COUNTING STAIRCASES IN INTEGER COMPOSITIONS

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ABSTRACT. The main theorem establishes the generating function $F$ which counts the number of times the staircase $1^+2^+3^+\cdots m^+$ fits inside an integer composition of $n$.

$$F = \frac{k_m - qx^m y k_{m-1}}{(1-q)x^\frac{(m+1)}{2} \left( \frac{y}{1-x} \right)^m + \frac{1-x-xy}{1-x} \left( k_m - qx^m y k_{m-1} \right)}.$$

where

$$k_m = \sum_{a=0}^{m-1} x^{m_j-(\frac{j}{2})} \left( \frac{y}{1-x} \right)^i.$$

Here $x$ and $y$ respectively track the composition size and number of parts, whilst $q$ tracks the number of such staircases contained.

1. Introduction

In several recent papers the notion of integer compositions of $n$ (represented as the associated bargraph) have been used to model certain problems in physics. See for example [2,7–9] where bargraphs are a representation of a polymer at an adsorbing wall subject to several forces.

In a paper by a current author et al (see [1]), the x-ray process was modelled using permutation matrices as a two dimensional analogue of the object being x-rayed, where the examining rays are modelled by diagonal lines with equation $x+y=n$ for positive integers $n$. The current paper is based instead on integer compositions as the object analogue and where the examining rays are represented by equation $x-y=n$ for non-negative integers $n$. Since this model is essentially parameterized by the degree to which the x-rays are contained inside an arbitrary composition, it translates naturally to obtaining a generating function which tracks the number of "staircases" which are contained inside particular integer compositions of $n$. More precisely, we will obtain a generating function which counts (with the exponent $s$ of $q$ as tracker) the number of times the staircase $1^+2^+3^+\cdots m^+$ ($m$ fixed) fits inside particular compositions. So the term of our generating function $n(a,b,s)x^ay^bq^s$ indicates that there are in total $n(a,b,s)$ compositions of $a$ with $b$ parts in which the staircases $1^+2^+3^+\cdots m^+$ occurs exactly $s$ times.
1.1. Definitions. A composition of a positive integer $n$ is a sequence of $k$ positive integers $a_1, a_2, \ldots, a_k$, each called a part such that $n = \sum_{i=1}^{k} a_i$; A staircase $1^+2^+3^+\cdots m^+$ is a word with $m$ sequential parts from left to right where for $1 \leq i \leq m$ the $i$th part $\geq i$.

See for example the staircase in Figure 1 below.

![Figure 1. The staircase $1^+2^+3^+4^+5^+$](image)

Much recent work has been done on various statistics relating to compositions. See, for example, [3, 5, 6] and [4] and references therein.

A particular composition may be represented as a bargraph (see [4] and [2]). For example the composition $4 + 3 + 1 + 2 + 3$ of 13 represented in Figure 2 as a bargraph, contains exactly one $1^+2^+3^+$ staircase, three $1^+2^+$ staircases and five $1^+$ staircases. It contains no others.

![Figure 2. The composition $4 + 3 + 1 + 2 + 3$ containing one staircase $1^+2^+3^+$ (coloured) and three $1^+2^+$ staircases](image)

In this paper, compositions (ie their associated bargraphs) are the analogue for a (2-dimensional) object to be x-rayed (as explained above). Across all possible compositions, the shapes are parameterized in a generating function by a marker variable $q$ which tracks the number of $1^+2^+3^+\cdots m^+$ staircases (again with $m$ fixed) that fit inside a composition. The generating function in question is defined as

\[(1) \quad F = \sum_{a \geq 1; b \geq 1; s \geq 0} n(a, b, s) x^a y^b q^s,\]

where $n(a, b, s)$ is the number of compositions of $a$ with $b$ parts that contain $s$ staircases $1^+2^+3^+\cdots m^+$. 
The main theorem arrived at by the end of the paper consists in establishing a formula for the generating function $F$ defined in equation (1). We state it here for completeness:

$$F = \frac{k_m - \frac{qxy}{1-x}k_{m-1}}{(1-q)x^{m+\frac{1}{2}}\left(\frac{y}{1-x}\right)^m + \frac{1-x-xy}{1-x}\left(k_m - \frac{qxy}{1-x}k_{m-1}\right)},$$

where $k_m = \sum_{\ell=0}^{m-1} x^{mj-(\ell)}\left(\frac{y}{1-x}\right)^j$. Prior to this main theorem, several lemmas present a set of recursions which are used in proving this result.

2. Proofs

2.1. Warmup: compositions containing words of the form $1^+2^+$ or $1^+2^+3^+$. Consider words which are of the form $1^+2^+$; i.e., words of two parts adjacent to each other from left to right with the first being a letter $> 0$ and the second being a letter $> 1$.

We let $F$ be the generating function for all words; $F_a$ be the generating function for all words starting with the letter $a$ and in general $F_{a_1a_2\cdots a_n}$ be the gf (generating function) for words starting with the letters $a_1a_2\cdots a_n$. So by definition

(2) 
$$F = 1 + \sum_{a \geq 1} F_a.$$

And we have the following recurrence:

(3) 
$$F_a = x^a y + F_{a1} + F_{a2} + F_{a3} + \cdots$$

Now $F_{a1} = x^a y F_1$ and $F_{ab} = q x^a y F_b$ for $b > 1$. So $F_a = x^a y (1 + F_1 + q F_2 + q F_3 + \cdots)$. Thus for all $a \geq 1$, we have $F_a = x^a y (1 - q)(1 + F_1) + q x^a y F$. As the second part of our warmup, we now examine the pattern $1^+2^+3^+$, i.e., we focus on compositions which contain this word sequence.

Extracting part of the first letter, we have

(4) 
$$F_a = x^{a-1} F_1.$$

From equation (2),

(5) 
$$F = 1 + \sum_{a \geq 1} F_a = 1 + \frac{1}{1-x} F_1.$$

Also

$$F_1 = xy + (F_{11} + F_{12} + F_{13} + \cdots) = xy + xy(F_1 + F_{12} + x F_{12} + x^2 F_{12} + \cdots) = xy + xy F_1 + \frac{1}{1-x} F_{12},$$

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where
\[ F_{12} = x^3y^2 + F_{121} + F_{122} + (F_{123} + \cdots) \]
\[ = x^3y^2 + x^3y^2F_1 + x^2yF_{12} + (qx^3yF_{12} + qx^4yF_{12} + \cdots) \]
\[ = x^3y^2 + x^3y^2F_1 + x^2yF_{12} + \frac{qx^3y}{1-x}F_{12}. \]
\[ (7) \]

The last three equations have three unknowns \( F, F_1, \) and \( F_{12} \) which we can solve for \( F \) using Cramer’s rule. However, instead, we try the general pattern.

2.2. The general pattern \( 1^+2^+3^+ \ldots m^+ \). As before, \( F_a = x^{a-1}F_1 \) and
\[ (8) \]
\[ F = 1 + \sum_{a \geq 1} F_a = 1 + \frac{1}{1-x}F_1. \]

Now
\[ F_1 = xy + (F_{11} + F_{12} + F_{13} + \cdots) \]
\[ = xy + xy(F_1 + F_{12} + xF_{12} + x^2F_{12} + \cdots) \]
\[ = xy + xyF_1 + \frac{1}{1-x}F_{12} \]
\[ (9) \]
and
\[ F_{12} = x^3y^2 + F_{121} + F_{122} + (F_{123} + \cdots) \]
\[ = x^3y^2 + x^3y^2F_1 + x^2yF_{12} + (F_{123} + xF_{123} + x^2F_{123} + \cdots) \]
\[ = x^3y^2 + x^3y^2F_1 + x^2yF_{12} + \frac{1}{1-x}F_{123}. \]
\[ (10) \]

Next, by a similar process
\[ (11) \]
\[ F_{123} = x^6y^3 + x^6y^3F_1 + x^5y^2F_{12} + x^3yF_{123} + \frac{1}{1-x}F_{1234}. \]

Proceeding in this way, we obtain in general for all \( j \leq m - 1 \)
\[ (12) \]
\[ F_{12\ldots j} = x^{(j+1)}y^j + x^{(j+1)}y^jF_1 + x^{(j+1)}y^jF_{12} + \frac{1}{1-x}F_{12\ldots j+1}. \]

with
\[ (13) \]
\[ F_{12\ldots m} = qx^m yF_{12\ldots m-1}. \]
To simplify the presentation we put $z = \frac{1}{1-y}$. Now, we rewrite equations (7)-(13) in matrix form. So we first define the matrix $A$ as

$$
\begin{pmatrix}
1 & z & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 1-x(\frac{1}{2})y & z & 0 & \cdots & \cdots & \cdots & 0 \\
0 & -x(\frac{3}{2})y^2 & 1-x(\frac{1}{2})y & z & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & -x(m-1)\cdot y^{m-2} & -x(m-1)\cdot y^{m-3} & -x(\frac{1}{2})y^{m-4} & \cdots & -x(\frac{1}{2})y^{m-2} & z & 0 \\
0 & -x(\frac{3}{2})y^m & -x(\frac{3}{2})y^{m-1} & -x(\frac{21}{4})y^m & \cdots & -x(\frac{21}{4})y^m & 1-x(\frac{21}{4})y & z \\
0 & 0 & 0 & 0 & \cdots & 0 & -qy^m & 1
\end{pmatrix}
$$

and $C$ to be the vector $\left( x(\frac{1}{2}), x(\frac{3}{2})y, x(\frac{3}{2})y^2, \ldots, x(m-1)\cdot y^{m-2}, x(\frac{3}{2})y^m, 0 \right)^T$. Then the matrix form of our equations is $AX = C$ where it is the first entry of matrix $X$ (the matrix of variables from equations (7)-(13)) that is our required generating function $F$. So defining $B$ as the matrix obtained from the above matrix $A$ by replacing its first column with the entries from $C$; i.e.

$$
\begin{pmatrix}
x(\frac{1}{2}) & z & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
x(\frac{3}{2})y & 1-x(\frac{3}{2})y & z & 0 & \cdots & \cdots & \cdots & 0 \\
x(\frac{3}{2})y^2 & -x(\frac{3}{2})y^2 & 1-x(\frac{1}{2})y & z & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
x(m-1)\cdot y^{m-2} & -x(m-1)\cdot y^{m-2} & -x(\frac{1}{2})y^{m-3} & -x(\frac{1}{2})y^{m-2} & \cdots & -x(\frac{1}{2})y^{m-2} & z & 0 \\
x(\frac{3}{2})y^m & -x(\frac{3}{2})y^m & -x(\frac{3}{2})y^m & -x(\frac{21}{4})y^m & \cdots & -x(\frac{21}{4})y^m & 1-x(\frac{21}{4})y & z \\
0 & 0 & 0 & 0 & \cdots & 0 & -qy^m & 1
\end{pmatrix}
$$

By Cramer’s rule, we obtain

$$
F = \frac{\det B}{\det A}.
$$

2.3. Equations for $\det A$ and $\det B$ in a form that can be solved recursively. Define the $mxm$ matrix $N_m$, to be the first $m$ rows and columns of the $(m+1)x(m+1)$ matrix $A$, but where the first column of $A$ has initially been replaced by the first $m$ entries of $C$. To simplify the notation further, we let $w_{ij} = x(\frac{i}{2})\cdot y^j$ and so explicitly written out,

$$
N_m := \begin{pmatrix}
x(\frac{1}{2})y^0 & z & 0 & 0 & \cdots & 0 \\
x(\frac{3}{2})y & 1-w_{21} & z & 0 & \cdots & \vdots \\
x(\frac{3}{2})y^2 & -w_{31} & 1-w_{31} & z & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
x(m-1)\cdot y^{m-1} & -w_{m1} & \cdots & \cdots & 1-w_{m1}
\end{pmatrix}.
$$

By cofactor expansions (initially along the last row of $B$), we obtain

$$
\det B = \det N_m + zqy^m \det N_{m-1}.
$$
And let $C_{m-1}$ be the $(m-1) \times (m-1)$ matrix obtained by deleting the first row and column of $N_m$. So, for example,

$$C_4 = \begin{pmatrix}
1 - w_{21} & z & 0 & 0 \\
-w_{31} & 1 - w_{32} & z & 0 \\
-w_{41} & -w_{42} & 1 - w_{43} & z \\
-w_{51} & -w_{52} & -w_{53} & 1 - w_{54}
\end{pmatrix}.$$ 

By employing cofactor expansions (also, initially along the last row of $A$), we obtain

$$\text{det } C = \text{det } C_{m-1} + wz^m \text{det } C_{m-2}. \tag{17}$$

Again, by employing co-factor expansions along the last row of $C_4$, we see that

$$\text{det } C_4 = (1 - w_{54}) \text{det } C_3 + zw_{53} \text{det } C_2 - w_{52}z^2 \text{det } C_1 + w_{51}z^3 \text{det } C_0,$$

where $\text{det } C_0 := 1$. In general, a cofactor expansion along the last row of $C_m$ yields for $m \geq 1$

$$\text{det } C_m = (1 - w_{m+1m}) \text{det } C_{m-1} + \sum_{a=1}^{m-1} (-1)^{m-1-j}w_{m+1j}z^{m-j} \text{det } C_{j-1}. \tag{18}$$

Once again making the replacement $w_{ij} = x^{(i/2)-(j/2)}y^{i-j}$, we have for $m \geq 1$

$$\text{det } C_m = (1 - x^m y) \text{det } C_{m-1} + \sum_{a=1}^{m-1} (-1)^{m-1-j}x^{(m+1/2)-(j/2)}y^{m+1-j}z^{m-j} \text{det } C_{j-1}. \tag{19}$$

Dropping $m$ by 1 and multiplying this equation by $-x^myz$, we obtain

$$-x^myz \text{det } C_{m-1}$$

$$= -x^myz(1 - x^{m-1} y) \text{det } C_{m-2} + \sum_{a=1}^{m-2} (-1)^{m-1-j}x^{(m+1/2)-(j/2)}y^{m+1-j}z^{m-j} \text{det } C_{j-1}.$$ 

By subtracting (18) from (17), we obtain

$$\text{det } C_m + x^m yz \text{det } C_{m-1}$$

$$= (1 - x^m y) \text{det } C_{m-1} + x^m yz(1 - x^{m-1} y) \text{det } C_{m-2} + x^{2m-1}y^2z \text{det } C_{m-2}.$$ 

Simplifying,

$$\text{det } C_m = (1 - x^m y(1 + z)) \text{det } C_{m-1} + x^m yz \text{det } C_{m-2}. \tag{19}$$

where $\text{det } C_{-1} := 1$; $\text{det } C_0 = 1$; $\text{det } C_1 = 1 - xy = 1 - w_{21}$.

For ease of notation in the remainder of the paper, we abbreviate $\text{det } C_m$ as $C_m$, and define the generating function $C(t) = \sum_{m \geq 0} C_mt^m$. By multiplying equation (19) by $t^m$ and then summing from 1 to infinity, we obtain

$$C(t) - 1 = tC(t) - (1 + z)xytC(xt) + x^2yt^2zC(xt) + xyzt.$$
Therefore
\begin{equation}
C(t) = \frac{1 + xyt}{1 - t} - xytC(xt) \frac{1 + z(1 - xt)}{1 - t}.
\end{equation}

Again to simplify the notation, substitute \( f(t) := \frac{1 + xyt}{1 - t} \) and \( \varphi(t) := -xyt\frac{1 + z(1 - xt)}{1 - t} \), and iterate the previous equation to obtain:
\begin{equation}
C(t) = f(t) + \varphi(t)C(xt) = f(t) + \varphi(t)f(xt) + \varphi(t)\varphi(xt)C(x^2t).
\end{equation}

Repeatedly iterating (assuming \(|x| < 1\)), we obtain
\[
C(t) = \sum_{j \geq 0} f(x^j t) \prod_{i=0}^{j-1} \varphi(x^i t)
= \sum_{j \geq 0} (-1)^j \left( \frac{1 + x^{j+1}yt}{1 - x^j t} \right) x^{(j+1)j} y^j t^j \prod_{i=0}^{j-1} \frac{1 + z(1 - x^{j+1}t)}{1 - x^i t}.
\]

Recall that \( z = \frac{1}{1-x} \) which implies \( 1 + z = \frac{-t}{1-x} \). Therefore,
\[
C(t) = \sum_{j \geq 0} (-1)^j \left( \frac{1 + x^{j+1}yt}{1 - x^j t} \right) x^{(j+1)j} y^j t^j \prod_{i=0}^{j-1} \frac{1 - \frac{x^i t}{1+2}}{1 - x^i t}
= \sum_{j \geq 0} \left( 1 + x^{j+1}yt \right) x^{\frac{j+3}{2}} y^j t^j \prod_{i=0}^{j-1} \frac{1 - x^i t}{1 - x^i t}
= \sum_{j \geq 0} \left( 1 + x^{j+1}yt \right) x^{\frac{j+3}{2}} y^j t^j \frac{1}{(1-x)^j (1-x^j t)}.
\]

For further notational simplification, we let
\[
f_j = \frac{(1 + x^{j+1}yt)x^{\frac{j+3}{2}} y^j t^j}{(1-x)^j (1-x^j t)}.
\]

Finally, substituting for the remaining \( z \) as above and using partial fractions
\[
f_j = \frac{x^{1 + \frac{j+3}{2}} y^{j+1} t^j}{(1-x)^{j+1}} + \frac{x^{\frac{j+3}{2}} y^j (1-x-xy)t^j}{(1-x)^{j+1}(1-x^j t)}
= \frac{x^{1 + \frac{j+3}{2}} y^{j+1} t^j}{(1-x)^{j+1}} + \frac{x^{\frac{j+3}{2}} y^j (1-x-xy)t^j}{(1-x)^{j+1}} \sum_{k \geq 0} x^k t^k.
\]

Hence the \( m \)th coefficient of \( C(t) \) is given by
\[
C_m = \frac{x^{\frac{m+2}{2}} y^{m+1}}{(1-x)^{m+1}} + \sum_{j=0}^{m} \frac{x^{\frac{j+3}{2}} - j^2 + jm y^j (1-x-xy)}{(1-x)^{j+1}}
\]

So, we obtain the following lemma.

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Lemma 2.1. The determinants $C_m$ of the matrices obtained from $N_{m+1}$ (see equation (2.3)) by deleting its first row and column are given by

$$C_m = x^{(m+2)} \left( \frac{y}{1-x} \right)^{m+1} + \frac{1 - x - xy}{1-x} \sum_{j=0}^{m} x^{(m+1)j-(\frac{j}{2})} \left( \frac{y}{1-x} \right)^j.$$

For initial cases, we have $\det N_1 = 1$ and $\det N_2 = 1 - xy - zxy$. By a cofactor expansion along the last row, we obtain for $m \geq 2$

$$\det N_m = (1 - x^{m-1}y) \det N_{m-1} + \sum_{a=1}^{m-2} (-1)^{m-j} x^{(m) - \left(\frac{m}{2}\right)} y^{m-j} z^{m-1-j} \det N_j + (-1)^{m-1} x^{(m)} y^{m-1} z^{m-1}.$$  \hspace{1cm} (23)

Dropping $m$ by 1 and multiplying this equation by $-x^{m-1}yz$ (a similar process to that used in a previous section), we obtain for $m \geq 3$

$$-x^{m-1}yz \det N_{m-1} = -x^{m-1}yz(1 - x^{m-2}y) \det N_{m-2} + \sum_{a=1}^{m-3} (-1)^{m-j} x^{(m) - \left(\frac{m}{2}\right)} y^{m-j} z^{m-1-j} \det N_j + (-1)^{m-1} x^{(m)} y^{m-1} z^{m-1}. $$  \hspace{1cm} (24)

Subtracting (24) from (23), we obtain

$$\det N_m + x^{m-1}yz \det N_{m-1} - x^{m-1}yz(1 - x^{m-2}y) \det N_{m-2} = x^{2m-3} y^2 z \det N_{m-2}.$$ 

Hence for $m \geq 2$,

$$\det N_m = (1 - x^{m-1}y(1+z)) \det N_{m-1} + x^{m-1}yz \det N_{m-2}$$  \hspace{1cm} (25)

with $\det N_0 = 0$ and $\det N_1 = 1$.

For the rest of the paper we simplify matters by abbreviating $N_m := \det N_m$ and now define the generating function $N(t) = \sum_{m \geq 0} N_m t^m$. By multiplying equation (25) by $t^m$, summing from 1 to infinity, we obtain

$$N(t) - t = tN(t) - y(1+z)tN(t) + xyzt^2 N(t)$$

with $N_{-1} := 0$. Hence

$$N(t) = \frac{t}{1-t} + \frac{xyzt^2 - y(1+z)t}{1-t} N(t).$$  \hspace{1cm} (26)
Repeatedly iterating (26) on \( t \) (while recalling that \( z = \frac{-1}{1-x} \), and assuming \( |x| < 1 \)), we obtain

\[
N(t) = \sum_{j \geq 0} x^j t \frac{j-1}{1-x t} \prod_{b=0}^{j-1} y x^i t \frac{x^{i+1} t}{1-x t} \prod_{b=0}^{j-1} y x^i t \\
= \sum_{j \geq 0} x^j t \frac{j-1}{1-x t} \prod_{b=0}^{j-1} y x^i t \frac{x^{i+1} t}{1-x t} \prod_{b=0}^{j-1} y x^i t \\
= \sum_{j \geq 0} \frac{x^{j+3j}}{(1-x t)(1-x)^j}.
\]

Thus, we have our final lemma.

**Lemma 2.2.** With \( N_m := \det N_m \) (see (2.3))

\[
(27) \quad N_m = \left[ r^m \right] N(t) = \sum_{j=0}^{m-1} x^{mj-(\frac{j}{2})} \left( \frac{y}{1-x} \right)^j.
\]

### 2.4. The generating function \( F \)

Finally, apply (15) and (16) to (14). Then, use lemma 2.1 and lemma 2.2, to obtain:

**Theorem 2.3.** The generating function \( F = \sum_{a \geq 1, b \geq 1, s \geq 0} n(a, b, s) x^a y^b q^s \) for the number of staircases \( 1^+ 2^+ 3^+ \cdots m^+ \) (tracked by the exponent of variable \( q \)) contained in particular compositions (of \( a \) with \( b \) parts) is given by

\[
(28) \quad F = \frac{N_m - q x^m y^{\frac{m}{2}} N_{m-1}}{(1-x) x^{(m+1)} \left( \frac{y}{1-x} \right)^m + 1-x-y q N_m - \frac{x^m y^{m+1}}{1-x} N_{m-1} \right).}
\]

For example, Theorem 2.3 with \( q = 1 \) yields \( F_{q=1} = \frac{1-x}{1-x-y} \), which is the generating function for the number of compositions of \( n \) with exactly \( m \) parts (see [4]).

By differentiating the generating function \( F \) with respect to \( q \) and then substituting \( q = 1 \), we obtain

\[
\frac{dF}{dq} \bigg|_{q=1} = \frac{(1-x-xy)^2}{(1-x)^2} \left( \frac{x^{m+1}}{2} \left( \frac{y}{1-x} \right)^m \right) - \sum_{j=0}^{m-1} x^{mj-(\frac{j}{2})} \left( \frac{y}{1-x} \right)^j - \sum_{j=1}^{m-1} x^{mj-(\frac{j}{2})} \left( \frac{y}{1-x} \right)^j
\]

\[
= \frac{x^{(m+1)} y^m}{(1-x)^2 (1-x)^{m-2}} - \frac{x^{(m+1)} y^m}{(1-x)^m (j+1) (1-x)^j}
\]

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Next, we extract coefficients; firstly of $[y^l]$ to obtain

\[
(\ell - m + 1) \frac{x^{\ell + \binom{m}{2}}}{(1 - x)^{\ell}} = (\ell - m + 1) \sum_{j \geq 0} \binom{\ell + j - 1}{j} x^{\ell + j + \binom{m}{2}},
\]

and then of $[x^n]$ which leads to the following result.

**Corollary 2.4.** The total number of staircases $1^+2^+3^+\cdots m^+$ in all compositions of $n$ with exactly $\ell$ parts is given by

\[
(\ell - m + 1) \left( \binom{n - 1}{\ell - 1} - \binom{m}{2} \right).
\]

**References**


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