ON SOME IDENTITIES AND GENERATING FUNCTIONS FOR K-PELL SEQUENCES AND CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we introduce a new operator in order to derive some properties of homogeneous symmetric functions. By making use of the proposed operator, we give some new generating functions for k-Fibonacci numbers, k-Pell numbers and product of sequences and Chebyshev polynomials of second kind.

1. **Introduction and Notations**

There are a lot of integer sequences such as Fibonacci, Pell, Lucas, etc. Pell and Pell-Lucas numbers are used by scientists for basic theories and their applications. For interest application of these numbers in science and nature [26], one can see [20, 21, 15, 23]. For instance, in science, authors gave sums of the generalized Pell numbers could be derived directly using a new matrix representation [28]. In [19], Horadam showed that some properties involving Pell numbers and gave the formula

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n,$$

for the Pell numbers. Also Ercolano [18], found the matrix A for generating the Pell sequence as follows

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

In [21], Horadam and Mahon obtained Simpson formula for the Pell-Lucas numbers as follows

$$Q_{n+1}Q_{n-1} - Q_n^2 = 8(-1)^{n+1}.$$

For the rest of this paper, for $n \geq 2$ the well known Pell $(P_n)_{n\geq 2}$ and Pell-Lucas numbers $(Q_n)_{n\geq 2}$ are defined by

$$P_n = 2P_{n-1} + P_{n-2} \text{ and } Q_n = 2Q_{n-1} + Q_{n-2},$$

with initial conditions given by $P_0 = 0,\ P_1 = 1$ and $Q_0 = Q_1 = 2$, respectively.

In [1, 3], Horadam gave some equations related to Pell numbers and generating functions for powers of a certain generalized sequences of numbers. Falcon, in [25], introduced the k-Lucas sequences by using a special sequence of squares of k-Fibonacci numbers. Recently, Catarino and Vasco have considered the k-Pell numbers, k-Pell

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Lucas numbers and have presented some properties involving these number sequences [16, 17].

In this contribution, we shall define a new useful operator denoted by $\delta^k_{p_1p_2}$ for which we can formulate, extend and prove new results based on our previous ones [6, 7, 12], in order to determine generating functions of the product of k-Fibonacci numbers, k-Pell numbers, and Chebyshev polynomials of second kind.

In order to render the work self-contained we give the necessary preliminaries. We recall some definitions and results.

Definition 1.1. [10] *Let $B = \{b_1, b_2, ...\}$ and $P = \{p_1, p_2, ...\}$be any two alphabets. We define $S_n(B-P)$ by the following form*

$$\frac{\Pi_{p\epsilon P}(1-pt)}{\Pi_{b\epsilon B}(1-bt)} = \sum_{n=0}^{\infty} S_n(B-P)t^n, \tag{1.1}$$

with the condition $S_n(B-P) = 0$ for $n < 0$.

Equation (1.1) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(B-P)t^n = \left(\sum_{n=0}^{\infty} S_n(B)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right),$$

where

$$S_n(B-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(B). \tag{1.2}$$

We know that the polynomial whose roots are P is written as

$$S_n(x-P) = \sum_{j=0}^{n} S_{n-j}(-P)x^n, \text{ with } card(P) = n.$$

On the other hand, if B has cardinality equal to 1, i.e., $B = \{x\}$, then (1.1) can be rewritten as follows [12]:

$$\sum_{n=0}^{\infty} S_n(x-P)t^n = \frac{\prod_{p\in P}(1-pt)}{(1-xt)} = 1 + \cdots + S_{n-1}(x-P)t^{n-1} + \frac{S_n(x-P)}{(1-xt)}t^n,$$

where $S_{n+k}(x-P) = x^k S_n(x-P)$ for all $k \geq 0$.

The summation is actually limited to a finite number of terms since $S_{-k}(\cdot) = 0$ for all $k > 0$. In particular, we have

$$\prod_{p\in P}(x-p) = S_n(x-P) = S_0(-P)x^n + S_1(-P)x^{n\ -1} + S_2(-P)x^{n\ -2} + \cdots,$$

where $S_k(-B)$ are the coefficients of the polynomials $S_n(x-P)$ for $0 \leq k \leq n$. These coefficients are zero for $k > n$.

For example, if all $p \in P$ are equal, i.e., $P = np$, then we have $S_n(x-np) = (x-p)^n$.

By choosing $p = 1$, i.e., $P = \left\{ \underbrace{1, 1, ...1}_{n} \right\}$, we obtain

$$S_k(-n) = (-1)^k \binom{n}{k} \text{ and } S_k(n) = \binom{n+k-1}{k}. \tag{1.3}$$

By combining (1.2) and (1.3), we obtain the following expression

$$S_n(B - nx) = S_n(B) - \binom{n}{1} S_{n-1}(B)x + \binom{n}{2} S_{n-2}(B)x^2 - \cdots + (-1)^n \binom{n}{n} x^n.$$

Definition 1.2. [9] *Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows*

$$\partial_{p_i p_{i+1}}(f) = \frac{f(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) - f(p_1, \cdots p_{i-1}, p_{i+1}, p_i, p_{i+2} \cdots p_n)}{p_i - p_{i+1}}.$$

Definition 1.3. *The symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by*

$$\delta^k_{p_1 p_2}(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in \mathbb{N}. \tag{1.5}$$

Proposition 1.4. [11] *Let $P = \{p_1, p_2\}$ an alphabet, we define the operator $\delta^k_{p_1 p_2}$ as follows*

$$\delta^k_{p_1 p_2} g(p_1) = S_{k-1}(p_1 + p_2) g(p_1) + p_2^k \partial_{p_1 p_2} g\ (p_1), \text{ for all } k \in \mathbb{N}.$$

Proposition 1.5. [6] *The relations*

$$\begin{aligned} 1) F_{k,-n} &= (-1)^{n+1} F_{k,n}, \\ 2) P_{k,-n} &= (-1)^{n+1} P_{k,n} \end{aligned}$$

hold for all $n \geq 0$.

2. The k-Pell Numbers and Properties

The k-Pell numbers have been defined in [16] for any number k as follows.

Definition 2.1. [16] *For any positive real number k, the k-Pell numbers, say $\{P_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by*

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1} \text{ for } n \geq 1, \tag{2.1}$$

with initial conditions $P_{k,0} = 1; P_{k,1} = 1$.

• If $k = 1$, the classical Pell numbers is obtained:
$P_0 = 0, P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$:
$\{P_n\}_{n \in \mathbb{N}} = \{0, 1, 2, 5, 8, 21, ...\}$

The well-known Binet's formula in the *Pell numbers theory* [17] allows us to express the k-Pell number in function of the roots r_1 and r_2 of the characteristic equation, associated to the recurrence relation (2.1):

$$r^2 = 2r + k. \tag{2.2}$$

Proposition 2.2. *(Binet's formula) The nth k-Pell number is given by*

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where $r_1,\ r_2$ are the roots of the characteristic equation (2.2) and $r_1 > r_2$.

Proof. The roots of the characteristic equation (2.2) are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$.

Note that, since $k > 0$, the
$r_2 < 0 < r_1$ and $|r_2| < |r_1|$,
$r_1 + r_2 = 2$ and $r_1.r_2 = -k$,
$r_1 - r_2 = 2\sqrt{1+k}$. □

If σ denotes the positive root of the characteristic equation, the general term may be written in the form [16]

$$P_{k,n} = \frac{\sigma^n - \sigma^{-n}}{\sigma + \sigma^{-1}}.$$

and the limit of the quotient of two terms is

$$\lim_{n\to\infty} \frac{P_{k,n+1}}{P_{k,n}} = \sigma.$$

3. **On the Generating Functions**

In our main result, we will combine all these results in a unified way such that they can be considered as a special case of the following Theorem.

Theorem 3.1. *Given two alphabets $P = \{p_1, p_2\}$ and $B = \{b_1, b_2, ..., b_n\}$, we have*

$$\sum_{n=0}^{\infty} S_n(B)\delta_{p_1p_2}^n(p_1)t^n = \frac{\sum\limits_{n=0}^{\infty} S_n(-B)\delta_{p_1p_2}(p_2^n)t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}. \tag{3.1}$$

Proof. By applying the operator $\delta_{p_1p_2}$ to the series $g(p_1t) = \sum_{n=0}^{\infty} S_n(B)p_1^n t^n$, we have

$$\begin{aligned}
\delta_{p_1p_2}g(p_1t) &= \delta_{p_1p_2}\left(\sum_{n=0}^{\infty} S_n(B)p_1^n t^n\right)\\
&= \frac{p_1\sum_{n=0}^{\infty} S_n(B)p_1^n t^n - p_2\sum_{n=0}^{\infty} S_n(B)p_2^n t^n}{p_1-p_2}\\
&= \sum_{n=0}^{\infty} S_n(B)\left(\frac{p_1^{n+1}-p_2^{n+1}}{p_1-p_2}\right)t^n\\
&= \sum_{n=0}^{\infty} S_n(B)\delta_{p_1p_2}^n(p_1)t^n.
\end{aligned}$$

Which is the left-hand side of (3.1). On the other part, setting

$$g(p_1t) = \frac{1}{\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n},$$

we have

$$\delta_{p_1p_2}g(p_1t) = \frac{p_1\prod_{b\in B}(1-bp_2)t - p_2\prod_{b\in B}(1-bp_1t)}{(p_1-p_2)\left(\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}.$$

Using the fact that : $\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n = \prod_{b\in B}(1-bp_1t)$, then

$$\begin{aligned}
\delta_{p_1p_2}g(p_1t) &= \frac{\sum_{n=0}^{\infty} S_n(-B)\frac{p_1p_2^n-p_2p_1^n}{p_1-p_2}t^n}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}\\
&= \frac{\sum_{n=0}^{\infty} S_n(-B)\delta_{p_1p_2}(p_2^n)t^n}{\left(\sum_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}.
\end{aligned}$$

This completes the proof. □

We now derive new generating functions of the products of some well-known polynomials. Indeed, we consider Theorem 1 in order to derive Fibonacci numbers and Chebychev polynomials of second kind and the symmetric functions.

Theorem 3.2. [8] *Given two alphabets $P = \{p_1, p_2\}$ and $B = \{b_1, b_2, b_3\}$, we have*

$$\sum_{n=0}^{\infty} S_n(B)\delta^n_{p_1p_2}(p_1)t^n = \frac{S_0(-B) - p_1p_2S_2(-B)t^2 - p_1p_2S_3(-B)\,(p_1+p_2)\,t^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}. \tag{3.2}$$

Case 1: For $p_1 = b_1 = 1$, $b_2 = y$ and $p_2 = x$, $b_3 = \alpha$ in Theorem 2, we propose the following new generating function

$$\sum_{n=0}^{\infty} \delta^n_{x1}(x)S_n(1+y+\alpha)t^n = \frac{1 - x(y+\alpha+\alpha y)t^2 - xy\alpha(1+x)t^3}{(1-t)\,(1-xt)\,(1-yt)\,(1-xyt)\,(1-\alpha t)(1-\alpha xt)}.$$

Remark 3.1. *For $\alpha = 0$, we obtain the following identity of Ramanujan* [12, 13]

$$\sum_{n=0}^{\infty} \delta^n_{x1}(x)S_n(1+y)t^n = \frac{1-xyt^2}{(1-t)\,(1-xt)\,(1-yt)\,(1-xyt)}.$$

Case 2: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = 1$, $p_1 - p_2 = k$ in Theorem 2, we derive a new generating function of both k-Fibonacci numbers and symmetric functions in several variables as follows

$$\sum_{n=0}^{\infty} S_n(B)F_{k,n}t^n = \frac{1 - S_2(-B)t^2 - kS_3(-B)t^3}{\prod i = 1^3\left(1 - kb_it - b_i^2t^2\right)}. \tag{3.3}$$

Replacing t by $(-t)$ in (3.3), we have the following corollary.

Corollary 3.2. [5] *We have the following generating function of both k-Fibonacci numbers at negative indices and symmetric functions in several variables as*

$$\sum_{n=0}^{\infty} S_n(B)F_{k,-n}t^n = \frac{-1 + S_2(-B)t^2 - kS_3(-B)t^3}{\prod_{i=1}^{3}\left(1 + kb_it - b_i^2t^2\right)}.$$

- Put $k = 1$ in the relationship (3.3) we have

$$\sum_{n=0}^{\infty} S_n(B)F_nt^n = \frac{1 + (b_1b_2 + b_1b_3 + b_2b_3)\,t^2 + b_1b_2b_3t^3}{\prod\limits_{i=1}^{3}\left(1 - b_it - b_i^2t^2\right)},$$

  representing a generating function of Fibonacci numbers and symmetric functions in several variables [8].

Setting $b_3 = 0$ and replacing b_2 by $(-b_2)$ in (3.3), and assuming $b_1 - b_2 = k$; $b_1b_2 = 1$; we deduce the following corollary.

Corollary 3.3. [9] *For $n \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F^2_{k,n}t^n = \frac{1-t^2}{1 - k^2t - 2(k^2+1)t^2 - k^2t^3 + t^4}.$$

Case 3: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = k,\ p_1 - p_2 = 2$ in Theorem 2, we derive a new generating function of both k-Pell numbers and symmetric functions in several variables as follows

$$\sum_{n\ =0}^{\infty} S_{n-1}(B)P_{k,n}t^n = \frac{t - kS_2(-B)t^3 - 2kS_3(-B)t^4}{\prod_{i=1}^{3}\left(1 - 2b_it - kb_i^2t^2\right)}. \tag{3.4}$$

Replacing t by $(-t)$ in (3.4) , we have the following corollary.

Corollary 3.4. *We have the following a new generating function of both k-Pell numbers at negative indices and symmetric functions in several variables as*

$$\sum_{n\ =0}^{\infty} S_{n-1}(B)P_{k,-n}t^n = \frac{t - kS_2(-B)t^3 + 2kS_3(-B)t^4}{\prod_{i=1}^{3}\left(1 - 2b_it - kb_i^2t^2\right)}.$$

- Put $k = 1$ in the relationship (3.4) we have

$$\sum_{n\ =0}^{\infty} S_{n-1}(B)P_nt^n = \frac{t - S_2(-B)t^3 - 2S_3(-B)t^4}{\prod_{i=1}^{3}\left(1 - 2b_it - b_i^2t^2\right)},$$

  which representing a new generating function of Pell numbers and symmetric functions in several variables.

Setting $b_3 = 0$ and replacing b_2 by $(-b_2)$ in (3.4), and assuming $b_1 - b_2 = 2$; $b_1b_2 = k$; we deduce the following corollary.

Corollary 3.5. *For $n \in \mathbb{N}$, the generating function of the product of k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} P_{k,n}^2t^n = \frac{t - k^2t^3}{1 - 4t - (2k^2 + 8k)t^2 - 4k^2t^3 + k^4t^4}.$$

We have the following theorems.

Theorem 3.3. *For $n \in \mathbb{N}$, The new generating function of the product of k-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} L_{k,n}^2t^n = \frac{4 - 3k^2t - 4(k^2 + 1)t^2 - k^2t^3}{1 - k^2t - 2(k^2 + 1)t^2 - k^2t^3 + t^4}. \tag{3.9}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} L_{k,n}^2 t^n &= \sum_{n=0}^{\infty}[(2+k^2)S_n(e_1+[-e_2])-kS_{n+1}(e_1+[-e_2])]\\ &\quad\times[(2+k^2)S_n(a_1+[-a_2])-kS_{n+1}(a_1+[-a_2])]t^n\\ &= (2+k^2)^2\sum_{n=0}^{\infty}S_n(e_1+[-e_2])S_n(a_1+[-a_2])t^n-\\ &\quad k(2+k^2)\sum_{n=0}^{\infty}S_{n+1}(e_1+[-e_2])S_n(a_1+[-a_2])t^n\\ &\quad -k(2+k^2)\sum_{n=0}^{\infty}S_{n+1}(a_1+[-a_2])S_n(e_1+[-e_2])t^n+\\ &\quad k^2\sum_{n=0}^{\infty}S_{n+1}(e_1+[-e_2])S_{n+1}(a_1+[-a_2])t^n\\ &= (2+k^2)^2\sum_{n=0}^{\infty}F_{k,n}^2t^n-k(2+k^2)\end{aligned}$$

$$\times\left[\frac{k+(a_1-a_2)t}{1-k(a_1-a_2)t-[(a_1-a_2)^2+2a_1a_2+k^2a_1a_2]t^2-k(a_1-a_2)a_1a_2t^3+a_1^2a_2^2t^4}\right]$$

$$-k(2+k^2)\left[\frac{k+(e_1-e_2)t}{1-k(e_1-e_2)t-[(e_1-e_2)^2+2e_1e_2+k^2e_1e_2]t^2-k(e_1-e_2)e_1e_2t^3+e_1^2e_2^2t^4}\right]$$

$$+k^2\left[\frac{k(a_1-a_2)+[(a_1-a_2)^2+a_1a_2+k^2a_1a_2]t+ka_1a_2(a_1-a_2)t^2-a_1^2a_2^2t^3}{1-k(a_1-a_2)t-[(a_1-a_2)^2+2a_1a_2+k^2a_1a_2]t^2-k(a_1-a_2)a_1a_2t^3+a_1^2a_2^2t^4}\right].$$

Since

$$\sum_{n=0}^{\infty}F_{k,n}^2t^n=\frac{1-t^2}{1-k^2t-2(k^2+1)t^2+k^2t^3+t^4}.$$

Therefore

$$\sum_{n=0}^{\infty}L_{k,n}^2t^n=\frac{4-3k^2t-4(k^2+1)t^2-k^2t^3}{1-k^2t-2(k^2+1)t^2-k^2t^3+t^4}.$$

This completes the proof. □

Theorem 3.4. *For $n\in\mathbb{N}$, The new generating function of the product of k-Pell Lucas numbers is given by*

$$\sum_{n=0}^{\infty}Q_{k,n}^2t^n=\frac{4-12t-4(4k+k^2)t^2-4k^2t^3}{1-4t-2(4k+k^2)t^2-4k^2t^3+k^4t^4}. \tag{3.10}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{k,n}^2 t^n &= \sum_{n=0}^{\infty} [S_{n+1}(e_1+[-e_2]) - (k+2)S_{n-1}(e_1+[-e_2])] \\ &\quad \times [S_{n+1}(a_1+[-a_2]) - (k+2)S_{n-1}(a_1+[-a_2])]t^n\end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} S_{n+1}(e_1+[-e_2])S_{n+1}(a_1+[-a_2])t^n - (k+2)\right) \times$$

$$\left(\sum_{n=0}^{\infty} S_{n+1}(e_1+[-e_2])S_{n-1}(a_1+[-a_2])t^n\right)$$

$$-(k+2)\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])t^n +$$

$$(k+2)^2 \sum_{n=0}^{\infty} S_{n-1}(e_1+[-e_2])S_{n-1}(a_1+[-a_2])t^n$$

$$= \left[\frac{2(a_1-a_2) + \left[k(a_1-a_2)^2 + ka_1a_2 + 4a_1a_2\right]t + 2ka_1a_2(a_1-a_2)t^2 - k^2a_1^2a_2^2t^3}{1 - 2(a_1-a_2)t - \left[k(a_1-a_2)^2 + 2ka_1a_2 + 4a_1a_2\right]t^2 - 2k(a_1-a_2)a_1a_2t^3 + k^2a_1^2a_2^2t^4}\right]$$

$$-(k+2)\left[\frac{(k+4)t + 2k(a_{1-}a_2)t^2 - k^2a_1a_2t^3}{1 - 2(a_1-a_2)t - \left[k(a_1-a_2)^2 + 2(k+2)a_1a_2\right]t^2 - 2k(a_1-a_2)a_1a_2t^3 + k^2a_1^2a_2^2t^4}\right]$$

$$-(k+2)\left[\frac{(k+4)t + 2k(e_{1-}e_2)t^2 - k^2e_1e_2t^3}{1 - 2(e_1-e_2)t - \left[k(e_1-e_2)^2 + 2(k+2)e_1e_2\right]t^2 - 2k(e_1-e_2)e_1e_2t^3 + k^2e_1^2e_2^2t^4}\right]$$

$$+(k+2)^2\sum_{n=0}^{\infty} P_{k,n}^2 t^n.$$

Since

$$\sum_{n=0}^{\infty} P_{k,n}^2 t^n = \frac{t - k^2t^3}{1 - 4t - 2(4k+k^2)t^2 - 4k^2t^3 + k^4t^4}.$$

Therefore

$$\sum_{n=0}^{\infty} Q_{k,n}^2 t^n = \frac{4 - 12t - 4(4k+k^2)t^2 - 4k^2t^3}{1 - 4t - 2(4k+k^2)t^2 - 4k^2t^3 + k^4t^4}.$$

This completes the proof. □

Case 4: Replacing p_1 by $2p_1$ and p_2 by $(-2p_2)$, and assuming that $4p_1p_2 = -1$ in Theorem 2 allows us to deduce the Chebyshev polynomials of second kind and the symmetric functions in several variables, as follows for $y = p_1 - p_2$,

$$\sum_{n=0}^{\infty} S_n(B)U_n(y)t^n = \frac{1 - S_2(-B)t^2 - S_3(-B)t^3}{\left(1 - 2b_1yt - b_1^2t^2\right)\left(1 - 2b_2yt - b_2^2t^2\right)\left(1 - 2b_3yt - b_3^2t^2\right)}. \tag{3.5}$$

Theorem 3.5. *The new generating function of the product of Chebyshev polynomials of first kind and the symmetric functions in several variables as*

(3.6)
$$\sum_{n=0}^{\infty} S_n(B)T_n(y)t^n = \frac{1 - yS_1(-B)t + S_2(-B)(2y^2-1)t^2 + S_3(-B)(y-4y^3-1)t^3}{\left(1-2b_1yt-b_1^2t^2\right)\left(1-2b_2yt-b_2^2t^2\right)\left(1-2b_3yt-b_3^2t^2\right)}.$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(B)T_n(y)t^n &= \sum_{n=0}^{\infty} S_n(B)(S_n((2p_1)+(-2p_2)) - yS_n((2p_1)+(-2p_2))t^n \\
&= \sum_{n=0}^{\infty} S_n(B)S_n((2p_1)+(-2p_2))t^n - \\
&\quad y\sum_{n=0}^{\infty} S_n(B)S_n((2p_1)+(-2p_2)t^n \\
&= \sum_{n=0}^{\infty} S_n(B)U_n(y)t^n - \frac{y}{2(p_1+p_2)}\sum_{n=0}^{\infty} S_n(B)((2p_1)^n-(-2p_2)^n)t^n.
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)t^n = \frac{1}{\prod_{b\in B}(1-bt)},$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(B)T_n(y)t^n &= \sum_{n=0}^{\infty} S_n(B)U_n(y)t^n - \frac{y}{2(p_1+p_2)}\left[\frac{1}{\prod_{b\in B}(1-2p_1bt)} - \frac{1}{\prod_{b\in B}(1+2p_2bt)}\right] \\
&= \frac{1 - yS_1(-B)t + S_2(-B)(2y^2-1)t^2 + S_3(-B)(y-4y^3-1)t^3}{\left(1-2b_1yt-b_1^2t^2\right)\left(1-2b_2yt-b_2^2t^2\right)\left(1-2b_3yt-b_3^2t^2\right)}.
\end{aligned}$$

This completes the proof. □

- Let $b_3 = 0$, by making the following restrictions: $p_1 - p_2 = k$, $p_1p_2 = 1$, $4b_1b_2 = -1$, and by replacing $(b_1 - b_2)$ by $2(b_1 - b_2)$ in (3.2), we get a new generating function, involving the product of k-Fibonacci numbers with Chebyshev polynomial of second kind as follows

$$\sum_{n=0}^{\infty} S_n(2b_1+[-2b_2])S_n(p_1+[-p_2])t^n$$

$$= \frac{1+t^2}{1-2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2+2k(b_1-b_2)t^3+t^4}.$$

Thus we conclude with the following corollary.

Corollary 3.6. *We have the following a new generating function of the product of k-Fibonacci numbers and Chebyshev polynomials of second kind as*

$$\sum_{n=0}^{\infty} F_{k,n}U_n(b_1-b_2)t^n = \frac{1+t^2}{1-2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2+2k(b_1-b_2)t^3+t^4}. \tag{3.7}$$

Put $k=2$ in the relationship (3.7) we get

$$\sum_{n=0}^{\infty} P_nU_{n-1}(b_1-b_2)t^n = \frac{t+t^3}{1-4(b_1-b_2)t+(6-4(b_1-b_2)^2)t^2+4(b_1-b_2)t^3+t^4},$$

which represents a new generating function, involving the product of Pell numbers with Chebyshev polynomials of second kind.

Theorem 3.6. *For $n \in \mathbb{N}$, The new generating of the product of $k-$Fibonacci numbers and Chebyshev polynomials of first kind as*

$$\sum_{n=0}^{\infty} F_{k,n}T_n(b_1-b_2)t^n = \frac{1-k(b_1-b_2)t+(1-2(b_1-b_2)^2)t^2}{1-2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2+2k(b_1-b_2)t^3+t^4}. \tag{3.8}$$

Proof. We have

$$\sum_{n=0}^{\infty} F_{k,n}T_n(b_1-b_2)t^n = \sum_{n=0}^{\infty} F_{k,n}(S_n(2b_1+[-2b_2])-(b_1-b_2)S_{n-1}(2b_1+[-2b_2]))t^n$$

$$= \sum_{n=0}^{\infty} F_{k,n}S_n(2b_1+[-2b_2])t^n-(b_1-b_2)\sum_{n=0}^{\infty} F_{k,n}S_{n-1}(2b_1+[-2b_2])t^n$$

$$= \sum_{n=0}^{\infty} F_{k,n}U_n(b_1-b_2)t^n-\frac{(b_1-b_2)}{2(b_1+b_2)}\sum_{n=0}^{\infty} F_{k,n}((2b_1)^n-(-2b_2)^n)t^n.$$

Since

$$\sum_{n=0}^{\infty} F_{k,n}t^n = \frac{1}{1-kt-t^2},$$

Therfore

$$\sum_{n=0}^{\infty} F_{k,n}T_n(b_1-b_2)t^n = \frac{1-k(b_1-b_2)t+(1-2(b_1-b_2)^2)t^2}{1-2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2+2k(b_1-b_2)t^3+t^4}.$$

This completes the proof. □

Put $k = 2$ in the relationship (3.7) we get

$$\sum_{n=0}^{\infty} P_n T_{n-1}(b_1 - b_2) t^n = \frac{t - k(b_1 - b_2)t^2 + (1 - 2(b_1 - b_2)^2)t^3}{1 - 4(b_1 - b_2)t + (6 - 4(b_1 - b_2)^2)t^2 + 4(b_1 - b_2)t^3 + t^4}.$$

which represents a new generating function, involving the product of Pell with Chebyshev polynomial of first kind.

4. **Conclusion**

In this paper, a new theorem has been proposed in order to determine the generating functions. The proposed theorem is based on the symmetric functions. The obtained results agree with the results obtained in some previous works.

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