The asymptotic volume of the Birkhoff polytope

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Abstract

Let $m, n \geq 1$ be integers. Define $\mathcal{T}_{m,n}$ to be the *transportation polytope* consisting of the $m \times n$ non-negative real matrices whose rows each sum to 1 and whose columns each sum to m/n. The special case $\mathcal{B}_n = \mathcal{T}_{n,n}$ is the much-studied *Birkhoff-von Neumann polytope* of doubly-stochastic matrices. Using a recent asymptotic enumeration of non-negative integer matrices (Canfield and McKay, 2007), we determine the asymptotic volume of $\mathcal{T}_{m,n}$ as $n \to \infty$ with $m = m(n)$ such that m/n neither decreases nor increases too quickly. In particular, we give an asymptotic formula for the volume of $\mathcal{B}_n$.

1 Introduction

Let $m, n \geq 1$ be integers. Define $\mathcal{T}_{m,n}$ to be the *transportation polytope* consisting of the $m \times n$ non-negative real matrices whose rows each sum to 1 and whose columns each sum to m/n. The special case $\mathcal{B}_n = \mathcal{T}_{n,n}$ is the famous *Birkhoff-von Neumann polytope* of doubly-stochastic matrices.

It is well known (see Stanley [8, Chap. 4] for basic theory and references) that $\mathcal{T}_{m,n}$ spans an $(m-1)(n-1)$-dimensional affine subspace of $\mathbb{R}^{m \times n}$. The vertices of $\mathcal{T}_{m,n}$ were described by Klee and Witzgall [7] and are moderately complicated. The special case of $\mathcal{B}_n$ is however very simple: the vertices are precisely the $n \times n$ permutation matrices.

Two types of volume are customarily defined for such polytopes. We can illustrate the difference using the example

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} z & 1-z \\ 1-z & z \end{pmatrix} \;\middle|\; 0 \leq z \leq 1 \right\} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right],$$

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where the last notation indicates a closed line-segment in $\mathbb{R}^{2\times 2}$. The length of this line-segment is the *volume* $\mathrm{vol}(B_2) = 2$. We can also consider the lattice induced by $\mathbb{Z}^{2\times 2}$ on the affine span of $\mathcal{B}_2$: this consists of the points $\left(\begin{smallmatrix} z & 1-z \\ 1-z & z \end{smallmatrix}\right)$ for integer z. The polytope $\mathcal{B}_2$ consists of a single basic cell of this lattice, so it has *relative volume* $\nu(\mathcal{B}_2) = 1$. In general, $\mathrm{vol}(\mathcal{T}_{m,n})$ is the volume in units of the ordinary $(m-1)(n-1)$-dimensional Lebesgue measure, while $\nu(\mathcal{T}_{m,n})$ is the volume in units of basic cells of the lattice induced by $\mathbb{Z}^{m\times n}$ on the affine span of $\mathcal{T}_{m,n}$. (For a thorough explanation of these matters, please consult [2].)

Lemma 1. *For $m, n \geq 2$, $\mathrm{vol}(\mathcal{T}_{m,n}) = m^{(n-1)/2} n^{(m-1)/2}\, \nu(\mathcal{T}_{m,n})$.*

Proof. This is established in [6, Theorem 3]. Also see the Appendix of [3]. □

Next, define the function $H_{m,n} : \mathbb{Z} \to \mathbb{Z}$ by

$$H_{m,n}(z) = \left| z\mathcal{T}_{m,n} \cap \mathbb{Z}^{m\times n} \right|.$$

Clearly $z\mathcal{T}_{m,n} \cap \mathbb{Z}^{m\times n}$ is the set of $m\times n$ non-negative integer matrices with row sums equal to z and column sums equal to zm/n. This set is non-empty when $zm/n \in \mathbb{Z}$; that is, when z is a multiple of $z_0 = n/\gcd(m,n)$. The base case $z = z_0$ corresponds to an expanded polytope $z_0\mathcal{T}_{m,n}$ whose vertices are integral [7, Cor. 1]. Therefore, by the celebrated theorem of Ehrhart (see [8]), there are constants $c_i(m,n)$ for $i = 0, 1, \ldots, (m-1)(n-1)$ such that

$$H_{m,n}(z) = \begin{cases} \displaystyle\sum_{i=0}^{(m-1)(n-1)} c_i(m,n) z^{(m-1)(n-1)-i}, & \text{if } z_0 \text{ divides } z; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

This is the *Ehrhart quasi-polynomial* of $\mathcal{T}_{m,n}$. Applying [8, Prop. 4.6.30] to $z_0\mathcal{T}_{m,n}$, we find that

$$\nu(\mathcal{T}_{m,n}) = c_0(m,n). \tag{2}$$

We turn now to asymptotics. Our main tool will be the following theorem of the present authors [4].

Theorem 1. *Suppose $m = m(n)$, $s = s(n)$ and $t = t(n)$ are positive integer functions such that $ms = nt$. Let $M(m,s;n,t)$ be the number of $m\times n$ non-negative integer matrices with row sums equal to s and column sums equal to t. Define $\lambda = \lambda(n)$ by $ms = nt = \lambda mn$. Let $a, b > 0$ be constants such that $a + b < \frac{1}{2}$. Suppose that $n \to \infty$ and that, for large n,*

$$\frac{(1+2\lambda)^2}{4\lambda(1+\lambda)} \left(1 + \frac{5m}{6n} + \frac{5n}{6m}\right) \leq a \log n. \tag{3}$$

Then

$$M(m,s;n,t) = \frac{\binom{n+s-1}{n-1}^m \binom{m+t-1}{m-1}^n}{\binom{mn+\lambda mn-1}{mn-1}} \exp\bigl(\tfrac{1}{2} + O(n^{-b})\bigr). \quad \square$$

Using this result, we can prove the following theorem concerning the volumes of $\mathcal{T}_{m,n}$ and $\mathcal{B}_n$.

Theorem 2. *Let $a, b > 0$ be constants such that $a + b < \frac{1}{2}$. Then*

$$\mathrm{vol}(\mathcal{T}_{m,n}) = \frac{1}{(2\pi)^{(m+n-1)/2} n^{(m-1)(n-1)}} \exp\Bigl(\frac{1}{3} + mn - \frac{(m-n)^2}{12mn} + O(n^{-b})\Bigr)$$

when $m, n \to \infty$ in such a way that $\max\bigl(\frac{m}{n}, \frac{n}{m}\bigr) \le \frac{6}{5} a \log n$. In particular, for any $\epsilon > 0$

$$\mathrm{vol}(\mathcal{B}_n) = \frac{1}{(2\pi)^{n-1/2} n^{(n-1)^2}} \exp\Bigl(\frac{1}{3} + n^2 + O(n^{-1/2+\epsilon})\Bigr)$$

as $n \to \infty$.

Proof. From (1) and (2), we have

$$\nu(\mathcal{T}_{m,n}) = \lim_{z\to\infty} \frac{H_{m,n}(z)}{z^{(m-1)(n-1)}} = \lim_{\lambda\to\infty} \frac{M(m,\lambda n; n, \lambda m)}{(\lambda n)^{(m-1)(n-1)}}\,, \tag{4}$$

where we restrict z to multiples of z_0 and λ to multiples of z_0/n. If $a' > a$ and $a' + b < \frac{1}{2}$, then the left side of (3) is less than $a' \log n$ for sufficiently large λ. Thus the conditions for Theorem 1 hold. It remains to apply that theorem to (4) using Stirling's formula, and to infer the value of $\mathrm{vol}(\mathcal{T}_{m,n})$ using Lemma 1. $\square$

It is of interest to note that the same asymptotic formula for the volume (except for the error term) follows from the estimate of $M(m,s;n,t)$ that Diaconis and Efron proposed without proof in 1985 [6].

Exact values of $\mathrm{vol}(\mathcal{B}_n)$ are known up to $n = 10$ [1]. In Table 1 we compare the exact values to the approximation given in Theorem 2. It appears that the true magnitude of the error term might be $O(n^{-1})$. This would indeed be the case if the well-tested conjecture made in [4] about the value of $M(n,s;n,t)$ was true. The same conjecture implies a value of $\mathrm{vol}(\mathcal{T}_{m,n})$ with relative error $O\bigl((m+n)^{-1}\bigr)$ for all m, n.

Recently, a summation with $O\bigl(n^n n!\bigr)$ terms was found for $\mathrm{vol}(\mathcal{B}_n)$ [5]. Whether it is useful for asymptotics remains to be seen.

n	estimate/actual
1	1.51345
2	1.20951
3	1.25408
4	1.22556
5	1.19608
6	1.17258
7	1.15403
8	1.13910
9	1.12684
10	1.11627

Table 1: Accuracy of Theorem 2 for $\mathrm{vol}(\mathcal{B}_n)$.

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