ON POLYNOMIALS ASSOCIATED TO PRODUCT AND COMPOSITION OF GENERATING FUNCTIONS

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To the memory of Prof. Mohamed Zitouni

Abstract. In this paper, we introduce a generalized family of numbers and polynomials of one or more variables attached to the formal composition $f.(g \circ h)$ of generating functions f, g and h. We give explicit formula and apply the obtained result to two special families of polynomials; the first concerns generalization of some polynomials applied to the theory of hyperbolic differential equations recently introduced and studied by M. Mihoubi and M. Sahari. The second concerns two variables Laguerre-based generalized Hermite-Euler polynomials introduced and should be updated to studied recently by N. U. Khan et al..

1. Introduction

As a new tradition, many people are interested by numbers and polynomials having applications on the hyperbolic differential equations, combinatorics, physics and engineering sciences. In the literature there is numerous works on polynomials and their generalizations. We invite the reader to consult [1, 2, 11, 12, 13] and references therein. Let

$$f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!},\ g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!} \text{ and } h(t) = \sum_{n\geq 0} h_n \frac{t^n}{n!}$$

three known formal exponential generating functions of numbers or polynomials of one or more variables. We consider the generating function $f(t)g \circ h(t) = \sum_{n\geq 0} L_n^{(f,g,h)} \frac{t^n}{n!}$ in order to compute explicit formula of $L_n^{(f,g,h)}$. For showing the importance of this generalization we study two kinds of polynomials. The first concerns polynomials dealing with the theory of hyperbolic differential equations recently introduced by M. Mihoubi and M. Sahari (see [10]); where we reproof the explicit formula therein. The second deals with three variables Laguerre polynomials investigated in the work [9] where we give the explicit formula too. It is well-known that the derivative at order n of the composition $g \circ h$ is given by Faà di Bruno formula:

$$(g \circ h)^{(n)}(t) = \sum_{k=1}^{n} B_{n,k}\left(h^{(1)}(t), \cdots, h^{(n-k+1)}(t)\right) g^{(k)} \circ h(t), \tag{1}$$

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where $B_{n,k} := B_{n,k}(x_1, x_2, \cdots)$ (see [3]) are exponential partial Bell polynomials given by

$$B_{n,k}(x_1, x_2, \cdots) = \frac{n!}{k!} \sum_{\pi_n(k)} \binom{k}{k_1, \cdots k_{n-k+1}} \prod_{r=1}^{n-k+1} \left(\frac{x_r}{r!}\right)^{k_r}, \tag{2}$$

with $\pi_n(k)$ is the set of all $(k_1, \cdots, k_{n-k+1}) \in \mathbb{N}^{n-k+1}$ such that $k_1 + \cdots + k_{n-k+1} = k$ and $k_1 + 2k_2 + \cdots + (n-k+2)k_{n-k+2} = n$. $B_{n,k}$ are generated by the function

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}(x_1, x_2, \cdots) \frac{t^n}{n!}. \tag{3}$$

According to polynomials $B_{n,k}$; the series expansion of the composition $g \circ h$ on neighborhood of zero is given by the following lemma, the proof is left as a sample exercise for the reader.

Lemma 1.1.

$$g \circ h(t) = g(h_0) + \sum_{n\geq 1} \sum_{k=1}^{n} B_{n,k}(h_1, h_2, \cdots) g^{(k)}(h_0) \frac{t^n}{n!}. \tag{4}$$

For $g(t) = t^\alpha$, the corresponding expression of $g \circ h$; if $h_0 \neq 0$ is given in the work [7]. We remember that the Cauchy product (see [6]) of f and g is

$$f(t)g(t) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}\right) \frac{t^n}{n!}. \tag{5}$$

With the combination of identity (4) Lemma 1.1 and the identity (5), the following theorem holds.

Theorem 1.2. *We have $L_0^{(f,g,h)} = g(h_0)f_0$ and for $n \geq 1$:*

$$L_n^{(f,g,h)} = g(h_0)f_n + \sum_{k=1}^{n} \sum_{i=1}^{k} \binom{n}{k} f_{n-k} B_{k,i}(h_1, \cdots, h_{k-i+1}) g^{(i)}(h_0). \tag{6}$$

Consequently we have

Corollary 1.3. *If $h_0 = 0$, we have $L_0^{(f,g,h)} = g_0 f_0$ and*

$$L_n^{(f,g,h)} = g_0 f_n + \sum_{k=1}^{n} \sum_{i=1}^{k} \binom{n}{k} f_{n-k} g_i B_{k,i}(h_1, \cdots, h_{k-i+1}). \tag{7}$$

2. Application to sequences of polynomials linked to the sequence of Bell polynomials.

Recently M. Mihoubi and M. Sahari (see [10]) studied polynomials $L_n^{(\alpha,\beta)}(x)$ defined by means of the generating function

$$(1-t)^{\alpha}\exp\left(x\left((1-t)^{\beta}-1\right)\right)=\sum_{n\geq 0}L_n^{(\alpha,\beta)}(x)\frac{t^n}{n!} \tag{8}$$

and proved that

$$L_n^{(\alpha,\beta)}(x)=\sum_{i=0}^{n}S_{\alpha,\beta}\left(n,i\right)x^i, \tag{9}$$

where

$$S_{\alpha,\beta}\left(n,i\right)=\frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\left\langle -\alpha-\beta j\right\rangle_n$$

and $\langle\alpha\rangle_n=\alpha\left(\alpha+1\right)\cdots\left(\alpha+n-1\right)$, which admits the reformulation $\langle\alpha\rangle_n=(-1)^n(\alpha)_n$; with $(\alpha)_n=\alpha\left(\alpha-1\right)\cdots\left(\alpha-n+1\right)$ is a falling number. In this section we prove this formula by using advanced algebraic combinatorics; our method is different of that given in [10] based on the Theorem [14, th. 7.50]. First we begin by an improvement of [10, Proposition.1]. Letting

$$f(t)=(1-t)^{\alpha}=\sum_{n\geq 0}(-1)^n(\alpha)_n\frac{t^n}{n!}$$

$$g(t)=e^t=\sum_{n\geq 0}\frac{t^n}{n!}$$

and

$$h(t)=x\left((1-t)^{\beta}-1\right)=\sum_{n\geq 1}(-1)^n(\beta)_n x\frac{t^n}{n!}.$$

Then

$$f(t)g\circ h(t)=(1-t)^{\alpha}\exp\left(x\left((1-t)^{\beta}-1\right)\right)$$

and $L_n^{(f,g,h)}=Ł_n^{(\alpha,\beta)}(x)$. Thereafter

$$f_n=(-1)^n(\alpha)_n,\ g_n=1,\ h_0=0\text{ and }h_n=(-1)^n(\beta)_n x.$$

The exponential partial Bell polynomials implicated in the expression of $Ł_n^{(\alpha,\beta)}(x)$ are

$$B_{k,i}\left((-1)^1(\beta)_1x,(-1)^2(\beta)_2x,\cdots\right)=\frac{k!}{i!}x^i(-1)^kB_{k,i}\left((\beta)_1,(\beta)_2,\cdots\right).$$

By means of Theorem 1.2, the following theorem holds.

Theorem 2.1.

$$L_n^{(\alpha,\beta)}(x)=(-1)^n(\alpha)_n+\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}(-1)^n(\alpha)_{n-k}B_{k,i}\left((\beta)_1,(\beta)_2,\cdots\right)x^i. \tag{10}$$

Remark 2.2. *For $i > k$, only we have $B_{k,i} = 0$. Then the expression* (10) *Theorem 2.1 can be written in the form*

$$L_n^{(\alpha,\beta)}(x) = (-1)^n(\alpha)_n + \sum_{k=1}^{n}\sum_{i=1}^{n}\binom{n}{k}(-1)^n(\alpha)_{n-k}B_{k,i}\left((\beta)_1,(\beta)_2,\cdots\right)x^i. \tag{11}$$

This formula is important, it helps us to prove [10, Proposition.1].

Now with the following lemma we give explicit formula of exponential partial Bell polynomials $B_{k,i}\left((\beta)_1,(\beta)_2,\cdots\right)$.

Lemma 2.3.

$$B_{k,i}\left((\beta)_1,(\beta)_2,\cdots\right) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\beta j)_n. \tag{12}$$

Proof. From the definition of exponential partial polynomials $B_{k,i}\left((\beta)_1,(\beta)_2,\cdots\right)$, we have

$$\frac{1}{k!}\left((1+t)^\beta - 1\right)^k = \sum_{n\geq k}B_{n,k}\left((\beta)_1,\cdots,(\beta)_{n-k+1}\right)\frac{t^n}{n!}$$

but

$$\left((1+t)^\beta - 1\right)^k = \sum_{j=0}^{k}\binom{k}{j}(1+t)^{\beta j}(-1)^{k-j}$$

and

$$\frac{1}{k!}\left((1+t)^\beta - 1\right)^k = \frac{1}{k!}\sum_{n\geq 0}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\beta j)_n\frac{t^n}{n!}.$$

Furthermore we conclude that

$$B_{k,i}\left((\beta)_1,(\beta)_2,\cdots\right) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\beta j)_n$$

□

According to identity (12) Lemma 2.3, the expression of $L_n^{(\alpha,\beta)}(x)$ becomes

$$L_n^{(\alpha,\beta)}(x) = (-1)^n(\alpha)_n + \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=0}^{i}\binom{n}{k}\binom{i}{j}\frac{1}{i!}(-1)^{n+i-j}(\beta j)_k(\alpha)_{n-k}x^i. \tag{13}$$

For $k = 0$ we have

$$\sum_{i=1}^{n}\left(\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\right)\frac{1}{i!}x^i = 0,$$

the above expression of $L_n^{(\alpha,\beta)}(x)$ becomes

$$L_n^{(\alpha,\beta)}(x) = (-1)^n(\alpha)_n + \sum_{k=0}^{n}\sum_{i=1}^{n}\sum_{j=0}^{i}\binom{n}{k}\binom{i}{j}\frac{1}{i!}(-1)^{n+i-j}(\beta j)_k(\alpha)_{n-k}x^i. \tag{14}$$

To write

$$L_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n} S_{\alpha,\beta}(n,i)x^i$$

we must prove the following lemma

Lemma 2.4.

$$S_{\alpha,\beta}(n,i) = \frac{1}{i!}\sum_{k=0}^{n}\sum_{j=0}^{i}\binom{n}{k}\binom{i}{j}(-1)^{n+i-j}(\beta j)_k(\alpha)_{n-k}. \tag{15}$$

Proof. We have $S_{\alpha,\beta}(n,0) = \langle\alpha\rangle_n = (-1)^n(\alpha)_n$. Since

$$(-1)^n\sum_{k=0}^{n}\binom{n}{k}(\beta j)_k(\alpha)_{n-k} = (-1)^n(\alpha+\beta j)_n = \langle -\alpha-\beta j\rangle_n$$

and according to the identity (14) the result follows. □

3. Application to two variable Laguerre polynomials

Laguerre polynomials $Ł_n(x,y)$ are defined by the generating function (see [4])

$$\frac{1}{1-yt}\exp\left(\frac{-xt}{1-yt}\right) = \sum_{n\geq 0} Ł_n(x,y)t^n, \tag{16}$$

which is equivalent (see [5]) to

$$\exp(yt)\, C_0(xt) = \sum_{n\geq 0} Ł_n(x,y)\frac{t^n}{n!}, \tag{17}$$

where

$$C_0(xt) = \sum_{n\geq 0}\frac{(-1)^n x^n}{n!}\frac{t^n}{n!}.$$

Then

$$\exp(yt)\, C_0(xt) = \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k x^k y^{n-k}}{k!}\frac{t^n}{n!}.$$

Finally the identity proved in [9] follows.

$$Ł_n(x,y) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k x^k y^{n-k}}{k!} \tag{18}$$

The generating function of $Ł_n(x,y)$ is written under the form

$$\frac{1}{1-yt}\exp\left(\frac{-xt}{1-yt}\right) = f(t)g\circ h(t);$$

where

$$f(t) = \frac{1}{1-yt} = \sum_{n\geq 0} n!y^n\frac{t^n}{n!},$$

$$g(t)=e^{t}=\sum_{n\geq 0}\frac{t^{n}}{n!}$$

and

$$h(t)=\frac{-xt}{1-yt}=-\sum_{n\geq 0}xy^{n}t^{n+1}=-\sum_{n\geq 1}n!xy^{n-1}\frac{t^{n}}{n!}.$$

Furthermore $f_n=n!y^n$, $g_n=1$, $h_0=0$ and $h_n=-n!xy^{n-1}$. According to identity (7) Corollary 1.3, we have already proved the following theorem

Theorem 3.1. *we have* $Ł_0(x,y)=1$ *and for* $n\geq 1$*;*

$$Ł_n(x,y)=y^n+\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{1}{k!}(-x)^i y^{n-i}B_{k,i}\left(1!,\cdots,(k-i+1)!\right). \tag{19}$$

To obtain the identity (18) we must compute $B_{k,i}\left(1!,\cdots,(k-i+1)!\right)$. In the first view L. Comtet (see Identity [3h] [3, Theorem B p.135]) obtained the following result

$$B_{k,i}\left(1!,\cdots,(k-i+1)!\right)=\binom{n-1}{k-1}\frac{n!}{k!}.$$

Here we give another reformulation of this identity. In one hand, we have

$$\frac{1}{k!}\left(\sum_{m\geq 1}t^m\right)^k=\sum_{n\geq k}B_{n,k}\left(1!,2!,\cdots,(n-k+1)!\right)\frac{t^n}{n!}.$$

In another hand we have

$$\frac{1}{k!}\left(\sum_{m\geq 1}t^m\right)^k=\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\frac{t^k}{k!}\sum_{n\geq 0}(-1)^n(-k)_n\frac{t^n}{n!}.$$

After little calculation, we will have

$$\frac{1}{k!}\left(\sum_{m\geq 1}t^m\right)^k=\sum_{n\geq 0}\binom{n}{k}(-1)^{n-k}(-k)_{n-k}\frac{t^n}{n!}.$$

But we know that

$$(-k)_n=(-1)^n\frac{(k+n-1)!}{(k-1)!}.$$

Finally we have

$$B_{k,i}\left(1!,\cdots,(k-i+1)!\right)=\binom{n}{k}\frac{(n-1)!}{(k-1)!}. \tag{20}$$

Returning back to identity (19) under Theorem 3.1 and after substitution we obtain

$$Ł_n(x,y)=y^n+\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{1}{k!}(-x)^i y^{n-i}\binom{k}{i}\frac{(k-1)!}{(i-1)!},$$

which can be written in the form

$$Ł_n(x,y) = y^n + \sum_{i=1}^{n}\sum_{k=i}^{n}(-x)^i y^{n-i}\frac{(k-1)!i}{(i!)^2(k-i)!}.$$

By using the well-known Hockey-stick identity we have

$$\sum_{k=i}^{n}\frac{i(k-1)!}{(k-i)!} = i!\sum_{k=i}^{n}\binom{k-1}{i-1} = i!\sum_{k=i-1}^{n-1}\binom{k}{i-1} = i!\binom{n}{i} = \frac{n!}{(n-i)!}.$$

Hence

$$Ł_n(x,y) = n!\sum_{i=0}^{n}\frac{(-x)^i y^{n-i}}{(i!)^2(n-i)!}.$$

Sometimes we are concerned by generating functions of the form $f.(g\circ h).(v\circ w)$ which contains four operations; two products and two compositions. Let $v(t) = \sum_{n\geq 0} v_n\frac{t^n}{n!}$, $w(t) = \sum_{n\geq 0} w_n\frac{t^n}{n!}$ and $L_{n,v,w}^{(f,g,h)}$ be the sequence generated by $f.(g\circ h).(v\circ w)$. According to identity (4) Lemma 1.1, we will have:

$$v\circ w(t) = v(w_0) + \sum_{n\geq 1}\sum_{k=1}^{n} B_{n,k}(w_1,w_2,\cdots)\, v^{(k)}(w_0)\frac{t^n}{n!}.$$

Let $G(t) = g\circ h(t) v\circ w(t)$, then $G(0) = g(h_0)v(w_0)$ and by means of Cauchy product of generating functions we have

$$\begin{aligned}G(t) = G(0) + \sum_{n\geq 1}\left(\sum_{k=1}^{n-1}\binom{n}{k}\sum_{i=1}^{k} B_{k,i}(h_1,h_2,\cdots)\, g^{(i)}(h_0)\sum_{i=1}^{n-k} B_{n-k,i}(w_1,w_2,\cdots)\, v^{(i)}(w_0)\right)\frac{t^n}{n!}\\ +g(h_0)\sum_{n\geq 1}\sum_{i=1}^{n} B_{n,i}(w_1,w_2,\cdots)\, v^{(i)}(w_0)\frac{t^n}{n!}\\ +v(w_0)\sum_{n\geq 1}\sum_{i=1}^{n} B_{n,i}(h_1,h_2,\cdots)\, g^{(i)}(h_0)\frac{t^n}{n!}.\end{aligned}$$

Writing $G(t) = \sum_{n\geq 0} G_n\frac{t^n}{n!}$, so $G_0 = g(h_0)v(w_0)$ and for $n\geq 1$ we have

$$\begin{aligned}G_n = \sum_{k=1}^{n-1}\binom{n}{k}\sum_{i=1}^{k}\sum_{j=1}^{n-k} B_{k,i}(h_1,h_2,\cdots)\, B_{n-k,j}(w_1,w_2,\cdots)\, g^{(i)}(h_0)v^{(j)}(w_0)\\ +g(h_0)\sum_{i=1}^{n} B_{n,i}(w_1,w_2,\cdots)\, v^{(i)}(w_0)\\ +v(w_0)\sum_{i=1}^{n} B_{n,i}(h_1,h_2,\cdots)\, g^{(i)}(h_0)\end{aligned}$$

Finally by means of Cauchy product of generating functions, we have

$$L_{s,v,w}^{(f,g,h)} = \sum_{n=0}^{s}\binom{s}{n} f_{s-n}G_n.$$

Hence

$$\begin{aligned}
L_{s,v,w}^{(f,g,h)} &= f_s g(h_0) v(w_0) \\
+\sum_1 \binom{s}{n}\binom{n}{k} f_{s-n} B_{k,i}\left(h_1,h_2,\cdots\right) B_{n-k,j}\left(w_1,w_2,\cdots\right) & g^{(i)}(h_0) v^{(j)}(w_0) \\
&+g(h_0)\sum_2 \binom{s}{n} f_{s-n} B_{n,i}\left(w_1,w_2,\cdots\right) v^{(i)}(w_0) \\
&+v(w_0)\sum_2 \binom{s}{n} f_{s-n} B_{n,i}\left(h_1,h_2,\cdots\right) g^{(i)}(h_0),
\end{aligned} \tag{21}$$

where

$$\sum_1 = \sum_{n=0}^{s}\sum_{k=1}^{n-1}\sum_{i=1}^{k}\sum_{j=1}^{n-k} \text{ and } \sum_2 = \sum_{n=0}^{s}\sum_{i=1}^{n}.$$

Recently, Khan, N. U. et al. introduced and studied Laguerre-based generalized Hermite-Euler polynomials ${}_LH^{E_s[\alpha,m-1]}(x,y,z)$ (see [9]); these are generated by the function

$$\left(\frac{2^m}{e^t+\sum_{n=0}^{m-1}\frac{t^n}{n!}}\right)^{\alpha} e^{yt+zt^2}C_0(xt) = \sum_{s\geq 0} {}_LH^{E_s[\alpha,m-1]}(x,y,z)\frac{t^s}{s!}.$$

and provide that (see [9, Theorem 2.3])

$${}_LH^{E_s[\alpha,m-1]}(x,y,z) = \sum_{r=0}^{s} E_{s-r}^{[m-1]}\,{}_LH^{E_s[\alpha-1,m-1]}(x,y,z),$$

where the numbers $E^{[m-1]}$ are generated by the function

$$\left(\frac{2^m}{e^t+\sum_{n=0}^{m-1}\frac{t^n}{n!}}\right)^{\alpha} = \sum_{s\geq 0} E^{[m-1]}\frac{t^s}{s!}.$$

The generating function of ${}_LH^{E_s[\alpha,m-1]}(x,y,z)$ obeys to the form below. One writes

$$\left(\frac{2^m}{e^t+\sum_{n=0}^{m-1}}\frac{t^n}{n!}\right)^{\alpha} e^{yt+zt^2}C_0(xt) = f(t)\left(g\circ h(t)\right)\left(v\circ w(t)\right),$$

with

$$f(t) = C_0(xt) = \sum_{n\geq 0}\frac{(-1)^n x^n}{n!}\frac{t^n}{n!},$$

$$h(t) = \frac{e^t+\sum_{n=0}^{m-1}\frac{t^n}{n!}}{2^m} = \frac{1}{2^m}\left(2\sum_{n=0}^{m-1}\frac{t^n}{n!}+\sum_{n\geq m}\frac{t^n}{n!}\right),$$

$g(t) = t^{-\alpha}, v(t) = e^t$ and $w(t) = yt + zt^2$. Thus $f_n = \frac{(-1)^n x^n}{n!}$, $h_n = 2^{-m+1}$ for $n \leq m-1$ and $h_n = 2^{-m}$ for $n \geq m$, $v_n = 1$, $w_0 = 0$, $w_1 = y$, $w_2 = x$ and $w_n = 0$ for $n \geq 3$. Then

$$
\begin{aligned}
{}_L H^{E_n[\alpha,m-1]}(x,y,z) &= \frac{(-1)^s x^s}{s!} 2^{(m-1)\alpha} \\
+\sum_1 \binom{s}{n}\binom{n}{k} \frac{(-1)^{s-n}x^{s-n}}{(s-n)!} 2^{(\alpha+i)(m-1)}(-\alpha)_i B_{k,i}\left(h_1,h_2,\cdots\right) B_{n-k,j}\left(y,z,0\cdots\right) \\
+2^{(m-1)\alpha}\sum_2 \binom{s}{n} \frac{(-1)^{s-n}x^{s-n}}{(s-n)!} B_{n,i}\left(y,z,0\cdots\right) \\
+\sum_2 \binom{s}{n} \frac{(-1)^{s-n}x^{s-n}}{(s-n)!} 2^{(\alpha+i)(m-1)}(-\alpha)_i B_{n,i}\left(h_1,h_2,\cdots\right).
\end{aligned}
$$

We have

$$B_{n,i}\left(y,z,0,\cdots\right) = \frac{n!}{i!} \sum_{\substack{i_1+i_2=i \\ i_1+2i_2=n}} \binom{i}{i_1,i_2} y^{i_1} z^{i_2} = \frac{n!}{i!}\sum_{j=0}^{i}\binom{i}{j} y^{n-2j} z^j.$$

and

$$B_{n,i}\left(h_1,h_2,\cdots\right) = \frac{n!}{i!}\sum_{\pi_n(i)} \binom{i}{i_1,\cdots i_{n-i+1}} \prod_{r=1}^{n-i+1}\left(\frac{h_r}{r!}\right)^{i_r}$$

If $n-i+1 \leq m-1$ we have

$$B_{n,i}\left(h_1,h_2,\cdots\right) = 2^{-i(m-1)} B_{n,i}\left(1,1,\cdots\right) = 2^{-i(m-1)} S(n,i).$$

If $n-i+1 \geq m$ we have

$$B_{n,i}\left(h_1,h_2,\cdots\right) = \frac{n!}{i!} 2^{-im} \sum_{\pi_n(i)} \binom{i}{i_1,\cdots i_{n-i+1}} \prod_{r=1}^{m-1}\left(\frac{2}{r!}\right)^{i_r} \prod_{r=1}^{n-i+1}\left(\frac{1}{r!}\right)^{i_r}.$$

Otheriwise we have

$$\binom{s}{n}\binom{n}{k}\frac{1}{(s-n)!}\frac{(n-k)!}{i!}\binom{i}{j} = \frac{s!}{((s-n)!)^2 k! j! (i-j)!}$$

and

$$\binom{s}{n}\frac{1}{(s-n)!}\frac{n!}{i!}\binom{i}{j} = \frac{s!}{((s-n)!)^2 j!(i-j)!}.$$

Then we have already proved the following theorem.

Theorem 3.2.

$$\frac{{}_{L}H^{E_n[\alpha,m-1]}(x,y,z)}{s!} = \frac{(-1)^s x^s}{(s!)^2} 2^{(m-1)\alpha}$$

$$+\sum_{1}\sum_{j=0}^{i} \frac{(-1)^{s-n} 2^{(\alpha+i)(m-1)} (-\alpha)_i}{((s-n)!)^2 k! j! (i-j)!} B_{k,i}\left(h_1, h_2, \cdots\right) x^{s-n} y^{n-k-2j} z^j$$

$$+2^{(m-1)\alpha} \sum_{2}\sum_{j=0}^{i} \frac{(-1)^{s-n}}{((s-n)!)^2 j! (i-j)!} x^{s-n} y^{n-2j} z^j$$

$$+\sum_{2} \frac{(-1)^{s-n} 2^{(\alpha+i)(m-1)} (-\alpha)_i}{n! ((s-n)!)^2} B_{n,i}\left(h_1, h_2, \cdots\right) x^{s-n}. \tag{22}$$

4. Conclusion

Using algebraic operations on generating functions such as multiplication and composition, we can build a large family of polynomials as extensions of well-known polynomials in the literature. In this work we are interested by numbers and polynomials generated by functions of the forms $f.(g \circ h)$ and $f.(g \circ h).(v \circ w)$; where f, g, h, v and w are generating functions and we use Bell polynomials to give the explicit formulae. The obtained results are applied to two kinds of polynomials. The first concerns polynomials dealing with the theory of hyperbolic differential equations recently introduced and investigated by M. Mihoubi and M. Sahari; where we give another proof of the explicit formula therein. The second deals with two variables Laguerre polynomials and Laguerre-based generalized Hermite-Euler polynomials investigated by N. U. Khan et al.; where we give the explicit formula too. These families are only an example of infinitely many numbers and polynomials generated by functions of types $f.(g \circ h)$ and $f.(g \circ h).(v \circ w)$.

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