CONVOLUTIONS BETWEEN BERNOULLI/EULER POLYNOMIALS AND PELL/LUCAS POLYNOMIALS

DONGWEI GUO AND WENCHANG CHU

Abstract. Between Bernoulli/Euler polynomials and Pell/Lucas polynomials, convolution sums are evaluated in closed form via the generating function method. Several interesting identities involving Fibonacci and Lucas numbers are shown as consequences including those due to Byrd (1975) and Frontczak (2020).

1. Introduction and Motivation

In 1985, Horadam and Mahon [14] introduced Pell and Lucas polynomials, that are defined, respectively, by the same recurrence relations

$$\mathrm{P}_n(y) = 2y\mathrm{P}_{n-1}(y) + \mathrm{P}_{n-2}(y),$$
$$\mathrm{Q}_n(y) = 2y\mathrm{Q}_{n-1}(y) + \mathrm{Q}_{n-2}(y);$$

with different initial conditions

$$\mathrm{P}_0(y) = 0 \quad \text{and} \quad \mathrm{P}_1(y) = 1,$$
$$\mathrm{Q}_0(y) = 2 \quad \text{and} \quad \mathrm{Q}_1(y) = 2y,$$

They are polynomial extensions of the following well–known numbers:

- Fibonacci number $F_n = \mathrm{P}_n(\frac{1}{2})$: $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$.
- Lucas number $L_n = \mathrm{Q}_n(\frac{1}{2})$: $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$ and $L_1 = 1$.

Letting α and β be two algebraic functions of y given by

$$\alpha := y + \sqrt{y^2+1} \quad \text{and} \quad \beta := y - \sqrt{y^2+1}, \tag{1}$$

then both polynomials can be expressed in Binet forms

$$\mathrm{P}_n(y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{Q}_n(y) = \alpha^n + \beta^n. \tag{2}$$

Similarly, for Fibonacci and Lucas numbers, we have

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi} \quad \text{and} \quad L_n = \phi^n + \psi^n, \tag{3}$$

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Corresponding author (W. Chu): chu.wenchang@unisalento.it.

where $\phi,\ \psi = \frac{1\pm\sqrt{5}}{2}$, with ϕ standing for the golden ratio.

For Pell/Lucas polynomials and Fibonacci/Lucas numbers, there are many interesting properties, that are collected in Koshy's monographs [15, 16]. Some summation formulae about them can be found in [6, 12, 17, 18]. Denoting by B_n and E_n the usual Bernoulli and Euler numbers, Byrd [3] discovered the following couple of beautiful identities

$$\sum_{k=0}^{n}(\sqrt{5})^k\binom{n}{k}B_kF_{n-k} = n\psi^{n-1}, \tag{4}$$

$$\sum_{k=0}^{n}(-\sqrt{5})^k\binom{n}{k}B_kF_{n-k} = n\phi^{n-1}; \tag{5}$$

where the former one was rediscovered by Zhang and Ma [20]. Another similar formula also due to Byrd [4] reads as

$$\sum_{k=0}^{n}\Big(\frac{\sqrt{5}}{2}\Big)^k\binom{n}{k}E_kL_{n-k} = 2^{1-n}. \tag{6}$$

Recently, Frontczak *et al.* [9, 10, 11] extended these identities to the cases when Bernoulli and Euler numbers are substituted by the corresponding polynomials. The authors [13] figured out similar identities when Fibonacci and Lucas numbers are replaced respectively by Pell and Lucas polynomials.

The purpose of the present paper is to generalize these results further to the convolution sums between Bernoulli/Euler polynomials and Pell/Lucas polynomials. Several remarkable new identities involving two independent variables x and y will be established. In the next section, we shall evaluate, in closed form, convolutions between Bernoulli/Euler polynomials and Pell polynomials. Then summation formulae about Lucas polynomials and Bernoulli/Euler polynomials will be examined in Section 3. Finally, the paper will end with Section 4, where further variant convolutions will briefly be commented.

There exists a vast literature (cf. [8]) on Bernoulli polynomials $B_k(x)$ and Euler polynomials $E_k(x)$, that are defined, respectively, by the exponential generating functions

$$\sum_{k=0}^{\infty}B_k(x)\frac{T^k}{k!} = \frac{Te^{Tx}}{e^T-1} \quad\text{and}\quad \sum_{k=0}^{\infty}E_k(x)\frac{T^k}{k!} = \frac{2e^{Tx}}{e^T+1}. \tag{7}$$

For recent formulae involving Bernoulli and Euler polynomials/numbers, the reader can refer to [2, 1, 8, 7, 5, 19]. Throughout the paper, the following exponential generating functions for Pell and Lucas polynomials will frequently be utilized

$$\sum_{k=0}^{\infty}\mathrm{P}_k(y)\frac{T^k}{k!} = \frac{e^{T\alpha}-e^{T\beta}}{\alpha-\beta}, \tag{8}$$

$$\sum_{k=0}^{\infty}\mathrm{Q}_k(y)\frac{T^k}{k!} = e^{T\alpha}+e^{T\beta}. \tag{9}$$

2. Convolution Sums Containing Pell Polynomials

Under the replacement $T \to T(\alpha-\beta)$, the generating functions of Bernoulli and Euler polynomials can be rewritten, respectively, as

$$\sum_{k=0}^{\infty}(\alpha-\beta)^k B_k(x)\frac{T^k}{k!}=\frac{T(\alpha-\beta)}{e^{T\alpha}-e^{T\beta}}e^{T(\beta+x\alpha-x\beta)}, \tag{10}$$

$$\sum_{k=0}^{\infty}(\alpha-\beta)^k E_k(x)\frac{T^k}{k!}=\frac{2e^{T(\beta+x\alpha-x\beta)}}{e^{T\alpha}+e^{T\beta}}. \tag{11}$$

They will be utilized in conjunction with the generating function (8) to prove four theorems about convolution sums between Bernoulli/Euler polynomials and Pell polynomials. For these theorems, by specifying variable y with

$$y=\frac{L_\lambda}{2}\quad\text{for odd }\lambda\quad\text{and}\quad y=\frac{\sqrt{-1}}{2}L_\lambda\quad\text{for even }\lambda, \tag{12}$$

and then making use of the equalities

$$\mathrm{P}_n\Big(\frac{L_{2\lambda-1}}{2}\Big)=\frac{F_{2\lambda n-n}}{F_{2\lambda-1}}\quad\text{and}\quad \mathrm{P}_n\Big(\frac{\sqrt{-1}}{2}L_{2\lambda}\Big)=\sqrt{-1}^{\,n-1}\frac{F_{2\lambda n}}{F_{2\lambda}}; \tag{13}$$

we shall deduce as consequences the corresponding four corollaries that evaluate, in closed form, the convolutions about Bernoulli/Euler polynomials and Fibonacci numbers (with their subscripts being multiple summation index).

Denote by $[T^n]\varphi(T)$ the coefficient of T^n in the formal power series $\varphi(T)$. Multiplying (8) with (10) and then extracting the coefficient of T^n across the resulting equation, we get the following binomial convolution

$$\begin{aligned}\sum_{k=0}^{n}(\alpha-\beta)^k\binom{n}{k}B_k(x)\mathrm{P}_{n-k}(y)&=n![T^n]\frac{e^{T\alpha}-e^{T\beta}}{\alpha-\beta}\times\frac{T(\alpha-\beta)}{e^{T\alpha}-e^{T\beta}}e^{T(\beta+x\alpha-x\beta)}\\&=n![T^{n-1}]e^{T(\beta+x\alpha-x\beta)},\end{aligned}$$

which gives rise to the formula as in the theorem below.

Theorem 1. *For $n\in\mathbb{N}$, letting α and β be as in* (1), *the following identity holds:*

$$\sum_{k=0}^{n}(\alpha-\beta)^k\binom{n}{k}B_k(x)\mathrm{P}_{n-k}(y)=n(\beta+x\alpha-x\beta)^{n-1}.$$

When y is assigned by (12), this theorem recovers the following identity due to Frontczak and Goy [10, Equation 14].

Corollary 2. *For $n\in\mathbb{N}$,*

$$\sum_{k=0}^{n}\big(\sqrt{5}F_\lambda\big)^k\binom{n}{k}B_k(x)F_{\lambda(n-k)}=nF_\lambda\Big\{\sqrt{5}xF_\lambda+\psi^\lambda\Big\}^{n-1}.$$

The particular case $\lambda = 1$ of this corollary is found by Frontczak [9, Equation 38]:

$$\sum_{k=0}^{n} 5^{\frac{k}{2}} \binom{n}{k} B_k(x) F_{n-k} = n\big(\sqrt{5}x + \psi\big)^{n-1},$$

which reduces, for $x = 0$, further to identity (4) due to Byrd [3].

Observe the generating function (8) is equivalent to the following one

$$\frac{e^{2T\alpha} - e^{2T\beta}}{\alpha - \beta} = \sum_{k=0}^{\infty} \mathrm{P}_k(y) \frac{(2T)^k}{k!}. \tag{14}$$

Then multiplying this with (10), we arrive at the binomial sum below

$$\sum_{k=0}^{n} (\alpha - \beta)^k \binom{n}{k} B_k(x) 2^{n-k} \mathrm{P}_{n-k}(y) = n![T^{n-1}]\Big\{e^{T(\alpha+\beta+x\alpha-x\beta)} + e^{T(2\beta+x\alpha-x\beta)}\Big\}.$$

This results in the following convolution formula.

Theorem 3. *For $n \in \mathbb{N}$, letting α and β be as in* (1)*, the following identity holds:*

$$\sum_{k=0}^{n} (\alpha - \beta)^k \binom{n}{k} B_k(x) 2^{n-k} \mathrm{P}_{n-k}(y) = n\big(\alpha + \beta + x\alpha - x\beta\big)^{n-1} + n\big(2\beta + x\alpha - x\beta\big)^{n-1}.$$

Corollary 4. *For $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n} \Big(\frac{\sqrt{5}}{2} F_\lambda\Big)^k \binom{n}{k} B_k(x) F_{\lambda(n-k)} = \frac{n}{2} F_\lambda \Big\{\frac{\sqrt{5}xF_\lambda + L_\lambda}{2}\Big\}^{n-1} + \frac{n}{2} F_\lambda \Big\{\frac{\sqrt{5}xF_\lambda}{2} + \psi^\lambda\Big\}^{n-1}.$$

Analogously, by multiplying (11) with (14), we can also derive the following formula.

Theorem 5. *For $n \in \mathbb{N}$, letting α and β be as in* (1)*, the following identity holds:*

$$\sum_{k=0}^{n} (\alpha - \beta)^k \binom{n}{k} E_k(x) 2^{n-k} \mathrm{P}_{n-k}(y) = \frac{(\alpha + \beta + x\alpha - x\beta)^n}{\sqrt{1+y^2}} - \frac{(2\beta + x\alpha - x\beta)^n}{\sqrt{1+y^2}}.$$

Corollary 6. *For $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n} \Big(\frac{\sqrt{5}}{2} F_\lambda\Big)^k \binom{n}{k} E_k(x) F_{\lambda(n-k)} = \frac{2}{\sqrt{5}} \Big\{\frac{\sqrt{5}xF_\lambda + L_\lambda}{2}\Big\}^{n} - \frac{2}{\sqrt{5}} \Big\{\frac{\sqrt{5}xF_\lambda}{2} + \psi^\lambda\Big\}^{n}.$$

Furthermore, it is trivial to see that

$$\frac{e^{3T\alpha} - e^{3T\beta}}{\alpha - \beta} = \sum_{k=0}^{\infty} \mathrm{P}_k(y) \frac{(3T)^k}{k!}.$$

Then multiplying this with (10), we arrive at the binomial sum below

$$\begin{aligned}&\sum_{k=0}^{n} (\alpha - \beta)^k \binom{n}{k} B_k(x) 3^{n-k} \mathrm{P}_{n-k}(y)\\ &= n![T^{n-1}]\Big\{e^{T(2\alpha+\beta+x\alpha-x\beta)} + e^{T(\alpha+2\beta+x\alpha-x\beta)} + e^{T(3\beta+x\alpha-x\beta)}\Big\}.\end{aligned}$$

This results in the following convolution formula.

Theorem 7. *For $n \in \mathbb{N}$, letting α and β be as in* (1), *the following identity holds:*

$$\sum_{k=0}^{n}(\alpha-\beta)^k\binom{n}{k}B_k(x)3^{n-k}\mathrm{P}_{n-k}(y) = n\big(2\alpha+\beta+x\alpha-x\beta\big)^{n-1} \\ +n\big(\alpha+2\beta+x\alpha-x\beta\big)^{n-1}+n\big(3\beta+x\alpha-x\beta\big)^{n-1}.$$

Corollary 8. *For $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n}\Big(\frac{\sqrt{5}}{3}F_\lambda\Big)^k\binom{n}{k}B_k(x)F_{\lambda(n-k)} = \frac{n}{3}F_\lambda\Big\{\frac{\sqrt{5}xF_\lambda+L_\lambda+\phi^\lambda}{3}\Big\}^{n-1} \\ +\frac{n}{3}F_\lambda\Big\{\frac{\sqrt{5}xF_\lambda+L_\lambda+\psi^\lambda}{3}\Big\}^{n-1}+\frac{n}{3}F_\lambda\Big\{\frac{\sqrt{5}xF_\lambda}{3}+\psi^\lambda\Big\}^{n-1}.$$

3. Convolution Sums Containing Lucas Polynomials

Instead of Pell polynomials, we shall derive four theorems about convolution sums involving Lucas polynomials. Then by assigning variable y as per (12) and employing the equalities

$$\mathrm{Q}_n\Big(\frac{L_{2\lambda-1}}{2}\Big)=L_{2\lambda n-n} \quad\text{and}\quad \mathrm{Q}_n\Big(\frac{\sqrt{-1}}{2}L_{2\lambda}\Big)=\sqrt{-1}^{\,n}L_{2\lambda n}, \tag{15}$$

we shall derive from each theorem a corresponding corollary concerning convolution sum about Lucas numbers.

Firstly, multiplying (11) with the generating function of Lucas polynomials (9) and then extracting the coefficient of T^n, we establish the following convolution formula.

Theorem 9. *For $n \in \mathbb{N}$, letting α and β be as in* (1), *the following identity holds:*

$$\sum_{k=0}^{n}(\alpha-\beta)^k\binom{n}{k}E_k(x)\mathrm{Q}_{n-k}(y)=2(\beta+x\alpha-x\beta)^n.$$

Corollary 10. *For $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n}\big(\sqrt{5}F_\lambda\big)^k\binom{n}{k}E_k(x)L_{\lambda(n-k)}=2\Big\{\sqrt{5}xF_\lambda+\psi^\lambda\Big\}^n.$$

Its special case $\lambda=1$ can be found in Frontczak [9, Equation 39]:

$$\sum_{k=0}^{n}5^{\frac{k}{2}}\binom{n}{k}E_k(x)L_{n-k}=2\big(\sqrt{5}x+\psi\big)^n,$$

which further gives, for $x=1/2$, identity (6) due to Byrd [4].

Analogously, multiplying (10) with

$$\sum_{k=0}^{\infty}\Big\{2^k\mathrm{Q}_k(y)-2(2y)^k\Big\}\frac{T^k}{k!}=\Big(e^{T\alpha}-e^{T\beta}\Big)^2,$$

and then extracting the coefficient of T^n, we get the expression

$$\sum_{k=0}^{n}(\alpha-\beta)^{n-k}\binom{n}{k}B_k(x)\{2^{n-k}Q_{n-k}(y)-2(2y)^{n-k}\}$$
$$=n(\alpha-\beta)\{(1+x)\alpha+(1-x)\beta\}^{n-1}-n(\alpha-\beta)\{2\beta+x(\alpha-\beta)\}^{n-1}.$$

Taking into account also that

$$\sum_{k=0}^{n}(\alpha-\beta)^{k}\binom{n}{k}B_k(x)(2y)^{n-k}=n![T^n]e^{2Ty}\times\frac{T(\alpha-\beta)}{e^{T(\alpha-\beta)}-1}e^{Tx(\alpha-\beta)}$$
$$=n![T^n]\frac{T(\alpha-\beta)}{e^{T(\alpha-\beta)}-1}\exp\left\{T(\alpha-\beta)\Big(x+\tfrac{y}{\sqrt{1+y^2}}\Big)\right\},$$

we find the following convolution formula.

Theorem 11. *For $n\in\mathbb{N}$, letting α and β be as in* (1), *the following identity holds:*

$$\sum_{k=0}^{n}(\alpha-\beta)^{k}\binom{n}{k}B_k(x)2^{n-k}Q_{n-k}(y)=2(\alpha-\beta)^nB_n\Big(x+\tfrac{y}{\sqrt{1+y^2}}\Big)$$
$$+n(\alpha-\beta)\Big\{(1+x)\alpha+(1-x)\beta\Big\}^{n-1}-n(\alpha-\beta)\Big\{x\alpha+(2-x)\beta\Big\}^{n-1}.$$

Corollary 12. *For $n\in\mathbb{N}$,*

$$\sum_{k=0}^{n}\Big(\frac{\sqrt5}{2}F_\lambda\Big)^k\binom{n}{k}B_k(x)L_{\lambda(n-k)}=2\Big(\frac{\sqrt5}{2}F_\lambda\Big)^nB_n\Big(x+\frac{L_\lambda}{\sqrt5F_\lambda}\Big)$$
$$+\frac{n\sqrt5}{2}F_\lambda\Big\{\frac{\sqrt5xF_\lambda+L_\lambda}{2}\Big\}^{n-1}-\frac{n\sqrt5}{2}F_\lambda\Big\{\frac{\sqrt5xF_\lambda}{2}+\psi^\lambda\Big\}^{n-1}.$$

Similarly, multiplying (11) with

$$\sum_{k=0}^{\infty}\big\{2^kQ_k(y)+2(2y)^k\big\}\frac{T^k}{k!}=\big(e^{T\alpha}+e^{T\beta}\big)^2,$$

we can express the convolution as

$$\sum_{k=0}^{n}(\alpha-\beta)^{k}\binom{n}{k}E_k(x)\{2^{n-k}Q_{n-k}(y)+2(2y)^{n-k}\}$$
$$=n![T^n]\frac{2}{e^{T(\alpha-\beta)}+1}e^{Tx(\alpha-\beta)}\times\big(e^{T\alpha}+e^{T\beta}\big)^2$$
$$=2n![T^n]\big\{e^{T((\alpha-\beta)x+\alpha+\beta)}+e^{T(2\beta+(\alpha-\beta)x)}\big\}.$$

Besides, it holds also that

$$\sum_{k=0}^{n}(\alpha-\beta)^{k}\binom{n}{k}E_k(x)(2y)^{n-k}=n![T^n]\frac{2e^{Tx(\alpha-\beta)}}{e^{T(\alpha-\beta)}+1}e^{2Ty}$$

$$= n![T^n]\frac{2e^{T(\alpha-\beta)\left(x+\frac{y}{\sqrt{1+y^2}}\right)}}{e^{T(\alpha-\beta)}+1}.$$

They lead us to the following identity.

Theorem 13. *For $n \in \mathbb{N}$, letting α and β be as in* (1), *the following identity holds:*

$$\sum_{k=0}^{n}(\alpha-\beta)^k\binom{n}{k}E_k(x)2^{n-k}Q_{n-k}(y) = 2\{\alpha+\beta+x(\alpha-\beta)\}^n \\ -2(\alpha-\beta)^n E_n\Big(x+\frac{y}{\sqrt{1+y^2}}\Big)+2\{2\beta+x(\alpha-\beta)\}^n.$$

Corollary 14. *For $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n}\Big(\frac{\sqrt{5}}{2}F_\lambda\Big)^k\binom{n}{k}E_k(x)L_{\lambda(n-k)} = 2\Big\{\frac{\sqrt{5}xF_\lambda+L_\lambda}{2}\Big\}^n+2\Big\{\frac{\sqrt{5}xF_\lambda}{2}+\psi^\lambda\Big\}^n \\ -2\Big(\frac{\sqrt{5}}{2}F_\lambda\Big)^n E_n\Big(x+\frac{L_\lambda}{\sqrt{5}F_\lambda}\Big).$$

Finally, rewriting, by replacing T with $3T$, the generating function (9) of Pell polynomials

$$\sum_{k=0}^{\infty}3^k Q_k(y)\frac{T^k}{k!} = e^{3T\alpha}+e^{3T\beta}$$

and then multiplying this with (11), we obtain the convolution formula below.

Theorem 15. *For $n \in \mathbb{N}$, letting α and β be as in* (1), *the following identity holds:*

$$\sum_{k=0}^{n}(\alpha-\beta)^k\binom{n}{k}E_k(x)3^{n-k}Q_{n-k}(y) = 2\Big\{(2+x)\alpha+(1-x)\beta\Big\}^n \\ -2\Big\{(1+x)\alpha+(2-x)\beta\Big\}^n+2\Big\{x\alpha+(3-x)\beta\Big\}^n.$$

Corollary 16. *For $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n}\Big(\frac{\sqrt{5}}{3}F_\lambda\Big)^k\binom{n}{k}E_k(x)L_{\lambda(n-k)} = 2\Big\{\frac{\sqrt{5}xF_\lambda+L_\lambda+\phi^\lambda}{3}\Big\}^n \\ -2\Big\{\frac{\sqrt{5}xF_\lambda+L_\lambda+\psi^\lambda}{3}\Big\}^n+2\Big\{\frac{\sqrt{5}xF_\lambda}{3}+\psi^\lambda\Big\}^n.$$

4. Further Comments

For Bernoulli and Euler polynomials, there exist two useful reciprocal relations (cf. [5, 19])

$$B_k(1-x) = (-1)^k B_k(x) \quad \text{and} \quad E_k(1-x) = (-1)^k E_k(x).$$

Then for all the identities established in the two preceding sections, by making the replacement $x \to 1-x$ and then making use of the above relations, we can transform them into formulae about alternating convolution sums.

For example, we have the following alternating sums corresponding to Theorem 1 and Corollary 2:

$$\sum_{k=0}^{n}(-1)^k(\alpha-\beta)^k\binom{n}{k}B_k(x)\mathrm{P}_{n-k}(y) = n(\alpha+x\beta-x\alpha)^{n-1},$$
$$\sum_{k=0}^{n}(-1)^k\big(\sqrt{5}F_\lambda\big)^k\binom{n}{k}B_k(x)F_{\lambda(n-k)} = nF_\lambda\Big\{\phi^\lambda-\sqrt{5}xF_\lambda\Big\}^{n-1}.$$

When $x=0$ and $\lambda=1$, this last identity becomes Byrd's (5).

Analogously for Theorem 9 and Corollary 10, there are the following counterparts:

$$\sum_{k=0}^{n}(-1)^k(\alpha-\beta)^k\binom{n}{k}E_k(x)\mathrm{Q}_{n-k}(y) = 2(\alpha-x\alpha+x\beta)^{n},$$
$$\sum_{k=0}^{n}(-1)^k\big(\sqrt{5}F_\lambda\big)^k\binom{n}{k}E_k(x)L_{\lambda(n-k)} = 2\Big\{\phi^\lambda-\sqrt{5}xF_\lambda\Big\}^{n}.$$

When $x=1/2$ and $\lambda=1$, the last formula yields the following companion of Byrd's identity (6):

$$\sum_{k=0}^{n}\Big(-\frac{\sqrt{5}}{2}\Big)^k\binom{n}{k}E_kL_{n-k} \;=\; 2^{1-n}.$$

For the remaining identities, the corresponding alternating convolution formulae can similarly be deduced. The reader can easily produce them whenever necessary.

Moreover, by examining the product of generating functions

$$\frac{e^{T\alpha^\lambda}-e^{T\beta^\lambda}}{\alpha-\beta}\times\frac{T(\alpha^\lambda-\beta^\lambda)}{e^{T\alpha^\lambda}-e^{T\beta^\lambda}}e^{T(x\alpha^\lambda-x\beta^\lambda+\beta^\lambda)},$$

we find that Theorem 1 can further be generalized by an extra integer parameter λ.

Theorem 17. *For $n\in\mathbb{N}$, letting α and β be as in* (1), *the following identities hold:*

$$\text{(a)}\ \sum_{k=0}^{n}\Big(2\mathrm{P}_\lambda(y)\sqrt{1+y^2}\Big)^k\binom{n}{k}B_k(x)\mathrm{P}_{\lambda(n-k)}(y)$$
$$= n\mathrm{P}_\lambda(y)\Big\{\mathrm{P}_{\lambda-1}(y)+(\beta+2x\sqrt{1+y^2})\mathrm{P}_\lambda(y)\Big\}^{n-1},$$
$$\text{(b)}\ \sum_{k=0}^{n}\Big(-2\mathrm{P}_\lambda(y)\sqrt{1+y^2}\Big)^k\binom{n}{k}B_k(x)\mathrm{P}_{\lambda(n-k)}(y)$$
$$= n\mathrm{P}_\lambda(y)\Big\{\mathrm{P}_{\lambda-1}(y)+(\alpha-2x\sqrt{1+y^2})\mathrm{P}_\lambda(y)\Big\}^{n-1}.$$

We remark that the above formula (b) is deduced by performing the replacement $x \to 1-x$ in formula (a), which reduces, when $\lambda = 1$, to the convolution identity in Theorem 1.

Similarly, taking into account the product

$$\Big(e^{T\alpha^{\lambda}} + e^{T\beta^{\lambda}}\Big) \times \frac{2e^{T(x\alpha^{\lambda} - x\beta^{\lambda} + \beta^{\lambda})}}{e^{T\alpha^{\lambda}} + e^{T\beta^{\lambda}}},$$

we extend Theorem 9, by an extra integer parameter λ, to the following one.

Theorem 18. *For $n \in \mathbb{N}$, letting α and β be as in* (1), *the following identities hold:*

$$\begin{aligned}
\text{(a)}\ \sum_{k=0}^{n} \Big(2\mathrm{P}_{\lambda}(y)\sqrt{1+y^2}\Big)^k \binom{n}{k} E_k(x)\mathrm{Q}_{\lambda(n-k)}(y) \\
= 2\Big\{\mathrm{P}_{\lambda-1}(y) + (\beta + 2x\sqrt{1+y^2})\mathrm{P}_{\lambda}(y)\Big\}^n, \\
\text{(b)}\ \sum_{k=0}^{n} \Big(-2\mathrm{P}_{\lambda}(y)\sqrt{1+y^2}\Big)^k \binom{n}{k} E_k(x)\mathrm{Q}_{\lambda(n-k)}(y) \\
= 2\Big\{\mathrm{P}_{\lambda-1}(y) + (\alpha - 2x\sqrt{1+y^2})\mathrm{P}_{\lambda}(y)\Big\}^n.
\end{aligned}$$

By employing the same technique, it is possible to extend other theorems in this paper by introducing the extra integer parameter λ. However, we shall not pursue them due to complexity.

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School of Mathematics and Statistics
Zhoukou Normal University
Zhoukou (Henan), China

Department of Mathematics and Physics
University of Salento (P. O. Box 193)
73100 Lecce, Italy

Email address: guo.dongwei2018@outlook.com
Email address: chu.wenchang@unisalento.it