TRIGONOMETRIC FORMULAE VIA TELESCOPING METHOD

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ABSTRACT. Motivated by the Monthly problem #11515, we prove further interesting formulae for trigonometric series by means of telescoping method.

The Monthly problem #11515 in [1] asks to evaluate the trigonometric series

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}x).$$

In order to highlight the telescopic approach, we reproduce Caro's recent proof. Considering the truncated series defined by

$$\Omega(m) := \sum_{n=1}^{m} 4^n \sin^4(2^{-n}x)$$

and then recalling the trigonometric relation

$$\sin^4 x = \sin^2 x - \frac{\sin^2 2x}{4}. \tag{1}$$

we may manipulate it as follows

$$\Omega(m) = \sum_{n=1}^{m} \Big\{ 4^n \sin^2(2^{-n}x) - 4^{n-1} \sin^2(2^{1-n}x) \Big\}.$$

Observing that the last sum fits into the telescopic scheme

$$\Omega(m) = \sum_{n=1}^{m} \{\omega_n - \omega_{n-1}\} = \omega_m - \omega_0 \quad \text{where} \quad \omega_n := 4^n \sin^2(2^{-n}x)$$

we get it evaluated in the following closed form

$$\Omega(m) = 4^m \sin^2(2^{-m}x) - \sin^2 x.$$

This is, in fact, a known result, which can be found in Gradshteyn–Ryzhik [3, 1.362.1, P37]. Then the limiting case $m \to \infty$ of $\Omega(m)$ leads to the closed expression

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}x) = x^2 - \sin^2 x. \tag{2}$$

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This solution is not an isolated example. In fact, the same approach can be employed to establish further similar trigonometric series formulae, including three known ones. The objective of this short paper is to present them in details.

§1. Define the finite trigonometric sum by

$$A(m) := \sum_{n=1}^{m} \frac{\sin^4(2^n x)}{4^n} \quad \text{and} \quad a_n := \frac{\sin^2(2^n x)}{4^n}.$$

According to the equality (1), we may analogously rewrite it as

$$A(m) = \sum_{n=1}^{m} \left\{ \frac{\sin^2(2^n x)}{4^n} - \frac{\sin^2(2^{n+1} x)}{4^{n+1}} \right\}$$

and then evaluate it by telescoping as follows

$$\begin{aligned} A(m) &= \sum_{n=1}^{m} \{a_n - a_{n+1}\} = a_1 - a_{m+1} \\ &= \frac{\sin^2 2x}{4} - \frac{\sin^2(2^{m+1} x)}{4^{m+1}}. \end{aligned}$$

Its limiting case $m \to \infty$ results in another trigonometric series identity

$$\sum_{n=1}^{\infty} \frac{\sin^4(2^n x)}{4^n} = \frac{\sin^2 2x}{4} = \sin^2 x \cos^2 x. \tag{3}$$

§2. Define the finite trigonometric sum by

$$B(m) := \sum_{n=1}^{m} \frac{\cos^4(2^n x)}{(-4)^n} \quad \text{and} \quad b_n := \frac{\cos(2^n x)}{(-4)^n}.$$

In view of the trigonometric relation

$$8 \cos^4 x = 3 + \cos 4x + 4 \cos 2x. \tag{4}$$

we may express $B(m)$ as

$$B(m) = 2 \sum_{n=1}^{m} \left\{ \frac{\cos(2^{n+2} x)}{(-4)^{n+2}} - \frac{\cos(2^{n+1} x)}{(-4)^{n+1}} \right\} + \frac{3}{8} \sum_{n=1}^{m} \left(\frac{-1}{4} \right)^n$$

and evaluate it by telescoping as follows:

$$\begin{aligned} B(m) &= 2\sum_{n=1}^{m}\big\{b_{n+2}-b_{n+1}\big\}+\frac{3}{40}\Big\{\frac{1}{(-4)^m}-1\Big\} \\ &= 2\big\{b_{m+2}-b_2\big\}+\frac{3}{40}\Big\{\frac{1}{(-4)^m}-1\Big\} \\ &= 2\Big\{\frac{\cos(2^{m+2}x)}{(-4)^{m+2}}-\frac{\cos 4x}{16}\Big\}+\frac{3}{40}\Big\{\frac{1}{(-4)^m}-1\Big\}. \end{aligned}$$

Its limiting case $m\to\infty$ gives the following trigonometric series identity

$$\sum_{n=1}^{\infty}\Big(\frac{-1}{4}\Big)^n\cos^4(2^n x) = -\frac{3}{40}-\frac{\cos 4x}{8}. \tag{5}$$

We remark that for equation (4), there is a companion relation

$$8\sin^4 x = 3+\cos 4x-4\cos 2x. \tag{6}$$

This was utilized in the initial version of this paper to show both (2) and (3). But the proofs are less elegant than that due to Caro [1].

§3. Define the finite trigonometric sum by

$$C(m) := \sum_{n=1}^{m} 3^n\sin^3(3^{-n}x) \quad\text{and}\quad c_n := 3^n\sin(3^{-n}x).$$

By invoking the trigonometric relation

$$4\sin^3 x = 3\sin x-\sin 3x. \tag{7}$$

we may reformulate $C(m)$ as

$$C(m) = \frac{3}{4}\sum_{n=1}^{m}\Big\{3^n\sin(3^{-n}x)-3^{n-1}\sin(3^{1-n}x)\Big\}$$

and then evaluate it, in closed form, by telescoping as

$$\begin{aligned} C(m) &= \frac{3}{4}\sum_{n=1}^{m}\big\{c_n-c_{n-1}\big\} = \frac{3}{4}\big\{c_m-c_0\big\} \\ &= \frac{3}{4}\Big\{3^m\sin(3^{-m}x)-\sin x\Big\}. \end{aligned}$$

Its limiting case $m\to\infty$ yields the following trigonometric series identity

$$\sum_{n=1}^{\infty}3^n\sin^3(3^{-n}x) = \frac{3}{4}\Big\{x-\sin x\Big\}. \tag{8}$$

§4. Define the finite trigonometric sum by

$$D(m) := \sum_{n=1}^{m} \frac{\sin^3(3^n x)}{3^n} \quad \text{and} \quad d_n := \frac{\sin(3^n x)}{3^n}.$$

Keeping in mind of the trigonometric equality (7), we may rewrite $D(m)$ as

$$D(m) = \frac{3}{4} \sum_{n=1}^{m} \left\{ \frac{\sin(3^n x)}{3^n} - \frac{\sin(3^{n+1} x)}{3^{n+1}} \right\}$$

and then evaluate it by telescoping as

$$\begin{aligned} D(m) &= \frac{3}{4} \sum_{n=1}^{m} \{d_n - d_{n+1}\} = \frac{3}{4}\{d_1 - d_{m+1}\} \\ &= \frac{3}{4}\left\{ \frac{\sin 3x}{3} - \frac{\sin(3^{m+1} x)}{3^{m+1}} \right\}. \end{aligned}$$

Its limiting case $m \to \infty$ results in the following trigonometric series identity

$$\sum_{n=1}^{\infty} \frac{\sin^3(3^n x)}{3^n} = \frac{\sin 3x}{4}. \tag{9}$$

§5. Define the finite trigonometric sum by

$$E(m) := \sum_{n=1}^{m} \frac{\cos^3(3^n x)}{(-3)^n} \quad \text{and} \quad e_n := \frac{\cos(3^n x)}{(-3)^n}.$$

Taking into account of the trigonometric equality

$$4\cos^3 x = 3\cos x + \cos 3x. \tag{10}$$

we may express $E(m)$ as

$$E(m) = \frac{3}{4} \sum_{n=1}^{m} \left\{ \frac{\cos(3^n x)}{(-3)^n} - \frac{\cos(3^{n+1} x)}{(-3)^{n+1}} \right\}$$

and then evaluate it by telescoping as

$$\begin{aligned} E(m) &= \frac{3}{4} \sum_{n=1}^{m} \{e_n - e_{n+1}\} = \frac{3}{4}\{e_1 - e_{m+1}\} \\ &= \frac{3}{4}\left\{ \frac{\cos 3x}{-3} - \frac{\cos(3^{m+1} x)}{(-3)^{m+1}} \right\}. \end{aligned}$$

Its limiting case $m \to \infty$ leads to the following trigonometric series identity

$$\sum_{n=1}^{\infty} \frac{\cos^3(3^n x)}{(-3)^n} = -\frac{\cos 3x}{4}. \tag{11}$$

By employing the same approach, we can further review the following three summation formulae of trigonometric functions, whose details will not be produced.

Gradshteyn–Ryzhik [3, 1.362.2, P38].

Trigonometric relation: $\sec^2 x = 4\csc^2 2x - \csc^2 x;$

Finite sum formula: $\sum_{n=1}^{m} \frac{1}{4^n} \sec^2 \frac{x}{2^n} = \csc^2 x - \frac{1}{4^m} \csc^2 \frac{x}{2^m};$

Infinite series identity: $\sum_{n=1}^{\infty} \frac{1}{4^n} \sec^2 \frac{x}{2^n} = \csc^2 x - \frac{1}{x^2}.$

Gradshteyn–Ryzhik [3, 1.371.1, P38].

Trigonometric relation: $\tan x = \cot x - 2\cot 2x;$

Finite sum formula: $\sum_{n=0}^{m} \frac{1}{2^n} \tan \frac{x}{2^n} = \frac{1}{2^m} \cot \frac{x}{2^m} - \cot x;$

Infinite series identity: $\sum_{n=0}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \frac{1}{x} - \cot x.$

Gradshteyn–Ryzhik [3, 1.371.2, P38].

Trigonometric relation: $\tan^2 x = 2 - \cot^2 x + 4\cot^2 2x;$

Finite sum formula: $\sum_{n=1}^{m} \frac{1}{4^n} \tan^2 \frac{x}{2^n} = \frac{2(4^m - 1)}{3 \cdot 4^m} + \cot^2 x - \frac{1}{4^m} \cot^2 \frac{x}{2^m};$

Infinite series identity: $\sum_{n=1}^{\infty} \frac{1}{4^n} \tan^2 \frac{x}{2^n} = \cot^2 x - \frac{1}{x^2} + \frac{2}{3}.$

In general, there may exist summation formulae for the trigonometric series of forms

$$U(\alpha, \beta) := \sum_{n=1}^{\infty} \alpha^n \sin(\beta^n x) \qquad \text{and} \qquad V(\alpha, \beta) := \sum_{n=1}^{\infty} \alpha^n \cos(\beta^n x).$$

It would be interesting to find other identities than those displayed in this paper.

Before concluding the paper, we present two further groups of infinite series identities involving the arctangent function. Their proofs can be carried out equally by means of the telescopic scheme.

§6. For the inverse tangent function, there is an elementary equality

$$\arctan x \pm \arctan y = \arctan \frac{x \pm y}{1 \mp xy} \quad \text{where} \quad |xy| < 1.$$

This can be employed to evaluate the following infinite series:

$$\sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1} = \frac{\pi}{4}, \qquad \text{Knopp [8, Exercise 102a]} \tag{12}$$

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \frac{3\pi}{4}, \qquad \text{Knopp [8, Exercise 102b]} \tag{13}$$

$$\sum_{n=0}^{\infty} \arctan \frac{2}{(2n+1)^2} = \frac{\pi}{2}. \qquad \text{Boros–Moll [2]} \tag{14}$$

$$\sum_{n=0}^{\infty} (-1)^n \arctan \frac{2n+3}{n^2+3n+1} = \frac{\pi}{4}, \qquad \text{Glasser–Klamkin [5].} \tag{15}$$

§7. Recall the Fibonacci numbers defined by

$$F_{n+1} = F_n + F_{n-1} \quad \text{with} \quad F_1 = F_2 = 1.$$

By combining the Cassini identity

$$F_{2n+1}^2 = 1 + F_{2n}F_{2n+2} \tag{16}$$

with the two inverse trigonometric relations

$$\left.\begin{aligned} \arcsin x &= \arctan \frac{x}{\sqrt{1-x^2}} \\ \arccos x &= \arctan \frac{\sqrt{1-x^2}}{x} \end{aligned}\right\} \quad \text{where} \quad 0 < x < 1$$

the interested reader may try to show the following remarkable formulae:

$$\sum_{n=1}^{\infty} \arctan \frac{1}{F_{2n+1}} = \frac{\pi}{4}, \qquad \text{Lehmer [7]} \tag{17}$$

$$\sum_{n=1}^{\infty} \arctan \frac{2F_{2n+1}}{F_{2n}F_{2n+2}} = \frac{\pi}{2}, \qquad \text{Guillot [6];} \tag{18}$$

$$\sum_{n=1}^{\infty} \arcsin \frac{2F_{2n+1}}{2 + F_{2n}F_{2n+2}} = \frac{\pi}{2}, \qquad \text{Guillot [6];} \tag{19}$$

$$\sum_{n=1}^{\infty} \arccos \frac{F_{2n}F_{2n+2}}{2 + F_{2n}F_{2n+2}} = \frac{\pi}{2}, \qquad \text{Guillot [6].} \tag{20}$$

Further infinite series identities concerning the inverse tangent function can be found in [2, 4, 9].

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