \prod_{n=3}^{\infty} (1 + w_n x^n)^{r_n}$. Continue this process inductively to define

$$1 + \sum_{n=j}^{\infty} C_{j,n} x^n = (1 + w_j x^j)^{r_j} [1 + \sum_{n=j+1}^{\infty} C_{j+1,n} x^n], \tag{26}$$

where $1 + \sum_{n=j}^{\infty} C_{j,n} x^n = \prod_{n=j}^{\infty} (1 + w_n x^n)^{r_n}$ and $1 + \sum_{n=j+1}^{\infty} C_{j+1,n} x^n = \prod_{n=j+1}^{\infty} (1 + w_n x^n)^{r_n}$. By comparing the coefficient of x^j on both sides of Equation (26) we discover that $w_j = \frac{C_{j,j}}{r_j}$ for all j. This fact, along with Equation (26), is the key to proving the following theorem.

Theorem 3.1. *Let j be any positive integer. Define $C_{j,0} = 1$ and $C_{j,N} = 0$ for $1 \leq N \leq j-1$. Let $\{r_n\}_{n=1}^{\infty}$ be a set of nonzero integers. Assume that $C_{j,N} \leq 0$ for all $j \leq N$. Then $C_{j+1,N} \leq 0$ whenever $j+1 \leq N$.*

Proof: Our proof involves two cases.

Case 1: Assume $r_j \geq 1$. Then $(1 + w_j x^j)^{r_j} = (1 + g_g x^j)^{r_j}$, and Equation (26) is equivalent to

$$1 + \sum_{n=j+1}^{\infty} C_{j+1,n} x^n = (1 + g_j x^j)^{-r_j} \left[1 + \sum_{n=j}^{\infty} C_{j,n} x^n\right]. \tag{27}$$

Newton's Binomial Theorem and $\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}$ implies that

$$\begin{aligned} 1+\sum_{n=j+1}^{\infty} C_{j+1,n}x^n &= \left[1+\sum_{k=1}^{\infty}\binom{-r_j}{k}(g_jx^j)^k\right]\left[1+\sum_{n=j}^{\infty}C_{j,n}x^n\right] \\ &= \left[1+\sum_{k=1}^{\infty}(-1)^k\binom{r_j+k-1}{k}(g_jx^j)^k\right]\left[1+\sum_{n=j}^{\infty}C_{j,n}x^n\right] \\ &= \left[1+\sum_{k=1}^{\infty}(-1)^k\binom{r_j+k-1}{k}\frac{C_{j,j}^k}{r_j^k}x^{jk}\right]\left[1+\sum_{n=j}^{\infty}C_{j,n}x^n\right], \end{aligned}$$

where the last equality uses the observation that $w_j = g_j = \frac{C_{j.j}}{r_j}$.

If we compare the coefficient of x^s on both sides of the previous equation we discover that

$$C_{j+1,s} = \sum_{n+jk=s}(-1)^k\frac{\binom{r_j+k-1}{k}}{r_j^k}C_{j,j}^kC_{j,n}. \tag{28}$$

Equation (28) may be rewritten as

$$C_{j+1,s} = A+B, \tag{29}$$

where

$$A := \sum_{\substack{n+jk=s\\ n\neq 0,j}}(-1)^k\frac{\binom{r_j+k-1}{k}}{r_j^k}C_{j,j}^kC_{j,n} \qquad B := \frac{\binom{r_j+\frac{s}{j}-1}{\frac{s}{j}}}{r_j^{\frac{s}{j}}}(-C_{j,j})^{\frac{s}{j}} - \frac{\binom{r_j+\frac{s}{j}-2}{\frac{s}{j}-1}}{r_j^{\frac{s}{j}-1}}(-C_{j,j})^{\frac{s}{j}}.$$

We begin by analyzing the structure of A. If $r_j \geq 1$, then $\frac{\binom{r_j+(k-1)}{k}}{r_j^k} = \frac{(r_j+k-1)(r_j+k-2)\ldots r_j}{k!r_j^k}$ is *always* positive. By hypothesis $C_{j,j} \leq 0$ and $C_{j,n} \leq 0$. Hence $C_{j,j}^kC_{j,n}$ is either zero or has a sign of $(-1)^{k+1}$. Therefore, $(-1)^k\frac{\binom{r_j+k-1}{k}}{r_j^k}C_{j,j}^kC_{j,n}$ is either zero or has a sign of $(-1)^k(-1)^{k+1} = -1$.

We now analyze the structure of B. Unless j is a multiple of s, B vanishes. So assume $\frac{s}{j} = \hat{k}$ where $\hat{k} > 1$. Then

$$
\begin{aligned}
B &= \frac{\binom{r_j+\hat{k}-1}{\hat{k}}}{r_j^{\hat{k}}}(-C_{j.j})^{\hat{k}}(-1)^{\hat{k}-1}\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}}C_{j,j}^{\hat{k}} \\
&= \frac{r_j+\hat{k}-1}{\hat{k}}\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}}}(-C_{j,j})^{\hat{k}} + (-1)^{\hat{k}-1}\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}}C_{j,j}^{\hat{k}} \\
&= (-1)^{\hat{k}-1}\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}}C_{j,j}^{\hat{k}}\left[-\frac{r_j+\hat{k}-1}{r_j\hat{k}}+1\right] \\
&= (-1)^{\hat{k}-1}\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}}C_{j,j}^{\hat{k}}\left[\frac{-r_j-\hat{k}+1+r_j\hat{k}}{r_j\hat{k}}\right] \\
&= (-1)^{\hat{k}-1}\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}}C_{j,j}^{\hat{k}}\left[\frac{(r_j-1)(\hat{k}-1)}{r_j\hat{k}}\right] \qquad (30)
\end{aligned}
$$

If $r_j \geq 1$ then $\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}}$ is positive. By hypothesis $C_{j,j} \leq 0$. Thus, the sign of $C_{j,j}^{\hat{k}}$ is either $(-1)^{\hat{k}}$ or zero, and $(-1)^{\hat{k}-1}\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}}C_{j,j}^{\hat{k}}$ is nonpositive. On the other hand, $r_j \geq 1$, with $\hat{k} > 1$, implies that $\frac{(r_j-1)(\hat{k}-1)}{r_j\hat{k}}$ is positive or zero. The representation of B provided by Equation (30) shows that B is either zero or negative.

Case 2: Assume $r_j \leq -1$; that is r_j is a negative integer which is represented as $-|r_j|$, and $(1+w_jx^j)^{r_j} = (1-h_jx^j)^{-|r_j|}$. Equation (26) is equivalent to

$$
\begin{aligned}
1+\sum_{n=j+1}^{\infty}C_{j+1,n}x^n &= (1-h_jx^j)^{|r_j|}\left[1+\sum_{n=j}^{\infty}C_{j,n}x^n\right] \\
&= \left[1+\sum_{k=1}^{\infty}\binom{|r_j|}{k}(-h_j)^kx^{jk}\right]\left[1+\sum_{n=j}^{\infty}C_{j,n}x^n\right] \\
&= \left[1+\sum_{k=1}^{\infty}(-1)^k\binom{|r_j|}{k}\left(\frac{C_{j,j}}{|r_j|}\right)^kx^{jk}\right]\left[1+\sum_{n=j}^{\infty}C_{j,n}x^n\right],
\end{aligned}
$$

where the last equality follows from the fact that $w_j = -h_j = \frac{C_{j,j}}{-|r_j|}$. If we compare the coefficient of x^s on both sides of the previous equation we find that

$$C_{j+1,s} = \sum_{jk+n=s} (-1)^k \frac{\binom{|r_j|}{k}}{|r_j|^k} C_{j,n} C_{j,j}^k. \tag{31}$$

Equation (31) may be written as

$$C_{j+1,s} = \overline{A} + \overline{B}, \tag{32}$$

where

$$\overline{A} := \sum_{\substack{n+jk=s \\ n\neq 0,j}} (-1)^k \frac{\binom{|r_j|}{k}}{|r_j|^k} C_{j,j}^k C_{j,n}, \qquad \overline{B} := \frac{\binom{|r_j|}{\frac{s}{j}}}{|r_j|^{\frac{s}{j}}} (-C_{j,j})^{\frac{s}{j}} + (-1)^{\frac{s}{j}-1} \frac{\binom{|r_j|}{\frac{s}{j}-1}}{|r_j|^{\frac{s}{j}-1}} C_{j,j}^{\frac{s}{j}}.$$

Since $|r_j|$ is a positive integer, $\frac{\binom{|r_j|}{k}}{r_j^k} \geq 0$. By hypothesis $C_{j,n}C_{j,j}^k$ is the product of $k+1$ nonpositive numbers and is either zero or has a sign of $(-1)^{k+1}$. Thus $(-1)^k C_{j,n} C_{j,j}^k$ is either zero or negative, and $\overline{A}$ is nonpositive.

It remains to show that $\overline{B}$ is also nonpositive. Notice that $\overline{B}$ only exists if $\frac{s}{j}$ is a positive integer, say $\frac{s}{j} = \hat{k}$. Then $\overline{B}$ becomes

$$\overline{B} = (-1)^{\hat{k}} \binom{|r_j|}{\hat{k}} \frac{C_{j,j}^{\hat{k}}}{|r_j|^{\hat{k}}} + (-1)^{\hat{k}-1} \binom{|r_j|}{\hat{k}-1} \frac{C_{j,j}^{\hat{k}}}{|r_j|^{\hat{k}-1}}$$

$$= (-1)^{\hat{k}} \frac{|r_j|}{\hat{k}} \binom{|r_j|-1}{\hat{k}-1} \frac{C_{j,j}^{\hat{k}}}{|r_j|^{\hat{k}}} + (-1)^{\hat{k}-1} \binom{|r_j|}{\hat{k}-1} \frac{C_{j,j}^{\hat{k}}}{|r_j|^{\hat{k}-1}} \tag{33}$$

$$= \frac{(-1)^{\hat{k}-1}}{|r_j|^{\hat{k}-1}} C_{j,j}^{\hat{k}} \left[-\frac{1}{\hat{k}} \binom{|r_j|-1}{\hat{k}-1} + \binom{|r_j|}{\hat{k}-1} \right]$$

$$= \frac{(-1)^{\hat{k}-1}}{|r_j|^{\hat{k}-1}} \binom{|r_j|-1}{\hat{k}-1} C_{j,j}^{\hat{k}} \left[-\frac{1}{\hat{k}} + \frac{|r_j|}{|r_j|-\hat{k}+1} \right]$$

$$= \frac{(-1)^{\hat{k}-1}}{|r_j|^{\hat{k}-1}} \binom{|r_j|-1}{\hat{k}-1} C_{j,j}^{\hat{k}} \left[\frac{(|r_j|+1)(\hat{k}-1)}{\hat{k}(|r_j|-\hat{k}+1)} \right]. \tag{34}$$

Since $|r_j|$ and $\hat{k}$ are positive integers $\binom{|r_j|-1}{\hat{k}-1} \geq 0$. By hypothesis $C_{j,j}^{\hat{k}}$ is either zero or has a sign of $(-1)^{\hat{k}}$. Thus $\frac{(-1)^{\hat{k}-1}}{|r_j|^{\hat{k}-1}} \binom{|r_j|-1}{\hat{k}-1} C_{j,j}^{\hat{k}}$ is nonpositive. It remains to analyze the sign of the rational expression inside the square bracket at (34). The sign of this expression depends only on the sign of $|r_j| - \hat{k} - 1$ since the other three factors are always nonnegative. If $|r_j| - \hat{k} + 1 > 0$, then $|r_j| + 1 > \hat{k}$, and the rational expression is nonnegative. If $|r_j| + 1 - \hat{k} < 0$, then $1 \leq |r_j| < \hat{k} - 1$, which in turn implies that $\binom{|r_j|-1}{\hat{k}-1} = 0$. So once again the quantity at (34) is nonpositive. Only one case remains, that of $|r_j| + 1 = \hat{k}$. Notice that $1 \leq |r_j| = \hat{k} - 1$. Then $\overline{B} = (-1)^{\hat{k}-1} \frac{C_{j,j}^{\hat{k}}}{|r_j|^{\hat{k}-1}}$, a quantity which is either zero or has a sign of $(-1)^{\hat{k}-1}(-1)^{\hat{k}} = -1$. In all three cases we have shown that $\overline{B}$ is nonpositive. $\square$

If we use the notation of [3], we may transform Theorem 3.1 into a theorem about the structure of the $C_{j+1,s}$. Define $\alpha = (j_1, j_2, \ldots, j_n)$ to be a vector with n components where each component is a positive integer. Let $\lambda = \lambda(\alpha)$ be the length of α, i.e. $\lambda = n$. Let $|\alpha|$ denote the sum of the components, namely $|\alpha| = \sum_{s=1}^{n} j_s$. The symbol $C_{j,\alpha}$ represents the expression $C_{j,j_1} C_{j,j_2} \ldots C_{j,j_n}$. For example if $\alpha = (2,3,4,3)$, then $\lambda = 4$, $|\alpha| = 12$, and $C_{j,(2,3,4,3)} = C_{j,2}C_{j,3}C_{j,4}C_{j,3} = C_{j,2}C_{j,3}^2 C_{j,4}$.

Theorem 3.2. *(Structure Property) Let j be a positive integer. Then*

$$C_{j+1,s} = \sum_{l} (-1)^{\lambda(\alpha(l))-1} |c(\alpha(l), j, s)| C_{j,\alpha(l)}, \tag{35}$$

where the sum is over all unordered sequences $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = s$ and at most one $j_i \neq j$. The expression $|c(\alpha(l), j, s)|$ denotes a rational expression in terms of j, s and $|r_j|$ which is nonnegative whenever $|r_j|$ is a positive integer. Furthermore, define $C_{j,\alpha(l)} = C_{j,j_1} C_{j,j_2} \ldots C_{j,j_\lambda}$. If $C_{j,s} \leq 0$ for all nonnegative integers j and all $s \geq j$, Equation (35) is equivalent to

$$C_{j+1,s} = -\sum |c(\alpha(l), j, s)| \left|C_{j,j_1}\right| \left|C_{j,j_2}\right| \cdots \left|C_{j,j_\lambda}\right|, \tag{36}$$

where the sum is over all unordered sequences $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = s$ and at most one $j_i \neq j$.

Proof. If $r_j \geq 1$, we have Equation (29) which says $C_{j+1,s} = A + B$, where

$$A := \sum_{\substack{n+jk=s \\ n \neq 0,j}} (-1)^k \frac{\binom{r_j+k-1}{k}}{r_j^k} C_{j,j} C_{j,n}, \qquad B := \frac{\binom{r_j + \frac{s}{j} - 1}{\frac{s}{j}}}{r_j^{\frac{s}{j}}} (-C_{j,j})^{\frac{s}{j}} + (-1)^{\frac{s}{j}-1} \frac{\binom{r_j + \frac{s}{j} - 2}{\frac{s}{j}}}{r_j^{\frac{s}{j}} - 1} C_{j,j}^{\frac{s}{j}}.$$

For A we represent $C_{j,j}^k C_{j,n}$ as $C_{j,\alpha(l)}$, and $\frac{\binom{r_j+k-1}{k}}{r_j^k}$ as $|c(\alpha(l), j, s)|$. Notice that $(-1)^k = (-1)^{\lambda(\alpha(l))-1}$. For B we combine via Equation (30), let $C_{j,j}^{\hat{k}} = C_{j,\alpha(l)}$, and le $|c(\alpha(l), j, s)| =$

$\frac{\binom{r_j+\hat{k}-2}{\hat{k}-1}}{r_j^{\hat{k}-1}} \frac{(r_j-1)(\hat{k}-1)}{r_j\hat{k}}$.

If $r_j \leq -1$, we have Equation (32) which says $C_{j+1,s} = \overline{A} + \overline{B}$, where

$$\bar{A} := \sum_{\substack{n+jk=s \\ n\neq 0,j}} (-1)^k \frac{\binom{|r_j|}{k}}{|r_j|^k} C_{j,j} C_{j,n}, \qquad B := \frac{\binom{|r_j|}{\frac{s}{j}}}{|r_j|^{\frac{s}{j}}} (-C_{j,j})^{\frac{s}{j}} + (-1)^{\frac{s}{j}-1} \frac{\binom{|r_j|}{\frac{s}{j}}}{|r_j|^{\frac{s}{j}} - 1} C_{j,j}^{\frac{s}{j}}.$$

For $\overline{A}$ we represent $C_{j,j}^k C_{j,n}$ as $C_{j,\alpha(l)}$ and $\frac{\binom{|r_j|}{k}}{|r_j|^k}$ as $|c(\alpha(l), j, s)|$. Notice that $(-1)^k = (-1)^{\lambda(\alpha(l))-1}$. For $\overline{B}$ we combine via Equation (34), let $C_{j,j}^{\hat{k}} = B_{j,\alpha(l)}$, and $|c(\alpha(l), j, s)| = \frac{\binom{|r_j|-1}{\hat{k}-1}}{|r_j|^{\hat{k}-1}} \frac{(|r_j|+1)(\hat{k}-1)}{\hat{k}(|r_j|-\hat{k}+1)} = \frac{(\hat{k}-1)(|r_j|+1)(|r_j|-1)(r_j-2)\ldots(|r_j|-\hat{k}+2)}{|r_j|^{\hat{k}-1}\hat{k}!}$ as long as $|r_j| \neq \hat{k}+1$. If $|r_j| = \hat{k}+1$, then $\overline{B} = (-1)^{\hat{k}-1} \frac{C_{j,j}^{\hat{k}}}{|r_j|^{\hat{k}-1}}$ and $C_{j,j}^{\hat{k}} = C_{j,\alpha(l)}$ while $|c(\alpha(l), j, s)| = \frac{1}{|r_j|^{\hat{k}-1}}$. □

If we take Equation (35) and iterate j times we discover that

$$C_{j+1,s} = \sum_l (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), j, s)| a_{\alpha(l)} = -\sum_l |c(\alpha(l), j, s)||a_{j_1}||a_{j_2}| \ldots |a_{j_\lambda}|, \tag{37}$$

where where the sum is over all $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = s$ and $|c(\alpha(l), j, s)|$ is a rational expression in j, s, and $\{|r_i|\}_{i=1}^{j}$ which is *nonnegative* whenever $|r_i|$ is a positive integer.

If $s = j+1$ Equation (37) becomes

$$\begin{aligned} C_{j+1,j+1} = r_{j+1} w_{j+1} &= \sum_l (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), j)| a_{\alpha(l)} \\ &= -\sum_l |c(\alpha(l), j)||a_{j_1}||a_{j_2}| \ldots |a_{j_\lambda}|, \end{aligned} \tag{38}$$

where the sum is over all unordered sequences $\alpha(l) = (j_1, j_2, \ldots j_\lambda)$ such that $|\alpha(l)| = j+1$. For $\{|r_i|\}_{i=1}^{j+1}$ a set of positive integers, the coefficient $|c(\alpha(l), j)|$ is nonnegative. If $r_{j+1} \geq 1$, Equation (38) implies that $w_{j+1} = g_{j+1}$ is *negative*. If $r_{j+1} \leq -1$, Equation (38) implies that $w_{j+1} = -h_{j+1}$ is *positive*. We explicitly list w_i for $1 \leq i \leq 6$.

$$w_1 = (-1)^0 \frac{1}{r_1} a_1, \qquad w_2 = (-1)^1 \frac{r_1 - 1}{2r_1 r_2} a_1^2 + (-1)^0 \frac{1}{r_2} a_2$$

$$w_3 = (-1)^2 \frac{r_1^2 - 1}{3r_1^2 r_3} a_1^3 + (-1)^1 \frac{1}{r_3} a_1 a_2 + (-1)^0 \frac{1}{r_3} a_3$$

$$w_4 = (-1)^1 \frac{r_2 - 1}{2r_2 r_4} a_2^2 + (-1)^2 \frac{1 + r_1(2r_2 - 1)}{2r_1 r_2 r_4} a_1^2 a_2 + (-1)^3 \frac{-2r_2 + 2r_1^3 r_2 - r_1^3 + 2r_1^2 - r_1}{8r_1^3 r_2 r_4} a_1^4 + (-1)^1 \frac{1}{r_4} a_1 a_3 + (-1)^0 \frac{1}{r_4} a_4$$

$$w_5 = (-1)^2 \frac{1}{r_5} a_1^2 a_3 + (-1)^1 \frac{1}{r_5} a_2 a_3 + (-1)^2 \frac{1}{r_5} a_1 a_2^2 + (-1)^3 \frac{1}{r_5} a_1^3 a_2 + (-1)^1 \frac{1}{r_5} a_1 a_4 + (-1)^4 \frac{r_1^4 - 1}{5r_1^4 r_5} a_1^5 + (-1)^0 \frac{1}{r_5} a_5$$

$$\begin{aligned} w_6 = {} & (-1)^2 \frac{1}{r_6} a_1^2 a_4 + (-1)^1 \frac{1}{r_6} a_2 a_4 + (-1)^1 \frac{r_3 - 1}{2r_3 r_6} a_3^2 \\ & + (-1)^3 \frac{-r_1^2 + 3r_1^2 r_3 + 1}{3r_1^2 r_3 r_6} a_1^3 a_3 + (-1)^2 \frac{2r_3 - 1}{r_3 r_6} a_1 a_2 a_3 \\ & + (-1)^2 \frac{r_2^2 - 1}{3r_2^2 r_6} a_2^3 + (-1)^3 \frac{-r_1 r_3 + 3r_1 r_2^2 r_3 - r_1 r_2^2 + r_3}{2r_1 r_2^2 r_3 r_6} a_1^2 a_2^2 + (-1)^0 \frac{1}{r_6} a_6 \\ & + (-1)^4 \frac{4r_2^2 - 4r_1^2 r_2^2 - 3r_1^2 r_3 + 6r_1 r_3 + 12r_1^2 r_2^2 r_3 - 3r_3}{12r_1^2 r_2^2 r_3 r_6} a_1^4 a_2 + (-1)^1 \frac{1}{r_6} a_1 a_5 \\ & + (-1)^5 \frac{12r_1^5 r_2^2 r_3 - 9r_1^3 r_3 + 3r_1^2 r_3 - 12r_2^2 r_3 - 3r_1^5 r_3 + 9r_3 r_1^4 - 4r_1^5 r_2^2 + 8r_2^2 r_1^3 - 4r_1 r_2^2}{72r_1^5 r_1^2 r_3 r_6} a_1^6 \end{aligned}$$

4. Convergence Criteria For Integer Power Products

Let $\{r_n\}_{n=1}^{\infty}$ be a set of nonzero integers. The structure of w_j provided by Equation (38) allows us to prove the following theorem.

Theorem 4.1. *Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$. Let $\{r_n\}_{n=1}^{\infty}$ be a given set of nonzero integers. Then $f(x)$ has $\mathbb{Z}$PPE*

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = \prod_{n=1}^{\infty} (1 + w_n x^n)^{r_n}. \tag{39}$$

Consider the auxiliary functions

$$C(x) = 1 - \sum_{n=1}^{\infty} |a_n| x^n = \prod_{n=1}^{\infty} \left(1 - \mathrm{sg}(r_n) \widehat{W}_n x^n\right)^{r_n}, \tag{40}$$

$$M(x) = 1 - \sum_{n=1}^{\infty} M_n x^n = \prod_{n=1}^{\infty} \left(1 - \mathrm{sg}(r_n) E_n x^n\right)^{r_n}. \tag{41}$$

where $\mathrm{sg}(r_n)$ *is defined via Equation (14). Assume that* $|a_n| \leq M_n$ *for all* n. *Then* $|w_n| \leq \widehat{W}_n \leq E_n$ *for all* n.

Proof: By Equation (38) we have

(42)
$$w_n = \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), n)| a_{\alpha(l)} = \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))+1} |c(\alpha(l), n)| a_{j_1} a_{j_2} \ldots a_{j_\lambda},$$

Equation (42) implies that

(43)
$$|w_n| = \left| \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l)+1)} |c(\alpha(l), n)| a_{j_1} a_{j_2} \ldots a_{j_\lambda} \right| \leq \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)| |a_{j_1}| |a_{j_2}| \ldots |a_{j_\lambda}|.$$

Equation (38) when applied to Equation (40) implies that

$$\begin{aligned} 0 \leq \widehat{W}_n &= \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(\alpha(l))} |c(\alpha(l), n)| (-|a_{j_1}|)(-|a_{j_2}|) \ldots (-|a_{j_\lambda}|) \\ &= \sum_{l:|\alpha(l)|=n} (-1)^{\lambda(2\alpha(l))} |c(\alpha(l), n)| (|a_{j_1}|)(|a_{j_2}|) \ldots (|a_{j_\lambda}|) \\ &= \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)| |a_{j_1}| |a_{j_2}| \ldots |a_{j_\lambda}|. \end{aligned} \tag{44}$$

Combining Equations (43) and (44) shows that $|w_n| \leq \widehat{W}_n$. Since $|a_n| \leq M_n$ we also have

$$0 \leq \widehat{W}_n = \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)| |a_{j_1}| |a_{j_2}| \ldots |a_{j_\lambda}| \leq \sum_{l:|\alpha(l)|=n} |c(\alpha(l), n)| M_{j_1} M_{j_2} \ldots M_{j_\lambda} = E_n,$$

where the last equality follows from Equation (38). Thus $\widehat{W}_n \leq E_n$. □

We now work with a particular case of $M(x)$, namely

$$M(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \prod_{n=1}^{\infty} \left(1 - \mathrm{sg}(r_n) E_n x^n\right)^{r_n}, \qquad s := \sup_{n \geq 1} |a_n|^{\frac{1}{n}}. \tag{45}$$

We want to determine when the $\mathbb{Z}$PPE of Equation (45) will absolutely convergent. Recall that

$$\log(1 - \mathrm{sg}(r_n) E_n x^n)^{r_n} = r_n \log\left(1 - \mathrm{sg}(r_n) E_n x^n\right) = -r_n \sum_{l=1}^{\infty} \frac{(\mathrm{sg}(r_n) E_n x^n)^l}{l}.$$

Then

(46)
$$\log \prod_{n=1}^{\infty} (1-\mathrm{sg}(r_n)E_n x^n)^{r_n} = \sum_{n=1}^{\infty} r_n \log(1-\mathrm{sg}(r_n)E_n x^n) = -\sum_{n=1}^{\infty} r_n \sum_{l=1}^{\infty} \frac{(\mathrm{sg}(r_n)E_n x^n)^l}{l}.$$

Equation (46) implies that if the double series is absolutely convergent, then both $\sum_{n=1}^{\infty} r_n \log(1-\mathrm{sg}(r_n)E_n x^n)$ and $r_n \log(1-\mathrm{sg}(r_n)E_n x^n)$ are absolutely convergent. Furthermore, the absolute convergence of the double series implies the absolute convergence of $\prod_{n=1}^{\infty} (1-\mathrm{sg}(r_n)E_n x^n)^{r_n}$ since

$$e^{\sum_{n=1}^{\infty} r_n \log(1-\mathrm{sg}(r_n)E_n x^n)} = e^{\sum_{n=1}^{\infty} \log(1-\mathrm{sg}(r_n)E_n x^n)^{r_n}} = \prod_{n=1}^{\infty} (1-\mathrm{sg}(r_n)E_n x^n)^{r_n}.$$

Thus it suffices to investigate the absolute convergence of $\sum_{n=1}^{\infty} r_n \log(1-\mathrm{sg}(r_n)E_n x^n)$.

If we take the logarithm of Equation (45) we find that

$$\sum_{n=1}^{\infty} r_n \log(1-\mathrm{sg}(r_n)E_n x^n) = \log\left(1-\sum_{n=1}^{\infty} s^n x^n\right). \tag{47}$$

Now

$$1-\sum_{n=1}^{\infty} s^n x^n = 1 - sx\sum_{n=0}^{\infty}(sx)^n = 1-\frac{sx}{1-sx} = \frac{1-2sx}{1-sx}.$$

Therefore,

$$\begin{aligned}\log\left(\frac{1-2sx}{1-sx}\right) &= \log(1-2sx)-\log(1-sx)\\ &= -\sum_{n=1}^{\infty}\frac{(2sx)^n}{n} + \sum_{n=1}^{\infty}\frac{(sx)^n}{n} = \sum_{n=1}^{\infty}\frac{1-2^n}{n}(sx)^n.\end{aligned}$$

By the Ratio Test we know that $\sum_{n=1}^{\infty} \frac{1-2^n}{n}(sx)^n$ absolutely converges whenever $\lim_{n\to\infty}\left|\frac{n(1-2^{n+1})}{(n+1)(1-2^n)}\right| |sx| < 1$. This is ensured by requiring $|x| < \frac{1}{2s}$.

We have shown that $\sum_{n=1}^{\infty} r_n \log(1-\mathrm{sg}(r_n)E_n x^n)$, and thus $\prod_{n=1}^{\infty} (1-\mathrm{sg}(r_n)E_n x^n)^{r_n}$ will be absolutely convergent whenever $|x| < \frac{1}{2s}$. We claim this information provides a lower bound on the range of absolute convergence for the ℤPPE of Equation (39) since

$$\begin{aligned}\left|\log\prod_{n=1}^{\infty}(1+w_nx^n)^{r_n}\right| &= \left|\sum_{n=1}^{\infty} r_n\log(1+w_nx^n)\right| \le \sum_{n=1}^{\infty}|r_n|\,|\log(1+w_nx^n)| \\ &= \sum_{n=1}^{\infty}|r_n|\left|\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(w_nx^n)^k}{k}\right| \le \sum_{n=1}^{\infty}|r_n|\sum_{k=1}^{\infty}\frac{(|w_n||x|^n)^k}{k} \\ &\le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|r_n|\frac{(E_n|x|^n)^k}{k},\end{aligned}$$

where the last inequality follows by Theorem 4.1. These calculations implies that if $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r_n\frac{(\mathrm{sg}(r_n)E_nx^n)^k}{k}$, and hence $\sum_{n=1}^{\infty} r_n\log\left(1-\mathrm{sg}(r_n)E_nx^n\right)$, are absolutely convergent, then $\sum_{n=1}^{\infty} r_n\log(1+w_nx^n)$ and $\prod_{n=1}^{\infty}(1+w_nx^n)^{r_n}$ will *also* be absolutely convergent. We summarize our conclusions in the following theorem.

Theorem 4.2. *Let $f(x) = 1+\sum_{n=1}^{\infty} a_nx^n$. Let $\{r_n\}_{n=1}^{\infty}$ be a given set of nonzero integers. Define $s := \sup_{n\ge1}|a_n|^{\frac{1}{n}}$. Then both $f(x)$ and its ℤPPE,*

$$f(x) = 1+\sum_{n=1}^{\infty} a_nx^n = \prod_{n=1}^{\infty}(1+w_nx^n)^{r_n},$$

and the auxiliary function, along with its ℤPPE,

$$M(x) = 1-\sum_{n=1}^{\infty} s^nx^n = \prod_{n=1}^{\infty}(1-\mathrm{sg}(r_n)E_nx^n)^{r_n}, \tag{48}$$

will be absolutely convergent whenever $|x| < \frac{1}{2s}$.

We now provide an asymptotic estimate for the majorizing GIPPE of Equation (48).

Theorem 4.3. *Let $f(x) = 1-\sum_{n=1}^{\infty} s^nx^n = \frac{1-2sx}{1-sx}$ where $s > 0$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of nonzero integers. For this particular $f(x)$ and its associated ℤPPE $\prod_{n=1}^{\infty}(1+w_nx^n)^{r_n}$ we have*

$$r_nw_n \sim \frac{(1-2^n)s^n}{n}, \quad n\to\infty. \tag{49}$$

To prove Theorem 4.3 we need the following lemma.

Lemma 4.4. *Let $f(x) = 1-\sum_{n=1}^{\infty} s^nx^n = \frac{1-2sx}{1-sx}$ where $s > 0$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of nonzero integers. For this particular $f(x)$ and its associated ℤPPE $\prod_{n=1}^{\infty}(1+w_nx^n)^{r_n}$ there exists α with $1<\alpha<2$ such that*

$$m|r_m|\,|w_m| \le \alpha 2^m s^m. \tag{50}$$

Proof: A straightforward calculation shows that $\frac{m|r_m||w_m|}{(2s)^m} \le 1.691$ whenever $1\le m\le 30$. To prove Equation (50) for arbitrary m assume inductively that $j|r_j|\,|w_j| \le \alpha 2^j s^j$ is true for $1 \leq j < m$. Our analysis shows that we may assume $m \geq 16$. Take Equation (24) and write it as

$$mD_m + \sum_{\substack{n|m \\ n \neq m}} (-1)^{\frac{m}{n}} n r_n w_n^{\frac{m}{n}} = m r_m w_m. \tag{51}$$

Since $f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n = \frac{1-2sx}{1-sx}$,

$$\log f(x) = \log\left(\frac{1-2sx}{1-sx}\right) = \sum_{k=1}^{\infty} \frac{-(2^k-1)s^k}{k} x^k = \sum_{m=1}^{\infty} D_m x^m,$$

and we deduce that that $D_m = \frac{-(2s)^m(1-2^{-m})}{m}$.

Take Equation (51) and write it as

$$m\left[D_m + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + \Delta\right] = m r_m w_m,$$

where

$$T_j := \frac{(-1)^{\frac{m}{j}} j r_j}{m} (w_j)^{\frac{m}{j}}, \qquad 1 \leq j \leq 7, \qquad \Delta := \frac{1}{m} \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} (-1)^{\frac{m}{n}} n r_n w_n^{\frac{m}{n}}.$$

The range of summation of Δ implies that $m \geq 16$. In order to prove Equation (50) it suffices to show that

$$\begin{aligned} \frac{m|r_m||w_m|}{(2s)^m} &= \frac{m}{(2s)^m} \left|D_m + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + \Delta\right| \\ &\leq \frac{m}{(2s)^m} \left[|D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta|\right] < 2, \end{aligned} \tag{52}$$

whenever $m \geq 16$. We must approximate $\frac{m}{(2s)^m}|D_m|$, $\frac{m}{(2s)^m}|T_j|$ for $1 \leq j \leq 7$, and $\frac{m}{(2s)^m}|\Delta|$. Begin with $\frac{m}{(2s)^m}|D_m|$ and observe that

$$\frac{m}{(2s)^m}|D_m| = \frac{m}{(2s)^m} \cdot \frac{(2s)^m(1-2^{-m})}{m} < 1. \tag{53}$$

We now work with $\frac{m}{(2s)^m}|T_j|$. Take the formulas for w_j provided at the end of previous section, let $a_i = -s^i$, and simplify the results to find that

$$w_1 = -\frac{s}{r_1} \qquad w_2 = -\frac{s^2(3r_1-1)}{2r_1r_2} \qquad w_3 = -\frac{s^3(7r_1^2-1)}{3r_1^2r_3} \qquad w_7 = -\frac{s^7(127r_1^6-1)}{7r_1^6r_7}$$

$$w_4 = -\frac{s^4(-9r_1^3+30r_1^3r_2+6r_1^2-2r_2-r_1)}{8r_1^3r_2r_4} \qquad w_5 = -\frac{s^5(31r_1^4-1)}{5r_1^4r_5}$$

$$w_6 = -\frac{s^6(-4r_1r_2^2-12r_2^2r_3+3r_1^2r_3+56r_2^2r_1^3+81r_3r_1^4+756r_1^5r_2^2r_3-196r_1^5r_2^2-81r_1^5r_3-27r_1^3r_3)}{72r_1^5r_2^2r_3r_6}.$$

We use this data to approximate $\frac{m}{(2s)^m}|T_j|$ for $1 \le j \le 7$. When doing the approximations recall that r_j is a *nonzero integer* for all j and that $m \ge 16$.

$$\frac{m}{(2s)^m}|T_1| = \frac{|r_1|}{(2s)^m}\left(\frac{s}{|r_1|}\right)^m = \frac{1}{2(2|r_1|)^{m-1}} \le \frac{1}{2^m} \le \frac{1}{2^{16}} \le 0.000016 \tag{54}$$

$$\frac{m}{(2s)^m}|T_2| = \frac{2|r_2|}{4^{\frac{m}{2}}}\left|\frac{3r_1-1}{2r_1r_2}\right|^{\frac{m}{2}} = 2|r_2|\left|\frac{3r_1-1}{8r_1r_2}\right|\left|\frac{3r_1-1}{8r_1r_2}\right|^{\frac{m}{2}-1} = \left|\frac{3r_1-1}{4r_1}\right|\left|\frac{3r_1-1}{8r_1r_2}\right|^{\frac{m}{2}-1}$$

$$\le \left(\frac{3}{4}+\frac{1}{4|r_1|}\right)\left(\frac{3+|r_1|^{-1}}{8|r_2|}\right)^{\frac{m}{2}-1} \le \left(\frac{1}{2}\right)^{\frac{16}{2}-1} \le \left(\frac{1}{2}\right)^7 = .0078125 \tag{55}$$

When approximating $\frac{m}{(2s)^m}|T_3|$ use the fact that $T_3 = 0$ if $3 \nmid m$.

$$\frac{m}{(2s)^m}|T_3| = \frac{3|r_3|}{2^m}\left|\frac{7r_1^2-1}{3r_1^2r_3}\right|^{\frac{m}{3}} = \frac{3|r_3|}{8^{\frac{m}{3}}}\left|\frac{7r_1^2-1}{3r_1^2r_3}\right|^{\frac{m}{3}} = 3|r_3|\left|\frac{7r_1^2-1}{24r_1^2r_3}\right|^{\frac{m}{3}}$$

$$= \left|\frac{7r_1^2-1}{8r_1^2}\right|\left|\frac{7r_1^2-1}{24r_1^2r_3}\right|^{\frac{m}{3}-1} \le \frac{7}{8}\left(\frac{8}{24}\right)^{\frac{m}{3}-1} \le \frac{7}{8}\left(\frac{1}{3}\right)^{\frac{18}{3}-1} = \frac{7}{8}\left(\frac{1}{3}\right)^5 \le 0.00361 \tag{56}$$

$$\frac{m}{(2s)^m}|T_4| = \frac{4|r_4|}{(2^4)^{\frac{m}{4}}}\left|\frac{-9r_1^3+30r_1^3r_2+6r_1^2-2r_2-r_1}{8r_1^3r_2r_4}\right|^{\frac{m}{4}} = 4|r_4|\left|\frac{-9r_1^3+30r_1^3r_2+6r_1^2-2r_2-r_1}{2^7r_1^3r_2r_4}\right|^{\frac{m}{4}}$$

$$= \left|\frac{-9r_1^3+30r_1^3r_2+6r_1^2-2r_2-r_1}{2^5r_1^3r_2}\right|\left|\frac{-9r_1^3+30r_1^3r_2+6r_1^2-2r_2-r_1}{2^7r_1^3r_2r_4}\right|^{\frac{m}{4}-1}$$

$$\le \frac{48}{2^5}\left(\frac{48}{2^7}\right)^{\frac{m}{4}-1} \le \frac{3}{2}\left(\frac{3}{8}\right)^{\frac{16}{4}-1} = \frac{3}{2}\left(\frac{3}{8}\right)^3 \le 0.08 \tag{57}$$

When approximating $\frac{m}{(2s)^m}|T_5|$ use the fact that $T_5 = 0$ if $5 \nmid m$.

$$\frac{m}{(2s)^m}|T_5| = \frac{5|r_5|}{(2^5)^{\frac{m}{5}}}\left|\frac{31r_1^4-1}{5r_1^4r_5}\right|^{\frac{m}{5}} = 5|r_5|\left|\frac{31r_1^4-1}{2^5 5r_1^4r_5}\right|^{\frac{m}{5}} = \left|\frac{31r_1^4-1}{2^5r_1^4}\right|\left|\frac{31r_1^4-1}{2^5 5r_1^4r_5}\right|^{\frac{m}{5}-1}$$

$$\leq \frac{31}{32}\left(\frac{32}{160}\right)^{\frac{m}{5}-1} \leq \frac{31}{32}\left(\frac{32}{160}\right)^{\frac{20}{5}-1} = \frac{31}{32}\left(\frac{32}{160}\right)^{3} = 0.00775 \tag{58}$$

When approximating $\frac{m}{(2s)^m}|T_6|$ use the fact that $T_6 = 0$ if $6 \nmid m$.

$$\frac{m}{(2s)^m}|T_6| =$$

$$\frac{6|r_6|}{(2^6)^{\frac{m}{6}}}\left|\frac{-4r_1r_2^2-12r_2^2r_3+3r_1^2r_3+56r_2^2r_1^3+81r_3r_1^4+756r_1^5r_2^2r_3-196r_1^5r_2^2-81r_1^5r_3-27r_1^3r_3}{72r_1^5r_2^2r_3r_6}\right|^{\frac{m}{6}}$$

$$= 6|r_6|\left|\frac{-4r_1r_2^2-12r_2^2r_3+3r_1^2r_3+56r_2^2r_1^3+81r_3r_1^4+756r_1^5r_2^2r_3-196r_1^5r_2^2-81r_1^5r_3-27r_1^3r_3}{3^2 2^9 r_1^5r_2^2r_3r_6}\right|^{\frac{m}{6}}$$

$$= \left|\frac{-4r_1r_2^2-12r_2^2r_3+3r_1^2r_3+56r_2^2r_1^3+81r_3r_1^4+756r_1^5r_2^2r_3-196r_1^5r_2^2-81r_1^5r_3-27r_1^3r_3}{3^1 2^8 r_1^5r_2^2r_3}\right| *$$

$$\left|\frac{-4r_1r_2^2-12r_2^2r_3+3r_1^2r_3+56r_2^2r_1^3+81r_3r_1^4+756r_1^5r_2^2r_3-196r_1^5r_2^2-81r_1^5r_3-27r_1^3r_3}{3^2 2^9 r_1^5r_2^2r_3r_6}\right|^{\frac{m}{6}-1}$$

(59)

$$\leq \frac{1216}{3^1 2^8}\left(\frac{1216}{3^2 2^9}\right)^{\frac{18}{6}-1} \leq 0.111$$

When approximating $\frac{m}{(2s)^m}|T_7|$ use the fact that $T_7 = 0$ if $7 \nmid m$.

$$\frac{m}{(2s)^m}|T_7| = \frac{7|r_7|}{(2^7)^{\frac{m}{7}}}\left|\frac{127r_1^6-1}{7r_1^6r_7}\right|^{\frac{m}{7}} = 7|r_7|\left|\frac{127r_1^6-1}{2^7 7r_1^6r_7}\right|^{\frac{m}{7}} = \left|\frac{127r_1^6-1}{2^7r_1^6}\right|\left|\frac{127r_1^6-1}{2^7 7r_1^6r_7}\right|^{\frac{m}{7}-1}$$

$$= \frac{127}{2^7}\left(\frac{128}{7^1 2^7 r_7}\right)^{\frac{m}{7}-1} \leq \frac{127}{2^7}\left(\frac{128}{7^1 2^7}\right)^{\frac{21}{7}-1} = \frac{127}{2^7}\left(\frac{128}{7^1 2^7}\right)^{2} \leq 0.021 \tag{60}$$

It remains to approximate $\frac{m}{(2s)^m}|\Delta|$. Here is where we make use of the induction hypothesis. We also use the fact the $\frac{m}{2} \geq n$ implies $\frac{m}{n} \geq 2$. By definition we have

$$\frac{m}{(2s)^m}|\Delta| \leq \frac{1}{(2s)^m} \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} n|r_n||w_n|^{\frac{m}{n}} \leq \frac{1}{(2s)^m} \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} n|r_n| \left(\frac{\alpha 2^n s^n}{n|r_n|}\right)^{\frac{m}{n}}$$

$$= \alpha \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} \frac{n|r_n|}{\alpha} \left(\frac{1}{\frac{n|r_n|}{\alpha}}\right)^{\frac{m}{n}}$$

$$= \alpha \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} \left(\frac{\alpha}{n|r_n|}\right)^{\frac{m}{n}-1} \leq \alpha \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} \left(\frac{\alpha}{n}\right)^{\frac{m}{n}-1} \leq \alpha \sum_{\substack{n|m \\ \frac{m}{2} \geq n \geq 8}} \left(\frac{\alpha}{8}\right)^{\frac{m}{n}-1}$$

$$(61) \qquad \leq \alpha \sum_{\frac{m}{n} \geq 2} \left(\frac{\alpha}{8}\right)^{\frac{m}{n}-1} = \alpha \left[\frac{\frac{\alpha}{8}}{1-\frac{\alpha}{8}}\right] = \alpha \left[\frac{\alpha}{8-\alpha}\right] \leq \alpha \left[\frac{2}{8-2}\right] = \frac{\alpha}{3} \leq \frac{2}{3}$$

We now take Equations (53) through Equation (61) and place them in $\frac{m|r_m||w_m|}{(2s)^m} \leq \frac{m}{(2s)^m}\left[|D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta|\right]$ to find that

$$\frac{m|r_m||w_m|}{(2s)^m} \leq \frac{m}{(2s)^m}\left[|D_m| + |T_1| + |T_2| + |T_3| + |T_4| + |T_5| + |T_6| + |T_7| + |\Delta|\right]$$
$$\leq 1 + 0.000016 + 0.0078125 + 0.00361 + 0.08 + 0.00775 + 0.111 + 0.021 + \frac{2}{3}$$
$$\leq 1.9 < 2.$$

Equation (52) is valid and our proof is complete. $\square$

Proof of Theorem 4.3: Equation (24) implies that

$$(62) \qquad kr_k w_k = kD_k + (-1)^k r_1 w_1^k + \sum_{\substack{n|k \\ \frac{k}{2} \geq n \geq 2}} (-1)^{\frac{k}{n}} n r_n w_n^{\frac{k}{n}}.$$

For $f(x) = 1 - \sum_{n=1}^{\infty} s^n x^n$ we have $D_k = \frac{-(2s)^k(1-2^{-k})}{k}$. Thus Equation (62) is equivalent to

$$(63) \qquad kr_k w_k = -(2s)^k(1-2^{-k}) + (-1)^k r_1 \left(-\frac{s}{r_1}\right)^k + \sum_{\substack{n|k \\ \frac{k}{2} \geq n \geq 2}} (-1)^{\frac{k}{n}} n r_n w_n^{\frac{k}{n}}.$$

Define

$$T_1 := (-2^k+1)s^k, \qquad T_2 := r_1\left(\frac{s_1}{r_1}\right)^k, \qquad \Delta := \sum_{\substack{n|k \\ \frac{k}{2}\geq n\geq 2}} (-1)^{\frac{k}{n}} n r_n w_n^{\frac{k}{n}}.$$

Equation (63) becomes $kr_kw_k = T_1 + T_2 + \Delta$. Lemma 4.4 implies there exist α with $1 < \alpha < 2$ such that

$$n\,|w_n| \leq n|r_n|\,|w_n| \leq \alpha 2^n s^n. \tag{64}$$

By definition

$$|\Delta| = \left|\sum_{\substack{n|k \\ k>n>1}} (-1)^{\frac{k}{n}} n r_n w_n^{\frac{k}{n}}\right| \leq \sum_{\substack{n|k \\ \frac{k}{2}\geq n\geq 2}} n|r_n||w_n|^{\frac{k}{n}} \leq \sum_{\substack{n|k \\ \frac{k}{2}\geq n\geq 2}} n|r_n|\left[\frac{\alpha 2^n s^n}{n|r_n|}\right]^{\frac{k}{n}}$$

$$= \alpha(2s)^k \sum_{\substack{n|k \\ \frac{k}{2}\geq n\geq 2}} \left(\frac{n|r_n|}{\alpha}\right)\frac{1}{\left(\frac{n|r_n|}{\alpha}\right)^{\frac{k}{n}}} = \alpha(2s)^k \sum_{\substack{n|k \\ \frac{k}{2}\geq n\geq 2}} \frac{1}{\left(\frac{n|r_n|}{\alpha}\right)^{\frac{k}{n}-1}}$$

$$\leq \alpha(2s)^k \sum_{\substack{n|k \\ \frac{k}{2}\geq n\geq 2}} \frac{1}{\left(\frac{n}{\alpha}\right)^{\frac{k}{n}-1}} \leq \alpha(2s)^k \sum_{\frac{k}{2}\geq n\geq 2} \frac{1}{\left(\frac{n}{\alpha}\right)^{\frac{k}{n}-1}}$$

$$= \alpha(2s)^k\left[\frac{1}{\left(\frac{2}{\alpha}\right)^{\frac{k}{2}-1}} + \frac{2\alpha}{k} + \sum_{\frac{k}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{\alpha}\right)^{\frac{k}{n}-1}}\right]$$

$$\leq \alpha(2s)^k\left[\frac{1}{\left(\frac{2}{\alpha}\right)^{\frac{k}{2}-1}} + \frac{2\alpha}{k} + \sum_{\frac{k}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k}{n}-1}}\right], \tag{65}$$

where the last equality reflects the fact that $\frac{1}{2} < \frac{1}{\alpha} < 1$.

Define $M := \sum_{\frac{k}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k}{n}-1}} = \dfrac{1}{\left(\frac{3}{2}\right)^{\frac{k}{3}-1}} + \dfrac{1}{\left(\frac{4}{2}\right)^{\frac{k}{4}-1}} + \dfrac{1}{\left(\frac{5}{2}\right)^{\frac{k}{5}-1}} + \sum_{\frac{k}{3}\geq n\geq 6}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k}{n}-1}}$ and $b(n,k) := -\log[\left(\frac{n}{2}\right)^{\frac{k}{n}-1}] = -(\frac{k}{n}-1)\log\frac{n}{2}$. Then

$$\frac{\partial b(n,k)}{\partial n} = \frac{k}{n^2}\log\frac{n}{2} - \frac{1}{n}(\frac{k}{n}-1) = \frac{k}{n}\left[\frac{1}{n}[\log\frac{n}{2}-1] + \frac{1}{k}\right] > 0, \quad n \geq 6 \tag{66}$$

Equation (66) shows that $b(n,k)$ is increasing in n whenever $n \geq 6$. Hence

$$b(n,k) < b(\frac{k}{3},k) = -(3-1)\log\left(\frac{k}{6}\right) = -2\log\left(\frac{k}{6}\right),$$

and each term in $\sum_{\frac{k}{3}\geq n\geq 6}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k}{n}-1}}$ satisfies $e^{b(n,k)}\leq e^{-2\log\left(\frac{k}{3}\right)}=\frac{36}{k^2}$. Therefore

$$\begin{aligned}
\sum_{\frac{k}{3}\geq n\geq 3}\frac{1}{\left(\frac{n}{2}\right)^{\frac{k}{n}-1}} &\leq \frac{1}{\left(\frac{3}{2}\right)^{\frac{k}{3}-1}}+\frac{1}{\left(\frac{4}{2}\right)^{\frac{k}{4}-1}}+\frac{1}{\left(\frac{5}{2}\right)^{\frac{k}{5}-1}}+\sum_{\frac{k}{3}\geq n\geq 6}\frac{36}{k^2}\\
&\leq \frac{1}{\left(\frac{3}{2}\right)^{\frac{k}{3}-1}}+\frac{1}{\left(\frac{4}{2}\right)^{\frac{k}{4}-1}}+\frac{1}{\left(\frac{5}{2}\right)^{\frac{k}{5}-1}}+k\frac{36}{k^2}\\
&= \frac{1}{\left(\frac{3}{2}\right)^{\frac{k}{3}-1}}+\frac{1}{\left(\frac{4}{2}\right)^{\frac{k}{4}-1}}+\frac{1}{\left(\frac{5}{2}\right)^{\frac{k}{5}-1}}+\frac{36}{k}. \qquad (67)
\end{aligned}$$

These calculations imply that $\lim_{k\to\infty}M=0$. By combining Equation (65) with Equation (67) we deduce that Th

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{\Delta}{(-2^k+1)s^k}\right| &= \lim_{k\to\infty}\frac{|\Delta|}{|(-1+2^{-k})|(2s)^k}\\
&= \lim_{k\to\infty}\frac{\alpha(2s)^k}{|(-1+2^{-k})|(2s)^k}\left[\frac{1}{\left(\frac{2}{\alpha}\right)^{\frac{k+1}{2}-1}}+\frac{2\alpha}{k}+M\right]=0.
\end{aligned}$$

Hence $\lim_{k\to\infty}\frac{\Delta}{(-2^k+1)s^k}=0.$

We return to Equation (63) and observe that

$$\begin{aligned}
r_k w_k &= \frac{T_1}{k}+\frac{T_2}{k}+\frac{\Delta}{k}\\
&= \frac{(-2^k+1)s^k}{k}+\frac{r_1\left(\frac{s}{r_1}\right)^k}{k}+\frac{\Delta}{k}\\
&= \frac{(-2^k+1)s^k}{k}\left[1-\frac{1}{r_1^{k-1}(-2^k+1)}-\frac{\Delta}{(-2^k+1)s^k}\right]\\
&= \frac{(-2^k+1)s^k}{k}\left[1+o(1)\right]=kD_k\left[1+o(1)\right]. \qquad \square
\end{aligned}$$

Remark 4.5. *Theorem 4.3 provides an asymptotic bound for the weights assigned to underlying colored multi-set of Equation (18).*

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