ON SOME IDENTITIES AND GENERATING FUNCTIONS FOR PELL-LUCAS NUMBERS

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Abstract. Generating functions for Pell and Pell-Lucas numbers are obtained. Applications are given for some results recently obtained by Mansour [12]; by using an alternative approach that considers the action of operator $\delta_{e_1e_2}^k$ to the series $\sum_{j=0}^{\infty} a_j(e_1z)^j$.

1. Introduction and the Main Result

Let us specify a second-order linear recurrence sequence $(U_n(a, b; p, q))_{n \geq 0}$, or briefly $(U_n)_{n \geq 0}$ by

$$U_{n+2} = pU_{n+1} + qU_n,$$

with $U_0 = a, U_1 = b$ and $n > 0$ (see [6, 11]) for example. This sequence was introduced in 1965 by Horadam [6, 7], which were generalized innumerable sequences, mostly depending on $p, q, a$ and $b$ (see [8]). Some examples of such sequences are the Pell number $(P_n)_{n \geq 0}$ and Pell-Lucas number sequences $(Q_n)_{n \geq 0}$; when one has $p = 2, q = b = 1, a = 0$ and $p = b = a = 2, q = 1$ respectively. In this paper we give some new generating functions for Pell and Pell-Lucas numbers.

In this contribution, we shall define a new useful operator denoted by $\delta_{e_1e_2}^k$ for which we can formulate, extend, and prove new results based on our previous ones [2, 3, 4].

In order to determine generating functions for Pell and Pell-Lucas numbers and Chebychev polynomials of second kind, we combine between our indicated past techniques and these presented polishing approaches.

In order to render the work self-contained, we give the necessary preliminary tools; we recall some definitions and results.

Definition 1.1. [1] Let $A$ and $B$ be any two alphabets, then we give $S_j(A - B)$ by the following form:

$$\prod_{b \in B}(1 - bz) \over \prod_{a \in A}(1 - az) = \sum_{j=0}^{\infty} S_j(A - B)z^j,$$

with the condition $S_j(A - B) = 0$ for $j < 0$. 

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Corollary 1.2. Taking $A = 0$ in (1.1) gives
\[ \prod_{b \in B} (1 - zb) = \sum_{j=0}^{\infty} S_j (-B)z^j. \]

Definition 1.3. [3] Given a function $g$ on $\mathbb{R}^n$, the divided difference operator is defined as follows
\[ \partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g(x_1, \cdots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}}. \]

Definition 1.4. [4] Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta_{e_1 e_2}^k$ is defined by
\[ \delta_{e_1 e_2}^k(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \text{ for all } j \in \mathbb{N}. \]

If $f(e_1) = e_1$, the operator (1.1) gives us
\[ \delta_{e_1 e_2}^k(f) = S_k(e_1 + e_2) = \partial_{e_1 e_2}^k(e_1^{k+1}). \]

Proposition 1.5. [5] Let $E = \{e_1, e_2\}$, we define the operator $\delta_{e_1 e_2}^k$ as follows:
\[ \delta_{e_1 e_2}^k f(e_1) = S_{k-1}(e_1 + e_2) f(e_1) + e_2^k \partial_{e_1 e_2} f(e_1), \text{ for all } k \in \mathbb{N}. \]

Proposition 1.6. [10] The relations
\[ \begin{align*}
1) \quad P_{-j} &= (-1)^{j+1}P_j, \\
2) \quad Q_{-j} &= (-1)^j Q_j
\end{align*} \]
hold for all $j \geq 0$.

In this paper, we shall combine all these above tools in a unified way. Then it seems that all our results can be treated as special cases of the following theorem.

Theorem 1.7. Given an alphabet $E = \{e_1, e_2\}$, two sequences $\sum_{j=0}^{+\infty} a_j z^j$, $\sum_{j=0}^{+\infty} b_j z^j$ such that
\[ \left( \sum_{j=0}^{+\infty} a_j z^j \right) \left( \sum_{j=0}^{+\infty} b_j z^j \right) = 1, \]
then
\[ \sum_{j=0}^{+\infty} a_j \partial_{e_1 e_2}^j(e_1^{k+j})z^j = \frac{k-1}{\sum_{j=0}^{+\infty} b_j e_1^j z^j} \left( \sum_{j=0}^{+\infty} b_j e_2^j z^j \right). \]

The paper is organized as follows. In Section 2 we give the proof our main Theorem. In Section 3 we give some applications for this Theorem.
2. Proof of main result

In this section, we present a proof of Theorem 1.6.

Let $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=0}^{\infty} b_j z^j$ be two sequences such that $(\sum_{j=0}^{\infty} a_j z^j) (\sum_{j=0}^{\infty} b_j z^j) = 1$.

In the first instance, since $f(e_1) = \sum_{j=0}^{\infty} a_j e_1^j z^j$, we have

$$\partial_{e_1 e_2} f(e_1) = \delta_{e_1 e_2}^{k_i} \left( \sum_{j=0}^{\infty} a_j e_1^j z^j \right) = \sum_{j=0}^{\infty} a_j \partial_{e_1 e_2} (e_1^{k_i} + j) z^j,$$

which is the left-hand side of (1.2). On the other hand, since $f(e_1) = \frac{1}{\sum_{j=0}^{\infty} b_j e_1^j z^j}$, we have

$$\partial_{e_1 e_2} f(e_1) = \frac{1}{e_1 - e_2} \left( \frac{1}{\sum_{j=0}^{\infty} b_j e_1^j z^j} - \frac{1}{\sum_{j=0}^{\infty} b_j e_2^j z^j} \right) = \frac{1}{e_1 - e_2} \left( \frac{\sum_{j=0}^{\infty} b_j e_2^j z^j - \sum_{j=0}^{\infty} b_j e_1^j z^j}{\sum_{j=0}^{\infty} b_j e_1^j z^j \sum_{j=0}^{\infty} b_j e_2^j z^j} \right) = \frac{\sum_{j=0}^{\infty} b_j \partial_{e_1 e_2} (e_1^{k_i} - e_2^{k_i}) z^j}{\left( \sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left( \sum_{j=0}^{\infty} b_j e_2^j z^j \right)} = \frac{\sum_{j=0}^{\infty} b_j S_j (e_1 + e_2) z^j}{\left( \sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left( \sum_{j=0}^{\infty} b_j e_2^j z^j \right)}.$$
By Proposition 1.5, it follows that
\[
\delta^k_{e_1e_2} f(e_1) = S_{k-1}(e_1 + e_2) f(e_1) + e_2^k \partial_{e_1e_2} f(e_1)
\]
\[
= S_{k-1}(e_1 + e_2) \left( \sum_{j=0}^{\infty} b_j S_{j-1}(e_1 + e_2) z^j \right) \left( \sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left( \sum_{j=0}^{\infty} b_j e_2^j z^j \right)
\]
\[
= \sum_{j=0}^{\infty} b_j \left[ e_2^j S_{k-1}(e_1 + e_2) - e_2^k S_{j-1}(e_1 + e_2) \right] z^j
\]
which can also be cast in the form
\[
\delta^k_{e_1e_2} f(e_1) = \sum_{j=0}^{k-1} b_j \left[ e_2^j S_{k-1}(e_1 + e_2) - e_2^k S_{j-1}(e_1 + e_2) \right] z^j
\]
\[
+ \sum_{j=k+1}^{\infty} b_j \left[ e_2^j S_{k-1}(e_1 + e_2) - e_2^k S_{j-1}(e_1 + e_2) \right] z^j
\]
\[
= \sum_{j=0}^{k-1} b_j (e_1 e_2)^j \partial_{e_1e_2} (e_1^{k-j}) z^j - (e_1 e_2 z)^k \sum_{j=0}^{\infty} b_j (e_1 e_2 z)^j (e_1^{j+1}) z^{j+1}
\]
\[
= \left( \sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left( \sum_{j=0}^{\infty} b_j e_2^j z^j \right)
\]
This completes the proof.

3. On the Generating Functions of Some Numbers and Polynomials

In this part, we derive several new generating functions of some known numbers and polynomials. Indeed, we consider our previous Theorem in order to derive Pell, Pell-Lucas numbers and Chebychev polynomials of the second kind.

Lemma 3.1. Given an alphabet \( E = \{e_1, e_2\} \), we have

\[
\sum_{j=0}^{\infty} \partial_{e_1e_2} (e_1^{j+1}) z^j = \frac{1}{1 - (e_1 + e_2)z + e_1 e_2 z^2}.
\]
Proof. Let $\sum_{j \geq 0} e_1^j z^j$ and $(1 - e_1 z)$ be two sequences such that $\sum_{j \geq 0} e_1^j z^j = \frac{1}{1 - e_1 z}$, the left-hand side of the formula (3.1) can be written as:

$$\delta_{e_1 e_2} \sum_{j=0}^{\infty} e_1^j z^j = \frac{e_1 \sum_{j=0}^{\infty} e_1^j z^j - e_2 \sum_{j=0}^{\infty} e_1^j z^j}{e_1 - e_2}$$

$$= \sum_{j=0}^{\infty} \frac{e_1^{j+1} - e_2^{j+1}}{e_1 - e_2} z^j$$

$$= \sum_{j=0}^{\infty} \partial_{e_1 e_2} (e_1^{j+1}) z^j.$$

while the right-hand side can be expressed as

$$\delta_{e_1 e_2} \left( \frac{1}{1 - e_1 z} \right) = \frac{e_1 \frac{1}{1 - e_1 z} - e_2 \frac{1}{1 - e_2 z}}{e_1 - e_2}$$

$$= \frac{e_1 (1 - e_2 z) - e_2 (1 - e_1 z)}{(e_1 - e_2)(1 - e_1 z)(1 - e_2 z)}$$

$$= \frac{e_1 - e_2}{(e_1 - e_2)(1 - (e_1 + e_2)z + e_1 e_2 z^2)}$$

$$= \frac{1}{1 - (e_1 + e_2)z + e_1 e_2 z^2}.$$

This completes the proof. □

Lemma 3.2. Given an alphabet $E = \{e_1, e_2\}$, we have

$$\sum_{j=0}^{\infty} \partial_{e_1 e_2} (e_1^{j+2}) z^j = \frac{e_1 + e_2 - e_1 e_2 z}{1 - (e_1 + e_2)z + e_1 e_2 z^2}.$$

Proof. Let $\sum_{j \geq 0} e_1^j z^j$ and $(1 - e_1 z)$ be two sequences such that $\sum_{j \geq 0} e_1^j z^j = \frac{1}{1 - e_1 z}$, the left-hand side of the formula (3.2), can be written as

$$\delta_{e_1 e_2}^2 \sum_{j=0}^{\infty} e_1^j z^j = \frac{e_1^2 \sum_{j=0}^{\infty} e_1^j z^j - e_2^2 \sum_{j=0}^{\infty} e_1^j z^j}{e_1 - e_2}$$

$$= \sum_{j=0}^{\infty} \frac{e_1^{j+2} - e_2^{j+2}}{e_1 - e_2} z^j$$

$$= \sum_{j=0}^{\infty} \partial_{e_1 e_2} (e_1^{j+2}) z^j.$$
while the right-hand side can be expressed as
\[
\delta_{e_1 e_2}^2 \left( \frac{1}{1 - e_1 z} \right) = \frac{e_1^2}{1 - e_1 z} - \frac{e_2^2}{1 - e_2 z}
\]
\[
= \frac{e_1^2(1 - e_2 z) - e_2^2(1 - e_1 z)}{(e_1 - e_2)(1 - e_1 z)(1 - e_2 z)}
\]
\[
= \frac{(e_1 - e_2)((e_1 + e_2)z + e_1 e_2 z^2)}{1 - (e_1 + e_2)z + e_1 e_2 z^2}
\]
This completes the proof. \[\square\]

This case consists of two related parts. **Firstly**, by replacing \(e_2\) by \((-e_2)\) in (3.1) and (3.2), we obtain

(3.3)
\[
\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+1})z^j = \frac{1}{1 - (e_1 - e_2)z - e_1 e_2 z^2 z'}
\]

(3.4)
\[
\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+2})z^j = \frac{e_1 - e_2 + e_1 e_2 z}{1 - (e_1 - e_2)z - e_1 e_2 z^2 z'}
\]

Choosing \(e_1\) and \(e_2\) such that \(\begin{cases} e_1 e_2 = 1 \\ e_1 - e_2 = 2 \end{cases}\) and substituting in (3.3) and (3.4), we obtain

(3.5)
\[
\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+1})z^j = \frac{1}{1 - 2z - z^2 z'}
\]

(3.6)
\[
\sum_{j=0}^{\infty} \partial_{e_1[-e_2]}(e_1^{j+2})z^j = \frac{2 + z}{1 - 2z - z^2}
\]

Multiplying the equation (3.5) by \((-2)\) and added to (3.6), we have

(3.7)
\[
\sum_{j=0}^{\infty} P_j z^j = \frac{z}{1 - 2z - z^2} \quad \text{with} \quad P_j = \partial_{e_1[-e_2]}(e_1^{j+2}) - 2 \partial_{e_1[-e_2]}(e_1^{j+1}),
\]

where (3.7) establishes the generating function for Pell numbers [12].

Multiplying (3.5) by (6) and adding to (3.6) multiplied by \((-2)\), gives the following generating function

(3.8)
\[
\sum_{j=0}^{\infty} Q_j z^j = \frac{2 - 2z}{1 - 2z - z^2} \quad \text{with} \quad Q_j = 6 \partial_{e_1[-e_2]}(e_1^{j+1}) - 2 \partial_{e_1[-e_2]}(e_1^{j+2}),
\]

Replacing \(z\) by \((-z)\) in (3.7) and (3.8), we have the following theorems.
Theorem 3.3. For $j \in \mathbb{N}$, the generating function of the Pell numbers is given by

$$\sum_{j=0}^{\infty} P_j z^j = \frac{z}{1 + 2z - z^2}. $$

Proof. The ordinary generating function associated is defined by

$$G(P, z) = \sum_{j=0}^{\infty} P_j z^j, $$

using the initial conditions, we have

$$\sum_{j=0}^{\infty} P_j z^j = P_0 + P_1 z + \sum_{j=2}^{\infty} P_j z^j$$

$$= z + \sum_{j=2}^{\infty} 2P_{j-1} z^j + \sum_{j=2}^{\infty} P_{j-2} z^j.$$ 

If we consider the case when $n = j - 2$ and $p = j - 1$, the right-hand side expand as

$$= z + 2z \sum_{j=1}^{\infty} P_j z^j + z^2 \sum_{j=2}^{\infty} P_{j-2} z^{j-2}$$

$$= z + 2z \sum_{n=0}^{\infty} P_n z^n + z^2 \sum_{p=0}^{\infty} P_p z^p,$$

which is equivalent to

$$(1 - 2z - z^2) \sum_{j=0}^{\infty} P_j z^j = z$$

$$\sum_{j=0}^{\infty} P_j z^j = \frac{z}{1 - 2z - z^2}. $$

Replacing $z$ by $(-z)$, we have

$$\sum_{j=0}^{\infty} (-1)^j P_j z^j = \frac{-z}{1 + 2z - z^2},$$

therefore

$$\sum_{j=0}^{\infty} (-1)^j+1 P_j z^j = \frac{z}{1 + 2z - z^2},$$

$$\sum_{j=0}^{\infty} P_j z^j = \frac{z}{1 + 2z - z^2}. $$

This completes the proof. □
Theorem 3.4. For \( j \in \mathbb{N} \), the generating function of the Pell-Lucas numbers is given by
\[
\sum_{j=0}^{\infty} Q_{-j} z^j = \frac{2 + 2z}{1 + 2z - z^2}.
\]

Proof. The ordinary generating function associated is defined by
\[
G(Q_j, z) = \sum_{j=0}^{\infty} Q_j z^j.
\]

Following the same procedure and using the initial conditions, we can write
\[
\sum_{j=0}^{\infty} Q_j z^j = Q_0 + Q_1 z + \sum_{j=2}^{\infty} Q_j z^j
= 2 + 2z + \sum_{j=2}^{\infty} 2Q_{j-1} z^j + \sum_{j=2}^{\infty} Q_{j-2} z^j,
\]
If we take \( n = j - 2 \) and \( p = j - 1 \). The right-hand side can be expanded as
\[
= 2 + 2z + 2z \sum_{j=1}^{\infty} Q_j z^j + z^2 \sum_{j=2}^{\infty} Q_{j-2} z^{j-2}
= 2 + 2z + 2z \left( \sum_{n=0}^{\infty} Q_n z^n - Q_0 \right) + z^2 \sum_{p=0}^{\infty} Q_p z^p,
\]
which is equivalent to
\[
(1 - 2z - z^2) \sum_{j=0}^{\infty} Q_j z^j = 2 - 2z
\]
\[
\sum_{j=0}^{\infty} Q_j z^j = \frac{2 - 2z}{1 - 2z - z^2}.
\]
Replacing \( z \) by \((-z)\), we have
\[
\sum_{j=0}^{\infty} (-1)^j Q_j z^j = \frac{2 + 2z}{1 + 2z - z^2},
\]
therefore
\[
\sum_{j=0}^{\infty} Q_{-j} z^j = \frac{2 + 2z}{1 + 2z - z^2}.
\]
This completes the proof. \( \square \)

Lastly, replacing \( e_1 \) by \( 2e_1 \) and \( e_2 \) by \((-2e_2)\) in (3.3) and (3.4), and under the condition \( 4e_1 e_2 = -1 \), we obtain, for \( x = e_1 - e_2 \),
\[
\sum_{j=0}^{\infty} U_j(x) z^j = \frac{1}{1 - 2xz + z^2}, \text{ such that } U_j(x) = \partial_{2e_1[-2e_2]}(e_1^{j+1}).
\]

(3.9)

\[
\sum_{j=0}^{\infty} \partial_{2e_1[-2e_2]}(e_1^{j+2}) z^j = \frac{2x - z}{1 - 2xz + z^2},
\]

which represents a generating function for Chebychev polynomials of second kind,[2] such that \(\partial_{2e_1[-2e_2]}(e_1^{j+2}) = (2x - z)U_j(e_1 - e_2).\)

Moreover, we deduce from (3.9), the following generating function:

\[
\sum_{j=0}^{\infty} \left[ \partial_{2e_1[-2e_2]}(e_1^{j+2}) - x\partial_{2e_1[-2e_2]}(e_1^{j+1}) \right] z^j = \frac{2x - (x + 1)z}{1 - 2xz + z^2}.
\]

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References


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