Centrosymmetric words avoiding 3-letter permutation patterns

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Abstract

A word is *centrosymmetric* if it is invariant under the reverse-complement map. In this paper, we give enumerative results on k-ary centrosymmetric words of length n avoiding a pattern of length 3 with no repeated letters.

1 Introduction

A sequence of positive integers $\sigma_1\sigma_2\ldots\sigma_n$ (for instance, a permutation or a word) is said to *avoid a pattern* $\tau = \tau_1\tau_2\ldots\tau_s$ if it does not contain any subsequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_s}$ which is order isomorphic to τ.

In the last two decades, the study of pattern avoidance has interested several research fields in combinatorics: many results have been found on permutations, which is still the most advanced area, and a considerable work has been made on words since 1998. Burstein [4] gave for the first time an explicit formula for the number of words avoiding a set of permutation patterns in S_3; later, Burstein and Mansour [5] extended the results to patterns with repeated letters.

Many improvement and generalizations have been made in the last years. In 2006 Firro and Mansour [7] presented a new method, the scanning-element algorithm (that we exploit in Section 5) to obtain an easier proof of Burstein's results for the pattern 123; in the same year, Mansour [11] applied the block decomposition method for the pattern 132. Brändén and Mansour [3] used finite automata theory to give a combinatorial explanation for the number of words avoiding patterns of length three; furthermore, Jelínek and Mansour [9] determined all the Wilf-equivalence classes of subsequence patterns of length at most six.

In this paper, we focus on centrosymmetric words, namely, words that are invariant under the reverse-complement map, or, equivalently, whose corresponding tableaux via the Robinson-Schensted-Knuth algorithm are invariant under Schützenberger's involution (see [8], [10] and [13] for more details). We find the generating functions for the number

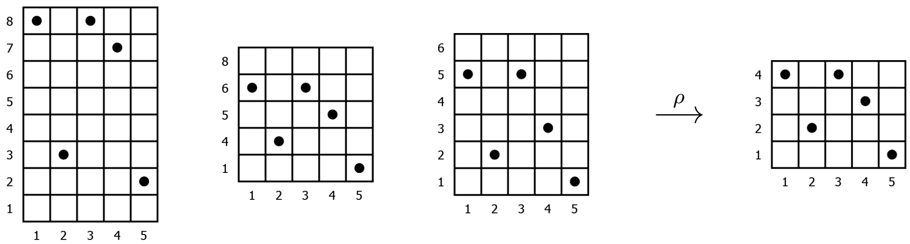


Figure 1. *The words* $83872 \in \mathcal{W}_{5,8}$, $64651 \in \mathcal{W}_{5,\{1,4,5,6,8\}}$ *and* $52531 \in \mathcal{W}_{5,6}$ *are all order isomorphic. Their common renormalization, as well as the representative of their equivalence class, is* $42431 \in \mathcal{W}_{5,4}$.

of centrosymmetric words avoiding a pattern of length 3 with no repeated letters, and, for the pattern 123, even the explicit formulae. In the special case of centrosymmetric surjective words, we also reobtain some of the results described in Egge's paper [6].

2 Preliminaries

Definition 2.1 *A* word of length n over a k-letter alphabet *(also said a* k-ary word of length n, or a *word of type* (n,k)*) is any map*

$$\sigma\colon \{1,2,\dots,n\} \longrightarrow \mathcal{A}_k,$$

where $\mathcal{A}_k$ *is a set of positive integers such that* $|\mathcal{A}_k| = k$.

We denote the set of all words of length n over $\mathcal{A}_k$ by $\mathcal{W}_{n,\mathcal{A}_k}$; in particular, when $\mathcal{A}_k = \{1,2,\dots,k\}$ we use the symbol $\mathcal{W}_{n,k}$ instead of $\mathcal{W}_{n,\{1,2,\dots,k\}}$. A word in $\mathcal{W}_{n,\mathcal{A}_k}$ will be represented either by the one-line notation

$$\sigma_1\sigma_2\dots\sigma_n,$$

or by the usual graphical representation, as in figure (1).

Definition 2.2 *Let* $\mathcal{W}_n$ *be the set of all words of length* n*:*

$$\mathcal{W}_n = \bigcup_{k\geq 1} \mathcal{W}_{n,\mathcal{A}_k}.$$

We say that two words $\sigma,\tau \in \mathcal{W}_n$ *are* order isomorphic *if, for every* $1 \leq a,b \leq n$*, we have*

$$\sigma_a * \sigma_b \quad \Longleftrightarrow \quad \tau_a * \tau_b$$

for any relation $* \in \{<,=,>\}$.

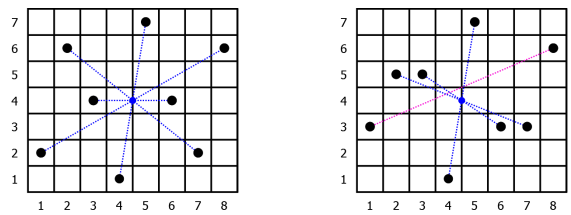
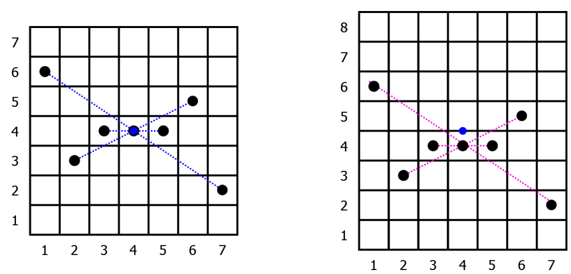


Figure 2. *The word* $26417426 \in \mathcal{W}_{8,7}$ *is centrosymmetric while* $35517336 \in \mathcal{W}_{8,7}$ *is not (figures above). Moreover, two words containing the same letters can be or not centrosymmetric depending on the alphabet: the word* $6344452 \in \mathcal{W}_{7,7}$ *is centrosymmetric while* $6344452 \in \mathcal{W}_{7,8}$ *is not (figures below).*



Roughly speaking, two words are order isomorphic if and only if, ignoring the row numbering, their graphical representations are identical, or, alternatively, one can be obtained from the other by erasing or adding a certain number of empty rows.

Observe that each word in a given equivalence class of order isomorphic words contains the same number d of different letters. Hence, it is natural to give the following definition.

Definition 2.3 *Given a word* $\sigma \in \mathcal{W}_n$ *containing exactly* d *different letters, we call* renormalization of σ *the only word* $\rho(\sigma) \in \mathcal{W}_{n,d}$ *which is order isomorphic to* σ. *This word will also be taken as the representative of the equivalence class that contains* σ.

Definition 2.4 *A word* $\sigma \in \mathcal{W}_{n,\mathcal{A}_k}$ *is said to* contain a pattern $\tau \in \mathcal{W}_{s,t}$, *where* $s \leq n$ *and* $t \leq k$, *if there exist indices* $1 \leq i_1 \leq \cdots \leq i_s \leq n$ *such that the subsequence* $\sigma_{i_1} \ldots \sigma_{i_s}$ *is order isomorphic to* $\tau = \tau_1 \ldots \tau_s$. *Otherwise,* σ *is said to* avoid τ.

For example, the word 325316 avoids the pattern 312 but contains 112, since the subsequence 336 is order isomorphic to 112. We denote the set of words in $\mathcal{W}_{n,\mathcal{A}_k}$ which avoid τ by the symbol $\mathcal{W}_{n,\mathcal{A}_k}(\tau)$.

Definition 2.5 *A word* $\sigma \in \mathcal{W}_{n,k}$ *is* centrosymmetric *if, for every* $i = 1, 2, \ldots, n$,

$$\sigma_i + \sigma_{n+1-i} = k + 1.$$

In this case, we say that σ_i *is the* conjugate letter *of* σ_{n+1-i}.

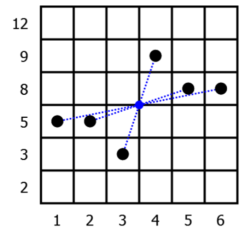


Figure 3. *The centrosymmetric word* $553988 \in \mathcal{C}_{6,\{2,3,5,8,9,12\}}$.

We denote the set of centrosymmetric words of length n over the alphabet $\{1, 2, \ldots, k\}$ by the symbol $\mathcal{C}_{n,k}$. Needless to say, a word is centrosymmetric if and only if its graphical representation is symmetric with respect to the center of the grid (see figure 2).

Observe that there are no centrosymmetric words with n odd and k even, since in this case there is no type (n, k) that satisfies $2\,\sigma_{(n+1)/2} = k + 1$. In all the other cases, the number of k-ary centrosymmetric words of length n is

$$|\mathcal{C}_{n,k}| = k^{\lfloor \frac{n}{2} \rfloor}.$$

We need the following generalization of centrosymmetry.

Definition 2.6 *Let $\mathcal{A}_k = \{a_1, a_2, \ldots, a_k\}$ be a k-letter alphabet, with $a_1 < a_2 < \cdots < a_k$, and let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathcal{W}_{n,\mathcal{A}_k}$, with $\sigma_i = a_{j(i)}$ for every $i = 1, \ldots, n$. The word σ is said to be* centrosymmetric *if, for every $i = 1, \ldots, n$,*

$$j(i) + j(n + 1 - i) = k + 1.$$

We denote the set of centrosymmetric words in $\mathcal{W}_{n,\mathcal{A}_k}$ by the symbol $\mathcal{C}_{n,\mathcal{A}_k}$.

3 Symmetries

Definition 3.1 *We call* reverse *and* complement*, respectively, the transformations*

$$r\colon \begin{array}{ccc} \mathcal{W}_{n,k} & \longrightarrow & \mathcal{W}_{n,k} \\ \sigma & \longmapsto & \sigma^r \end{array} \qquad c\colon \begin{array}{ccc} \mathcal{W}_{n,k} & \longrightarrow & \mathcal{W}_{n,k} \\ \sigma & \longmapsto & \sigma^c \end{array}$$

such that $(\sigma^r)_i = \sigma_{n+1-i}$ and $(\sigma^c)_i = k + 1 - \sigma_i$.

The transformations r and c generate the group $G = \langle r, c \rangle = \{id, r, c, rc\}$, which is isomorphic to the group of symmetries of a rectangle D_4 (the dihedral group). The action of G on $\mathcal{W}_{n,k}$ allows us to state that, for every $\sigma \in \mathcal{W}_{n,k}$, $\tau \in \mathcal{W}_{s,t}$ and $\varphi \in G$

$$\sigma \text{ avoids } \tau \quad \Longleftrightarrow \quad \varphi(\sigma) \text{ avoids } \varphi(\tau).$$

Hence, in order to find the cardinality of $\mathcal{C}_{n,k}(\tau)$ for every permutation pattern τ of length three it is sufficient to study the sets $\mathcal{C}_{n,k}(123)$ and $\mathcal{C}_{n,k}(132)$.

We finally remark that centrosymmetric words are exactly those words which are invariant under the reverse-complement (rc) operation. Then, for every $\sigma \in \mathcal{C}_{n,k}$,

$$\sigma \text{ avoids } \tau \quad \Longleftrightarrow \quad \sigma \text{ avoids } \tau^{rc}$$

and hence

$$|\mathcal{C}_{n,k}(\tau)| = |\mathcal{C}_{n,k}(\tau^{rc})| = |\mathcal{C}_{n,k}(\tau, \tau^{rc})|. \tag{1}$$

4 Surjective words

From definition (2.1), it is natural to define a *surjective word* of length n over a k-letter alphabet $\mathcal{A}_k$ to be a surjective map $\sigma\colon \{1, 2, \ldots, n\} \to \mathcal{A}_k$. In other terms, a surjective word is a word in $\mathcal{W}_{n,\mathcal{A}_k}$ which contains all the letters of its alphabet. We denote the set of surjective words in $\mathcal{W}_{n,\mathcal{A}_k}$ by $\mathcal{S}_{n,\mathcal{A}_k}$, and the set of centrosymmetric surjective words in $\mathcal{W}_{n,\mathcal{A}_k}$ by $\mathcal{CS}_{n,\mathcal{A}_k}$. As usual, when $\mathcal{A}_k = \{1, 2, \ldots, k\}$, we use the symbols $\mathcal{S}_{n,k}$ and $\mathcal{CS}_{n,k}$, respectively. Obviously, $\mathcal{S}_{n,\mathcal{A}_k} \neq \varnothing$ if and only if $n \geq k \geq 1$. We also observe that the set S_n of permutations of length n can be seen as the set $\mathcal{S}_{n,n}$ of surjective words with length equal to the cardinality of the alphabet.

Now, we focus on centrosymmetric words. Let $c_{n,k}$ be the number of centrosymmetric words of type (n, k) and $cs_{n,k}$ the number of centrosymmetric surjective words of the same type:

$$c_{n,k} = |\mathcal{C}_{n,k}|, \qquad cs_{n,k} = |\mathcal{CS}_{n,k}|.$$

Proposition 4.1 *For $m, h \geq 1$ the following relations hold:*

$$\begin{aligned} &(i) \quad c_{2m,2h} = \sum_{i=1}^{h} \binom{h}{i} cs_{2m,2i} \\ &(ii) \quad c_{2m,2h+1} = \sum_{i=1}^{h} \binom{h}{i} cs_{2m,2i} + \sum_{i=0}^{h} \binom{h}{i} cs_{2m,2i+1} \\ &(iii) \quad c_{2m+1,2h+1} = \sum_{i=0}^{h} \binom{h}{i} cs_{2m+1,2i+1} \end{aligned} \tag{2}$$

Proof. Let d be the number of distinct letters contained in a fixed word $\sigma \in \mathcal{C}_{n,k}$, and consider the graphical representation. In case (i) and (iii), σ is obtained from its renormalization $\rho(\sigma) \in \mathcal{CS}_{n,d}$, which has the same type parities of σ, by adding a suitable number (possibly zero) of pairs of empty rows, and relabelling the alphabet letters from 1 to k. To preserve centrosymmetry, every pair consists of two rows inserted at the same distance from the center. In the remaining case (ii), the renormalization $\rho(\sigma) \in \mathcal{CS}_{n,d}$ of an even-odd centrosymmetric word σ (i.e. a word with even length over an odd cardinality alphabet) can either be even-even, or even-odd. In both cases, to obtain σ from $\rho(\sigma)$ we have to add again some pairs of empty rows, while in the first case a further central empty row is needed. □

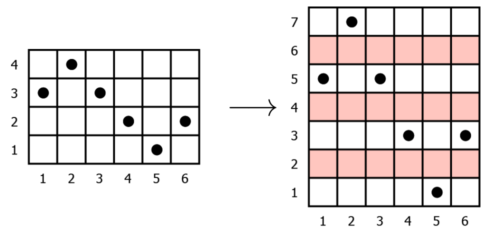
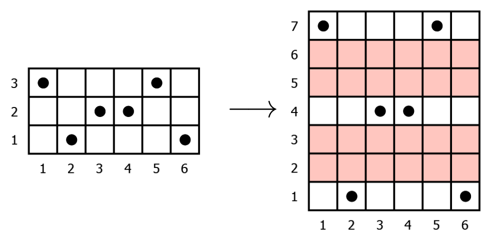


Figure 4. *Two words in $\mathcal{C}_{6,7}$ obtained from their renormalizations by adding the central row and another pair (figure above) or two pairs of rows (figure below).*



By definition, a word σ contains a given pattern τ whenever its renormalization $\rho(\sigma)$ contains τ. This implies that relations (2) hold also for restricted words. Hence, to get the desired enumeration of $\mathcal{C}_{n,k}(\tau)$ it is sufficient to obtain the number $|\mathcal{CS}_{n,k}(\tau)|$ of centrosymmetric surjective words avoiding τ, and then use (2).

Obviously, the same can be done for the related generating functions. Define

$$c^{\tau}_{n,k} = |\mathcal{C}_{n,k}(\tau)| \qquad \text{and} \qquad cs^{\tau}_{n,k} = |\mathcal{CS}_{n,k}(\tau)|,$$

and let

$$C^{\tau}(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} c^{\tau}_{n,k}\, x^n y^k$$

be the ordinary generating function for the number of centrosymmetric words avoiding the pattern τ. Defining the series

$$\begin{aligned}
C^{\tau}_{ee}(x,y) &= \sum_{\substack{n\geq 0\\ n \text{ even}}}\sum_{\substack{k\geq 0\\ k \text{ even}}} c^{\tau}_{n,k}\, x^n y^k, & CS^{\tau}_{ee}(x,y) &= \sum_{\substack{n\geq 0\\ n \text{ even}}}\sum_{\substack{k\geq 0\\ k \text{ even}}} cs^{\tau}_{n,k}\, x^n y^k,\\
C^{\tau}_{eo}(x,y) &= \sum_{\substack{n\geq 0\\ n \text{ even}}}\sum_{\substack{k\geq 0\\ k \text{ odd}}} c^{\tau}_{n,k}\, x^n y^k, & CS^{\tau}_{eo}(x,y) &= \sum_{\substack{n\geq 0\\ n \text{ even}}}\sum_{\substack{k\geq 0\\ k \text{ odd}}} cs^{\tau}_{n,k}\, x^n y^k, \qquad (3)\\
C^{\tau}_{oo}(x,y) &= \sum_{\substack{n\geq 0\\ n \text{ odd}}}\sum_{\substack{k\geq 0\\ k \text{ odd}}} c^{\tau}_{n,k}\, x^n y^k, & CS^{\tau}_{oo}(x,y) &= \sum_{\substack{n\geq 0\\ n \text{ odd}}}\sum_{\substack{k\geq 0\\ k \text{ odd}}} cs^{\tau}_{n,k}\, x^n y^k,
\end{aligned}$$

by inverse binomial transform (see e.g. [14]), relations (2) imply that the following relations hold:

$$C^{\tau}_{ee}(x,y) = \frac{1}{1-y^2}\, CS^{\tau}_{ee}\left(x, \frac{y}{\sqrt{1-y^2}}\right),$$

$$C^{\tau}_{eo}(x,y) = \frac{y}{1-y^2}\, CS^{\tau}_{ee}\left(x, \frac{y}{\sqrt{1-y^2}}\right) + \frac{1}{\sqrt{1-y^2}}\, CS^{\tau}_{eo}\left(x, \frac{y}{\sqrt{1-y^2}}\right) - \frac{y}{1-y^2},$$

$$C^{\tau}_{oo}(x,y) = \frac{1}{\sqrt{1-y^2}}\, CS^{\tau}_{oo}\left(x, \frac{y}{\sqrt{1-y^2}}\right).$$

Hence, since

$$C^{\tau}(x,y) = C^{\tau}_{ee}(x,y) + C^{\tau}_{eo}(x,y) + C^{\tau}_{oo}(x,y),$$

we finally obtain that

$$\begin{aligned} C^{\tau}(x,y) &= \frac{1}{1-y}\, CS^{\tau}_{ee}\left(x, \frac{y}{\sqrt{1-y^2}}\right) + \frac{1}{\sqrt{1-y^2}}\, CS^{\tau}_{eo}\left(x, \frac{y}{\sqrt{1-y^2}}\right) + \\ &\quad + \frac{1}{\sqrt{1-y^2}}\, CS^{\tau}_{oo}\left(x, \frac{y}{\sqrt{1-y^2}}\right) - \frac{y}{1-y^2}. \end{aligned} \tag{4}$$

5 The pattern 123

In order to compute the cardinality of the set $\mathcal{CS}_{n,k}(123)$ we exploit the scanning-element algorithm, presented in [7].

Denote by $cs^{123}_{n,k,i}$ the number of centrosymmetric surjective words avoiding 123 and starting with the letter i. Of course, we have

$$cs^{123}_{n,k} = \sum_{i=1}^{k} cs^{123}_{n,k,i}.$$

Observe that a word $\sigma \in \mathcal{CS}_{n,k}$ starting with σ_1 such that $1 \leq \sigma_1 \leq \left\lceil \frac{k}{2} \right\rceil - 1$ contains the pattern 123. In fact, by surjectivity there exists a letter σ_x such that the subsequence $\sigma_1\sigma_x\sigma_n$ is a 123-pattern. Hence, the preceding formula reduces to

$$cs^{123}_{n,k} = \sum_{i=\left\lceil \frac{k}{2} \right\rceil}^{k} cs^{123}_{n,k,i}.$$

Now, our first goal is to find a recurrence relation for the terms $cs^{123}_{n,k,i}$. In order to do this, we consider the number $cs^{123}_{n,k,i,j}$ (with $n \geq 4$ and $k \geq 3$) of centrosymmetric surjective words, that avoid 123, with first letter i and second letter j. By the preceding considerations, we have

$$cs^{123}_{n,k,i} = \sum_{j=\left\lceil \frac{k}{2} \right\rceil - 1}^{k} cs^{123}_{n,k,i,j}. \tag{5}$$

The case $j = \lceil \frac{k}{2} \rceil - 1$ holds only in some special cases, that we present in the next result.

Theorem 5.1 *For $n \geq 4$ and $k \geq 3$, if $cs^{123}_{n,k,i,j} \neq 0$ then only one of the following cases holds:*

(i) $\lceil \frac{k}{2} \rceil \leq i < j = k$,

(ii) $k \geq i = j \geq \lceil \frac{k}{2} \rceil$,

(iii) $k \geq i > j \geq \lceil \frac{k}{2} \rceil$,

(iv) *n is even, $i = \lceil \frac{k}{2} \rceil$ and $j = \lceil \frac{k}{2} \rceil - 1$,*

(v) *n and k are even, $i = \frac{k}{2} + 1$ and $j = \frac{k}{2} - 1$.*

Moreover, for each one of the preceding cases we have the following recurrences:

(i) $cs^{123}_{n,k,i,j} = cs^{123}_{n-2,k,i} + cs^{123}_{n-2,k-2,i-1}$,

(ii) $cs^{123}_{n,k,i,j} = cs^{123}_{n-2,k,i}$,

(iii) $cs^{123}_{n,k,i,j} = \begin{cases} cs^{123}_{n-2,k,\frac{k}{2}} & \text{if } i = \frac{k}{2}+1,\ j = \frac{k}{2} \text{ and } n,k \text{ are even} \\ cs^{123}_{n-2,k,j} + cs^{123}_{n-2,k-2,j-1} & \text{otherwise} \end{cases}$,

(iv) $cs^{123}_{n,k,\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil - 1} = \begin{cases} cs^{123}_{n-2,k-2,\frac{k-2}{2}} & \text{if } n \text{ and } k \text{ are even} \\ cs^{123}_{n-2,k-1,\frac{k-1}{2}} & \text{if } n \text{ is even and } k \text{ is odd} \end{cases}$,

(v) $cs^{123}_{n,k,\frac{k}{2}+1,\frac{k}{2}-1} = cs^{123}_{n-2,k-2,\frac{k-2}{2}}$.

Proof. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ be a word in $\mathcal{CS}_{n,k}(123)$ with $\sigma_1 = i$ and $\sigma_2 = j$, respectively. Recall that $i \geq \lceil \frac{k}{2} \rceil$ and $j \geq \lceil \frac{k}{2} \rceil - 1$.

Suppose $i < j$. If $j \neq k$, surjectivity implies that the sequence ijk is a 123-pattern: hence, $j = k$. In this case, the symbol k in the second position of σ and its conjugate 1 in the $(n-1)$-th position cannot form any 123 pattern. Hence, the number $cs^{123}_{n,k,i,j}$ is completely determined by the subsequence $\tilde{\sigma} = \sigma_1\sigma_3\ldots\sigma_{n-2}\sigma_n$, whose symbols can belong either to the alphabet $\{2, \ldots, k-1\}$, or $\{1, \ldots, k\}$. Obviously, in the first case we can renormalize $\tilde{\sigma}$, so its first letter becomes $i-1$ and its alphabet $\{1, \ldots, k-2\}$. All these considerations prove recurrence (i).

The case $i = j$ is quite simple: when σ has two identical starting letters, it avoids 123 if and only if its subword $\tilde{\sigma} = \sigma_2\ldots\sigma_{n-1}$ does. Furthermore, surjectivity is preserved between σ and $\tilde{\sigma}$, and hence recurrence (ii) is proved.

Now, suppose that $i > j$. In this case, the only fact that there is no 123-pattern with first letter j implies that no 123-pattern with first letter i occurs. The same is true for the conjugates of i and j, and hence the number $cs^{123}_{n,k,i,j}$ is completely determined by the subword $\tilde{\sigma} = \sigma_2\sigma_3\ldots\sigma_{n-1}$. To preserve surjectivity, $\tilde{\sigma}$ may either be over the alphabet $\{1, \ldots, k\} \setminus \{\sigma_1, \sigma_n\}$, or $\{1, \ldots, k\}$. This leads to the second recurrence of (iii). Observe that, for particular values of i and j only one of the alphabets holds. In the even-even case, if $i = \frac{k}{2} + 1$ and $j = \frac{k}{2}$ the subword $\tilde{\sigma}$ can only be over $\{1, \ldots, k\}$: this proves the first recurrence of (iii). The last case $j = \lceil \frac{k}{2} \rceil - 1$ holds if and only if $i = \lceil \frac{k}{2} \rceil$ (with n even) or $i = \frac{k}{2} + 1$ (with both n and k even), otherwise a 123-pattern occurs. For the same reason, there cannot be any letter equal to i or its conjugate in the subword $\tilde{\sigma}$, hence the only $\{1, \ldots, k\} \setminus \{\sigma_1, \sigma_n\}$ alphabet is allowed for $\tilde{\sigma}$. This yields recurrences (iv) and (v). □

Theorem (5.1) and relation (5), for $n \geq 4$, $k \geq 3$ (n and k not odd and even, respectively) and for every i such that $\lceil \frac{k}{2} \rceil \leq i \leq k$, yield the following recurrence relation for $cs^{123}_{n,k,i}$:

$$\begin{aligned} cs^{123}_{n,k,i} &= cs^{123}_{n-2,k,\lceil \frac{k}{2} \rceil} + cs^{123}_{n-2,k-2,\lceil \frac{k}{2} \rceil -1} + cs^{123}_{n-2,k,i} + \\ &+ \delta_{\lceil \frac{k}{2} \rceil +1 \leq i \leq k-1} (cs^{123}_{n-2,k,i} + cs^{123}_{n-2,k-2,i-1}) + \\ &+ \delta_{i \geq \lceil \frac{k}{2} \rceil +2} \sum_{j=\lceil \frac{k}{2} \rceil +1}^{i-1} (cs^{123}_{n-2,k,i} + cs^{123}_{n-2,k-2,i-1}) + \delta_{i=\lceil \frac{k}{2} \rceil} \varphi(n,k), \end{aligned} \tag{6}$$

where

$$\delta_A = \begin{cases} 1 & \text{if } A \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi(n,k) = \begin{cases} cs^{123}_{n-2,k-2,\frac{k}{2}-1} & \text{if } n \text{ and } k \text{ are even} \\ 0 & \text{if } n \text{ is even and } k \text{ is odd} \\ cs^{123}_{n-2,k-1,\frac{k-1}{2}} & \text{if } n \text{ and } k \text{ are odd} \end{cases} .$$

Now, the base cases of the recurrence are needed. First of all, we set

$$cs^{123}_{0,0,0} = 1, \quad cs^{123}_{n,0,0} = 0 \quad \text{and} \quad cs^{123}_{0,k,i} = 0 \tag{7}$$

for every $n, k \geq 1$ and for every i such that $1 \leq i \leq k$. If $k = 1$ or $k = 2$ it is straightforward that

$$cs^{123}_{n,1,1} = 1 \qquad \forall\, n \geq 1, \tag{8}$$

$$cs^{123}_{n,2,1} = cs^{123}_{n,2,2} = \begin{cases} 2^{\frac{n}{2}-1} & \text{if } n \geq 2,\ n \text{ even} \\ 0 & \text{if } n \geq 1,\ n \text{ odd.} \end{cases} \tag{9}$$

Moreover, we recall that, for every i, $cs^{123}_{n,k,i} = 0$ if $k > n$. Hence, the only remaining base cases necessary for the recurrence are

$$cs^{123}_{3,3,1} = cs^{123}_{3,3,2} = 0 \quad \text{and} \quad cs^{123}_{3,3,3} = 1.$$

Now we are ready to use the scanning-element algorithm. We will show its application only for the even-even case, since the remaining two cases can be treated analogously. First of all, we consider

$$P^{123}_{ee}(x,y,v) = \sum_{\substack{n \geq 0 \\ n \text{ even}}} \sum_{\substack{k \geq 0 \\ k \text{ even}}} cs^{123}_{n,k,\frac{k}{2}}\, x^n y^k v^{\frac{k}{2}}.$$

Recurrence (6), for every $n, k \geq 4$ and for $i = \frac{k}{2}$, yields

$$cs^{123}_{n,k,\frac{k}{2}} = 2\, cs^{123}_{n-2,k,\frac{k}{2}} + 2\, cs^{123}_{n-2,k-2,\frac{k-2}{2}}.$$

Hence, multiplying by $x^n y^k v^{\frac{k}{2}}$, summing over all even $n, k \geq 4$ and solving the resulting equation we get

$$P^{123}_{ee}(x,y,v) = \frac{1-2x^2-x^2y^2v}{1-2x^2-2x^2y^2v}. \tag{10}$$

Now, define

$$CS^{123}_{n,k}(v) = \sum_{i=\frac{k}{2}}^{k} cs^{123}_{n,k,i}\, v^i$$

and

$$CS^{123}_{ee}(x,y,v) = \sum_{\substack{n\geq 0\\ n \text{ even}}} \sum_{\substack{k\geq 0\\ k \text{ even}}} \sum_{i=\frac{k}{2}}^{k} cs^{123}_{n,k,i}\, x^n y^k v^i = \sum_{\substack{n\geq 0\\ n \text{ even}}} \sum_{\substack{k\geq 0\\ k \text{ even}}} CS^{123}_{n,k}(v)\, x^n y^k.$$

Multiplying recurrence (6) by v^i and summing over all i from $\frac{k}{2}$ to k we obtain

$$\begin{aligned} CS^{123}_{n,k}(v) &= \frac{2-v}{1-v}\, CS^{123}_{n-2,k}(v) - \frac{v^{k+1}}{1-v}\, CS^{123}_{n-2,k}(1) + \frac{v}{1-v}\, CS^{123}_{n-2,k-2}(v) \\ &\quad -\frac{v^{k+1}}{1-v}\, CS^{123}_{n-2,k-2}(1) + cs^{123}_{n-2,k-2,\frac{k-2}{2}}\, v^{\frac{k}{2}} - cs^{123}_{n-2,k,k}\, v^k. \end{aligned} \tag{11}$$

Observe that

$$cs^{123}_{n-2,k,k} = \sum_{i=\frac{k}{2}}^{k} cs^{123}_{n-4,k,i} + \sum_{i=\frac{k}{2}}^{k} cs^{123}_{n-4,k-2,i} = CS^{123}_{n-4,k}(1) + CS^{123}_{n-4,k-2}(1). \tag{12}$$

Substituting (12) in (11), multiplying (11) by $x^n \left(\frac{y}{v}\right)^k$ and summing over all possible even $n, k \geq 4$ we get

$$\begin{aligned} &\left(1 - \frac{2-v}{1-v}\, x^2 - \frac{x^2y^2}{v(1-v)}\right) CS^{123}_{ee}\left(x, \frac{y}{v}, v\right) = \\ &\quad = \left(-\frac{vx^2}{1-v} - \frac{vx^2y^2}{1-v} - x^4 - x^4y^2\right) CS^{123}_{ee}(x,y,1) + \frac{x^2y^2}{v}\, P^{123}_{ee}\left(x, \frac{y}{v}, v\right) + (x^2-1)^2, \end{aligned} \tag{13}$$

where

$$P^{123}_{ee}\left(x, \frac{y}{v}, v\right) \overset{(10)}{=} \frac{v-2x^2v-x^2y^2}{v-2x^2v-2x^2y^2}$$

and obviously

$$CS^{123}_{ee}(x,y,1) = CS^{123}_{ee}(x,y).$$

Now, we apply the *kernel method* (for further details and examples, see [2] and [12]). The coefficient of $CS_{ee}^{123}\left(x,\frac{y}{v},v\right)$ in (13) vanishes for

$$v_{\pm}(x,y)=\frac{1-2x^2\pm\sqrt{\Delta(x,y)}}{2(1-x^2)},$$

where

$$\Delta(x,y)=1-4x^2(1-x^2)(1+y^2).$$

Hence, substituting $v=v_-(x,y)$ in (13) and solving the resulting equation we finally obtain that

$$CS_{ee}^{123}(x,y)=\frac{y^2+\sqrt{\Delta(x,y)}}{(1+y^2)\sqrt{\Delta(x,y)}}. \tag{14}$$

The same method, used for the even-odd and the odd-odd case, yields

$$CS_{eo}^{123}(x,y)=\frac{y(1-2x^2)(1-\sqrt{\Delta(x,y)})}{2(1-x^2)(1+y^2)\sqrt{\Delta(x,y)}} \tag{15}$$

and

$$CS_{oo}^{123}(x,y)=\frac{y(1-\sqrt{\Delta(x,y)})}{2x(1-x^2)(1+y^2)}. \tag{16}$$

The desired generating function $C^{123}(x,y)$ is now immediately obtained by (4):

$$C^{123}(x,y)=1-\frac{y}{1-y^2}+\frac{y}{2x(1-x)}-\frac{y(1-2x-y)\sqrt{1+y-2x^2}}{2x(1-x)\sqrt{1-y^2}\sqrt{1-y-2x^2}}.$$

Moreover, evaluating the coefficients of the series expansions of the generating functions (14), (15) and (16), we get the following explicit formulae for the number of surjective centrosymmetric words avoiding 123:

$$\begin{aligned}
cs_{2m,2h}^{123} &= \sum_{i=0}^{m}(-1)^{m-i}\binom{2i}{i}\binom{i}{m-i}\binom{i-1}{h-1}, \quad m\geq 1,\ h\geq 1,\\
cs_{2m,2h+1}^{123} &= \sum_{i=1}^{m}(-1)^{m-i}\binom{2i-1}{m-1}\binom{m}{i}\binom{i-1}{h}, \quad m\geq 1,\ h\geq 0,\\
cs_{2m+1,2h+1}^{123} &= \sum_{i=0}^{m}(-1)^{m-i}\frac{1}{i+1}\binom{2i}{i}\binom{i}{m-i}\binom{i}{h}, \quad m\geq 0,\ h\geq 0.
\end{aligned} \tag{17}$$

As already observed, the set $\mathcal{S}_{n,n}$ corresponds to the set S_n of permutations of length n. We recall that, as Egge already proved in [6] (Theorems 2.12 and 2.17), centrosymmetric permutations avoiding 123 are counted by the central binomial coefficients and the Catalan numbers, according to the parity of their length (for a bijective proof, see also [1]). We submit that this result can be reobtained from the preceding formulae by setting $m=h$ in the first and last of (17).

Now, using relations (2) we finally obtain the formulae for the number of centrosymmetric words avoiding 123:

$$\begin{aligned} c^{123}_{2m,2h} &= \sum_{i=0}^{m} (-1)^{m-i} \binom{2i}{i}\binom{i}{m-i}\binom{i+h-1}{h-1}, && m \geq 1,\ h \geq 1, \\ c^{123}_{2m,2h+1} &= \sum_{i=1}^{m} (-1)^{m-i} \frac{m+2h}{2i}\binom{2i}{i}\binom{i}{m-i}\binom{i+h-1}{h}, && m \geq 1,\ h \geq 0, \\ c^{123}_{2m+1,2h+1} &= \sum_{i=0}^{m} (-1)^{m-i} \frac{1}{i+1}\binom{2i}{i}\binom{i}{m-i}\binom{i+h}{h}, && m \geq 0,\ h \geq 0. \end{aligned} \tag{18}$$

6 The pattern 132

Theorem 6.1 *A word $\sigma \in \mathcal{W}_{n,k}$, with $n, k \geq 6$, n and k not odd and even respectively, is a centrosymmetric surjective word that avoids 132 if and only if σ is of one of the following types:*

(i) $\sigma = k\alpha 1$, where α is a centrosymmetric surjective word of length $n-2$, that avoids 132, either over the alphabet $\{2, \ldots, k-1\}$, or $\{1, \ldots, k\}$;

(ii) $\sigma = \beta\alpha\beta'$, where

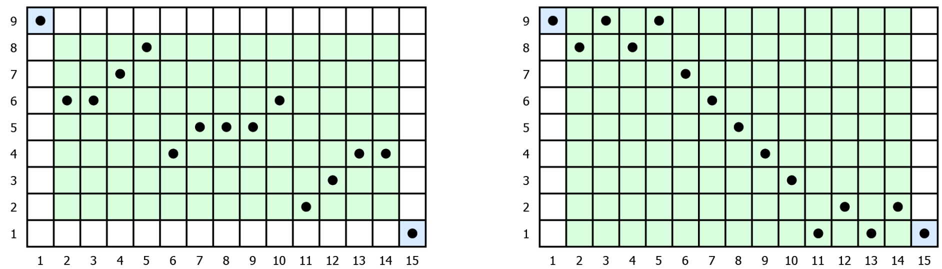
- *(a) $\beta = \beta_1 \ldots \beta_j$ is a binary word over the alphabet $\{k-1, k\}$, with $\beta_1 = k-1$ and $\beta_j = k$, such that it is the longest prefix of σ of this kind;*
- *(b) α is a (possibly empty) centrosymmetric surjective word, that avoids 132, either over the alphabet $\{3, \ldots, k-2\}$, or $\{2, \ldots, k-1\}$;*
- *(c) $\beta' = \rho(\beta)^{rc}$;*

(iii) $\sigma = \gamma\alpha\gamma'$, where

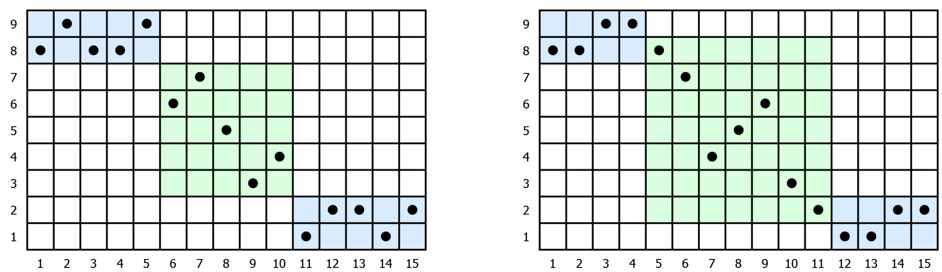
- *(a) $\gamma = \gamma_1\gamma_2 \ldots \gamma_j$ is a surjective weakly increasing word over the alphabet $\{\gamma_1, \ldots, \gamma_j\}$, with $\lfloor \frac{k}{2} \rfloor + 1 \leq \gamma_1 \leq k-2$ and $\gamma_j = k$, such that it is the longest prefix of σ of this kind;*
- *(b) α is a (possibly empty) centrosymmetric surjective word, that avoids 132, either over the alphabet $\{k+2-\gamma_1, \ldots, \gamma_1-1\}$ (if $\gamma_1 > \lfloor \frac{k}{2} \rfloor + 1$), or $\{k+1-\gamma_1, \ldots, \gamma_1\}$;*
- *(c) $\gamma' = \rho(\gamma)^{rc}$;*

(iv) σ is a weakly increasing centrosymmetric surjective word over the alphabet $\{1, \ldots, k\}$.

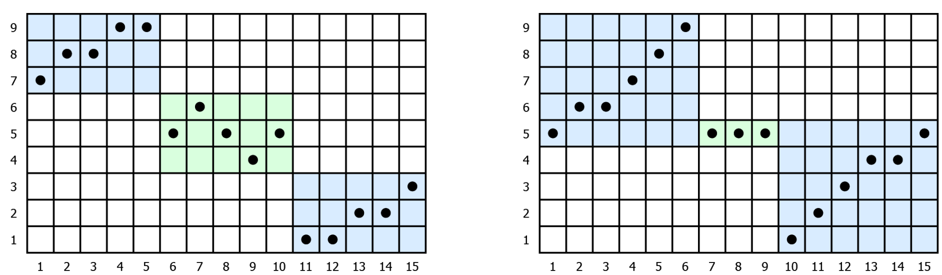
Proof. It is easily checked that a word σ either of type (i), (ii), (iii) or (iv) is a centrosymmetric surjective word that avoids 132. To prove the converse, let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ be a centrosymmetric surjective word over the alphabet $\{1, \ldots, k\}$ and avoiding 132. First



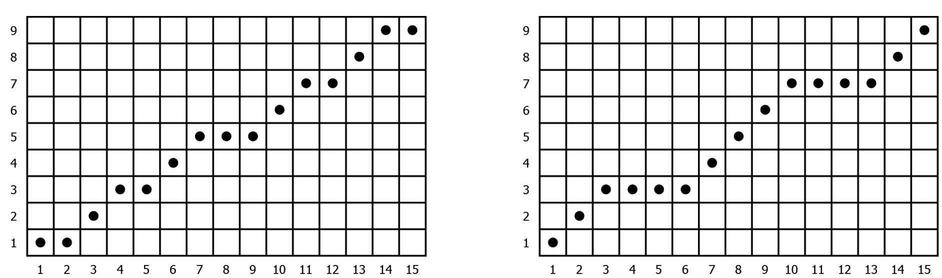
(a) *Type (i)*



(b) *Type (ii)*



(c) *Type (iii)*



(d) *Type (iv)*

Figure 5. *Some examples of the four types of words descripted in Theorem (6.1).*

of all, observe that we can't have $2 \leq \sigma_1 \leq \lfloor \frac{k}{2} \rfloor$, otherwise $\sigma_n = k + 1 - \sigma_1$ and hence the sequence $\sigma_1 k \sigma_n$ would be an occurrence of 132. Therefore, we analyse the remaining cases for σ_1.

(*i*) If $\sigma_1 = k$ then, by centrosymmetry, we have $\sigma_n = 1$. Hence, the subword $\sigma_2\sigma_3 \ldots \sigma_{n-1}$ is a surjective word either over the alphabet $\{2, \ldots, k-1\}$ or $\{1, \ldots, k\}$ that avoids 132.

(*ii*) If $\sigma_1 = k - 1$, let j be the rightmost position such that $\sigma_j = k$. Then, the prefix $\beta = \sigma_1 \ldots \sigma_j$ contains only the letters $k-1$ and k: otherwise, it would contain an occurrence of the pattern 213, and hence, by centrosymmetry, σ would contain the pattern 132. It is immediately checked that $j \leq \lfloor \frac{n}{2} \rfloor$ and that $\sigma = \beta\alpha\beta'$, where $\beta' = \rho(\beta)^{rc}$ and, if $j \neq \frac{n}{2}$, α is a centrosymmetric surjective word either over the alphabet $\{3, \ldots, k-2\}$, or $\{2, \ldots, k-1\}$. If $j = \frac{n}{2}$, α is necessarily the empty word.

(*iii*) If $\lfloor \frac{k}{2} \rfloor + 1 \leq \sigma_1 \leq k - 2$, then σ_1 is necessarily the first letter of a weakly increasing subword $\gamma = \gamma_1\gamma_2 \ldots \gamma_j$, where $j \leq \lfloor \frac{n}{2} \rfloor$ is the rightmost position such that $\gamma_j = k$. If not, γ would contain a 21 pattern that yields a 213 by the surjectivity of σ. Moreover, in order to avoid both 132 and 213 (see relation (1)), we have $\sigma_i \leq \gamma_1$ for every $i > j$. Hence, $\sigma = \gamma\alpha\gamma'$, where $\gamma' = \rho(\gamma)^{rc}$ and, if $j \neq \frac{n}{2}$, α is a centrosymmetric surjective word, that avoids 132, either over the alphabet $\{k + 2 - \gamma_1, \ldots, \gamma_1 - 1\}$ (if $k + 2 - \gamma_1 \leq \gamma_1 - 1$, i.e. $\gamma_1 > \lfloor \frac{k}{2} \rfloor + 1$), or $\{k + 1 - \gamma_1, \ldots, \gamma_1\}$. As in the previous case, if $j = \frac{n}{2}$, α is necessarily the empty word.

(*iv*) If $\sigma_1 = 1$ then $\sigma_n = k$, so, in order to avoid both 132 and 213, the subword $\sigma_2 \ldots \sigma_{n-1}$ cannot contain any 21 pattern. Therefore, σ is weakly increasing.

□

Theorem (6.1) allows us to determine a recurrence relation for the number $cs^{132}_{n,k}$ of k-ary centrosymmetric surjective words of length n that avoid 132. In order to do this, it is convenient to analyse the first values of n and k. We recall that $cs^{132}_{n,k} = 0$ if n is odd and k is even, and we set

$$cs^{132}_{0,0} = 1, \quad cs^{132}_{n,0} = 0 \quad \text{and} \quad cs^{132}_{0,k} = 0 \qquad \forall n, k \geq 1.$$

- For $k = 1$ or $k = 2$, we have

$$cs^{132}_{n,1} = 1 \qquad \forall\, n \geq 1, \tag{19}$$

$$cs^{132}_{n,2} = \begin{cases} 2^{\frac{n}{2}} & \text{if } n \geq 2,\ n \text{ even} \\ 0 & \text{if } n \geq 1,\ n \text{ odd.} \end{cases} \tag{20}$$

- If $k = 3$ and $n \geq 3$, a word $\sigma \in \mathcal{CS}_{n,3}(132)$ can be either of type (*i*), (*ii*) or (*iv*) of Theorem (6.1).

  – If σ is of type (*i*) then its central subword α belongs either to $\mathcal{CS}_{n-2,3}(132)$, or $\mathcal{CS}_{n-2,1}(132)$; then, there are $cs^{132}_{n-2,3} + 1$ of such words.

– If σ is of type (ii) then the central subword is necessarily $\alpha = 22\dots2$, so we have

$$\sum_{n_\beta=2}^{\lfloor\frac{n}{2}\rfloor} 2^{n_\beta-2} = 2^{\lfloor\frac{n}{2}\rfloor-1} - 1$$

of such words σ.

– It is easily checked that there are $\lceil\frac{n}{2}\rceil - 1$ words σ of type (iv).

Hence, the preceding considerations lead to the recurrence

$$cs^{132}_{n,3} = cs^{132}_{n-2,3} + 2^{\lfloor\frac{n}{2}\rfloor-1} + \left\lceil\frac{n}{2}\right\rceil - 1,$$

which yields

$$cs^{132}_{n,3} = \begin{cases} \binom{\frac{n}{2}}{2} + 2^{\frac{n}{2}} - 2 & \text{if } n \text{ is even} \\ \binom{\frac{n-1}{2}}{2} + 2^{\frac{n-1}{2}} + \dfrac{n-3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

- If $n, k \geq 4$, denote by $cs^{132}_{n,k\,(i)}$, $cs^{132}_{n,k\,(ii)}$, $cs^{132}_{n,k\,(iii)}$ and $cs^{132}_{n,k\,(iv)}$ the number of elements of $\mathcal{CS}_{n,k}(132)$ of type (i), (ii), (iii) and (iv), respectively (see Theorem 6.1). Of course, we have

$$cs^{132}_{n,k} = cs^{132}_{n,k\,(i)} + cs^{132}_{n,k\,(ii)} + cs^{132}_{n,k\,(iii)} + cs^{132}_{n,k\,(iv)}.$$

We deduce a recurrence formula for each one of the sequences $cs^{132}_{n,k\,(\cdot)}$.

(i) It is immediately verified that

$$cs^{132}_{n,k\,(i)} = cs^{132}_{n-2,k} + cs^{132}_{n-2,k-2}.$$

(ii) Fix the length n_β of the subword β. There are $2^{n_\beta-2}$ of such subwords, and each one matches with $cs^{132}_{n-2n_\beta,k-2} + cs^{132}_{n-2n_\beta,k-4}$ surjective subwords α. Hence,

$$cs^{132}_{n,k\,(ii)} = \sum_{n_\beta=2}^{\lfloor\frac{n}{2}\rfloor} 2^{n_\beta-2}\left(cs^{132}_{n-2n_\beta,k-2} + cs^{132}_{n-2n_\beta,k-4}\right).$$

(iii) Of course, $cs^{132}_{n,k\,(iii)} \neq 0$ if and only if $n \geq 6$ and $k \geq 5$. Fix the length n_γ and the number of symbols k_γ of the subword γ. We have $3 \leq n_\gamma \leq \lfloor\frac{n}{2}\rfloor$ and $3 \leq k_\gamma \leq \min\{n_\gamma, \lceil\frac{k}{2}\rceil\}$. There are $\binom{n_\gamma-1}{n_\gamma-k_\gamma}$ of such subwords, and each of them is associated with:

* $cs^{132}_{n-2n_\gamma,k-2k_\gamma+2} + cs^{132}_{n-2n_\gamma,k-2k_\gamma}$ subwords α, if $k \geq 6$ and $k_\gamma \leq \lfloor\frac{k}{2}\rfloor$;
* the only subword $\alpha = \frac{k+1}{2}\frac{k+1}{2}\cdots\frac{k+1}{2}$, if $k \geq 5$ is odd and $k_\gamma = \frac{k+1}{2}$.

Hence, for $n \geq 6$ and $k = 5$, we have $k_\gamma = 3$ and then

$$cs^{132}_{n,5\,(iii)} = \sum_{n_\gamma=3}^{\lfloor \frac{n}{2} \rfloor} \binom{n_\gamma - 1}{n_\gamma - 3} = \binom{\lfloor \frac{n}{2} \rfloor}{3},$$

while, for $n, k \geq 6$,

$$\begin{aligned} cs^{132}_{n,k\,(iii)} &= \sum_{n_\gamma=3}^{\lfloor \frac{n}{2} \rfloor} \sum_{k_\gamma=3}^{\min\{n_\gamma, \lfloor \frac{k}{2} \rfloor\}} \binom{n_\gamma - 1}{n_\gamma - k_\gamma} \left(cs^{132}_{n-2n_\gamma, k-2k_\gamma+2} + cs^{132}_{n-2n_\gamma, k-2k_\gamma} \right) + \\ &\quad + \delta_{(k\,\mathrm{odd})} \sum_{n_\gamma = \lceil \frac{k}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} \binom{n_\gamma - 1}{n_\gamma - \lceil \frac{k}{2} \rceil} = \\ &= \sum_{n_\gamma=3}^{\lfloor \frac{n}{2} \rfloor} \sum_{k_\gamma=3}^{\min\{n_\gamma, \lfloor \frac{k}{2} \rfloor\}} \binom{n_\gamma - 1}{n_\gamma - k_\gamma} \left(cs^{132}_{n-2n_\gamma, k-2k_\gamma+2} + cs^{132}_{n-2n_\gamma, k-2k_\gamma} \right) + \\ &\quad + \delta_{(k\,\mathrm{odd})} \binom{\lfloor \frac{n}{2} \rfloor}{\frac{k+1}{2}}. \end{aligned}$$

(iv) The number of weakly increasing centrosymmetric surjective words is trivially

$$cs^{132}_{n,k\,(iv)} = \binom{\lceil \frac{n}{2} \rceil - 1}{\lceil \frac{k}{2} \rceil - 1}.$$

Hence, for $n, k \geq 4$, we conclude that

$$\begin{aligned} cs^{132}_{n,k} = \; & cs^{132}_{n-2,k} + cs^{132}_{n-2,k-2} + \sum_{n_\beta=2}^{\lfloor \frac{n}{2} \rfloor} 2^{n_\beta - 2} \left(cs^{132}_{n-2n_\beta, k-2} + cs^{132}_{n-2n_\beta, k-4} \right) + \\ & + \delta_{(n \geq 6 \wedge k \geq 6)} \sum_{n_\gamma=3}^{\lfloor \frac{n}{2} \rfloor} \sum_{k_\gamma=3}^{\min\{n_\gamma, \lfloor \frac{k}{2} \rfloor\}} \binom{n_\gamma - 1}{n_\gamma - k_\gamma} \left(cs^{132}_{n-2n_\gamma, k-2k_\gamma+2} + cs^{132}_{n-2n_\gamma, k-2k_\gamma} \right) + \\ & + \delta_{(k \geq 5 \wedge k\mathrm{odd})} \binom{\lfloor \frac{n}{2} \rfloor}{\frac{k+1}{2}} + \binom{\lceil \frac{n}{2} \rceil - 1}{\lceil \frac{k}{2} \rceil - 1}. \end{aligned}$$

Further computations in the special cases $k = 4$ and $k = 5$ yield

$$cs^{132}_{n,4} = \binom{\frac{n}{2}}{2} + n\,2^{\frac{n}{2}-2} - 1 \qquad \text{if } n \text{ is even, } n \geq 2,$$

$$cs^{132}_{n,5} = \begin{cases} n\,2^{\frac{n-4}{2}} + \dfrac{n^4 + 4n^3 - 100n^2 - 208n + 384}{384} & \text{if } n \text{ is even, } n \geq 2 \\ (n+7)\,2^{\frac{n-5}{2}} + \dfrac{n^4 - 58n^2 - 384n - 327}{384} & \text{if } n \text{ is odd, } n \geq 1. \end{cases}$$

n/k	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	1	2	0	0	0	0	0	0	0
3	0	1	0	2	0	0	0	0	0	0
4	0	1	4	3	4	0	0	0	0	0
5	0	1	0	6	0	4	0	0	0	0
6	0	1	8	9	14	6	8	0	0	0
7	0	1	0	13	0	19	0	8	0	0
8	0	1	16	20	37	28	42	12	16	0
9	0	1	0	25	0	59	0	52	0	16

Table 1. *The first values for $cs^{132}_{n,k}$. The values for $6 \leq k \leq n \leq 9$ have been computed directly.*

Routine calculations yield the following recurrence.

Formula 6.2 *The number $cs^{132}_{n,k}$ of k-ary centrosymmetric surjective words of length n that avoid 132, with $n \geq 10$ and $k \geq 6$, is given by the following recurrence relation:*

$$\begin{aligned} cs^{132}_{n,k} &= 6cs^{132}_{n-2,k} + 2cs^{132}_{n-2,k-2} - 14cs^{132}_{n-4,k} - 9cs^{132}_{n-4,k-2} + 16cs^{132}_{n-6,k} + \\ &\quad 15cs^{132}_{n-6,k-2} + cs^{132}_{n-6,k-4} - 9cs^{132}_{n-8,k} - 11cs^{132}_{n-8,k-2} - 2cs^{132}_{n-8,k-4} + \\ &\quad 2cs^{132}_{n-10,k} + 3cs^{132}_{n-10,k-2} - cs^{132}_{n-10,k-6}. \end{aligned}$$

Multiplying the above equation by $x^n y^k$, summing over all even n and k (with $n \geq 10$ and $k \geq 6$) and solving the resulting equation we obtain that

$$CS^{132}_{ee}(x,y) = \frac{\begin{array}{c} 1 - 6x^2 + 14x^4 - 16x^6 + 9x^8 - 2x^{10} + \\ +x^4y^2 - 3x^6y^2 + 3x^8y^2 - x^{10}y^2 - x^6y^4 + x^8y^4 + x^{10}y^4 \end{array}}{(1 - 2x^2 + x^4 + x^4y^2)(1 - 4x^2 + 5x^4 - 2x^6 - 2x^2y^2 + 4x^4y^2 - x^6y^2 + x^6y^4)}. \tag{21}$$

Similarly,

$$CS^{132}_{eo}(x,y) = \frac{x^2y\,(1 + x^2y^2 - 5x^2 + 9x^4 - 2x^4y^2 - 7x^6 + 2x^8 - x^8y^4 + x^8y^2)}{(1 - 2x^2 + x^4 + x^4y^2)(1 - 4x^2 + 5x^4 - 2x^6 - 2x^2y^2 + 4x^4y^2 - x^6y^2 + x^6y^4)} \tag{22}$$

and

$$CS^{132}_{oo}(x,y) = \frac{xy\,(1 - 3x^2 + 2x^4)}{1 - 4x^2 + 5x^4 - 2x^6 - 2x^2y^2 + 4x^4y^2 - x^6y^2 + x^6y^4}. \tag{23}$$

Even in this case, in the series expansions of (21) and (23) the coefficients of the monomials associated to words of length equal to their alphabet cardinality agree with Egge's result (see [6]). Hence, for every $m \geq 0$ we have

$$cs^{132}_{2m,2m} = 2^m \qquad \text{and} \qquad cs^{132}_{2m+1,2m+1} = 2^m.$$

Using relation (4), we finally obtain that the generating function for the number of centrosymmetric words that avoid 132 is the rational function

$$C^{132}(x,y) = \frac{\alpha(x,y)}{\beta(x,y)},$$

where

$$\begin{aligned}\alpha(x,y) = \; & 1 - 16x^6 - 2x^{10} + 8x^4y^3 - 5x^4y - 17x^6y^3 \\ & +9x^8 - 6x^2 - 41x^4y^2 + 14x^4 + xy + 45x^6y^2 + 18x^2y^2 \\ & -3y^2 - 43x^6y^4 - 24x^8y^2 + 22x^8y^4 + 5x^{10}y^2 - 3x^{10}y^4 - 7x^7y \\ & -18x^2y^4 + 3y^4 + 40x^4y^4 - 8x^5y^7 + 5x^3y^7 - xy^7 + 4x^7y^7 + 3xy^5 + 25x^5y^5 \\ & -15x^3y^5 - 4x^9y^3 + 2x^9y^5 + 15x^3y^3 - 3xy^3 - 15x^7y^5 + 9x^5y + 18x^7y^3 + 2x^9y \\ & -5x^3y - 26x^5y^3 - 13x^4y^6 - 7x^8y^6 + 6x^2y^6 - y^6 - 3x^{10}y^3 - x^2y^3 \\ & +7x^6y^5 + 13x^8y^3 + x^6y^7 - 6x^8y^5 - x^2y^5 - x^4y^5 - 2x^4y^7 \\ & +x^2y^7 + 14x^6y^6 + 9x^6y - 7x^8y + x^2y + 2x^{10}y\end{aligned}$$

and

$$\begin{aligned}\beta(x,y) = \; & (1-y^2)(1-2x^2+x^4+2x^2y^2-y^2) \\ & (1-2x^6+3x^6y^2-6x^4y^2+x^4y^4+5x^4+6x^2y^2-2x^2y^4-4x^2+y^4-2y^2).\end{aligned}$$

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