IRREDUCIBLE FACTORS OF THE q-LAH NUMBERS OVER $\mathbb{Z}$

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ABSTRACT. In this paper, we first give a new q-analogue of the Lah numbers. Then we show the irreducible factors of the q-Lah numbers over $\mathbb{Z}$.

1. INTRODUCTION

In combinatorics, the Lah numbers are used to count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets. These numbers were first discovered by Ivo Lah in 1955. Usually, the Lah numbers were denoted by $L(n,k)$ and defined as

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}. \tag{1}$$

It is straightforward that $L(n,k)$ can also be represented as

$$L(n,k) = \binom{n}{k}\frac{(n-1)!}{(k-1)!}. \tag{2}$$

Another kind of numbers related to Lah numbers is r-Lah numbers, which were denoted by $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r$. The r-Lah numbers were used to count the number of partitions of the set $\{1,2,\cdots,n\}$ into k nonempty ordered lists, such that the numbers $1,2,\cdots,r$ are in distinct lists. The r-Lah numbers have the following explicit formula

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \binom{n-r}{k-r}\frac{(n+r-1)!}{(k+r-1)!} = \binom{n+r-1}{k+r-1}\frac{(n-r)!}{(k-r)!}. \tag{3}$$

There are some results on r-Lah numbers recently, see for example [1, 2, 3].

In this paper, we would like to introduce the q-analogue of the Lah numbers given by (1), which we call them q-Lah numbers. Before introducing the q-Lah numbers, several notations need to be introduced.

The basic notation of this paper is the quantum factorial symbol, which is defined as

$$(x;q)_n := \prod_{k=0}^{n-1}(1-xq^k), \quad 0<q<1.$$

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Let x be a real number, the q-real number of x is defined as

$$[x]_q = \frac{1-q^x}{1-q}.$$

In particular, when k is a positive integer, $[k]_q = 1 + q + \cdots + q^k$ is called q-positive integer. It is clear that $[k]_q$ is irreducible over $\mathbb{Z}$. The k-th order factorial of the q-number $[x]_q$ is defined as

$$[x]_{k,q} = [x]_q[x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)^k}.$$

In particular, $[k]_q! = [1]_q[2]_q \cdots [k]_q$ is called the q-factorial. The q-binomial coefficient is defined as

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{[x]_{k,q}}{[k]_q} = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

In particular, for a positive integer n,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

With these concepts, we can now define the q-Lah numbers as

$$L_q(n,k) := \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{[n]_q!}{[k]_q!}. \tag{4}$$

Actually, at the end of Wagner's beautiful paper [4], Wagner also gave three kinds of q-Lah numbers as follows,

$$\hat{L}_q(n,k) := \frac{[n]_q!}{k!}\binom{n-1}{k-1},$$

$$\tilde{L}_q(n,k) := \frac{n!}{k!}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q,$$

and

$$L_q(n,k) := q^{\frac{k(k-1)}{2}}\tilde{L}_q(n,k).$$

The reason why we use the q-Lah numbers defined by (4) instead of these three kinds of q-Lah number is though these three kinds of q-Lah numbers are q-analogues of the Lah numbers, they do not generalize $L(n,k)$ "completely". In other words, these three kinds of q-Lah numbers do not hold the property like Property 2.2 we will show in next section. However, both Lah numbers and r-Lah numbers hold the same property. In the remaining of the paper, the term q-Lah numbers and the notation $L_q(n,k)$ refer to the q-Lah numbers defined by (4).

In next section, we will discuss the basic properties of the q-Lah numbers $L_q(n,k)$. Next, we would like to introduce another concept, the n-cyclotomic polynomial. Let $\Phi_n(q)$ be the n-cyclotomic polynomial,

$$\Phi_n(q) = \prod_{\substack{0\leqslant m<n\\ \gcd(m,n)=1}} (q - \mathrm{e}^{2\pi m i/n}).$$

It is well-known that $\Phi_n(q) \in \mathbb{Z}[q]$ is the irreducible polynomial for $\mathrm{e}^{2\pi i/n}$. The polynomial $x^n - 1$ has the following factorization into irreducible polynomials over $\mathbb{Z}$:

$$x^n - 1 = \prod_{j|n} \Phi_j(x). \tag{5}$$

With these definitions and notations in hand, we can now start our discussion.

2. Basic Properties of q-Lah Numbers

In these section, we would like to introduce two basic properties of q-Lah numbers.

Property 2.1. *The relation between q-Lah numbers and Lah numbers is*

$$\lim_{q\to 1} L_q(n,k) = L(n,k).$$

This relation can be obtained by the following two limits.

$$\begin{aligned}
&\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \lim_{q\to 1} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \\
=&\lim_{q\to 1} \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})} \\
=&\lim_{q\to 1} \frac{(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})}{(1+q)\cdots(1+q+\cdots q^{k-1})(1+q)\cdots(1+q+\cdots+q^{n-k+1})} \\
=&\frac{1\times 2\times\cdots\times(n-1)\times n}{1\times 2\times\cdots\times(k-1)\times k\times 1\times 2\times\cdots\times(n-k+1)\times(n-k)} \\
=&\frac{n!}{k!(n-k)!} = \binom{n}{k}.
\end{aligned}$$

$$\begin{aligned}
&\lim_{q\to 1}[n]_q! = \lim_{q\to 1}[1]_q[2]_q\cdots[n]_q \\
=&\lim_{q\to 1} \frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^n}{1-q} \\
=&\lim_{q\to 1}(1+q)\cdots(1+q+\cdots+q^{n-1}) \\
=&1\times 2\times\cdots\times n = n!.
\end{aligned}$$

By Property 2.1, we can now claim that q-Lah numbers are q-analogue of Lah numbers.

Just like (2) and (3) for Lah numbers and r-Lah numbers, q-Lah numbers also have the same relation, which can be stated as the following property.

Property 2.2. *The q-Lah numbers have the following formula*

$$L_q(n,k) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{[n]_q!}{[k]_q!} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[n-1]_q!}{[k-1]_q!}. \tag{6}$$

Proof. By the definitions showed in last section, we have

$$\begin{aligned} L_q(n,k) &= \frac{(q;q)_{n-1}}{(q;q)_{k-1}(q;q)_{n-k}} \frac{(q;q)_n}{(q;q)_k} \frac{(1-q)^k}{(1-q)^n} \\ &= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \frac{(q;q)_{n-1}}{(q;q)_{k-1}} \frac{(1-q)^{k-1}}{(1-q)^{n-1}} \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[n-1]_q!}{[k-1]_q!}. \end{aligned}$$

Thus, (6) holds true. ☐

3. Irreducible Factors of the q-Lah Numbers

In this section, we will mainly focus on irreducible factors of the q-Lah numbers over $\mathbb{Z}$. It is clear that the q-Lah numbers always have the following irreducible factors over $\mathbb{Z}$,

$$[2]_q, \cdots, [n]_q.$$

This is because $L_q(n,k) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{[n]_q!}{[k]_q!}$, $[n]_q! = \prod_{i=1}^n [i]_q = \prod_{i=2}^n [i]_q$ and we have mentioned above that $[i]_q$ is irreducible over $\mathbb{Z}$. So, we call $[2]_q, \cdots, [n]_q$ the trivial irreducible factors of the q-Lah numbers over $\mathbb{Z}$. By these trivial irreducible factors, we get the following congruence

$$L_q(n,k) \equiv 0 \pmod{\prod_{i=2}^{n} [i]_q}.$$

Except for these trivial irreducible factors, q-Lah numbers also have some other irreducible factors under certain condition.

Theorem 3.1. *Suppose $2k \le n+1$. Not including these trivial irreducible factors mentioned above, the q-Lah numbers $L_q(n,k)$ have at least other k irreducible factors over $\mathbb{Z}$.*

Proof. By the definition of q-Lah numbers, we have

$$L_q(n,k) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{[n]_q!}{[k]_q!}.$$

Since [5, Theorem 2] is important to our statement. Let us recall it here. Let $\lfloor x \rfloor$ stands for the largest integer less than or equal to x. Then by (5), we can get that

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \\ &= \frac{\prod_{i=1}^{\infty}\big(\Phi_i(q)\big)^{\lfloor n/i \rfloor}}{\prod_{i=1}^{\infty}\big(\Phi_i(q)\big)^{\lfloor k/i \rfloor}\prod_{i=1}^{\infty}\big(\Phi_i(q)\big)^{\lfloor (n-k)/i \rfloor}} \\ &= \prod_{i=1}^{n}\big(\Phi_i(q)\big)^{\lfloor n/i \rfloor - \lfloor k/i \rfloor - \lfloor (n-k)/i \rfloor}.\end{aligned}$$

Suppose $n-k+1 \leq i \leq n$. When $2k \leq n$, we have that $i \geq k+1$. So,

$$\left\lfloor \frac{n}{i} \right\rfloor = 1, \quad \left\lfloor \frac{k}{i} \right\rfloor = \left\lfloor \frac{n-k}{i} \right\rfloor = 0.$$

These give us that when $2k \leq n$, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has at least k irreducible factors over $\mathbb{Z}$: $\Phi_{n-k+1}(q)$, $\Phi_{n-k+2}(q)$, $\cdots$, $\Phi_n(q)$. By this statement, we can say that when $2k \leq n+1$, $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ has at least $k-1$ irreducible factors over $\mathbb{Z}$: $\Phi_{n-k+1}(q), \Phi_{n-k+2}(q), \cdots, \Phi_{n-1}(q)$.

There are two ways to show that $\Phi_n(q)$ is also an irreducible factor of $L_q(n,k)$. The first way is as follows. By (5), we can obtain that

$$(q;q)_n = (-1)^n \prod_{i=1}^{n} \Phi_i^{\lfloor n/i \rfloor}(q)$$

which implies that $\Phi_n(q)$ is also an irreducible factor of $L_q(n,k)$.

Another way to show this is to use Property 2.2. By Property 2.2, we have

$$L_q(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[n-1]_q!}{[k-1]_q!}.$$

Then according to [5, Theorem 2], we know that $\Phi_n(q)$ is also an irreducible factor of $L_q(n,k)$.

To sum up, we can say that $\Phi_{n-k+1}(q), \Phi_{n-k+2}(q), \cdots, \Phi_n(q)$ are k irreducible factors of $L_q(n,k)$ in this case. □

According to this conclusion, we can derive the following corollary at once.

Corollary 3.2. *For $2k \leq n+1$, we have the following congruence*

$$L_q(n,k) \equiv 0 \pmod{\prod_{i=0}^{k-1} \Phi_{n-i}(q)}.$$

Here, we would like to introduce another irreducible factor of a very special $L_q(n,k)$.

Theorem 3.3. *For the special case $n = 2k$. In addition to these trivial irreducible factors and the k irreducible factors mentioned in Theorem 3.1, $1 + q^k$ is also an irreducible factor.*

Proof. Since [6, Proposition 3.7]

$$1 + q^k = \frac{1 + q + \cdots + q^{k-1} + q^k(1 + q + \cdots + q^{k-1})}{(1 + q + \cdots + q^{k-1})} = \frac{1 + \cdots + q^{2k-1}}{1 + \cdots + q^{k-1}} = \frac{(1 - q^{2k})}{(1 - q^k)}.$$

Thus,

$$(1 + q^k) \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q = \frac{(1 - q^{2k})}{(1 - q^k)} \frac{(q;q)_{2k-1}}{(q;q)_k (q;q)_{k-1}} = \frac{(q;q)_{2k}}{(q;q)_k^2} = \begin{bmatrix} 2k \\ k \end{bmatrix}_q .$$

Furthermore, by the definition of Gaussian polynomials, it is clear that $\begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q = \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}_q$. These give us that

$$\begin{bmatrix} 2k \\ k \end{bmatrix}_q = (1 + q^k) \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q = (1 + q^k) \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}_q .$$

Then the conclusion follows from the above formula and Property 2.2. □

With this we can get the following congruence

$$L_q(2k, k) \equiv 0 \pmod{(1 + q^k)}.$$

Theorem 3.4. *Let $\{x\}$ denote the fractional part of x. $\Phi_i(q)$ $(1 \le i \le n)$ is an irreducible factor of $L_q(n, k)$ if and only if $\{k/i\} > \{n/i\}$.*

Proof. It is straightforward that $\Phi_i(q)$ $(1 \le i \le n)$ is an irreducible factor of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ if and only if $\{k/i\} > \{n/i\}$. So, we can get that $\Phi_i(q)$ $(1 \le i \le n)$ is an irreducible factor of $L_q(n, k)$ if and only if $\{k/i\} > \{n/i\}$ because $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a factor of $L_q(n, k)$. □

According to this theorem, we have

Corollary 3.5. *The congruence*

$$L_q(n, k) \equiv 0 \pmod{\prod_{i=1}^{n} \Phi_i(q)}$$

holds if and only if $\{k/i\} > \{n/i\}$.

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