MIXED r-STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. The Stirling number of the second kind $\left\{ {n \atop k} \right\}$ counts the number of ways to partition a set of n labeled balls into k non-empty unlabeled cells. We extend this problem and give a new statement of the r-Stirling numbers of the second kind and r-Bell numbers. We also introduce the *r-mixed Stirling number of the second kind and r-mixed Bell numbers*. As an application of our results we obtain a formula for the number of ways to write an integer $m > 0$ in the form $m_1 \cdot m_2 \cdot \ldots \cdot m_k$, where $k \geqslant 1$ and m_i's are positive integers greater than 1.

1. INTRODUCTION

Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$, is the number of partitions of a set with n distinct elements into k disjoint non-empty sets. The recurrence relation

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\},$$

is valid for $k > 0$, but we also require the definitions

$$\left\{ {0 \atop 0} \right\} = 1,$$

and

$$\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0,$$

For $n > 0$. As an alternative definition, we can say that the $\left\{ {n \atop k} \right\}$'s are the unique numbers satisfying

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)(x-2)\ldots(x-k+1). \tag{1.1}$$

An introduction on Stirling numbers can be found in [5, 9]. Bell numbers, denoted by B_n, is the number of all partitions of a set with n distinct elements into disjoint non-empty sets. Thus $B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}$. These numbers also satisfy the recurrence relation $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. See [3, 10, 9] .

Values of B_n are given in Sloane's on-line Encyclopaedia of Integer Sequences [12] as the sequence A000110. The sequence A008277 also gives the triangle of Stirling numbers of the second kind. There are a number of well-known results associated with them in [3, 8, 5].

In this paper we consider the following new problem.

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- $\star$ Consider $b_1 + b_2 + \ldots + b_n$ balls with b_1 balls labeled 1, b_2 balls labeled 2, ..., b_n balls labeled n and $c_1 + c_2 + \ldots + c_k$ cells with c_1 cells labeled 1, c_2 cells labeled 2, ..., c_k cells labeled k. Evaluate the number of ways to partition the set of these balls into cells of these types.

In the present paper, we just consider the following two special cases of the mentioned problem: $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathbb{N}, c_1 = \ldots = c_k = 1$.

As an application of the above problem, for a positive integer m we evaluate the number of ways to write m in the form $m_1 m_2 \ldots m_k$, where $k \geqslant 1$ and m_i's are positive integers.

2. Mixed Bell and Stirling Numbers of second kinds

Definition 2.1. A multiset is a pair (A, m) where A is a set and $m : A \to \mathbb{N}$ is a function. The set A is called the set of underlying elements of (A, m). For each $a \in A$, $m(a)$ is called the multiplicity of a.

A formal definition for a multiset can be found in [1]. Let $A = \{1, 2, \ldots, n\}$ and $m(i) = b_i$ for $i = 1, 2, \ldots, n$. We denote the multiset (A, m) by $\mathcal{A}(b_1, \ldots, b_n)$. Under this notation, the problem of partitioning $b_1 + b_2 + \ldots + b_n$ balls with b_1 balls labeled 1, b_2 balls labeled 2, ..., b_n balls labeled n into $c_1 + c_2 + \ldots + c_k$ cells with c_1 cells labeled 1, c_2 cells labeled 2, ..., c_k cells labeled k can be stated as follows.

Definition 2.2. Let $\mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ and $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. Then the number of ways to partition $\mathcal{B}$ balls into $\mathcal{C}$ non-empty cells is denoted by $\left\{ {\mathcal{B} \atop \mathcal{C}} \right\}$. These numbers are called the *mixed partition numbers* . If cells are allowed to be empty, then we denote the number of ways to partition these balls into these cells by $\left\{ {\mathcal{B} \atop \mathcal{C}} \right\}_0$.

The following result is straightforward.

Proposition 2.3. *Let $\mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ and $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. Then*

$$\left\{ {\mathcal{B} \atop \mathcal{C}} \right\}_0 = \sum_{1 \leqslant i \leqslant k, 0 \leqslant j_i \leqslant c_i} \left\{ {\mathcal{B} \atop J} \right\},$$

where $J = \mathcal{A}(j_1, \ldots, j_k)$.

3. Two Special Cases

In this section we consider two special cases. The first case is $b_1 = b_2 = \ldots = b_n = 1$ and $c_1, \ldots, c_k \in \mathbb{N}$ and the second case is $b_1, \ldots, b_n \in \mathbb{N}$ and $c_1 = \ldots = c_k = 1$.

Note that if $b_1 = b_2 = \ldots = b_n = 1$ and $c_1 = k, c_2 = \ldots = c_k = 0$ then $\left\{ {\mathcal{B} \atop \mathcal{C}} \right\} = \left\{ {n \atop k} \right\}$ and $\left\{ {\mathcal{B} \atop \mathcal{C}} \right\}_0 = \sum_{i=1}^{k} \left\{ {n \atop i} \right\}$. We denote $\sum_{i=1}^{k} \left\{ {n \atop i} \right\}$ by $\left\{ {n \atop k} \right\}_0$. Moreover, if $b_1 = b_2 = \ldots = b_n = 1$ and $c_1 = n, c_2 = \ldots = c_k = 0$ then $\left\{ {\mathcal{B} \atop \mathcal{C}} \right\}_0 = B_n$.

Definition 3.1. Let n, k and r be positive integers, $b_1 = b_2 = \ldots = b_n = 1$ and $c_1 = r, c_2 = \ldots = c_k = 1$. Then we denote $\left\{ {\mathcal{B} \atop \mathcal{C}} \right\}$ by $S(n, k, r)$. These numbers are called the *mixed Stirling numbers of the second kind*. In this case $\left\{ {\mathcal{B} \atop \mathcal{C}} \right\}_0$ is also denoted by $B_0(n, k, r)$ and is called *mixed Bell numbers*.

Let us illustrate this definition by an example.

Example 3.2. We evaluate $B_0(2,2,2)$. Suppose that our balls are $\circledast$ and $\circledcirc$, and our cells are $(\ ), (\ )$ and $[\ ]$. The partitions are

$$\begin{array}{ccc} (\circledast\circledcirc) & (\quad) & [\quad], \\ (\circledast) & (\circledcirc) & [\quad], \\ (\circledast) & (\quad) & [\circledcirc], \\ (\circledcirc) & (\quad) & [\circledast], \\ (\quad) & (\quad) & [\circledast\circledcirc]. \end{array}$$

Thus $B_0(2,2,2) = 5$.

Proposition 3.3. *Let n, k and r be positive integers and $\left\{ {0 \atop k} \right\}_0 = 1$. Then*

$$B_0(n,k,r) = \sum_{\ell=0}^{n} \binom{n}{\ell} \left\{ {\ell \atop r} \right\}_0 (k-1)^{n-\ell}.$$

Proof. Choose ℓ balls in $\binom{n}{\ell}$ ways and put them in r cells in $\left\{ {\ell \atop r} \right\}_0$ ways. We then have $n-\ell$ different balls and $k-1$ different cells. Each of these balls has $k-1$ choices. Thus the number of ways to put the remaining balls into the remaining cells is $(k-1)^{n-\ell}$. □

Proposition 3.4. *Let n, k and r be positive integers. Then*

$$S(n,k,r) = \sum_{\ell=r}^{n-k+1} \binom{n}{\ell} \left\{ {\ell \atop r} \right\} \left\{ {n-\ell \atop k-1} \right\} (k-1)!.$$

Proof. Choose $\ell \geqslant r$ balls in $\binom{n}{\ell}$ ways and put them in r non-empty cells in $\left\{ {\ell \atop r} \right\}$ ways. We then have $n-\ell$ different balls and $k-1$ different cells. Thus the number of ways to put the remaining balls into the remaining cells is $(k-1)!\left\{ {n-\ell \atop k-1} \right\}$. Note that we should have $k-1 \leqslant n-\ell$. □

Proposition 3.5. *Let n, k and r be positive integers. Then*

$$S(n,k,r) = \sum_{0 \leqslant s \leqslant r, 0 \leqslant t \leqslant k-1} (-1)^{t+\varepsilon_s} \binom{k-1}{t} B_0(n,k-t,r-s),$$

where

$$\varepsilon_s = \begin{cases} 0 & \textit{if } s = 0 \\ 1 & \textit{otherwise} \end{cases}$$

Proof. Let

$$\begin{aligned} E_i &= \text{the set of all partitions in which } i \text{ cells labeled 1 are empty}, \quad i = 1, \ldots, r; \\ F_j &= \text{the set of all partitions in which the cell labeled } j \text{ is empty}, \quad j = 2, \ldots, k. \end{aligned}$$

Then $S(n,k,r) = B_0(n,k,r) - |(\cup_{i=1}^{r} E_i) \cup (\cup_{j=2}^{k} F_j)| + |\cup_{i=1}^{r} E_i| + |\cup_{j=2}^{k} F_j|$. We have

$$\begin{aligned} |E_s \cap F_{j_1} \cap \ldots \cap F_{j_t}| &= B_0(n,k-t,r-s), \quad 1 \leqslant s \leqslant r, 1 \leqslant t \leqslant k-1, \\ |E_s| &= B_0(n,k,r-s), \quad 1 \leqslant s \leqslant r, \\ |F_{j_1} \cap \ldots \cap F_{j_t}| &= B_0(n,k-t,r), \quad 1 \leqslant t \leqslant k-1. \end{aligned}$$

Now the Inclusion Exclusion Principle implies the result. □

Proposition 3.6. *Let n, k and r be positive integers. Then*

$$S(n,k,r) = S(n-1,k,r-1) + (k-1)S(n-1,k-1,r) + (k-1+r)S(n-1,k,r).$$

with the initial values $S(0,k,r) = 1$ and $S(n-1,k,0) = (k-1)!\left\{{n-1 \atop k-1}\right\}$.

Proof. We have one ball labeled 1 and there are three cases for this ball:

Case I. This ball is the only ball of a cell labeled 1. There are $S(n-1,k,r-1)$ ways for partitioning the remaining balls into the remaining cells.
Case II. This ball is the only ball of a cell labeled $j, 2 \leqslant j \leqslant k$. There are $k-1$ ways for choosing a cell and $S(n-1,k-1,r)$ ways for partitioning the remaining balls into the remaining cells.
Case III. This ball is not alone in any cell. In this case we put the remaining balls into all cells in $S(n-1,k,r)$ ways and then our ball has $k-1+r$ different choices, since all cells are now different after putting different balls into them.

□

These results can be easily extended to the following general facts.

Theorem 3.7. *Let $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ and $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. Then*

$$\left\{{\mathcal{B} \atop \mathcal{C}}\right\} = \sum_{\ell_1+\ldots+\ell_k=n} \frac{n!}{\ell_1!\ldots\ell_k!} \left\{{\ell_1 \atop c_1}\right\}\left\{{\ell_2 \atop c_2}\right\}\cdots\left\{{\ell_k \atop c_k}\right\}.$$

Theorem 3.8. *Let $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ and $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. Then*

$$\left\{{\mathcal{B} \atop \mathcal{C}}\right\} = \sum_{1\leqslant i\leqslant k, 0\leqslant j_i\leqslant c_i} (-1)^{\sharp(j_1,\ldots,j_k)} \left\{{\mathcal{B} \atop J}\right\},$$

where $\sharp(j_1, \ldots, j_k)$ is the number of i's such that $j_i \neq 0$ and $J = \mathcal{A}(j_1, \ldots, j_k)$.

Theorem 3.9. *Let $b_1 = \ldots = b_n = 1, c_1, \ldots, c_k \in \mathbb{N}, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n), \mathcal{C} = \mathcal{A}(c_1, \ldots, c_k), \mathcal{B}' = \mathcal{A}(b_2, \ldots, b_n)$ and $\mathcal{C}_j = \mathcal{A}(c_1, \ldots, c_{j-1}, c_j - 1, c_{j+1}, \ldots, c_k)$. Then*

$$\left\{{\mathcal{B} \atop \mathcal{C}}\right\} = (c_1 + \ldots + c_k)\left\{{\mathcal{B}' \atop \mathcal{C}}\right\} + \sum_{j=1}^{k} \left\{{\mathcal{B}' \atop \mathcal{C}_j}\right\}$$

Theorem 3.10. *Let $b_1, \ldots, b_n \in \mathbb{N}, c_1 = \ldots = c_k = 1, \mathcal{B} = \mathcal{A}(b_1, \ldots, b_n)$ and $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$. Then*

$$\left\{{\mathcal{B} \atop \mathcal{C}}\right\}_0 = \prod_{j=1}^{n} \binom{b_j + k - 1}{k-1}.$$

Proof. The number of ways to partition b_j balls labeled j into k labeled cells is $\binom{b_j+k-1}{k-1}$. Now the result is obvious. □

Theorem 3.11. *Let $b_1,\dots,b_n \in \mathbb{N}, c_1 = \dots = c_k = 1, \mathcal{B} = \mathcal{A}(b_1,\dots,b_n)$ and $\mathcal{C} = \mathcal{A}(c_1,\dots,c_k)$. Then*

$$\left\{ \begin{matrix} \mathcal{B} \\ \mathcal{C} \end{matrix} \right\} = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{n} \binom{b_j + k - i - 1}{k - i - 1}.$$

4. Application to r- Stirling Numbers of the second kind and Multiplicative Partitioning

The *r-Stirling numbers of the second kind* were introduced by Broder [4] for positive integers n,k,r. The r-Stirling numbers of the second kinds $\left\{ {n \atop k} \right\}_r$ are defined as the number of partitions of the set $\{1,2,\dots,n\}$ that have k non-empty disjoint subsets such that the elements $1,2,\dots,r$ are in distinct subsets. They satisfy the recurrence relations

i. $\left\{ {n \atop k} \right\}_r = 0 \qquad n < r,$
ii. $\left\{ {n \atop k} \right\}_r = k\left\{ {n-1 \atop k} \right\}_{r-1} + \left\{ {n-1 \atop k-1} \right\}_r \qquad n > r,$
iii. $\left\{ {n \atop k} \right\}_r = \left\{ {n \atop k} \right\}_{r-1} - (r-1)\left\{ {n-1 \atop k} \right\}_{r-1} \qquad n \geqslant r \geqslant 1.$

The identity 1.1 extends into

$$(x+r)^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r x(x-1)(x-2)\dots(x-k+1).$$

The ordinary Stirling numbers of the second kind are identical to both 0-Stirling and 1-Stirling numbers. We give a theorem in which we express the r-Stirling numbers in terms of the Stirling numbers.

Theorem 4.1. *For positive integers n,k and r*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = S(n-r, r+1, k-r).$$

Proof. Firstly, we put the numbers of $\{1,2,\dots,r\}$ into r cells as singletons. Now, we partitions $n-r$ elements to k non-empty cells such that r cells are labeled. The number of partitions of these elements is equal to $S(n-r, r+1, k-r)$. □

Corollary 4.2. *Let n,k and r be positive integers. Then*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = \sum_{l=k-r}^{n-2r} \binom{n-r}{l} \left\{ \begin{matrix} l \\ k-r \end{matrix} \right\} \left\{ \begin{matrix} n-r+l \\ r \end{matrix} \right\} r!.$$

Corollary 4.3. *For positive integer n,k and r*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_{r-1} + r\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_r + k\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_r.$$

Proof. It is clear by Proposition 3.6. □

Definition 4.4. Let n,k and r be positive integers. The *r-mixed Stirling number of second kind* is the number of non-empty partitions of the set $\{1,2,\dots,n\}$ to $\mathcal{C}=\mathcal{A}(c_1,\dots,c_k)$ such that the elements $1,2,\dots,r$ are in distinct cells. We denote the r-mixed Stirling number of second kind by $\{{n \atop k}\}_r^{\mathcal{C}}$.

Example 4.5. We evaluate $\{{4 \atop 2}\}_2^{\mathcal{A}(2,1)}$. Suppose that set of our balls is $\{1,2,3,4\}$ and our cells are $(\ ),(\ )$ and $[\ ]$. The partitions are

$$\begin{array}{lllll}
(1,4)\ (2)\ [3], & (1)\ (2,4)\ [3], & (1)\ (2)\ [3,4], & (1,4)\ (3)\ [2], & (1)\ (3,4)\ [2],\\
(1)\ (3)\ [2,4], & (3,4)\ (2)\ [1], & (3)\ (2,4)\ [1], & (3)\ (2)\ [1,4], & (1,3)\ (2)\ [4],\\
(2,3)\ (4)\ [1], & (1)\ (2,4)\ [3], & (1)\ (2,3)\ [4], & (2)\ (4)\ [1,3], & (1)\ (4)\ [2,3],
\end{array}$$

so that $\{{4 \atop 2}\}_2^{\mathcal{A}(2,1)}=15$.

Theorem 4.6. *Let n,k and r be positive integers. If $c_1=t, c_2=\dots=c_k=1$, then*

$$\left\{{n \atop k}\right\}_r^{\mathcal{C}}=\sum_{i=0}^{\min\{t,r\}}\sum_{\ell=k-1+t-r}^{n-r}\binom{n-r}{\ell}\binom{r}{i}\binom{k-1}{r-i}(r-i)!(n-r-\ell)^r S(\ell,k-1+t-r,t-i).$$

Proof. We have n balls with labels 1 to n and k cells such that t cells are unlabeled. We partitions this balls in two steps in such a way that the numbers of $\{1,2,\dots,r\}$ are put into r cells as singletons.

Step I. Let i be the number of the t cells of the first kind which contains some of the balls $1,2,\dots,r$. Then $0\leqslant i\leqslant \min\{t,r\}$. We choose i balls of the balls $1,2,\dots,r$ and put them into i cells of the t cells of the first kind in $\binom{r}{i}$ ways. Then we choose $r-i$ cells of the $k-1$ different cells and put the remaining $r-i$ balls into them in $\binom{k-1}{r-i}(r-i)!$ ways. Thus if N is the number of partitions of these balls into cells in this way, then

$$N=\sum_{i=1}^{r}\binom{r}{i}\binom{k-1}{r-i}(r-i)!.$$

Step II. Now, we have $n-r$ different balls and $t+k-1$ cells among which there are $t-i$ cells are of the first kind and the remaining cells are different. Note that we now have $k-1+t-r$ empty cells. Prior to anything, we fill these empty cells by ℓ balls of the $n-r$ balls. Thus $k-1+t-r\leqslant \ell\leqslant n-r$. Choose ℓ balls in $\binom{n-r}{\ell}$ ways and put them into cells in such a way that there are no empty set. The number of ways is $S(\ell,k-1+t-r,t-i)$, by Proposition 3.4. Then put the remaining $n-r-\ell$ different balls into r cells which contains the balls $1,2,\dots,r$.

□

Corollary 4.7. *Let n,k and r be positive integers. If $c_1,\dots,c_k\in\mathbb{N}$ and $\mathcal{C}=\mathcal{A}(c_1,\dots,c_k)$, then*

$$\left\{{n \atop k}\right\}_r^{\mathcal{C}}=\sum_{i_1+\dots+i_k=r}\frac{r!}{i_1!\dots i_k!}\left\{{n-r \atop \mathcal{C}}\right\}.$$

The *r-Bell numbers* $B_{n,r}$ with parameters $n \geqslant r$ is the number of the partitions of a set $\{1,2,\ldots,n\}$ such that the r elements $1,2,\ldots,r$ are distinct cells in each partition. Hence

$$B_{n,r} = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r.$$

It obvious that $B_n = B_{n,0}$. The name of r-Stirling numbers of second kind suggests the name for r-Bell numbers with polynomials

$$B_{n,r}(x) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\} x^k,$$

which is called the r-Bell polynomials [11].

Theorem 4.8. *Let n, k and r be positive integers. Then*

$$B_{n,r} = \sum_{k=0}^{n} \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} \left\{ {\ell \atop k-r} \right\}_0 r^{n-r-\ell}.$$

Proof. Put the numbers $1,2,\ldots,r$ in r cells as singletons. Now, we partition $n-r$ elements into k cells such that r cells are labeled. Thus

$$\begin{aligned} B_{n,r} &= \sum_{k=0}^{n} B_0(n-r, r+1, k-r) \\ &= \sum_{k=0}^{n} \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} \left\{ {\ell \atop k-r} \right\}_0 r^{n-r-\ell}. \end{aligned}$$

□

Definition 4.9. Let n, k and r be positive integers. The *r-mixed Bell number* is the number of partitions of the set $\{1,2,\ldots,n\}$ to $\mathcal{C} = \mathcal{A}(c_1,\ldots,c_k)$ such that the elements $1,2,\ldots,r$ are in distinct cells. We denote the r-mixed Bell number by $B^{\mathcal{C}}_{n,r}$.

Theorem 4.10. *Let n, k and r be positive integers. If $c_1 = t, c_2 = \ldots = c_k = 1$, then*

$$B^{\mathcal{C}}_{n,r} = \sum_{i=1}^{r} \binom{r}{i} \binom{k-i}{r-i} (r-i)! B_0(n-r, k+i-1, t-i).$$

Proof. We have n balls with labels 1 to n and k cells such that t cells are unlabeled. We partition these balls into k cells such that the numbers $1,2,\ldots,r$ into r cells are as singletons. There are $\binom{r}{i}$ ways to choose element $i = 1,2,\ldots,r$. The number of partitions of these balls is equal to

$$\sum_{i=1}^{r} \binom{r}{i} \binom{k-i}{r-i} (r-i)!.$$

By Proposition 3.3, the number of partitions of $n-r$ labeled balls into $k+i-1$ cells such that $t-i$ cells are unlabeled is equal to $B_0(n-r, k+i-1, t-i)$. □

Corollary 4.11. *Let n, k and r be positive integers, then*

$$B^{\mathcal{C}}_{n,r} = \sum_{k=r}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}^{\mathcal{C}}_{r}.$$

Proposition 4.12. *Let n and k be positive integers. If $\mathcal{C} = \mathcal{A}(c_1, \ldots, c_k)$, then*

$$B^{\mathcal{C}}_{n,r} = \sum_{i_1+\ldots+i_k=r} \frac{r!}{i_1! \ldots i_k!} \left\{ \begin{matrix} n-r \\ \mathcal{C} \end{matrix} \right\}_{0}.$$

We now give an application of Theorem 3.10 and Theorem 3.11 in multiplicative partitioning.

Let n be a positive integer. A multiplicative partitioning of n is a representation of n as a product of positive integers. Since the order of parts in a partition does not count, they are registered in decreasing order of magnitude.

Theorem 4.13. *Let $m = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$ be a positive integer, where p_i's are different prime numbers. Then*

i. *the number of ways to write m as the form $m_1 \ldots m_k$, where $k \geqslant 1$ and m_i's are positive integers is*
$$\prod_{j=1}^{n} \binom{\alpha_j + k - 1}{k-1};$$
ii. *the number of ways to write m as the form $m_1 \ldots m_k$, where $k \geqslant 1$ and m_i's are positive integers greater than 1 is*
$$\sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{n} \binom{\alpha_j + k - i - 1}{k-i-1}.$$

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