IDENTITIES INVOLVING SUM OF DIVISORS, INTEGER PARTITIONS AND COMPOSITIONS

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Abstract. In this paper we show some identities come from the q-identities of Euler, Jacobi, Gauss and Rogers-Ramanujan. Some of these identities relating the function sum of divisor of a positive integer n and the number of integer partitions of n. One of the most intriguing result found here is given by the next equation, for $n > 0$.

$$\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l} = p_1(n),$$

where $\sigma_1(n)$ is the sum of all positive divisors of n, $p_1(n)$ is the number of integer partitions of n, and $C(n)$ is the set of integer compositions of n. In the last section we show seven applications, one of them is a series expansion for

$$\frac{(q^{a_1};q^{b_1})_\infty (q^{a_2};q^{b_2})_\infty \cdots (q^{a_k};q^{b_k})_\infty}{(q^{c_1};q^{d_1})_\infty (q^{c_2};q^{d_2})_\infty \cdots (q^{c_r};q^{d_r})_\infty},$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_r, d_1, \ldots d_r$ are positive integers, and $|q| < 1$.

1. Introduction

In Alegri [1], the authors found some new identities as

$$\sum_{\substack{w_1+\ldots+w_m \in C(n) \\ w_i \geq 2}} \frac{(-1)^m p_2(w_1-2)\cdots p_2(w_m-2)}{(2m+1)!} = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k p_{2k}(n-2k)}{(2k+1)!}$$

where $p_k(n)$ is the number of k-colored partitions of n, and $C(n)$ is the set of the integer compositions of n. This and other identities were obtained by a simple technique that the authors employed which is based on obtaining the coefficient of q^n on an infinite product. In this work, we use related techniques to find some identities coming from q-hypergeometric series, as defined in Bailey [9], which are expressed as an infinite product.

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As the title of the paper suggests, an important function for our purposes is the sum of positive divisors function $\sigma_x(n)$, for a complex x, is an arithmetic function given by:

$$\sigma_x(n) = \sum_{\substack{d|n \\ d>0}} d^x.$$

In section 2 we obtain results involving the function that counts the number of integer partitions in at most k colors. This function is widely studied as in recent articles like Chern and Fu [12] and Fu and Tang [14].

An integer partition[1] of n is a non-increasing sequence of natural numbers whose sum is n. The partitions of $n = 4$ are given next: 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. A k-colored partition of n is an integer partition of n wherein each part appear colored with one of k available colors. For example, if $k = 2$, and the colors are black and red, the 2-colored partitions of $n = 4$ are: 4, 4, $3+1$, $3+1$, $3+1$, $3+1$, $2+2$, $2+2$, $2+2$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $2+1+1$, $1+1+1+1$, $1+1+1+1$, $1+1+1+1$, $1+1+1+1$ and $1+1+1+1$.

Let $p_k(n)$ denote by the number of k-colored partition of an integer n. The generating function for $p_k(n)$ is given by the next infinite product.

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty^k}$$

where the q-Pochhammer symbol is defined by:

$$(a,q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

Taking the limit $n \to \infty$, we have:

$$(a,q)_\infty = \lim_{n\to\infty} (a,q)_n.$$

One result involving colored integer partitions found here is the next, for $n \geq 1$.

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m+1} \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} p_{m+1-k}(n) = \frac{\sigma_1(n)}{n}$$

Another useful concept here is the integer compositions. An integer composition of a positive integer n is an ordered collection of positive integers whose sum is n. The set of compositions of n is denoted by $C(n)$. There are 8 integer compositions of 4:

$$C(4) = \{4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1\}.$$

One of the first mathematicians who made extensive use of integer compositions was Major Percy A. MacMahon as in [24, 25]. More recent results involving integer compositions can be found in Heubach [21] and Sills [32].

[1]More information about integer partitions can be found in Andrews [2, 3, 4, 6]

In this paper, some of our results were obtained using identities of Euler, Gauss, Jacobi and Rogers-Ramanujan. The first three ones are particular cases of the triple product identity as given next.

Theorem 1 (The Triple Product Identity)**.** *For $|q| < 1$ and $x \in \mathbb{C} - \{0\}$,*

$$(x;q)_\infty (q/x;q)_\infty (q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{n(n-1)/2} x^k.$$

The proof of this theorem can be found in Andrews [4]. For our purposes it is important to find the coefficient of q^n in

$$\left(\sum_{n=1}^{\infty} a_n q^n \right)^l$$

for a positive integer l and a sequence of non-zero complex numbers $(a_n)_{n\geq 1}$. The multinomial coefficients is

$$\binom{l}{k_1, \ldots, k_{n-1}} = \frac{l!}{k_1! k_2! \cdots k_n!}$$

where $k_n = l - (k_1 + k_2 + \ldots + k_{n-1})$, $0 \leq k_i \leq l$, $1 \leq i \leq n$.

This number is found in the expansion of $(x_1 + x_2 + \ldots + x_r)^l$, as in theorem 3.7 of Charalambides[11], but our problem is quite different, because we have an infinite series instead of a polynomial one. For this intent, we will use integer compositions of n and we have the next result.

Proposition 1. *For $n \geq l$, the coefficient of q^n in the expansion of*

$$\left(\sum_{n=1}^{\infty} a_n q^n \right)^l$$

is equal to

$$\sum_{w_1 + w_2 + \ldots + w_l \in C(n)} a_{w_1} a_{w_2} \cdots a_{w_l}$$

Two of the most famous q-hypergeometric identities are the first and second Rogers-Ramanujan identities, which are respectively:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+n)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.2}$$

for $|q| < 1$.

The two last theorems of the penultimate section of this paper make use of these identities. Analytical proofs and combinatorial interpretations for the first and second Rogers-Ramanujan identities can be found in Andrews [6], Ramanujan [28], Rogers [29, 30] and Sills [33].

In the papers of Heim et al, [17, 18, 19], the authors obtained results involving the function

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z}=\prod_{n=1}^{\infty}(1-q^n)^{-z},$$

for $\tau\in H=\{b\in\mathbb{C}|Im\{b\}>0\}$, $z\in\mathbb{C}$, and the η function as given in Ono [27]. Considering the Fourier expansion of the function above, we get

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z}=\sum_{n=0}^{\infty}P_n(z)q^n.$$

As explained in [17], closed formulas for the polynomials $P_n(z)$ are not yet known. In the second section of this paper we will exhibit a formula for these polynomials depending on integer compositions, $z\in\mathbb{C}$, and sum of divisors function, σ_1. This formula is, for $n>0$,

$$P_n(z)=\sum_{l=1}^{n}\frac{z^l}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}$$

The problem of finding these polynomials is posed by Morris Newman in [26]. Properties of $P_n(z)$ are explored in [26], as well as by S. Ramanujan (see [20]). For $z=k$, a positive integer, we have a very interesting combinatorial interpretation. In fact, $P_n(k)$ is equal to the number of colored partitions of n where each part can be colored in up to k colors. Results for this class of partitions can be found in Chern and Fu, [12], [14], respectively.

Since for $\text{Im}\{z\}>0$, $q=e^{2\pi iz}$ and $|q|<1$, we can write the Dedekind eta function, as found in Ono [27], Bailey [9], Apostol [8] and Berndt [10], as

$$\eta(z)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}\frac{1}{1-q^n}=\sum_{n=0}^{\infty}P_n(-1)q^n=1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^l}{l!}\sum_{w_1+w_2+\cdots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}q^n$$

By the Euler pentagonal number theorem,

$$\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n+1)/2}=\prod_{n=1}^{\infty}(1-q^n), \tag{1.3}$$

for $|q|<1$, we can write, for $n>0$:

$$\sum_{l=1}^{n}\frac{(-1)^l}{l!}\sum_{w_1+w_2+\cdots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}=\begin{cases}(-1)^k, & \text{if } n=k(3k\pm1)/2\\ 0, & \text{elsewhere}\end{cases}$$

In the last section we will show that

$$\frac{\eta(2az)}{\eta(az)} = q^{\frac{a}{24}} \prod_{n=1}^{\infty}(1+q^{an}) = q^{\frac{a}{24}} + q^{\frac{a}{24}} \sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{1}{l!}\sum_{w_1+w_2+\cdots+w_l\in C(n)} d_{w_1}d_{w_2}\cdots d_{w_l}q^n,$$

wherein

$$d_n = \sum_{d|n,\ d>0}\frac{(-1)^{d+1}}{d}.$$

It is evident that the previous quotient is related to integer partitions into distinct parts. In this paper we will obtain some results for this class of integer partitions. Still in this article we will obtain an expansion for Gaussian polynomials

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)(1-q)(1-q^2)\cdots(1-q^{n-k})},$$

defined for $|q|<1$ and $k\le n$. A nice expansion for these is well known:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \sum_{j=0}^{\infty} p(j|\le kparts, each \le n-k)q^j,$$

ours is given by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = 1+\sum_{j=1}^{\infty}\sum_{l=1}^{j}\frac{1}{l!}\sum_{w_1+w_2+\cdots+w_l\in C(n)} g_{w_1}g_{w_2}\cdots g_{w_l}q^j,$$

where

$$g_j = \sum_{\substack{d|j,\ d>0 \\ \frac{j}{d}\le k}}\frac{1}{d} - \sum_{\substack{e|j,\ e>0 \\ \frac{j}{e}\ge n-k+1}}\frac{1}{e}.$$

In the paper "Some new infinite families of η-function identities", Leininger and Milne [23], found beautiful expansions for $(q;q)_\infty^{k^2+2}$ (Section 2), $(q;q)_\infty^{k^2}$ (Section 3) and a conjecture for $(q;q)_\infty^{k^2-2}$ (Section 4).

For the first product we have

$$(q;q)_\infty^{k^2+2} = 1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^l(k^2+2)^l}{l!}\sum_{w_1+w_2+\cdots w_l\in C(n)}\frac{\sigma_1(w_1)\cdots\sigma_1(w_l)}{w_1\cdots w_l}q^n.$$

For the second and third products, the expansions are:

$$(q;q)_\infty^{k^2} = 1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^l k^{2l}}{l!}\sum_{w_1+w_2+\cdots w_l\in C(n)}\frac{\sigma_1(w_1)\cdots\sigma_1(w_l)}{w_1\cdots w_l}q^n,$$

$$(q;q)_\infty^{k^2-2} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{(-1)^l(k^2-2)^l}{l!} \sum_{w_1+w_2+\cdots w_l \in C(n)} \frac{\sigma_1(w_1)\cdots\sigma_1(w_l)}{w_1\cdots w_l} q^n.$$

In this article we will find more general expansions than the previous ones. We will deal with expansions like $(q^a;q^b)_\infty^{-z}$, where a and b are positive integers and $z, q \in \mathbb{C}$, $|q| < 1$. In this case, we will show that

$$(q^a;q^b)_\infty^{-z} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{z^l}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} \rho_{a,b}(w_1)\rho_{a,b}(w_2)\cdots\rho_{a,b}(w_l) q^n,$$

where

$$\rho_{a,b}(n) = \sum_{\substack{d|n,\ d>0 \\ \frac{n}{d} \equiv a mod b}} \frac{1}{d}.$$

In the ante-penultimate result of this paper, we will prove next the equation

$$\frac{(q^{a_1};q^{b_1})_\infty(q^{a_2};q^{b_2})_\infty\cdots(q^{a_k};q^{b_k})_\infty}{(q^{c_1};q^{d_1})_\infty(q^{c_2};q^{d_2})_\infty\cdots(q^{c_r};q^{d_r})_\infty} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{z^l}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} H(w_1)H(w_2)\cdots H(w_l) q^n,$$

where

$$H(n) = \rho_{a_1,b_1}(n) - \rho_{a_2,b_2}(n) - \ldots \rho_{a_k,b_k}(n) + \rho_{c_1,c_1}(n) + \rho_{c_2,d_2}(n) + \ldots + \rho_{c_r,d_r}(n)$$

2. Jacobi

The q-identity utilized here, due to Jacobi, is the following.

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = \prod_{n=1}^{\infty}(1-q^n)^3 \tag{2.1}$$

for $|q| < 1$. This identity can be found in Jacobi [22] and Andrews [4]. Using the previous equation, we can state the following result.

Theorem 2. *For $n \geq 1$, we have*

$$\sum_{l=1}^{n} \frac{(-1)^l 3^l}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_n)}{w_1 w_2\cdots w_l}$$
$$= \begin{cases} (-1)^k(2k+1), & \text{if } n = k(k+1)/2 \\ 0, & \text{elsewhere} \end{cases}$$

Proof. For $q \in \mathbb{R}$, $|q| < 1$, we have

$$ln\left(\prod_{n=1}^{\infty}(1-q^n)^3\right) = 3\sum_{n=1}^{\infty} ln(1-q^n),$$

and expanding $ln(1-q^n)$ into Taylor series, we get

$$ln(1-q^n) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1}(-1)^{m+1}q^{n(n+1)/2} = -\sum_{m=1}^{\infty} \frac{q^{nm}}{m}.$$

Thus,

$$ln\left(\prod_{n=1}^{\infty}(1-q^n)^3\right) = -3\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{nm}}{m} = -3\left(\sum_{n=1}^{\infty} q^n + \frac{q^{2n}}{2} + \frac{q^{3n}}{3} + \frac{q^{4n}}{4} + \dots\right)$$

We are interested in finding the coefficient of q^n in

$$\sum_{n=1}^{\infty} q^n + \frac{q^{2n}}{2} + \frac{q^{3n}}{3} + \frac{q^{4n}}{4} + \dots$$

One may note that

$$\sum_{n=1}^{\infty} q^n + \frac{q^{2n}}{2} + \frac{q^{3n}}{3} + \dots = q + (\frac{1}{2}+1)q^2 + (\frac{1}{3}+1)q^3 + (\frac{1}{2}+\frac{1}{4}+1)q^4 + (\frac{1}{5}+1)q^5 + (\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+1)q^6 + \dots$$

and generally the coefficient of q^n in this sum is given by the sum of reciprocals of the positive divisors of n. It is well-known that this sum is given by $\frac{\sigma_1(n)}{n}$, and then

$$ln\left(\prod_{n=1}^{\infty}(1-q^n)^3\right) = -3\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n,$$

so that the identity of Jacobi, (2.1), can be rewritten as

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = exp\left(-3\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right),$$

wherein $exp(z) = e^z$. By the Taylor expansion of the complex exponential function of the previous equation, we have

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = \sum_{l=0}^{\infty}\frac{(-1)^l3^l}{l!}\left(\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)^l.$$

Using the proposition 1, the following equation is true.

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = 1 + \sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^l3^l}{l!}\left(\sum_{w_1+w_2+\dots+w_l\in C(n)}\frac{\sigma_{w_1}\sigma_{w_2}\cdots\sigma_{w_l}}{w_1w_2\cdots w_l}\right)q^n$$

We conclude the proof of the theorem by comparing the coefficient of q^n in both sides of the previous equation.

□

For the next result, we shall consider the derivative of the equation (2.1) regarding the variable q as given next.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n(n+1)(2n+1)}{2}q^{n(n+1)/2-1} = exp\left(-3\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)\frac{d}{dq}\left[-3\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right]$$
$$= \left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\right)\left(-3\sum_{n=1}^{\infty}\sigma_1(n)q^{n-1}\right)$$
$$= -3\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\right)\left(\sum_{n=0}^{\infty}\sigma_1(n+1)q^n\right)$$

Writing

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = \sum_{n=0}^{\infty}c_nq^n,$$

where

$$c_n = \begin{cases}(-1)^{k+1}3(2k+1), & \text{if } n = k(k+1)/2\\ 0, & \text{elsewhere}\end{cases}$$

we have

$$\sum_{n=1}^{\infty}\sum_{k+l=n}c_k\sigma_1(l+1)q^n = \sum_{n=1}^{\infty}(-1)^n\frac{n(n+1)(2n+1)}{2}q^{n(n+1)/2-1}.$$

Comparing the coefficient of q^n in both sides of the previous equation, we can state the next result.

Theorem 3. *For $n \geq 1$, we have*

$$\sum_{k+l=n}c_k\sigma_1(l+1) = \begin{cases}(-1)^kk(2k+1)(2k+1)/2, & \textit{if } n = k(k+1)/2-1\\ 0, & \textit{elsewhere}\end{cases}$$

3. Euler

One equation useful to obtain the next result is the well-known Euler identity as given next.

For $q \in \mathbb{R}$, $|q| < 1$,

$$\sum_{n=0}^{\infty}p_1(n)q^n = \sum_{n=0}^{\infty}\frac{q^n}{(q;q)_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n}.$$

This equation can be found in Euler [13], Weil [34] and Andrews and Eriksson [6], in the latter, a proof with combinatorial arguments is provided. The first result in this section is the following.

Theorem 4. *For $n \geq 1$, we have*
$$\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l} = p_1(n).$$

Proof. Since $q \in \mathbb{R}$, $|q| < 1$, we have

$$ln\left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right) = -\sum_{n=1}^{\infty} ln(1-q^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{q^{nm}}{m} = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n.$$

Thus, we can rewrite this Euler identity as

$$\sum_{n=0}^{\infty} p_1(n) q^n = exp\left(\sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n\right) = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n\right)^l.$$

As we known,

$$\left(\sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n\right)^l = \sum_{j=l}^{\infty} \sum_{w_1+w_2+\ldots+w_l \in C(j)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l} q^j$$

therefore,

$$\sum_{n=1}^{\infty} p_1(n) q^n = \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l} q^n,$$

and the conclusion of the proof coming from comparing the coefficient q^n in both sides of the last equation. □

For example, if $n = 5$, we have $\sigma_1(1) = 1$, $\sigma_1(2) = 3$, $\sigma_1(3) = 4$, $\sigma_1(4) = 7$, $\sigma_1(5) = 6$ and then,

$$\sum_{l=1}^{5} \frac{1}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(5)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l} = \frac{1}{1!}\frac{\sigma_1(5)}{5} + \frac{1}{2!}\left(2\frac{\sigma_1(4)}{4} + 2\frac{\sigma_1(3)}{3}\frac{\sigma_1(2)}{2}\right)$$
$$+\frac{1}{3!}\left(3\frac{\sigma_1(3)}{3} + 3\frac{\sigma_1(2)}{2}\frac{\sigma_1(2)}{2}\right) + \frac{1}{4!}\left(4\frac{\sigma_1(2)}{2}\right) + \frac{1}{5!}(\sigma_1(1))^5 = 7 = p_1(5).$$

In the other hand, for $|q| < 1/10$, we have $|\sum_{n=0}^{\infty} p_1(n) q^n| < 1$. Actually, for $|q| < 1/10$ it is a fact that $|\sum_{n=0}^{\infty} p_1(n) q^n| \leq 0,2$. Thus, we can consider the Taylor series expansion of $ln(\sum_{n=0}^{\infty} p_1(n) q^n) = ln(1 + \sum_{n=1}^{\infty} p_1(n) q^n)$. Here we get:

$$ln\left(1 + \sum_{n=1}^{\infty} p_1(n) q^n\right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1}\left(1 + \sum_{n=1}^{\infty} p_1(n) q^n\right)^{m+1}$$

Particularly,

$$\left(1+\sum_{n=1}^{\infty}p_1(n)q^n\right)^{m+1}=\left(\frac{1}{(q;q)_\infty}-1\right)^{m+1}=\sum_{k=0}^{m+1}\binom{m+1}{k}\frac{(-1)^k}{(q;q)_\infty^{m+1-k}}.$$

Since the infinite product $\frac{1}{(q;q)_\infty^{m+1-k}}$ is the generating function for the number of colored partitions of an integer n into at most $m+1-k$ colors, we can write

$$\frac{(-1)^k}{(q;q)_\infty^{m+1-k}}=\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\sum_{n=0}^{\infty}p_{m+1-k}(n)q^n=\sum_{n=0}^{\infty}\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}p_{m+1-k}(n)q^n.$$

So we get

$$ln\left(\sum_{n=0}^{\infty}p_1(n)q^n\right)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\frac{(-1)^m}{m+1}\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}p_{m+1-k}(n)q^n$$

By the previous theorem, we know that

$$ln\left(\frac{1}{(q;q)_\infty}\right)=\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n.$$

Therefore, the next result is valid.

Theorem 5.

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}p_{m+1-k}(n)=\frac{\sigma_1(n)}{n}$$

□

The last two theorems in this section making use of equation (1.3). The first one is given next.

Theorem 6. *For $n>0$, we have*

$$\sum_{l=1}^{n}\frac{(-1)^l}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}=\begin{cases}(-1)^k, & \text{if } n=k(3k\pm1)/2\\ 0, & \text{elsewhere}\end{cases}$$

Proof. Since for $q\in\mathbb{R}$, $|q|<1$, we have

$$ln\left(\prod_{n=1}^{\infty}(1-q^n)\right)=-\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n,$$

then

$$\sum_{n=0}^{\infty}(-1)^nq^{n(3n+1)/2}=exp\left(-\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)=\sum_{l=0}^{\infty}\frac{1}{l!}\left(-\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)^l.$$

As we know

$$\left(-\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)^l=\sum_{n=1}^{\infty}(-1)^l\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}q^n,$$

we have

$$\sum_{n=0}^{\infty}(-1)^n q^{n(3n+1)/2}=1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^l}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}q^n.$$

Comparing the coefficient o q^n in both sides of the previous equation, we have the expected result.

□

Deriving both sides of equation (1.3), we have

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{n(3n+1)}{2}q^{n(3n+1)/2-1}=\frac{d}{dq}\left[exp\left(-\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)\right]$$
$$=\left(\sum_{n=-\infty}^{\infty}(-1)^{n+1}q^{n(3n+1)/2}\right)\left(\sum_{n=0}^{\infty}\sigma_1(n+1)q^n\right).$$

Writing

$$\left(\sum_{n=-\infty}^{\infty}(-1)^{n+1}q^{n(3n+1)/2}\right)=\sum_{n=0}^{\infty}d_nq^n,$$

where

$$d_n=\begin{cases}(-1)^{k+1}, & \text{if } n=k(3k\pm 1)/2\\ 0, & \text{elsewhere}\end{cases}$$

the coefficient of q^n, $n>0$ in

$$\left(\sum_{n=0}^{\infty}d_nq^n\right)\left(\sum_{n=0}^{\infty}\sigma_1(n+1)q^n\right)$$

is

$$\sum_{k+l=n}d_k\sigma_1(l+1)=\begin{cases}(-1)^j(3j+1)/2, & \text{if } n=j(3j\pm 1)/2-1\\ 0, & \text{elsewhere}\end{cases}$$

The last theorem of this section is a formula for the polynomials $P_n(z)$, $z\in\mathbb{R}$, as considered in Newman [26] and Heim [17]. These polynomials are found in

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z}=\prod_{n=1}^{\infty}(1-q^n)^{-z}=\sum_{n=0}^{\infty}P_n(z)q^n.$$

Theorem 7. *For $n>0$, we have*

$$P_n(z)=\sum_{l=1}^{n}\frac{z^l}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}$$

Proof. For $q\in\mathbb{R}$, $|q|<1$, we have

$$ln\left(\prod_{n=1}^{\infty}(1-q^n)^{-z}\right)=-z\sum_{n=1}^{\infty}ln(1-q^n),$$

opening $ln(1-q^n)$ as made before, we get

$$ln\left(\prod_{n=1}^{\infty}(1-q^n)^{-z}\right)=z\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n,$$

and then

$$\sum_{n=0}^{\infty}P_n(z)q^n=exp\left(z\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)=\sum_{l=0}^{\infty}\frac{z^l}{l!}\left(\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)^l.$$

Using the proposition 1 for

$$\left(\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)^l,$$

we get

$$\sum_{n=0}^{\infty}P_n(z)q^n=\sum_{l=0}^{\infty}\frac{z^l}{l!}\left(\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}q^n\right)^l=1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}q^n,$$

and comparing the coefficient of q^n in the previous equation, we have the expected result.

□

For instance,

$$P_n(1)=\sum_{l=1}^{n}\frac{1}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}=p_1(n),$$

for all $n>0$, and

$$P_n(-1)=\sum_{l=1}^{n}\frac{(-1)^l}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1w_2\cdots w_l}=\begin{cases}(-1)^k, & \text{if } n=k(3k\pm 1)/2k\\ 0, & \text{elsewhere}\end{cases}$$

4. Gauss

The main equation of this section is, for $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2. \tag{4.1}$$

The equation (4.1) is due to Gauss and can be found in Gauss [15, 16] and Andrews [4]. Using this equation, we can state the next result.

Theorem 8. *For $n > 0$, and*

$$f_n = \begin{cases} \sum\limits_{\substack{d|n \\ d \text{ is odd}}} \dfrac{2}{d}, & \text{if } n \text{ is odd} \\ \sum\limits_{\substack{dx|n \\ d \text{ is the greatest power of } 2 \text{ such that } d|n,\ x \text{ is odd}}} -\dfrac{2}{dx} + \sigma_1(n/2), & \text{if } n \text{ is even} \end{cases}$$

we have

$$\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} f_{w_1} f_{w_2} \cdots f_{w_l} = \begin{cases} 2, & \text{if } n \text{ is a perfect square} \\ 0, & \text{elsewhere} \end{cases}$$

Proof. Since $q \in \mathbb{R}$, $|q| < 1$, we get

$$\begin{aligned} ln\left(\prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2\right) &= \sum_{n=1}^{\infty} \left[ln(1 - q^{2n}) + 2ln(1 + q^{2n-1})\right] \\ &= \sum_{n=1}^{\infty} \left[-\sum_{m=1}^{\infty} \frac{q^{2nm}}{m} + 2\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} q^{m(2n-1)}\right] \\ &= \sum_{n=0}^{\infty} \sigma_1(n+1) q^{2n+2} + 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} q^{m(2n-1)} \end{aligned}$$

Now, we will analyze the last term of the previous equation.

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}q^{m(2n-1)}=q-\frac{1}{2}q^2+\left(1+\frac{1}{3}\right)q^3-\frac{1}{4}q^4+\left(1+\frac{1}{5}\right)q^5+$$
$$+\left(-\frac{1}{2}-\frac{1}{6}\right)q^6+\left(1+\frac{1}{7}\right)q^7-\frac{1}{8}q^8+\left(1+\frac{1}{3}+\frac{1}{9}\right)q^9+\left(-\frac{1}{2}-\frac{1}{10}\right)q^{10}$$
$$+\left(1+\frac{1}{11}\right)q^{11}+\left(-\frac{1}{4}-\frac{1}{12}\right)q^{12}+\left(1+\frac{1}{13}\right)q^{13}+\left(-\frac{1}{2}-\frac{1}{14}\right)q^{14}$$
$$+\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{15}\right)q^{15}-\frac{1}{16}q^{16}+\left(1+\frac{1}{17}\right)q^{17}+\left(-\frac{1}{2}-\frac{1}{6}-\frac{1}{18}\right)q^{18}+\dots$$

It is easy to see that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}q^{m(2n-1)}=\sum_{n=1}^{\infty}e_nq^n$$

wherein

$$e_n=\begin{cases}\displaystyle\sum_{\substack{d|n\\ d\ is\ odd}}\frac{1}{d}, & \text{if n is odd}\\ \displaystyle\sum_{\substack{dx|n\\ d\ is\ the\ greatest\ power\ of\ 2\ such\ that\ d|n,\ x\ is\ odd}}-\frac{1}{dx}, & \text{if n is even}\end{cases}$$

Thus, considering

$$f_n=\begin{cases}\displaystyle\sum_{\substack{d|n\\ d\ is\ odd}}\frac{2}{d}, & \text{if n is odd}\\ \displaystyle\sum_{\substack{dx|n\\ d\ is\ the\ greatest\ power\ of\ 2\ such\ that\ d|n,\ x\ is\ odd}}-\frac{2}{dx}+\sigma_1(n/2), & \text{if n is even}\end{cases}$$

we have

$$\sum_{n=-\infty}^{\infty}q^{n^2}=1+\sum_{n=1}^{\infty}2q^{n^2}=exp\left(\sum_{n=1}^{\infty}f_nq^n\right)=1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{1}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}f_{w_1}f_{w_2}\cdots f_{w_l}q^n$$

and the result of the theorem is valid.

$\square$

5. Rogers-Ramanujan

For the first theorem we use the equation (1.1), and we denote by $r_1(n)$ the number of partitions of n, whose difference for consecutive parts is at least 2 (= equal to the number of partitions of n into parts congruent to ± 1 mod 5).

Theorem 9. *For $n > 0$, and*

$$\tau_n = \sum_{\substack{x|n \\ \frac{n}{x} \equiv \pm 1 mod 5}} \frac{1}{x}$$

we have

$$r_1(n) = \sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} \tau_{w_1}\tau_{w_2}\cdots\tau_{w_l}$$

Proof. Since,

$$ln\left(\prod_{n=0}^{\infty}\left(\frac{1}{q^{5n+1}}\right)\left(\frac{1}{q^{5n+4}}\right)\right) = \sum_{n=0}^{m=1}\left(\frac{q^{(5n+1)m}}{m} + \frac{q^{(5n+4)m}}{m}\right) = (q+q^6+q^{11}+\ldots)+(q^4+q^9+q^{14}+\ldots$$
$$= \frac{1}{2}(q^2+q^{12}+q^{22}+\ldots)+\frac{1}{2}(q^8+q^{18}+q^{28}+\ldots)$$
$$+\frac{1}{3}(q^3+q^{18}+q^{33}+\ldots)+\frac{1}{3}(q^{12}+q^{27}+q^{42}+\ldots)+\ldots,$$

it is easy to confirm that the coefficient of q^n, $n > 0$, in the previous infinite sum is

$$\tau_n = \sum_{\substack{x|n \\ \frac{n}{x} \equiv \pm 1 mod 5}} \frac{1}{x}.$$

Thus,

$$\sum_{n=0}^{\infty} r_1(n)q^n = exp\left(\sum_{n=1}^{\infty}\tau_n q^n\right) = \sum_{l=0}^{\infty}\frac{1}{l!}\left(\sum_{n=1}^{\infty}\tau_n q^n\right)^l = \sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{1}{l!}\sum_{w_1+w_2+\ldots+w_l\in C(n)}\tau_{w_1}\tau_{w_2}\cdots\tau_{w_l}q^n,$$

the conclusion of the proof comes from the comparison of the coefficient of q^n in the previous equation. □

For the last theorem of this section we use the equation (1.2), and we denote by $r_2(n)$ the number of partitions of n, whose difference for consecutive parts is at least 2, and the small part is greater than 1 (= equal to the number of partitions of n into parts congruent to ± 2 mod 5).

Theorem 10. *For $n > 0$, and*

$$Y_n = \sum_{\substack{x|n \\ \frac{n}{x} \equiv \pm 2 mod 5}} \frac{1}{x}$$

we have

$$r_2(n) = \sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} Y_{w_1} Y_{w_2} \cdots Y_{w_l}$$

The proof of this theorem is similar to the theorem (9), and it is omitted here.

6. Dedekind Eta Function, Gaussian Polynomials and more

Let denote by $p_d(n)$ the number of integer partitions of a non-negative integer n into distinct parts. The generating function for the sequence $(p_d(n))_{n\geq 0}$ is

$$\sum_{n=0}^{\infty} p_d(n) q^n = \prod_{n=1}^{\infty} (1+q^n),$$

for $|q| < 1$ and $p_d(0) = 1$.

Considering the logarithm of the previous infinite product, we have

$$\ln\left(\prod_{n=1}^{\infty}(1+q^n)\right) = \sum_{n=1}^{\infty} \ln(1+q^n).$$

Since $|q| < 1$, using the Taylor expansion of $\ln(1+q^n)$, we can wright the previous series as

$$\sum_{n=1}^{\infty} \ln(1+q^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} q^{mn} = \sum_{n=1}^{\infty} q^n - \frac{q^{2n}}{2} + \frac{q^{3n}}{3} - \frac{q^{4n}}{4} + \ldots.$$

It is easy to see that the coefficient of q^n in the last series is

$$d_n = \sum_{d|n,\ d>0} \frac{(-1)^{d+1}}{d}. \tag{6.1}$$

That way, $\sum_{n=0}^{\infty} p_d(n) q^n = \exp(\sum_{n=1}^{\infty} d_n q^n)$, where $\exp(z) = e^z$, and expanding this exponential function into Taylor series around $z_0 = 0$, we have:

$$\sum_{n=0}^{\infty} p_d(n) q^n = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{n=1}^{\infty} d_n q^n\right)^l.$$

Using Proposition 1, we can expand $(\sum_{n=1}^{\infty} d_n q^n)^l$, for l a positive integer like

$$\left(\sum_{n=1}^{\infty} d_n q^n\right)^l = \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} d_{w_1} d_{w_2} \cdots d_{w_l}\right) q^n.$$

Comparing the coefficients of q^n in the previous equation, we get:

$$p_d(n) = \sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} d_{w_1} d_{w_2} \cdots d_{w_l} \tag{6.2}$$

We know that the number of partitions of n into odd parts is equal to the number of partitions of n into odd parts. Writing this fact in terms of generating functions we have:

$$\sum_{n=1}^{\infty} p_d(n) = \sum_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}.$$

Considering the logarithm $\ln(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}})$, for $|q| < 1$, we get:

$$\ln\left(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}\right) = -\sum_{n=1}^{\infty} \ln(1-q^{2n-1}) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{q^{(2n-1)m}}{m}$$
$$= \sum_{n=1}^{\infty} q^{2n-1} + \frac{q^{2(2n-1)}}{2} + \frac{q^{3(2n-1)}}{3} + \frac{q^{4(2n-1)}}{4} + \ldots$$

One may note that if n is a power of 2, the coefficient of q^n in the last series is $\frac{1}{n}$. If n is odd, such coefficient is $\frac{\sigma_1(n)}{n}$. For the case where $n = 2^j \alpha_1 \alpha_2 \cdots \alpha_m$, where $2 \nmid \alpha_i$, the coefficient of q^n is

$$\sum_{d' \mid \alpha_1\alpha_2\cdots\alpha_m} \frac{1}{2^j d'}.$$

Summarizing, we have

$$\prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} = \exp\left(\sum_{n=1}^{\infty} c_n q^n\right),$$

where

$$c_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is a power of 2} \\ \frac{\sigma_1(n)}{n}, & \text{if } n \text{ is odd} \\ \displaystyle\sum_{d' \mid \alpha_1\alpha_2\cdots\alpha_m} \frac{1}{2^j d'}, & \text{if } n = 2^j \alpha_1\alpha_2\cdots\alpha_m,\ 2 \nmid \alpha_i \end{cases} \tag{6.3}$$

Again, using proposition 1, we get

$$(6.4) \qquad \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} = 1 + \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} c_{w_1} c_{w_2} \cdots c_{w_l} q^n.$$

Comparing the coefficients of q^n, $n > 0$ in (6.2) and (6.4), we can establish the following result.

Theorem 11.

$$p_d(n) = \sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} c_{w_1} c_{w_2} \cdots c_{w_l} = \sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} d_{w_1} d_{w_2} \cdots d_{w_l},$$

where c_n and d_n are as given in (6.3) and (6.1).

□

The next result we use an expansion for $\eta(\tau)$, where $\mathrm{Im}\{\tau\} > 0$, $q = e^{2\pi i \tau}$, and $|q| < 1$. Since $q^{-\frac{1}{24}} \eta(\tau) = (q;q)_\infty$, by the same procedure done in the previous theorem, we have

$$q^{-\frac{1}{24}} \eta(\tau) = exp\left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \right),$$

and we can write

$$\eta(\tau) = q^{\frac{1}{24}} + \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_2(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l} q^{n+\frac{1}{24}}.$$

For $a \in \mathbb{C} - \{0\}$, we have

$$\eta(a\tau) = q^{\frac{a}{24}} + \sum_{n=1}^{\infty} \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_2(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l} q^{an+\frac{a}{24}}.$$

By the definition of the Dedekind eta function, the next quotient is

$$\frac{\eta(2a\tau)}{\eta(a\tau)} = q^{\frac{a}{24}} \prod_{n=1}^{\infty} (1+q^{an}).$$

Since $|q| < 1$, considering the expansion of $\ln(\prod_{n=1}^{\infty}(1+q^{an}))$ in Taylor series, we have

$$\ln\left(\prod_{n=1}^{\infty} (1+q^{an}) \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} q^{anm}}{m}.$$

The coefficient of q^{an} in the previous series is

$$d_n = \sum_{d|n,\ d>0} \frac{(-1)^{d+1}}{d}. \tag{6.5}$$

Thus, the next result is true.

Theorem 12.

$$\frac{\eta(2a\tau)}{\eta(a\tau)} = q^{\frac{a}{24}} + q^{\frac{a}{24}} \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} d_{w_1} d_{w_2} \cdots d_{w_l} \right) q^{an}$$

□

For instance, if $a = 4$,

$$\frac{\eta(8\tau)}{\eta(4\tau)} = q^{\frac{1}{6}} + q^{\frac{1}{6}} \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} d_{w_1} d_{w_2} \cdots d_{w_l} \right) q^{4n},$$

and

$$\eta(8\tau) = q^{\frac{1}{6}} \eta(4\tau) + q^{\frac{1}{6}} \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} d_{w_1} d_{w_2} \cdots d_{w_l} \right) q^{4n} \eta(4\tau).$$

Considering

$$a_n = \begin{cases} \displaystyle\sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_2(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l}, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases}$$

and

$$b_n = \begin{cases} \displaystyle\sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} d_{w_1} d_{w_2} \cdots d_{w_l}, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \end{cases}$$

we have

$$\eta(8\tau) = \left(\sum_{n=0}^{\infty} b_n q^{4n+\frac{1}{6}} \right) \left(\sum_{n=0}^{\infty} a_n q^{4n+\frac{1}{6}} \right),$$

and since

$$\eta(8\tau) = q^{\frac{1}{3}} \sum_{n=0}^{\infty} a_n q^{8n},$$

we get

$$\left(\sum_{n=0}^{\infty} b_n q^{4n}\right)\left(\sum_{n=0}^{\infty} a_n q^{4n}\right) = \sum_{n=0}^{\infty} a_n q^{8n},$$

thus

$$\sum_{\substack{i+j \in C(8n) \\ i,j \equiv 0 (mod 4)}} b_i a_j = a_n.$$

More generally, we can state the next corollary.

Corollary 1. *If k is a positive integer, then*

$$\sum_{\substack{i+j \in C(2kn) \\ i,j \equiv 0 (mod k)}} b_i a_j = a_n.$$

The next result is a Taylor expansion for the Gaussian polynomials. For $k \leq n$, and $|q| < 1$, we have

$$\ln\left(\left[\begin{array}{c} n \\ k \end{array}\right]_q\right) = \sum_{j=1}^{n} \ln(1-q^j) - \sum_{j=1}^{k} \ln(1-q^j) - \sum_{j=1}^{n-k} \ln(1-q^j)$$
$$= -\sum_{j=1}^{n}\sum_{m=1}^{\infty} \frac{q^{jm}}{m} + \sum_{i=1}^{k}\sum_{m=1}^{\infty} \frac{q^{im}}{m} + \sum_{r=1}^{n-k}\sum_{m=1}^{\infty} \frac{q^{rm}}{m}$$

The coefficient of q^j in the previous series is

$$g_j = \sum_{\substack{d|j, d>0 \\ \frac{j}{d} \leq k}} \frac{1}{d} - \sum_{\substack{e|j, e>0 \\ \frac{j}{e} \geq n-k+1}} \frac{1}{e} \tag{6.6}$$

Thence,

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = exp\left(\sum_{j=1}^{\infty} g_j q^j\right),$$

and by the Proposition 1, we have the next theorem.

Theorem 13. *For $k \leq n, |q| < 1$, and g_j as in (6.6),*

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = 1 + \sum_{j=1}^{\infty}\left(\sum_{l=1}^{j} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} g_{w_1} g_{w_2} \cdots g_{w_l}\right) q^j$$

□

Obviously, we get:

$$p(j|\leq k parts, each \leq n-k) = \sum_{l=1}^{j} \frac{1}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} g_{w_1} g_{w_2} \cdots g_{w_l}.$$

In the next result, we get an expansion for infinite products like $(q^a;q^b)_\infty^{-z}$, for a and b positive integers, $z \in \mathbb{C}$ and $|q|<1$. In fact, these expansions generalize the expansion of $\eta(\tau)$, as seen earlier in this article. Here,

$$\ln\left((q^a;q^b)_\infty^{-z}\right) = z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{(a+nb)m}}{m} = z\sum_{n=1}^{\infty} q^{a+nb} + \frac{q^{2(a+nb)}}{2} + \frac{q^{3(a+nb)}}{3} + \frac{q^{4(a+nb)}}{4} + \dots,$$

thus, the coefficient of q^n in the previous series is

$$\rho_{a,b}(n) = \sum_{\substack{d|n, d>0 \\ \frac{n}{d} \equiv a (mod b)}} \frac{1}{d}. \tag{6.7}$$

By using the Proposition 1, we have proved the next result.

Theorem 14.

$$(q^a;q^b)_\infty^{-z} = 1 + \sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\frac{z^l}{l!}\sum_{w_1+w_2+\cdots+w_l \in C(n)} \rho_{a,b}(w_1)\rho_{a,b}(w_2)\cdots\rho_{a,b}(w_l)\right) q^n$$

□

For instance, by $a=1$, $b=5$ and $z=1$, we have that the number of partitions of n into parts $\equiv 1(mod 5)$ is equal to

$$\sum_{l=1}^{n}\frac{1}{l!}\sum_{w_1+w_2+\cdots+w_l \in C(n)} \rho_{1,5}(w_1)\rho_{1,5}(w_2)\cdots\rho_{1,5}(w_l).$$

In next result, we will obtain a Taylor expansion for

$$\frac{(q^{a_1};q^{b_1})_\infty (q^{a_2};q^{b_2})_\infty \cdots (q^{a_k};q^{b_k})_\infty}{(q^{c_1};q^{d_1})_\infty (q^{c_2};q^{d_2})_\infty \cdots (q^{c_r};q^{d_r})_\infty},$$

where $a_1,\dots,a_k,b_1,\dots,b_k,c_1,\dots,c_r,d_1,\dots d_r$ are positive integers, and $|q|<1$. Obviously, we can obtain such expansion applying $k+r$ times the previous theorem, but we believe that the expansion obtained here is more simple. As usual,

$$
\begin{aligned}
&\ln\left(\frac{(q^{a_1};q^{b_1})_\infty(q^{a_2};q^{b_2})_\infty\cdots(q^{a_k};q^{b_k})_\infty}{(q^{c_1};q^{d_1})_\infty(q^{c_2};q^{d_2})_\infty\cdots(q^{c_r};q^{d_r})_\infty}\right)\\
&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)\left[\frac{q^{m(a_1+nb_1)}}{m}+\cdots+\frac{q^{m(a_k+nb_k)}}{m}\right]+\left[\frac{q^{m(c_1+nd_1)}}{m}+\ldots+\frac{q^{m(c_r+nd_r)}}{m}\right]\\
&=\sum_{n=1}^{\infty}H(n)q^n,
\end{aligned}
$$

wherein

$$H(n)=\rho_{c_1,d_1}(n)+\rho_{c_2,d_c}(n)+\cdots+\rho_{c_r,d_r}(n)-\rho_{a_1,b_1}(n)-\rho_{a_2,b_2}(n)-\ldots-\rho_{a_k,b_k}(n),$$

and $\rho_{a,b}(n)$ as in (6.7). In this way, we get the following result.

Theorem 15. *For $a_1,\ldots,a_k,b_1,\ldots,b_k,c_1,\ldots,c_r,d_1,\ldots d_r$ positive integers, and $|q|<1$,*

$$\frac{(q^{a_1};q^{b_1})_\infty(q^{a_2};q^{b_2})_\infty\cdots(q^{a_k};q^{b_k})_\infty}{(q^{c_1};q^{d_1})_\infty(q^{c_2};q^{d_2})_\infty\cdots(q^{c_r};q^{d_r})_\infty}=1+\sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\frac{1}{l!}\sum_{w_1+w_2+\cdots+w_l\in C(n)}H(w_1)H(w_2)\cdots H(w_l)\right)q^n$$

□

The first Rogers-Selberg identity is given as follows.

$$\frac{(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}$$

This equation and others of the "sum=product" type are compiled in the paper of Slater [31] in 1952. Using our last theorem, we can write

$$
\begin{aligned}
\frac{(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty}{(q^2;q^2)_\infty}&=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}\\
&=1+\sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\frac{1}{l!}\sum_{w_1+w_2+\cdots+w_l\in C(n)}H(w_1)H(w_2)\cdots H(w_l)\right)q^n,
\end{aligned}
$$

where in the coefficient of q^n the $H(n)'s$ are given by

$$H(n)=\sum_{\substack{d|n,d>0\\ \frac{n}{d}\equiv 0(mod 2)}}\frac{1}{d}-\sum_{\substack{a|n,a>0\\ \frac{n}{a}\equiv 3(mod 7)}}\frac{1}{a}-\sum_{\substack{b|n,b>0\\ \frac{n}{b}\equiv 4(mod 7)}}\frac{1}{b}-\sum_{\substack{c|n,c>0\\ \frac{n}{c}\equiv 0(mod 7)}}\frac{1}{c}.$$

Another example, in Entry 11.3.1 (pp. 6,16) of the Ramanujan's Lost Notebook (Part I), available in [5].

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q)_{2n}} = \frac{(q^6;q^{12})_\infty^2 (q^{12};q^{12})_\infty}{(q;q)_\infty}.$$

This identity was first proved by Sister Slater [31], and appears in her list as identity (29). A combinatorial interpretation for the previous identity and more similar identities can be found in Andrews and Lewis, [7]. Using the aforementioned identity, we can assert that the coefficient of q^n in this identity is equal to

$$c_n = \sum_{l=1}^{n} \frac{1}{l!} \sum_{w_1+\cdots+w_l \in C(n)} a_{w_1} \cdots a_{w_l},$$

where

$$a_j = \frac{\sigma_1(j)}{j} - \sum_{\substack{d|j \\ \frac{j}{d} \equiv 6 (mod 12)}} \frac{2}{d} - \sum_{\substack{e|j \\ \frac{j}{e} \equiv 0 (mod 12)}} \frac{1}{e}$$

Let consider the first Rogers-Ramanujan identity as given in equation (1.1).

As we know, the coefficient of q^n in equation (1.1) is equal to the the number of partitions of n, whose difference for consecutive parts is at least 2 (= number of partitions of n into parts congruent to $\pm 1 (mod 5)$). Let denote by $r_1(n)$ this number. For $|q| < \frac{1}{10}$, is a fact that $|\sum_{n=1}^{\infty} p(n) q^n| < 1$. Since $r_1(n) \leq p(n)$, we have $|\sum_{n=1}^{\infty} r_1(n) q^n| < 1$. Thus we can consider the Taylor expansion for $\ln(1 + \sum_{n=1}^{\infty} r_1(n) q^n)$. As we did before, we will find the coefficient of q^n in the expansion of the aforementioned logarithm.

$$\ln\left(1 + \sum_{n=1}^{\infty} r_1(n) q^n\right) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{n=1}^{\infty} r_1(n) q^n\right)^m,$$

for $|q| < \frac{1}{10}$. The coefficient of q^n in $(\sum_{n=1}^{\infty} r_1(n) q^n)^m$ is equal to the number of partitions of n, whose difference for consecutive parts is at least 2 and those parts appears colored up to m colors (= number of partitions of n into colored parts, up to m colours, congruent to $\pm 1 (mod 5)$). We denote this number by $r_1^m(n)$. That way we can rewrite the previous equation as

$$\ln\left(1 + \sum_{n=1}^{\infty} r_1(n) q^n\right) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{n=1}^{\infty} r_1^m(n) q^n,$$

and thus the coefficient of q^n in this logarithm is $\sum_{m=1}^{\infty} \frac{(-1)^{m+1} r_1^m(n)}{m}$. On the other side, we know that the coefficient of q^n in

$$\ln\left(\frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}\right)$$

is $\rho_{1,5}(n) + \rho_{4,5}(n)$, in which $\rho_{a,b}(n)$ is as in (6.7). Thus we have

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}r_1^m(n)=\rho_{1,5}(n)+\rho_{4,5}(n).$$

More generally, considering an identity of type "sum-product", as

$$1+\sum_{n=1}^{\infty}h_nq^n=\frac{(q^{a_1};q^{b_1})_\infty(q^{a_2};q^{b_2})_\infty\cdots(q^{a_k};q^{b_k})_\infty}{(q^{c_1};q^{d_1})_\infty(q^{c_2};q^{d_2})_\infty\cdots(q^{c_r};q^{d_r})_\infty}, \tag{6.8}$$

where $a_1,\ldots,a_k,b_1,\ldots,b_k,c_1,\ldots,c_r,d_1,\ldots d_r$ are positive integers, and $|q|<s<1$, in such a way that $|\sum_{n=1}^{\infty}h_nq^n|<1$, we can establish our penultimate result in this paper. For this purpose, we should deal with the expansion of $(\sum_{n=1}^{\infty}h_nq^n)^m$, in which by the Proposition 1 is

$$\left(\sum_{n=1}^{\infty}h_nq^n\right)^m=\sum_{n=1}^{\infty}\sum_{w_1+w_2+\cdots+w_m\in C(n)}h_{w_1}h_{w_2}\cdots h_{w_m}q^n.$$

We denote by h_n^m the coefficient of q^n in the last series.

$$h_n^m=\sum_{w_1+w_2+\cdots+w_m\in C(n)}h_{w_1}h_{w_2}\cdots h_{w_m}. \tag{6.9}$$

Thus we have just proved the result as follows.

Theorem 16.

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}h_n^m}{m}=\rho_{c_1,d_1}(n)+\rho_{c_2,d_c}(n)+\ldots+\rho_{c_r,d_r}(n)-\rho_{a_1,b_1}(n)-\rho_{a_2,b_2}(n)-\ldots-\rho_{a_k,b_k}(n),$$

where h_n^m is as in (6.9), h_n as in (6.8) and $\rho_{a,b}(n)$ as in (6.7).

In our last result we will obtain an expansion to

$$f(z,q)=\left(1+\sum_{n=1}^{\infty}a_nq^n\right)^z,$$

where $z\in\mathbb{C}$, and the series

$$1+\sum_{n=1}^{\infty}a_nq^n$$

is convergent for $|q|<r<1$, in order that

$$\left|\sum_{n=1}^{\infty}a_nq^n\right|<1.$$

Under the above conditions, we have

$$\ln(f(z,q)) = z \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(1 + \sum_{n=1}^{\infty} a_n q^n\right)^m,$$

where, by Proposition 1, we can write

$$\begin{aligned}
\ln(f(z,q)) &= z \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{n \geq m} \left(\sum_{w_1+\cdots+w_m \in C(n)} a_{w_1} \cdots a_{w_m} \right) q^n \\
&= z \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{m=1}^{n} \left(\sum_{w_1+\cdots+w_m \in C(n)} a_{w_1} \cdots a_{w_m} \right) q^n.
\end{aligned}$$

Denoting b_n by

$$b_n = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m} \sum_{w_1+\cdots+w_m \in C(n)} a_{w_1} \cdots a_{w_m},$$

we get

$$f(z,q) = \exp\left(\sum_{n=1}^{\infty} b_n q^n\right) = 1 + \sum_{l=1}^{\infty} \frac{z^l}{l!} \left(\sum_{n=1}^{\infty} b_n q^n\right)^l.$$

Using Proposition 1 again, we conclude that

$$f(z,q) = 1 + \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{z^l}{l!} \sum_{w_1+\cdots+w_l \in C(n)} b_{w_1} \cdots b_{w_l} \right) q^n.$$

Summarizing what has been proposed so far, we can establish the following theorem.

Theorem 17. *Considering the series*

$$\sum_{n=1}^{\infty} a_n q^n$$

convergent to $|q| < r < 1$, *in order that*

$$\left| \sum_{n=1}^{\infty} a_n q^n \right| < 1,$$

we have for all $z \in \mathbb{C}$ *the following expansion:*

$$\left(1+\sum_{n=1}^{\infty} a_n q^n\right)^z = 1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{z^l}{l!}\sum_{w_1+\cdots+w_l\in C(n)}\left(\sum_{m=1}^{w_1}\frac{(-1)^{m+1}}{m}\sum_{u_1+\cdots+u_m\in C(w_1)} a_{u_1}\cdots a_{u_m}\right)$$
$$\cdots\left(\sum_{m=1}^{w_l}\frac{(-1)^{m+1}}{m}\sum_{u_1+\cdots+u_m\in C(w_l)} a_{u_1}\cdots a_{u_m}\right) q^n$$

For example, consider $a_n = \frac{p_1(n)}{F_{n+1}}$, where $(F_{n+1})_{n\geq 0}$ is the sequence of Fibonacci. In Andrews and Eriksson, [6], in Chapter 3, the authors prove using combinatorial arguments that $p_1(n) \leq F_{n+1}$. Thus $a_n \leq 1$, $\forall n > 0$, and for $|q| < \frac{1}{10}$ the power series $\sum_{n=0}^{\infty} a_n q^n$ is convergent and yet $|\sum_{n=0}^{\infty} a_n q^n| < 1$. By the previous theorem, we have:

$$\left(1+\sum_{n=1}^{\infty}\frac{p_1(n)}{F_{n+1}} q^n\right)^z = 1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{z^l}{l!}\sum_{w_1+\cdots+w_l\in C(n)}\left(\sum_{m=1}^{w_1}\frac{(-1)^{m+1}}{m}\sum_{u_1+\cdots+u_m\in C(w_1)}\frac{p_1(u_1)}{F_{u_1+1}}\cdots\frac{p_1(u_m)}{F_{u_m+1}}\right)$$
$$\cdots\left(\sum_{m=1}^{w_l}\frac{(-1)^{m+1}}{m}\sum_{u_1+\cdots+u_m\in C(w_l)}\frac{p_1(u_1)}{F_{u_1+1}}\cdots\frac{p_1(u_m)}{F_{u_m+1}}\right) q^n$$

In another specific case, for $a(n) = p_1(n)$, $n > 0$, for $z = k$, where k is a positive integer, and $|q| < \frac{1}{10}$, we have that the coefficient of q^n in

$$(q;q)^{-k} = \left(1+\sum_{n=1}^{\infty} p_1(n) q^n\right)^k,$$

is equal to $p_k(n)$, the number of partitions of n in up to k colors. Using the previous theorem, we can establish that

$$p_k(n) = \sum_{l=1}^{n}\frac{k^l}{l!}\sum_{w_1+\cdots+w_l\in C(n)}\left(\sum_{m=1}^{w_1}\frac{(-1)^{m+1}}{m}\sum_{u_1+\cdots+u_m\in C(w_1)} p_1(u_1)\cdots p_1(u_m)\right)$$
$$\cdots\left(\sum_{m=1}^{w_l}\frac{(-1)^{m+1}}{m}\sum_{u_1+\cdots+u_m\in C(w_l)} p_1(u_1)\cdots p_1(u_m)\right)$$

As we known,

$$p_k(n) = P_n(k) = \sum_{l=1}^{n}\frac{k^l}{l!}\sum_{w_1+w_2+\cdots+w_l\in C(n)}\frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2\cdots w_l}.$$

Thus, we can establish the following corollary.

Corollary 2.

$$p_k(n) = \sum_{l=1}^{n} \frac{k^l}{l!} \sum_{w_1+\cdots+w_l \in C(n)} \left(\sum_{m=1}^{w_1} \frac{(-1)^{m+1}}{m} \sum_{u_1+\cdots+u_m \in C(w_1)} p_1(u_1)\cdots p_1(u_m) \right)$$
$$\cdots \left(\sum_{m=1}^{w_l} \frac{(-1)^{m+1}}{m} \sum_{u_1+\cdots+u_m \in C(w_l)} p_1(u_1)\cdots p_1(u_m) \right)$$
$$= \sum_{l=1}^{n} \frac{k^l}{l!} \sum_{w_1+w_2+\cdots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l}.$$

As a final comment, we believe that many identities like the ones described by Slater in [31] can reveal new identities involving sums of reciprocals of divisors as those found here.

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