STATISTICS ON PERMUTATIONS

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Abstract

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be any permutation of length n, we say a descent $\pi_i\pi_{i+1}$ is a *lower*, *middle*, *upper* if there exists $j > i+1$ such that $\pi_j < \pi_{i+1}$, $\pi_{i+1} < \pi_j < \pi_i$, $\pi_i < \pi_j$, respectively. Similarly, we say a rise $\pi_i\pi_{i+1}$ is a *lower*, *middle*, *upper* if there exists $j > i+1$ such that $\pi_j < \pi_i$, $\pi_i < \pi_j < \pi_{i+1}$, $\pi_{i+1} < \pi_j$, respectively. In this paper we give an explicit formula for the generating function for the number of permutations of length n according to number of upper, middle, lower rises, and upper, middle, lower descents. This allows us to recover several known results in the combinatorics of permutation patterns as well as many new results. For example, we give an explicit formula for the generating function for the number of permutations of length n having exactly m middle descents.

Keywords: Statistics; Kernel method; Generating functions; Descents; Rises.

1. Introduction

Almost a hundred years ago, MacMahon [6] started the theory of permutation statistics by studying the number *descents* in a permutation. This statistic still play an important role in the theory. The *descent set* (respectively, *rise set*) of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is the set of indices i for which $\pi_i > \pi_{i+1}$ (respectively, $\pi_i < \pi_{i+1}$), and the number of *descents* (respectively, *rises*) in a permutation π is the cardinality of the descent set (respectively, rise set). The distribution of the number of descents (rises) in the set of permutations of length n is given by the Eulerian numbers $A(n,k)$. More precisely, the number of permutations of length n with exactly k descents (rises) is given by the Eulerian number $A(n,k)$, see [3].

By the definition, $\pi_i\pi_{i+1}$ is a descent (respectively, rise) in a permutation $\pi_1\pi_2\cdots\pi_n$ if and only if $\pi_i > \pi_{i+1}$ (respectively, $\pi_i < \pi_{i+1}$). In this paper we consider the refinement of the notion of a descent (rise) by fixing an element π_j of the right side of π_{i+1}, namely $j > i+1$, such that either $\pi_j < m = \min\{\pi_i, \pi_{i+1}\}$, $\pi_j > M = \max\{\pi_i, \pi_{j+1}\}$, or $m \le \pi_j \le M$. We provide exact formulas for the distribution generating functions of our new statistics.

Let S_n denote the set of permutations of length n (permutations of $[n] = \{1,2,\ldots,n\}$). For any $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, we say a descent $\pi_i\pi_{i+1}$ is a *lower*, *middle*, *upper* if there exists $j > i+1$ such that $\pi_j < \pi_{i+1}$, $\pi_{i+1} < \pi_j < \pi_i$, $\pi_i < \pi_j$, respectively. Similarly, we say that a rise $\pi_i\pi_{i+1}$ is a *lower*, *middle*, *upper* if there exists $j > i+1$ such that $\pi_j < \pi_i$, $\pi_i < \pi_j < \pi_{i+1}$, $\pi_{i+1} < \pi_j$, respectively. We define the following six statistics:

- $Des_L(\pi) = \{i \mid \pi_i\pi_{i+1}$ is a lower descent$\}$, $des_L(\pi) = \#Des_L(\pi)$,
- $Des_M(\pi) = \{i \mid \pi_i\pi_{i+1}$ is a middle descent$\}$, $des_M(\pi) = \#Des_M(\pi)$,
- $Des_U(\pi) = \{i \mid \pi_i\pi_{i+1}$ is a upper descent$\}$, $des_U(\pi) = \#Des_U(\pi)$,
- $Ris_L(\pi) = \{i \mid \pi_i\pi_{i+1}$ is a lower rise$\}$, $ris_L(\pi) = \#Ris_L(\pi)$,
- $Ris_M(\pi) = \{i \mid \pi_i\pi_{i+1}$ is a middle rise$\}$, $ris_M(\pi) = \#Ris_M(\pi)$,
- $Ris_U(\pi) = \{i \mid \pi_i\pi_{i+1}$ is a upper rise$\}$, $ris_U(\pi) = \#Ris_U(\pi)$.

For example, if $\pi = 316425 \in S_6$, then $Des_L(\pi) = \{3\}$, $Des_M(\pi) = \{1,3\}$, $Des_U(\pi) = \{1,4\}$, $Ris_L(\pi) = \emptyset$, $Ris_M(\pi) = \{2\}$, and $Ris_U(\pi) = \emptyset$. Then $des_L(\pi) = 1$, $des_M(\pi) = des_U(\pi) = 2$, $ris_L(\pi) = ris_U(\pi) = 0$ and $ris_M(\pi) = 1$.

Clearly, $des_X(\pi) = ris_X(\pi^c)$ for each permutation π of length n and $X \in \{L, M, U\}$, where $\pi^c = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n)$ denote the complement of π.

Our main goal is to find an explicit formula for the generating function

$$\mathbb{G}(x) = \sum_{n\geq 0}\left(x^n \sum_{\pi\in S_n}\prod_{X\in\{L,M,U\}}(d_X^{des_X(\pi)} r_X^{ris_X(\pi)})\right),$$

see the next section. In Section 3 we present several application for our main goal, Theorem 2.8. In particular, we show that the generating function for the number of permutations of length n with exactly one middle descent is given by

$$\frac{1-\sqrt{2\sqrt{1-4x}-1}}{2x} - \frac{1}{\sqrt{1-4x}},$$

and the generating function for number of permutations of length n with exactly two middle descent is given by

$$\frac{1-\sqrt{2\sqrt{2\sqrt{1-4x}-1}-1}}{2x} + \frac{x}{(1-4x)\sqrt{1-4x}} - \frac{1}{\sqrt{1-4x}\sqrt{2\sqrt{1-4x}-1}}.$$

2. Derivation an explicit formula for $\mathbb{G}(x)$

Denote the generating function for the number of permutations π of length n with $\pi_1\cdots\pi_m = i_1\cdots i_m$ according to the statistics des_X and ris_X, $X \in \{L, M, U\}$, by $f(n|i_1\cdots i_m) = f(n; d_L, d_M, d_U, r_L, r_M, r_U|i_1\cdots i_m)$, that is,

$$f(n|i_1\cdots i_m) = \sum_{\pi=i_1i_2\cdots i_m\pi'\in S_n}\prod_{X\in\{L,M,U\}}\left(d_X^{des_X(\pi)} r_X^{ris_X(\pi)}\right).$$

At first, let us present a recurrence relation for the polynomials $f(n|i)$.

Proposition 2.1. *Let $n \geq 3$ and $3 \leq i \leq n-2$. Then*

$$\begin{aligned}
f(n|1) &= r_U f(n-1|1) + r_U r_M \sum_{j=2}^{n-2} f(n-1|j) + r_M f(n-1|n-1),\\
f(n|2) &= d_U f(n-1|1)\\
&\quad + r_U r_L f(n-1|2) + r_U r_M r_L \sum_{j=3}^{n-2} f(n-1|j) + r_M r_L f(n-1|n-1),\\
f(n|i) &= d_U d_M f(n-1|1) + d_U d_M d_L \sum_{j=2}^{i-2} f(n-1|j) + d_U d_L f(n-1|i-1)\\
&\quad + r_U r_L f(n-1|i) + r_U r_M r_L \sum_{j=i+1}^{n-2} f(n-1|j) + r_M r_L f(n-1|n-1),\\
f(n|n-1) &= d_U d_M f(n-1|1) + d_U d_M d_L \sum_{j=2}^{n-3} f(n-1|j) + d_U d_L f(n-1|n-2)\\
&\quad + r_L f(n-1|n-1),\\
f(n|n) &= d_M f(n-1|1) + d_M d_L \sum_{j=2}^{n-2} f(n-1|j) + d_L f(n-1|n-1).
\end{aligned}$$

Proof. By the definitions we have that

$$f(n|i) = \sum_{j=1}^{i-1} f(n|ij) + \sum_{j=i+1}^{n} f(n|ij),$$

for all $n \geq 3$. Now let us find the equation of $f(n|1)$. It is not hard to see that $f(n|12) = r_U f(n-1,1)$, $f(n|1j) = r_U r_M f(n-1|j-1)$ for all $j = 3, 4, \ldots, n-1$, and $f(n|1n) = r_M f(n-1|n-1)$. Therefore,

$$f(n|1) = r_U f(n-1|1) + r_U r_M \sum_{j=2}^{n-2} f(n-1|j) + r_M f(n-1|n-1).$$

All the other equations can be obtained by using similar arguments as above. □

The above proposition generates quickly the polynomials $f(n|i)$, see Table 1.

n	$f(n\|1)$	$f(n\|2)$	$f(n\|3)$	$f(n\|4)$
1	1			
2	1	1		
3	$r_U + r_M$	$d_U + r_L$	$d_M + d_L$	
4	$r_U^2 + r_U r_M + r_U r_M r_L + r_M d_L + r_M d_M + r_U r_M d_U$	$d_U r_U + d_U r_M + r_L r_U d_U + r_U r_L^2 + r_M r_L d_M + r_M r_L d_L$	$d_U d_M r_U + r_L d_L + d_U^2 d_L + d_U d_L r_L + r_L d_M + d_U d_M r_M$	$d_M r_U + d_M r_M + d_U d_M d_L + d_M d_L + r_L d_M d_L + d_L^2$

TABLE 1. The polynomials $f(n|i)$ for all $1 \leq i \leq n \leq 4$.

Define $f(n) = f(n; d_L, d_M, d_U, r_L, r_M, r_U)$ to be the generating function for the number of permutations of length n according to the statistics des_X and ris_X, that is,

$$f(n) = \sum_{\pi \in S_n} \prod_{X \in \{L,M,U\}} \left(d_X^{des_X(\pi)} r_X^{ris_X(\pi)} \right).$$

Clearly, for all $n \geq 1$, $f(n) = \sum_{i=1}^{n} f(n|i)$. For example, Table 1 gives

$$\begin{aligned}
f(1) &= 1, \\
f(2) &= 2, \\
f(3) &= \textstyle\sum_{X \in \{U,L,M\}} (r_X + d_X), \\
f(4) &= (r_U + r_M)(d_U d_M + d_U + d_M + r_U) + (r_M + d_L)(r_M r_L + r_L + r_M + d_L) \\
&\quad + (d_U + r_L)(r_U r_L + r_U r_M + d_L d_U + d_L d_M).
\end{aligned}$$

Now, we introduce the multivariate generating function

$$F(x, v, w) = F(x, v, w; d_L, d_M, d_U, r_L, r_M, r_U)$$

for the number of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ of length n according to the statistics des_X and ris_X, $X \in \{L, M, U\}$, and π_1 (the leftmost element of π) as

$$F(x, v, w) = \sum_{n \geq 0} F_n(v, w) x^n = \sum_{n \geq 0} x^n \left(\sum_{i=1}^{n} f(n|i) v^{n-i} w^{i-1} \right).$$

Clearly,

$$f(n|n) = F_n(0, 1), \quad f(n|1) = F_n(1, 0) \text{ and } f(n) = F_n(1, 1), \quad \text{for all } n \geq 1. \tag{2.1}$$

In this section we interest to find an explicit formula for the generating function $F(x, v, w)$. In order to do that let us write a recurrence relation for the polynomials $F_n(v, w)$. Proposition 2.1 gives

$$\underbrace{f(n|1)}_{\langle 0 \rangle} = \underbrace{r_U f(n-1|1)}_{\langle 1 \rangle} + \underbrace{r_U r_M \sum_{j=2}^{n-2} f(n-1|j)}_{\langle 6 \rangle} + \underbrace{r_M f(n-1|n-1)}_{\langle 7 \rangle},$$

$$\begin{aligned}
\underbrace{f(n|2)}_{\langle 0 \rangle} &= \underbrace{d_U f(n-1|1)}_{\langle 2 \rangle} \\
&\quad + \underbrace{r_U r_L f(n-1|2)}_{\langle 5 \rangle} + \underbrace{r_U r_M r_L \sum_{j=3}^{n-2} f(n-1|j)}_{\langle 9 \rangle} + \underbrace{r_M r_L f(n-1|n-1)}_{\langle 8 \rangle},
\end{aligned}$$

$$\begin{aligned}
\underbrace{f(n|i)}_{\langle 0 \rangle} &= \underbrace{d_U d_M f(n-1|1)}_{\langle 3 \rangle} + \underbrace{d_U d_M d_L \sum_{j=2}^{i-2} f(n-1|j)}_{\langle 11 \rangle} + \underbrace{d_U d_L f(n-1|i-1)}_{\langle 10 \rangle} \\
&\quad + \underbrace{r_U r_L f(n-1|i)}_{\langle 5 \rangle} + \underbrace{r_U r_M r_L \sum_{j=i+1}^{n-2} f(n-1|j)}_{\langle 9 \rangle} + \underbrace{r_M r_L f(n-1|n-1)}_{\langle 8 \rangle},
\end{aligned}$$

$$\underbrace{f(n|n-1)}_{\langle 0\rangle} = \underbrace{d_Ud_Mf(n-1|1)}_{\langle 3\rangle} + \underbrace{d_Ud_Md_L\sum_{j=2}^{n-3}f(n-1|j)}_{\langle 11\rangle} + \underbrace{d_Ud_Lf(n-1|n-2)}_{\langle 10\rangle} + \underbrace{r_Lf(n-1|n-1)}_{\langle 14\rangle},$$

$$\underbrace{f(n|n)}_{\langle 0\rangle} = \underbrace{d_Mf(n-1|1)}_{\langle 4\rangle} + \underbrace{d_Md_L\sum_{j=2}^{n-2}f(n-1|j)}_{\langle 12\rangle} + \underbrace{d_Lf(n-1|n-1)}_{\langle 13\rangle}.$$

Multiplying the equation of $f(n|i)$ in the above system by $v^{n-i}w^{i-1}$ and summing over all possibly values $i = 1, 2, \ldots, n$, we obtain that the contribution of all the terms assigned in the above system by $\langle a\rangle$, $a = 0, 1, \ldots 14$, in the polynomial $F_n(v, w)$ is given by

- Case $a = 0$: $\sum_{i=1}^{n} f(n|i)v^{n-i}w^{i-1} = F_n(v, w)$,
- Case $a = 1$: $r_Uv^{n-1}f(n-1|1) = r_Uv^{n-1}F_{n-1}(1, 0)$,
- Case $a = 2$: $d_Uv^{n-2}wf(n-1|1) = d_Uv^{n-2}wF_{n-1}(1, 0)$,
- Case $a = 3$:

$$\begin{aligned} d_Ud_M\sum_{i=3}^{n-1} v^{n-i}w^{i-1}f(n-1|1) &= d_Ud_Mv^{n-3}w^2F_{n-1}(1,0)\sum_{i=3}^{n-1}\left(\tfrac{w}{v}\right)^{i-3} \\ &= d_Ud_MF_{n-1}(1,0)\frac{vw^2(v^{n-3}-w^{n-3})}{v-w}, \end{aligned}$$

- Case $a = 4$: $d_Mv^0w^{n-1}f(n-1|1) = d_Mw^{n-1}F_{n-1}(1, 0)$,
- Case $a = 5$:

$$\begin{aligned} &r_Ur_L\sum_{i=2}^{n-2} f(n-1|i)v^{n-i}w^{i-1} \\ &\quad = r_Ur_Lv\left(F_{n-1}(v,w) - w^{n-2}F_{n-1}(0,1) - v^{n-2}F_{n-1}(1,0)\right), \end{aligned}$$

- Case $a = 6$:

$$r_Ur_Mv^{n-1}\sum_{j=2}^{n-2} f(n-1|j) = r_Ur_Mv^{n-1}\left(F_{n-1}(1,1) - F_{n-1}(0,1) - F_{n-1}(1,0)\right),$$

- Case $a = 7$: $r_Mv^{n-1}f(n-1|n-1) = r_Mv^{n-1}F_{n-1}(0, 1)$,
- Case $a = 8$:

$$r_Mr_L\sum_{i=2}^{n-2} v^{n-i}w^{i-1}f(n-1|n-1) = r_Mr_LF_{n-1}(0,1)\tfrac{v^2w(v^{n-3}-w^{n-3})}{v-w},$$

- Case $a = 9$:

$$
\begin{aligned}
& r_U r_M r_L \sum_{i=2}^{n-3} v^{n-i} w^{i-1} \sum_{j=i+1}^{n-2} f(n-1|j) \\
&\quad = r_U r_M r_L v^{n-2} w \sum_{j=3}^{n-2} \left(\sum_{i=0}^{j-3} w^i / v^i \right) f(n-1|j) \\
&\quad = r_U r_M r_L \tfrac{v^{n-1} w}{v-w} \sum_{j=2}^{n-2} (1 - w^{j-2}/v^{j-2}) f(n-1|j) \\
&\quad = r_U r_M r_L \tfrac{v^{n-1} w}{v-w} (F_{n-1}(1,1) - F_{n-1}(1,0) - F_{n-1}(0,1))) \\
&\qquad\quad - r_U r_M r_L \tfrac{v^2}{v-w} (F_{n-1}(v,w) - v^{n-2} F_{n-1}(1,0) - w^{n-2} F_{n-1}(0,1)),
\end{aligned}
$$

- Case $a = 10$:

$$
\begin{aligned}
& d_U d_L \sum_{i=3}^{n-1} v^{n-i} w^{i-1} f(n-1|i-1) \\
&\quad = d_U d_L w \sum_{i=2}^{n-2} f(n-1|i) v^{n-1-i} w^{i-1} \\
&\quad = d_U d_L w (F_{n-1}(v,w) - v^{n-2} F_{n-1}(1,0) - w^{n-2} F_{n-1}(0,1)),
\end{aligned}
$$

- Case $a = 11$: Similarly as in the case $a = 9$ we have that

$$
\begin{aligned}
& d_U d_M d_L \sum_{i=2}^{n-3} v^{n-2-i} w^{i+1} \sum_{j=2}^{i} f(n-1|j) \\
&\quad = d_U d_M d_L \tfrac{w^2}{v-w} (F_{n-1}(v,w) - v^{n-2} F_{n-1}(1,0) - w^{n-2} F_{n-1}(0,1))) \\
&\qquad\quad - d_U d_M d_L \tfrac{v w^{n-1}}{v-w} (F_{n-1}(1,1) - F_{n-1}(1,0) - F_{n-1}(0,1)),
\end{aligned}
$$

- Case $a = 12$:

$$
d_M d_L w^{n-1} \sum_{i=2}^{n-2} f(n-1|i) = d_M d_L w^{n-1} (F_{n-1}(1,1) - F_{n-1}(1,0) - F_{n-1}(0,1)),
$$

- Case $a = 13$: $d_L w^{n-1} f(n-1|n-1) = d_L w^{n-1} F_{n-1}(0,1)$,
- Case $a = 14$: $r_L v w^{n-2} f(n-1|n-1) = r_L v w^{n-2} F_{n-1}(0,1)$.

Therefore, adding the above contributions, Cases $a = 1, 2, \ldots, 14$, we obtain a recurrence relation for the polynomials $F_n(v,w)$:

$$
\begin{aligned}
F_n(v,w) &= \left(r_U r_L v + d_U d_L w - r_U r_M r_L \tfrac{v^2}{v-w} + d_U d_M d_L \tfrac{w^2}{v-w} \right) F_{n-1}(v,w) \\
&+ \left(r_U r_M v^{n-1} + d_M d_L w^{n-1} + r_U r_M r_L \tfrac{v^{n-1} w}{v-w} - d_U d_M d_L \tfrac{w^{n-1} v}{v-w} \right) F_{n-1}(1,1) \\
&+ \Bigg(r_u v^{n-1} + d_U w v^{n-2} + d_U d_M \tfrac{v w^2 (v^{n-3} - w^{n-3})}{v-w} + d_M w^{n-1} \\
&\qquad - r_U r_L v^{n-1} - r_U r_M v^{n-1} + r_U r_M r_L v^{n-1} - d_U d_L w v^{n-2} \\
&\qquad\qquad - d_U d_M d_L \tfrac{w^2 v (v^{n-3} - w^{n-3})}{v-w} - d_M d_L w^{n-1} \Bigg) F_{n-1}(1,0) \\
&+ \Bigg(r_M v^{n-1} - r_U r_L v w^{n-2} - r_U r_M v^{n-1} + r_M r_L \tfrac{v^2 w (v^{n-3} - w^{n-3})}{v-w} \\
&\qquad - r_U r_M r_L \tfrac{v^2 w (v^{n-3} - w^{n-3})}{v-w} - d_U d_L w^{n-1} + d_U d_M d_L w^{n-1} \\
&\qquad\qquad - d_M d_L w^{n-1} + d_L w^{n-1} + r_L v w^{n-2} \Bigg) F_{n-1}(0,1),
\end{aligned}
\tag{2.2}
$$

with the initial conditions $F_0(v,w)=F_1(v,w)=1$ and $F_2(v,w)=v+w$. Multiplying (2.2) by x^n and summing over all possibly values $n\geq 3$ we obtain the following result.

Proposition 2.2. *Define*

$$K(x,v,w)=1-x\left(r_Ur_Lv-\tfrac{v^2r_Ur_Mr_L}{v-w}+d_Ud_Lw+\tfrac{w^2d_Ud_Md_L}{v-w}\right),$$
$$A_{11}(x,v,w)=-r_Ur_Mx\left(1+\tfrac{wr_L}{v-w}\right),$$
$$B_{11}(x,v,w)=-d_Md_Lx\left(1-\tfrac{vd_U}{v-w}\right),$$
$$A_{10}(x,v,w)=-x\left(r_U(1-r_M)(1-r_L)+\tfrac{wd_U}{v}\left(1+\tfrac{wd_M}{v-w}\right)(1-d_L)\right),$$
$$B_{10}(x,v,w)=-d_Mx(1-d_L)\left(1-\tfrac{vd_U}{v-w}\right),$$
$$A_{01}(x,v,w)=-r_Mx(1-r_U)\left(1+\tfrac{wr_L}{v-w}\right),$$
$$B_{01}(x,v,w)=-x\left(d_L(1-d_U)(1-d_M)+\tfrac{vr_L}{w}\left(1-\tfrac{vr_M}{v-w}\right)(1-r_U)\right),$$

and

$$\begin{aligned}&H(x,v,w)\\&\quad=1+x+(v+w)x^2-x(1+x)\left(r_Ur_Lv+d_Ud_Lw-\tfrac{v^2r_Ur_Mr_L}{v-w}+\tfrac{w^2d_Ud_Md_L}{v-w}\right)\\&\quad-x(1+xv)\left(r_U(1-r_M)(1-r_L)+r_M(1+\tfrac{wr_L}{v-w})+\tfrac{wd_U}{v}(1-d_L)(1+\tfrac{wd_M}{v-w})\right)\\&\quad-x(1+xw)\left(d_L(1-d_M)(1-d_U)+d_M(1-\tfrac{vd_U}{v-w})+\tfrac{vr_L}{w}(1-r_U)(1-\tfrac{vr_M}{v-w})\right).\end{aligned}$$

Then the generating function $F(x,v,w)$ satisfies the

$$\begin{aligned}&K(x,v,w)F(x,v,w)\\&\quad+A_{11}(x,v,w)F(xv,1,1)+B_{11}(x,v,w)F(xw,1,1)\\&\quad+A_{10}(x,v,w)F(xv,1,0)+B_{10}(x,v,w)F(xw,1,0)\\&\quad+A_{01}(x,v,w)F(xv,0,1)+B_{01}(x,v,w)F(xw,0,1)=H(x,v,w).\end{aligned}\tag{2.3}$$

Now, let us express the generating functions $F(x,1,0)$ and $F(x,0,1)$ in terms of $F(x,1,1)$.

Lemma 2.3. *Define*

$$\begin{aligned}r(x)&=1+(1-r_U-d_L+d_Md_L)x+(1-d_L-d_M(r_M-d_L))(1-r_U)x^2\\&\qquad+(r_M-d_L)(1-d_M)(1-r_U)x^3,\\r'(x)&=xr_M(r_U-xd_L(r_u-d_M)),\\s(x)&=1+(1-r_U-d_L+r_Ur_M)x+(1-r_U-r_M(d_M-r_U))(1-d_L)x^2\\&\qquad+(d_M-r_U)(1-d_L)(1-r_M)x^3,\\s'(x)&=xd_M(d_L-xr_U(d_L-r_M)),\\t(x)&=(1-xr_U(1-r_M))(1-xd_L(1-d_M))-x^2d_Mr_M(1-r_U)(1-d_L).\end{aligned}$$

Then

$$F(x,1,0)=\frac{r(x)}{t(x)}+\frac{r'(x)}{t(x)}F(x,1,1)\ \textit{and}\ F(x,0,1)=\frac{s(x)}{t(x)}+\frac{s'(x)}{t(x)}F(x,1,1).\tag{2.4}$$

Proof. Applying (2.2) for $v=1$ and $w=0$ we obtain that

$$F_n(1,0) = r_U(1-r_M)F_{n-1}(1,0) + r_Ur_MF_{n-1}(1,1) + r_M(1-r_U)F_{n-1}(0,1)$$

with the initial conditions $F_0(v,w)=F_1(v,w)=1$ and $F_2(v,w)=v+w$. Rewriting the above recurrence relation in terms of generating functions we arrive at

$$\begin{aligned}(1-xr_U(1-r_M))F(x,1,0) - xr_M(1-r_U)F(x,0,1)\\ = 1+x(1+x)(1-r_U)(1-r_M)+xr_Ur_MF(x,1,1).\end{aligned} \tag{2.5}$$

Now applying (2.2) for $v=0$ and $w=1$ we obtain that

$$F_n(0,1) = d_L(1-d_M)F_{n-1}(0,1) + d_Md_LF_{n-1}(1,1) + d_M(1-d_L)F_{n-1}(1,0)$$

with the initial conditions $F_0(v,w)=F_1(v,w)=1$ and $F_2(v,w)=v+w$. Rewriting the above recurrence relation in terms of generating functions we arrive at

$$\begin{aligned}(1-xd_L(1-d_M))F(x,0,1) - xd_M(1-d_L)F(x,1,0)\\ = 1+x(1+x)(1-d_L)(1-d_M)+xd_Md_LF(x,1,1).\end{aligned} \tag{2.6}$$

Solving the system equations (2.5) and (2.6) we get the desired result. ∎

Using the expressions of the generating functions $F(x,1,0)$ and $F(x,0,1)$ as described in (2.4) together with Proposition 2.2, we obtain that the generating function $F(x,v,w)$ satisfies

$$\begin{aligned}&K(x,v,w)F(x,v,w)\\ &\quad+\left(A_{11}(x,v,w)+A_{10}(x,v,w)\tfrac{r'(xv)}{t(xv)}+A_{01}(x,v,w)\tfrac{s'(xv)}{t(xv)}\right)F(xv,1,1)\\ &\quad+\left(B_{11}(x,v,w)+B_{10}(x,v,w)\tfrac{r'(xw)}{t(xw)}+B_{01}(x,v,w)\tfrac{s'(xw)}{t(xw)}\right)F(xw,1,1)\\ &\quad\quad= H(x,v,w)-\tfrac{A_{10}(x,v,w)r(xv)+A_{01}(x,v,w)s(xv)}{t(xv)}-\tfrac{B_{10}(x,v,w)r(xw)+B_{01}(x,v,w)s(xw)}{t(xw)},\end{aligned} \tag{2.7}$$

where the rational functions $K(x,v,w)$, $A_{ij}(x,v,w)$ and $B_{ij}(x,v,w)$ are given in the statement of Proposition 2.2, and the polynomials $r(x)$, $r'(x)$, $s(x)$, $s'(x)$ and $t(x)$ are given in the statement of Lemma 2.3.

Define $I\!\!F(x,w)$ to be the generating function $F(x,1,w)$, that is,

$$I\!\!F(x,w) = I\!\!F(x,w;r_U,r_M,r_L,d_U,d_M,d_L) = F(x,1,w;r_U,r_M,r_L,d_U,d_M,d_L).$$

Then (2.7) for $v=1$ gives

Theorem 2.4. *Let*

$$\begin{aligned}K(x,w) &= K(x,1,w),\\ A(x,w) &= A_{11}(x,1,w)+A_{10}(x,1,w)\tfrac{r'(x)}{t(x)}+A_{01}(x,1,w)\tfrac{s'(x)}{t(x)},\\ B(x,w) &= B_{11}(x,1,w)+B_{10}(x,1,w)\tfrac{r'(xw)}{t(xw)}+B_{01}(x,1,w)\tfrac{s'(xw)}{t(xw)},\\ H(x,w) &= H(x,1,w)-\tfrac{A_{10}(x,1,w)r(x)+A_{01}(x,1,w)s(x)}{t(x)}-\tfrac{B_{10}(x,1,w)r(xw)+B_{01}(x,1,w)s(xw)}{t(xw)},\end{aligned}$$

where the rational functions $K(x,v,w)$, $A_{ij}(x,v,w)$ and $B_{ij}(x,v,w)$ are given in the statement of Proposition 2.2, and the polynomials $r(x)$, $r'(x)$, $s(x)$, $s'(x)$ and $t(x)$ are given in the statement of Lemma 2.3. Then the generating function $I\!\!F(x,w)$ satisfies

$$K(x,w)I\!\!F(x,w)+A(x,w)I\!\!F(x,1)+B(x,w)I\!\!F(xw,1) = H(x,w). \tag{2.8}$$

The above functional equation, namely (2.8), can be solved systematically using the kernel method (see [1, 5]). In this case, if we assume that there is a small branch $w = u(x)$ such that $K(x, u(x)) = 0$ then we obtain

$$A(x, u(x))\mathbb{G}(x) + B(x, u(x))\mathbb{G}(xu(x)) = H(x, u(x)),$$

where $\mathbb{G}(x) = \mathbb{G}(x; r_U, r_M, r_L, d_U, d_M, d_L) = \mathbb{F}(x, 1; r_U, r_M, r_L, d_U, d_M, d_L)$. Define,

$$u(x) = \frac{1 - xr_U r_L(1 - r_M)}{1 - x(r_U r_L - d_U d_L)} \mathcal{C}\left(\frac{xd_U d_L(1 - d_M)(1 - xr_U r_L(1 - r_M))}{1 - x(r_U r_L - d_U d_L))^2}\right), \tag{2.9}$$

and $\mathcal{C}(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. Note that $u(x)$ is the zero of the following quadric equation

$$K(x, u) = (1 - xr_L r_U(1 - r_M)) - (1 - x(r_L r_U - d_U d_L))u + xd_U d_L(1 - d_M)u^2 = 0.$$

Hence, Theorem 2.4 gives a functional equation that $\mathbb{G}(x)$ satisfies.

Theorem 2.5. *Let $H(x, w), A(x, w), B(x, w)$ defined as in the statement of Theorem 2.4. Then, the generating function*

$$\mathbb{G}(x) = \sum_{n\geq 0} f(n)x^n = \sum_{n\geq 0}\left(x^n \sum_{\pi\in S_n} \prod_{X\in\{L,M,U\}} (d_X^{des_X(\pi)} r_X^{ris_X(\pi)})\right)$$

satisfies

$$A(x, u(x))\mathbb{G}(x) + B(x, u(x))\mathbb{G}(xu(x)) = H(x, u(x)).$$

Note that the functional equation in Theorem 2.5 can be solved easily if hold either $A(x, u(x)) = 0$ or $B(x, u(x)) = 0$. In general, to obtain an explicit formula for the generating function $\mathbb{G}(x)$ we need the following lemmas.

Lemma 2.6. *Let $Q(x)$ be any generating function satisfies $f(x) = p(x)+q(x)f(xu(x))$. Then*

$$f(x) = \sum_{j\geq 0} p(w_j(x)) \prod_{i=0}^{j-1} q(w_i(x)),$$

where $w_0(x) = x$ and $w_j(x) = w_{j-1}(xu(x))$ for $j \geq 1$.

Proof. Applying the equation $f(x) = p(x) + q(x)f(xu(x))$ infinite number of times we get the desired result. □

Let $H(x, w), A(x, w), B(x, w)$ defined as in the statement of Theorem 2.4. Define

$$P(x) = \frac{H(x, u(x))}{A(x, u(x))} \text{ and } Q(x) = -\frac{B(x, u(x))}{xA(x, u(x))}.$$

Then the following result holds immediately from the definitions (can be checked using any mathematical programming such as Maple and Mathematica).

Lemma 2.7. *The generating functions $P(x)$ and $Q(x)$ are analytical functions at $x = 0$ with $P(0) = 1$ and $\lim_{x\to 0} Q(x) \neq 0$.*

Theorem 2.5 together with Lemmas 2.6-2.7 states an explicit formula for the generating function $\mathbb{G}(x)$.

Theorem 2.8. *The generating function*

$$\mathcal{G}(x)=\sum_{n\geq 0} f(n)x^n=\sum_{n\geq 0}\left(x^n\sum_{\pi\in S_n}\prod_{X\in\{L,M,U\}}(d_X^{des_X(\pi)}r_X^{ris_X(\pi)})\right)$$

is given by

$$\mathcal{G}(x)=\sum_{j\geq 0} x^j P(w_j(x))\prod_{i=0}^{j-1}Q(w_i(x)),$$

where $w_0(x)=x$ *and* $w_j(x)=w_{j-1}(xu(x))$ *for* $j\geq 1$.

Note that $u(0)=1$ and $w_j(0)=0$ for all $j\geq 0$. Thus $P(w_j(0))=1$ and $\lim_{x\to 0}Q(w_j(x))\neq 0$ for all $j\geq 0$. Hence, the expression of the generating function $\mathcal{G}(x)$ its useful when we calculate the first terms of the power series $\mathcal{G}(x)$.

3. Counting permutations according to a statistic in $\{ris_U, ris_M, ris_L, des_U, des_M, des_L\}$

Let f be any statistic defined on the set of permutations of length n, that is, f is a function from the set of permutations S_n to the set of nonnegative integer numbers. Define

$$K_f(x;q)=\sum_{n\geq 0}x^n\sum_{\pi\in S_n}q^{f(\pi)}=\sum_{m\geq 0}K_{f;m}(x)q^m.$$

In this subsection we study the generating function $K_f(x;q)$, where either $f=ris_X$ or $f=des_X$ with $X\in\{U,M,L\}$. Indeed by using simple symmetric operations, the reversal (that is, $\pi_1\pi_2\cdots\pi_n\mapsto\pi_n\cdots\pi_2\pi_1$) and the complement (that is, $\pi_1\pi_2\cdots\pi_n\mapsto(n+1-\pi_1)(n+2-\pi_2)\cdots(n+1-\pi_n)$), we obtain that

$$K_{ris_U}(x;q)=K_{des_L}(x;q),\quad K_{ris_M}(x;q)=K_{des_M}(x;q),\quad K_{ris_L}(x;q)=K_{des_U}(x;q).$$

Now, we are ready to present an explicit formula for the generating function $K_{des_X}(x;q)$, where $X\in\{U,M,L\}$.

3.1. The generating function $K_{des_L}(x;q)$.

Let $r_U=r_M=r_L=d_U=d_M=1$ and $d_L=q$. Then $u(x)=\frac{1}{1-x(1-q)}$, $A(x,u(x))=\frac{1}{1-q}$, $B(x,u(x))=q-\frac{x(1-q)}{1-x(1-q)}$, and $H(x,u(x))=1$. Then Theorem 2.5 gives that the generating function $K_{des_L}(x;q)$ satisfies

$$K_{des_L}(x;q)=1-q+\left(q+\frac{x(1-q)}{1-x(1-q)}\right)K_{des_L}\left(\frac{x}{1-x(1-q)};q\right).$$

Applying the above functional equation infinity number of times, we get the following result.

Corollary 3.1. *We have*

$$K_{des_L}(x;q)=(1-q)\sum_{j\geq 0}\prod_{i=1}^{j}\left(q+\frac{x(1-q)}{1-ix(1-q)}\right).$$

Corollary 3.1 for $q=0$ gives

$$K_{des_L;0}(x)=\sum_{n\geq 0}\frac{x^n}{\prod_{j=0}^{n}(1-jx)},$$

which recovers the well known enumeration of permutations of length n with no lower descent (that is, avoid the pattern 32−1, or equivalently, there no i,j such that $i+1<j$ and $\pi_i>\pi_{i+1}>\pi_j$) by the n-th Bell number (see [2]).

Now let us find an explicit formula for $K_{des_L;1}(x)$. Differentiating $K_{des_L}(x;q)$, see Corollary 3.1, respect to q we obtain

$$\begin{aligned}&\frac{\partial}{\partial q}K_{des_L}(x;q)\\&=(1-q)\sum_{j\geq 0}\left(\prod_{i=1}^{j}\left(q+\frac{x(1-q)}{1-ix(1-q)}\right)\sum_{i=1}^{j}\frac{1-\frac{x}{(1-i(1-q)x)^2}}{q+\frac{x(1-q)}{1-ix(1-q)}}\right)-\sum_{j\geq 0}\prod_{i=1}^{j}\left(q+\frac{x(1-q)}{1-ix(1-q)}\right).\end{aligned}$$

which implies that the generating function for the number of permutations of length n with exactly one lower descent is given by

$$K_{des_L;1}(x)=\frac{\partial}{\partial q}K_{des_L}(x;q)\bigg|_{q=0}=\sum_{j\geq 0}\frac{x^j}{\prod_{i=0}^{j}(1-ix)}\left(\sum_{i=1}^{j}\frac{(1-ix)^2-x}{x(1-ix)}-1\right).$$

3.2. **The generating function $K_{des_U}(x;q)$.** Let $r_U=r_M=r_L=d_L=d_M=1$ and $d_U=q$. Then $u(x)=\frac{1}{1-x(1-q)}$, $A(x,u(x))=\frac{1-x(1-q)}{1-q}$, $B(x,u(x))=-\frac{q+x(1-q)^2}{1-q}$, and $H(x,u(x))=1-x(1-q)$. Then Theorem 2.5 gives that the generating function $K_{des_U}(x;q)$ satisfies

$$K_{des_U}(x;q)=1-q+\left(q+\frac{x(1-q)}{1-x(1-q)}\right)K_{des_U}\left(\frac{x}{1-x(1-q)};q\right).$$

Applying the above functional equation infinity number of times, we get the following result.

Corollary 3.2. *We have*

$$K_{des_U}(x;q)=K_{des_L}(x;q)=(1-q)\sum_{j\geq 0}\prod_{i=1}^{j}\left(q+\frac{x(1-q)}{1-ix(1-q)}\right).$$

3.3. **The generating function $K_{des_M}(x;q)$.** Let $r_U=r_M=r_L=d_U=d_L=1$ and $d_M=q$. Then from the definitions we have $u(x)=\mathcal{C}(x(1-q))$, $A(x,u(x))=\frac{1}{(1-q)\mathcal{C}^2(x(1-q))}$, $B(x,u(x))=-\frac{q}{(1-q)\mathcal{C}(x(1-q))}$, and $H(x,u(x))=\frac{1}{\mathcal{C}(x(1-q))}$, where $\mathcal{C}(t)=\frac{1-\sqrt{1-4t}}{2t}$ is the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. Then Theorem 2.5 gives that the generating function $K_{des_M}(x;q)$ satisfies

$$K_{des_M}(x;q)=(1-q)\mathcal{C}(x(1-q))+q\mathcal{C}(x(1-q))K_{des_M}(x\mathcal{C}(x(1-q));q).$$

Applying the above functional equation infinity number of times, we get the following result.

Corollary 3.3. *We have*

$$K_{des_M}(x;q)=(1-q)\sum_{j\geq 0}q^j p_j(x;q),$$

where $p_0(x;q)=\mathcal{C}(x(1-q))$ *and* $p_j(x;q)=p_{j-1}(x;q)\mathcal{C}(x(1-q)p_{j-1}(x;q))$ *for all* $j\geq 1$.

Corollary 3.3 for $q=0$ gives

$$K_{des_M;0}(x)=\mathcal{C}(x)=\frac{1-\sqrt{1-4x}}{2x},$$

which recovers the well known enumeration of permutations of length n that avoid the pattern $31-2$ (that is, there no i,j such that $i+1<j$ and $\pi_i>\pi_j>\pi_{i+1}$) by the n-th Catalan number (see [2]).

Now let us find an explicit formula for $K_{des_M;1}(x)$. Differentiating $K_{des_M}(x;q)$, see Corollary 3.3, respect to q we obtain

$$\begin{aligned}\frac{\partial}{\partial q}K_{des_M}(x;q)\Big|_{q=0} &= -p_0(x;0)+p_1(x;0)+\frac{\partial}{\partial q}p_0(x;q)\Big|_{q=0}\\ &= -\mathcal{C}(x)+\mathcal{C}(x)\mathcal{C}(x\mathcal{C}(x))-\frac{1}{\sqrt{1-4x}}+\mathcal{C}(x)\\ &= \mathcal{C}(x)\mathcal{C}(x\mathcal{C}(x))-\frac{1}{\sqrt{1-4x}},\end{aligned}$$

which implies that the generating function for the number of permutations of length n with exactly one middle descent is given by (see Sequence A122892 in [7])

$$K_{des_M;1}(x)=\mathcal{C}(x)\mathcal{C}(x\mathcal{C}(x))-\frac{1}{\sqrt{1-4x}}=\frac{1-\sqrt{2\sqrt{1-4x}-1}}{2x}-\frac{1}{\sqrt{1-4x}}.$$

Differentiating $K_{des_M}(x;q)$ exactly twice, we get that the generating function for the number of permutations of length n with exactly twice middle descent is given by

$$K_{des_M;2}(x)=\frac{\sqrt{1-4x}^3\sqrt{2\sqrt{1-4x}-1}\left(1-\sqrt{2\sqrt{2\sqrt{1-4x}-1}-1}\right)+2x^2\sqrt{2\sqrt{1-4x}-1}-2x(1-4x)}{2x\sqrt{1-4x}^3\sqrt{2\sqrt{1-4x}-1}}.$$

In general, by Faá di Bruno's formula for the m-th derivative of the composition of two functions[1] and Corollary 3.3 we can compute the formula for $K_{des_M;m}(x)$ by the following result.

Theorem 3.4. *The generating function* $K_{des_M;m}(x)$ *is given by*

$$\sum_{j=0}^{m}\frac{p_j^{(m-j)}(x;0)}{(m-j)!}-\sum_{j=1}^{m}\frac{p_{j-1}^{(m-j)}(x;0)}{(m-j)!},$$

where the d*-th derivative of* $p_j(x;q)$ *at* $q=0$*, namely* $p_j^{(d)}(x;0)$*, satisfies the following recurrence relation*

$$p_j^{(d)}(x;0)=\sum_{i=0}^{d}\sum_{k_1+2k_2+\cdots+ik_i=i}\frac{d!p_{j-1}^{(d-i)}(x;0)\mathcal{C}^{(k_1+\cdots+k_i)}(xp_{j-1}(x;0))}{(m-i)!k_1!k_2!\cdots k_i!}\prod_{p=1}^{i}\left(\frac{x(p_{j-1}^{(p)}(x;0)-p_{j-1}^{(p-1)}(x;0))}{p!}\right)^{k_p}$$

with the initial condition $p_0^{(d)}(x;0)=(-x)^d\mathcal{C}^{(d)}(x)$ *for all* $d,j\geq 0$.

[1] see http://mathworld.wolfram.com/FaadiBrunosFormula.html

4. Further results and open questions

In this section we describe several applications of Theorem 2.8 which generalize several known results. In Section 3 showed that (see [2])

- the number of permutations of length n that avoid the generalized pattern either $32-1$ or $21-3$ is given by the n-th Bell number, and
- the number of permutations of length n that avoid the generalized pattern $31-2$ is given by the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Ones can refine these results by studying the generating function for the number of permutations without either upper, middle, or lower descent according to the length of the permutation and the statistic $f \in \{ris_U, ris_M, ris_L, des_U, des_M, des_L\}$. For instance, Theorem 2.8 for $des_L = 0$ (that is, the case of permutations avoiding the generalized pattern $32-1$) gives the following result.

Corollary 4.1. *We have*

(1) *The generating function* $g(x) = \mathbb{G}(x; q, 1, 1, 1, 1, 0)$ *satisfies*

$$g(x) = \frac{q}{q + x(1-q)} + \frac{x}{(1-xq)(q + x(1-q))} g\left(\frac{x}{1-xq}\right).$$

(2) *The generating function* $g(x) = \mathbb{G}(x; 1, q, 1, 1, 1, 0)$ *satisfies*

$$g(x) = 1 + \frac{x}{1-x} g\left(\frac{x(1-x+xq)}{1-x}\right).$$

(3) *The generating function* $g(x) = \mathbb{G}(x; 1, 1, q, 1, 1, 0)$ *satisfies*

$$g(x) = 1 + \frac{x}{1-x} g\left(\frac{x}{1-xq}\right).$$

(4) *The generating function* $g(x) = \mathbb{G}(x; 1, 1, 1, q, 1, 0)$ *satisfies*

$$g(x) = \frac{1-xq}{1-x} + \frac{x(x+q-2xq)}{(1-x)^2} g\left(\frac{x}{1-x}\right).$$

(5) *The generating function* $g(x) = \mathbb{G}(x; 1, 1, 1, 1, q, 0)$ *satisfies*

$$g(x) = \frac{1-x}{1-2x+xq} + \frac{1-2x+xq}{(1-x)^2} g\left(\frac{x}{1-x}\right).$$

Also, Theorem 2.8 for $des_U = 0$ (that is, the case of permutations avoiding the generalized pattern $21-3$) gives the following result.

Corollary 4.2. *We have*

(1) *The generating function* $g(x) = \mathbb{G}(x; q, 1, 1, 0, 1, 1)$ *satisfies*

$$g(x) = \frac{(1-xq)^2 + (q-1)x^3(q-2+(2q-1)x)}{(1-xq)^2 + x^2(q-1)} + \frac{x(1-xq)(1+x^2(q-1))}{(1-xq)^2 + x^2(q-1))} g\left(\frac{x}{1-xq}\right).$$

(2) *The generating function* $g(x) = \mathbb{G}(x; 1, q, 1, 0, 1, 1)$ *satisfies*

$$g(x) = \frac{1 + (q-2)x(1 + (q-1)x)}{1 + (q-2)x + (q-1)^2x^2} + \frac{x(1 + (q-1)x)^2}{1 + (q-2)x + (q-1)^2x^2} g\left(\frac{x(1 + (q-1)x)}{1-x}\right).$$

(3) *The generating function* $g(x) = \mathbb{G}(x; 1, 1, q, 0, 1, 1)$ *satisfies*

$$g(x) = \frac{1}{q + (1-q)x} + \frac{x}{(1-xq)(q + (1-q)x)} g\left(\frac{x}{1-xq}\right).$$

(4) *The generating function* $g(x) = \mathbb{G}(x; 1, 1, 1, 0, q, 1)$ *satisfies*

$$g(x) = \frac{1-x}{1-2x+xq} + \frac{xq}{1-2x+xq} g\left(\frac{x}{1-x}\right).$$

(5) *The generating function* $g(x) = \mathbb{G}(x; 1, 1, 1, 0, 1, q)$ *satisfies*

$$g(x) = \frac{1}{1-x+xq} + \frac{x(q+x-xq)}{(1-x)(1-x+xq)} g\left(\frac{x}{1-x}\right).$$

Finally, Theorem 2.8 for $des_M = 0$ (that is, the case of permutations avoiding the generalized pattern $31-2$) gives the following result.

Corollary 4.3. *We have*

(1) *The generating function* $\mathbb{G}(x; q, 1, 1, 1, 0, 1)$ *is given by*

$$g(x) = \frac{q-1}{q(1-x)} + \frac{1}{q(1-x+xq)} \mathcal{C}\left(\frac{xq}{(1-x+xq)^2}\right).$$

(2) *The generating function* $\mathbb{G}(x; 1, q, 1, 1, 0, 1)$ *is give by*

$$1 + \frac{x}{1-2x} \mathcal{C}\left(\frac{x^2q}{1-2x}\right).$$

(3) *The generating function* $\mathbb{G}(x; 1, 1, q, 1, 0, 1)$, $g(x) = \mathbb{G}(x; 1, 1, 1, q, 0, 1)$ *and* $g(x) = \mathbb{G}(x; 1, 1, 1, 1, 0, q)$ *are given by*

$$g(x) = \frac{1 - x - 2q + xq + \sqrt{(1 + x - xq)^2 - 4x}}{2(1 - q - 2x + xq)}.$$

Applying the above results, Corollaries 4.1-4.3, for $q = 0$ we get explicit formula for the generating functions for the number of permutations of length n avoiding two generalized patterns of length three from the set $GP = \{12-3, 13-2, 21-3, 23-1, 31-2, 32-1\}$, see [4]. More generally, ones can use Theorem 2.8 to obtain the generating function for the number of permutations avoiding a set of pattern $T \subseteq GP$.

We end this section by presenting two open problems:

- Theorem 3.4 gives an explicit formula for the generating function $K_{des_M;m}(x)$ for the number of permutations π of length n having $des_M(\pi) = m$. Can ones give explicit formula for $K_{des_U;m}(x)$? We remark that by Corollaries 3.1-3.2 we have that $K_{des_U;m}(x) = K_{des_L;m}(x)$.

- Corollary 3.1 gives

$$K_{des_L}(x;q) = (1-q)\sum_{j\geq 0}\prod_{i=1}^{j}\left(q + \frac{x(1-q)}{1-ix(1-q)}\right).$$

This formula it is not "nice" when we assume $q \neq 1$. For example, the substitution $q = -1$ is not legal in this generating function, otherwise we get $2\sum_{j\geq 0}(-1)^j \frac{1-2(j+1)x}{1-2x}$ which does not converge. In order to restrict this problem, we need to write the generating function $K_{des_L}(x;q)$ as $\sum_{n\geq 0} a_n(q)x^n$ and then we can substitute $q = -1$. But this it is extremely hard! Thus, can ones find another formula for $K_{des_L}(x;q)$ that is suitable for substitutions $q = 0, 1, -1$?

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