

Asymptotic probability distributions of some permutation statistics for the wreath product $C_r \wr \mathfrak{S}_n$

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Abstract

We present in this work results on some distributions of permutation statistics of random elements of the wreath product $G_{r,n} = C_r \wr \mathfrak{S}_n$. We consider the distribution of the descent number, the flag major index, the excedance, and the number of fixed points, over the whole group $G_{r,n}$ or over the subclasses of derangements and involutions. We compute the mean, variance and moment generating function, and establish the asymptotic distributions of these statistics.

Subject Class: Primary: 05A15; Secondary: 05A16, 05A30

Keywords: Derangements, flag major index, wreath product, asymptotic normality

1 Introduction

An active area of research in combinatorics is permutations enumeration. The enumeration can be over certain permutation groups or over special subclasses of them. See, for instance, Brenti [8] for a sample of results in the former case, and Brenti [7] and Foata-Zeilberger [20] for the latter case. The enumeration can be done exactly, asymptotically or probabilistically. In the latter approach, one considers a random variable X taking values of a certain statistic of random permutations. The quantities of interest are the statistical parameters of the random variable X .

Recently, there is a resurgence of interest in the probabilistic approach to permutation enumeration. See, e.g., [4, 5, 10, 17, 25] for a sample of results in this direction. Among the mentioned work, Chen and Wang [10] studied the limiting distributions of q -derangement numbers and computed several statistical quantities, namely, the mean, variance and the moment generating function of a random variable taking values the major index of random derangements in the symmetric group \mathfrak{S}_n , and the flag major index of random derangements in the hyperoctahedral group B_n .

More generally, one can consider the wreath product $G_{r,n} := C_r \wr \mathfrak{S}_n$, where $r, n \geq 1$. See Section 2 for definitions of undefined terms. Some well studied classes of groups are particular cases of $G_{r,n}$. For instance, when $r = 1$, $G_{r,n}$ is precisely the symmetric group \mathfrak{S}_n , and when $r = 2$, $G_{r,n}$ is the hyperoctahedral group B_n . Proofs of results for generic $r \geq 1$ will lead to a unified theory, which includes the results on \mathfrak{S}_n and B_n as special cases.

The goal of this work is to study the distribution of certain statistics on $G_{r,n}$, and obtain formulas of statistical parameters and asymptotic information of the distribution. The organization of this paper is as follows. In the next section, we gather some definitions which will be adhered to in the sequel. Starting from Section 3, we consider the probability distribution of descents over $G_{r,n}$, flag major index over $G_{r,n}$, excedances on involutions in $G_{r,n}$, excedance on derangements in $G_{r,n}$, and in the final section, the number of fixed points of elements of $G_{r,n}$.

2 Notations and preliminaries

We collect in this section notations and results that will be needed in the sequel.

Let \mathbb{N} , \mathbb{Z} , \mathbb{Q} and \mathbb{R} denote, as usual, the sets of all nonnegative integers, integers, rational numbers and real numbers, respectively. Let $n \in \mathbb{N}$. Denote by $[n]$ the interval of integers $\{1, 2, \dots, n\}$ (in particular, $[0] = \emptyset$).

If S is a finite set, then its cardinality is denoted by $\#S$.

For $n \in \mathbb{N}$, we define the q -integer $[n]_q := 1 + q + q^2 + \dots + q^{n-1}$ and the q -factorial $[n]_q! := [1]_q [2]_q \dots [n]_q$. It is clear that $[n]_q = n$ and $[n]_q! = n!$ when $q = 1$.

Let r, n be positive integers. Let \mathfrak{S}_n denote the symmetric group on n letters. Any element π of \mathfrak{S}_n is represented as the word $\pi(1)\pi(2)\dots\pi(n)$.

Let $C_r := \mathbb{Z}/r\mathbb{Z}$ be the cyclic group of order r , whose elements are represented by those of $\{0, 1, 2, \dots, r-1\}$. Denote by $G_{r,n} := C_r \wr \mathfrak{S}_n$, where \wr is the wreath product with respect to the usual action of the symmetric group \mathfrak{S}_n by permutations of $[n]$. Elements of $G_{r,n}$ are represented as $\pi \times \mathbf{z}$, where $\pi = \pi(1)\pi(2)\dots\pi(n) \in \mathfrak{S}_n$ and $\mathbf{z} = (z_1, z_2, \dots, z_n)$ is a n -tuple of integers such that $z_i \in C_r$ for $i = 1, 2, \dots, n$. The product of elements $\pi \times \mathbf{z}$ and $\tau \times \mathbf{w}$ of $G_{r,n}$ is defined as: $(\pi \times \mathbf{z}) \cdot (\tau \times \mathbf{w}) = \pi\tau \times (\mathbf{w} + \tau(\mathbf{z}))$, where $\pi\tau = \pi \circ \tau$ is evaluated from right to left, $\tau(\mathbf{z}) = (z_{\tau(1)}, z_{\tau(2)}, \dots, z_{\tau(n)})$ and the addition is coordinatewise modulo r .

It is easy to see that the identity element of $G_{r,n}$ is $12\dots n \times (0, 0, \dots, 0)$, where $12\dots n$ is the identity element of \mathfrak{S}_n .

Let $p = \pi \times \mathbf{z} \in G_{r,n}$. An integer $i \in [n]$ is a *fixed point* of p if $\pi(i) = i$ and $z_i = 0$, and $i \in [n]$ is a *descent* of p if $z_i > z_{i+1}$, or $z_i = z_{i+1}$ and $\pi(i) > \pi(i+1)$, where $z_{n+1} := 0$ and $\pi(n+1) := n+1$. Note that one can consider $p \in G_{r,n}$ as an r -colored permutation, whose letters can be suitably linearly ordered. The choice of this linear order follows that in [18].

An *excedance* in $p = \pi \times \mathbf{z} \in G_{r,n}$ is an integer $i \in [n]$ such that $\pi(i) > i$, or $\pi(i) = i$ and $z_i > 0$. An element $p = \pi \times \mathbf{z}$ of $G_{r,n}$ is a *derangement* if it has no fixed points. Denote by $\mathcal{D}_{r,n}$ the set of derangements in $G_{r,n}$. Denote by $\text{exc}(p)$ the number of excedances of $p \in G_{r,n}$.

The *flag major index* of $p = \pi \times \mathbf{z} \in G_{r,n}$ is defined by

$$\text{fmaj}(p) = r \sum_{i=1}^{n-1} i \chi(i \in D(p)) + \sum_{j=1}^n z_j,$$

where $\chi(P) = 1$ if the statement P is true, and 0 otherwise, and $D(p)$ denotes the descent set of p .

For enumerative and geometric results on $G_{r,n}$ by descents and excedances, see the work of Steingrímsson [27] (where $G_{r,n}$ is instead denoted by S_n^r).

Let S be some set of r -colored permutations and let $\text{stat}: S \rightarrow \mathbb{R}$ be a certain statistic on S . We denote by X_S^{stat} the random variable taking values $\text{stat}(p)$ of random r -colored permutations p from S , and similarly by μ_S^{stat} , $(\sigma_S^{\text{stat}})^2$ and $M_{X_S^{\text{stat}}}(t)$ the mean, variance and moment generating function of X_S^{stat} , respectively.

For $n \in \mathbb{P}$ and $0 \leq k \leq D_n \in \mathbb{N}$, let $b(n, k) \in [0, \infty)$ and $b_n := b(n, 0) + \dots + b(n, D_n) > 0$. We say that the array $\{b(n, k): n \geq 1, 0 \leq k \leq D_n\}$ satisfies a *central limit theorem with mean μ_n and variance σ_n^2* provided

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sum_{k \leq \lfloor (x)_n \rfloor} \frac{b(n, k)}{b_n} - \Phi(x) \right| = 0,$$

where $(x)_n := x\sigma_n + \mu_n$. Equivalently, we say that $\{b(n, k)\}$ is *asymptotically normal*; we also say that the array $\{b(n, k): n \geq 1, 0 \leq k \leq D_n\}$ satisfies a *local limit theorem* on $S \subseteq \mathbb{R}$ if and only if

$$\lim_{n \rightarrow \infty} \sup_{x \in S} \left| \frac{b(n, \lfloor (x)_n \rfloor)}{b_n/\sigma_n} - \phi(x) \right| = 0,$$

where $\Phi(x)$ and $\phi(x)$ are, respectively, the cumulative distribution function and the probability density function of the standard normal distribution $N(0, 1)$, that is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \quad \text{and} \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

where $x \in \mathbb{R}$.

We shall need the following theorem of Canfield [9, Theorem II].

Theorem 2.1 *Suppose that the array $\{b(n, k): n \geq 1, 0 \leq k \leq D_n\}$ of real numbers, which satisfies the central limit theorem, is also log-concave with no internal zeros. If $\sigma_n \rightarrow \infty$ as $n \rightarrow \infty$, then the array $\{b(n, k): n \geq 1, 0 \leq k \leq D_n\}$ satisfies a local limit theorem on $S = \mathbb{R}$.*

3 The distribution of descent numbers

Denote by $D_{r,n}(t) := \sum_{p \in G_{r,n}} t^{\text{des}(p)}$ the generating function of r -colored permutations by descents. The first four members of $D_{r,n}(t)$ are as follows:

$$\begin{aligned} D_{r,1}(t) &= 1 + (r-1)t, \\ D_{r,2}(t) &= 1 + (r^2 + 2r - 2)t + (r-1)^2 t^2, \\ D_{r,3}(t) &= 1 + (r^3 + 3r^2 + 3r - 3)t + (4r^3 - 6r + 3)t^2 + (r-1)^3 t^3, \\ D_{r,4}(t) &= 1 + (r^4 + 4r^3 + 6r^2 + 4r - 4)t + (11r^4 + 12r^3 - 6r^2 - 12r + 6)t^2 \\ &\quad + (11r^4 - 12r^3 - 6r^2 + 12r - 4)t^3 + (r-1)^4 t^4. \end{aligned}$$

It is easy to see that $D_{r,n}(t)$'s are symmetric only when $r = 1, 2$. Steingrímsson [27] had shown that

$$D_r(t, x) := \sum_{n \geq 0} D_{r,n}(t) \frac{x^n}{n!} = \frac{(1-t)e^{x(1-t)}}{1 - te^{x(1-t)}}. \quad (3.1)$$

Theorem 3.1 *The mean $\mu_{G_{r,n}}^{\text{des}}$ and variance $(\sigma_{G_{r,n}}^{\text{des}})^2$ of the random variable $X_{G_{r,n}}^{\text{des}}$ are given by*

$$\mu_{G_{r,n}}^{\text{des}} = \frac{rn + r - 2}{2r} \quad \text{and} \quad (\sigma_{G_{r,n}}^{\text{des}})^2 = \frac{n+1}{12}.$$

PROOF. Replacing in (3.1) x by x/r , we have

$$\sum_{n \geq 0} D_{r,n}(t) \frac{x^n}{r^n n!} = \frac{(1-t)e^{x(1-t)/r}}{1 - te^{x(1-t)/r}}. \quad (3.2)$$

Differentiating (3.2) with respect to t , followed by letting $t \rightarrow 1$, we have

$$\sum_{n \geq 0} \frac{(D_{r,n})'(1)}{r^n n!} x^n = \frac{2(r-1)x + (2-r)x^2}{2r(1-x)^2}.$$

Since $\#G_{r,n} = r^n n!$, we see that the preceding identity is exactly the ordinary generating function of the mean of descent numbers. Expanding the right side as a formal power series in x ,

$$\frac{2(r-1)x + (2-r)x^2}{2r(1-x)^2} = \sum_{n \geq 1} \left(\frac{rn + r - 2}{2r} \right) x^n,$$

we obtain that

$$\mu_{G_{r,n}}^{\text{des}} = \frac{rn + r - 2}{2r}.$$

Differentiating (3.2) twice with respect to t , followed by letting $t \rightarrow 1$, we obtain

$$\sum_{n \geq 0} \frac{D''_{r,n}(1)}{r^n n!} x^n = \frac{(r^2 - 6r + 6)x^4 + (-4r^2 + 18r - 12)x^3 + (6r^2 - 12r + 6)x^2}{6r^2(1-x)^3}.$$

Expanding the right side as a formal power series in x ,

$$\frac{(r^2 - 6r + 6)x^4 + (-4r^2 + 18r - 12)x^3 + (6r^2 - 12r + 6)x^2}{6r^2(1-x)^3} = \sum_{n \geq 2} \frac{(12 - 2r^2) + (r-12)rn + 3r^2 n^2}{12r^2} x^n$$

we have that for $n \geq 2$,

$$\frac{D''_{r,n}(1)}{r^n n!} = \frac{(12 - 2r^2) + (r-12)rn + 3r^2 n^2}{12r^2}.$$

so that

$$(\sigma_{G_{r,n}}^{\text{des}})^2 = \frac{D''_{r,n}(1)}{r^n n!} + \mu_{G_{r,n}}^{\text{des}} - (\mu_{G_{r,n}}^{\text{des}})^2 = \frac{n+1}{12},$$

which is independent of r . □

Since $D_{r,n}(t)/r^n n!$ is the probability generating function of $X_{G_{r,n}}^{\text{des}}$, it follows that the moment generating function (mgf) of $X_{G_{r,n}}^{\text{des}}$ is given by

$$M_{X_{G_{r,n}}^{\text{des}}}(t) = \frac{D_{r,n}(e^t)}{r^n n!}.$$

From (3.2), we see that

$$\sum_{n \geq 0} M_{X_{G_{r,n}}^{\text{des}}}(t) x^n = \frac{(1-e^t)e^{x(1-e^t)/r}}{1-e^t e^{x(1-e^t)}},$$

which is the ordinary generating function of the moment generating functions of $X_{G_{r,n}}^{\text{des}}$.

Steingrímsson [27, Lemma 16] obtained a recurrence relation for the coefficients of $D_{r,n}(t)$ from which the following recurrence relation for $D_{r,n}(t)$ is readily obtained:

$$D_{r,n}(t) = ((rn-1)t+1)D_{r,n-1}(t) + rt(1-t)D'_{r,n-1}(t). \quad (3.3)$$

With this recurrence relation, one readily deduces the following, which unifies and generalizes the types *A* and *B* cases.

Theorem 3.2 *For $r, n \geq 1$, the polynomial $D_{r,n}(t)$ is simply real-rooted and $D_{r,n}(t)$ interlaces $D_{r,n+1}(t)$.*

PROOF. Since the case $r=1$ is well known [15, p. 292, Exercise 3], we only consider the case that $r \geq 2$. (For $r=2$, a different proof of the real-rootedness has been given by Brenti [8].) The proof proceeds by induction on n . It is clear that $D_{r,1}(t) = 1 + (r-1)t$ is simply real-rooted. Since $D_{r,2}(t) = 1 + (r^2 + 2r - 2)t + (r-1)^2 t^2$ has discriminant

$$(r^2 + 2r - 2)^2 - 4(r-1)^2 = r^4 + 4r^2(r-1) > 0,$$

$D_{r,2}(t) = 0$ has two real zeros. Since $D_{r,2}(-1/(r-1)) = -r^2/(r-1) < 0$, the only zero of $D_{r,1}(t)$ lies in between those of $D_{r,2}(t)$, i.e., $D_{r,1}(t)$ interlaces $D_{r,2}(t)$. Thus, the case $n=1$ holds.

Assume that the result holds for $n \geq 1$ and let $t_{r,n,1} < t_{r,n,2} < \dots < t_{r,n,n} < 0$ be the zeros of $D_{r,n}(t)$. Let also $t_{r,n,0} := -\infty$ and $t_{r,n,n+1} := 0$. Setting $t = t_{r,n,i}$ in the recurrence relation (3.3) with $n+1$ in place of n , we have

$$D_{r,n+1}(t_{r,n,i}) = rt_{r,n,i}(1-t_{r,n,i})D'_{r,n}(t_{r,n,i}).$$

By the induction hypothesis, $D_{r,n}(t)$ is simply real-rooted so that $D'_{r,n}(t)$ interlaces $D_{r,n}(t)$. By a standard argument, we have $\text{sgn } D'_{r,n}(t_{r,n,i}) = (-1)^{n-i}$ so that

$$\text{sgn } D_{r,n+1}(t_{r,n,i}) = -\text{sgn } D'_{r,n}(t_{r,n,i}) = (-1)^{n+1-i}, \quad i = 1, 2, \dots, n.$$

Since $\deg D_{r,n+1}(t) = n+1$ and all coefficients of $D_{r,n+1}(t)$ are positive,

$$\text{sgn } D_{r,n+1}(t_{r,n,0}) = (-1)^{n+1} \text{ and } \text{sgn } D_{r,n+1}(t_{r,n,n+1}) = 1.$$

By the intermediate-value theorem, there exist $t_{r,n+1,i} \in (t_{r,n,i-1}, t_{r,n,i})$ for which $D_{r,n+1}(t_{r,n+1,i}) = 0$ for $i = 1, 2, \dots, n+1$. This proves that $D_{r,n+1}(t)$ is simply real-rooted and that $D_{r,n}(t)$ interlaces $D_{r,n+1}(t)$. \square

The next corollary is an immediate consequence of the Aissen-Schoenberg-Whitney Theorem [6, Theorem 2.2.4].

Corollary 3.3 *For $r, n \geq 1$, the sequence of coefficients of $D_{r,n}(t)$ is unimodal, log-concave and has no internal zeros.*

If the probability generating function is real-rooted, then it is called a *PF* distribution. According to Pitman [24, p. 286], any *PF* distribution on $\{0, 1, \dots, n\}$ with mean μ_n and variance σ_n^2 is asymptotically normal if and only if $\sigma_n \rightarrow \infty$ so that the proof of the next theorem is immediate.

Theorem 3.4 *Let $X_{G_{r,n}}^{\text{des}}$ be the random variable taking values $\text{des}(\pi)$ of random elements π in $G_{r,n}$. Then the standardized random variable $Z_{G_{r,n}}^{\text{des}} := (X_{G_{r,n}}^{\text{des}} - \mu_{G_{r,n}}^{\text{des}})/\sigma_{G_{r,n}}^{\text{des}}$ converges to a random variable having the standard normal distribution $N(0, 1)$ as $n \rightarrow +\infty$.*

By Corollary 3.3 and Theorem 3.4, the next theorem is immediate from Theorem 2.1.

Theorem 3.5 *The sequence of coefficients $\{D_{r,n,0}, D_{r,n,1}, \dots, D_{r,n,n}\}$ of $D_{r,n}(t)$ satisfies a local limit theorem.*

Since des and exc are equidistributed over $G_{r,n}$ (see, e.g., [27]), the above central and local limit theorems also hold for the statistic exc over the whole group $G_{r,n}$.

4 The distribution of flag major indices

We consider in this section the probability distribution of flag major index on $G_{r,n}$.

Denote by $\text{fmaj}(p)$ the flag major index of $p \in G_{r,n}$. It is well known that the generating function of $G_{r,n}$ by the flag major index is equal to

$$P_{r,n}(q) := \sum_{p \in G_{r,n}} q^{\text{fmaj}(p)} = [r]_q [2r]_q \cdots [nr]_q,$$

which is precisely the Poincaré series of $G_{r,n}$ [21, Theorem 1.4].

With this explicit formula, one can imitate those proofs in [10] to obtain analogous results for the flag major index over the whole group $G_{r,n}$.

It is a routine, albeit tedious, exercise to compute the following sums:

$$\begin{aligned} \sum_{1 \leq i \neq j \leq n} \frac{(ir-1)(jr-1)}{4} &= \left[\frac{r}{2} \binom{n}{2} + \frac{(r-1)n}{2} \right]^2 - \frac{r^2(n-1)n(2n-1)}{24} - \binom{r}{2} \binom{n}{2} - \frac{(r-1)^2 n}{4}, \\ \sum_{i=1}^n \frac{(ir-2)(ir-1)}{3} &= \frac{r^2(n-1)n(2n-1)}{18} + \frac{r(2r-3)}{3} \binom{n}{2} + \frac{(r-2)(r-1)n}{3}. \end{aligned} \quad (4.1)$$

Theorem 4.1 *The mean $\mu_{G_{r,n}}^{\text{fmaj}}$ and variance $(\sigma_{G_{r,n}}^{\text{fmaj}})^2$ of the random variable $X_{G_{r,n}}^{\text{fmaj}}$ are given by*

$$\mu_{G_{r,n}}^{\text{fmaj}} = \frac{n(rn+r-2)}{4}, \quad \text{and} \quad (\sigma_{G_{r,n}}^{\text{fmaj}})^2 = \frac{2r^2n^3 + 3r^2n^2 + (r^2-6)n}{72}.$$

PROOF. Since

$$P'_{r,n}(q) = \sum_{i=1}^n [r]_q \cdots [(i-1)r]_q (1+2q+\cdots+(ir-1)q^{ir-2}) [(i+1)r]_q \cdots [nr]_q,$$

we have

$$\begin{aligned} P'_{r,n}(1) &= \sum_{i=1}^n r(2r) \cdots (i-1)r \left(\frac{(ir-1)ir}{2} \right) (i+1)r \cdots nr \\ &= r^n n! \sum_{i=1}^n \frac{ir-1}{2} \\ &= r^n n! \left(\frac{r}{2} \binom{n}{2} + \frac{(r-1)n}{2} \right) \end{aligned}$$

so that

$$\mu_{G_{r,n}}^{\text{fmaj}} = \frac{P'_{r,n}(1)}{r^n n!} = \frac{r}{2} \binom{n}{2} + \frac{(r-1)n}{2} = \frac{n(rn+r-2)}{4}.$$

Differentiating $P_{r,n}(q)$ twice with respect to q , we have

$$\begin{aligned} P''_{r,n}(q) &= 2 \sum_{1 \leq i < j \leq n} [r]_q \cdots [(i-1)r]_q (1+2q+\cdots+(ir-1)q^{ir-2}) \\ &\quad \times [(i+1)r]_q \cdots [(j-1)r]_q (1+2q+\cdots+(jr-1)q^{jr-2}) \\ &\quad \times [(j+1)r]_q \cdots [nr]_q \\ &\quad + \sum_{i=1}^n [r]_q \cdots [(i-1)r]_q (2 \cdot 1 + 3 \cdot 2q + \cdots + (ir-1)(ir-2)q^{ir-3}) \\ &\quad \times [(i+1)r]_q \cdots [nr]_q \end{aligned}$$

so that

$$\begin{aligned} P''_{r,n}(1) &= 2 \sum_{1 \leq i < j \leq n} r(2r) \cdots (i-1)r \left(\frac{(ir-1)ir}{2} \right) \times (i+1)r \cdots (j-1)r \left(\frac{(jr-1)jr}{2} \right) (j+1)r \cdots (nr) \\ &\quad + \sum_{i=1}^n r(2r) \cdots (i-1)r \left(\frac{(ir-2)(ir-1)ir}{3} \right) (i+1)r \cdots (nr) \\ &= r^n n! \left(\sum_{1 \leq i < j \leq n} \frac{(ir-1)(jr-1)}{4} + \sum_{i=1}^n \frac{(ir-2)(ir-1)}{3} \right). \end{aligned}$$

Using (4.1) and simplifying, we obtain

$$\frac{P''_{r,n}(1)}{r^n n!} = \frac{9r^2 n^4 + (22r^2 - 36r)n^3 + (15r^2 - 72r + 36)n^2 + (2r^2 - 36r + 60)n}{144}.$$

Finally, we have

$$(\sigma_{G_{r,n}}^{\text{fmaj}})^2 = \frac{P''_{r,n}(1)}{r^n n!} + \mu_{G_{r,n}}^{\text{fmaj}} - (\mu_{G_{r,n}}^{\text{fmaj}})^2 = \frac{2r^2 n^3 + 3r^2 n^2 + (r^2 - 6)n}{72}.$$

□

The moment generating function is given by

$$M_{X_{G_{r,n}}^{\text{fmaj}}}(t) = \frac{P_{r,n}(e^t)}{r^n n!} = \frac{[r]_{e^t} [2r]_{e^t} \cdots [nr]_{e^t}}{r^n n!}. \quad (4.2)$$

Theorem 4.2 *The moment generating function is equal to*

$$M_{X_{G_{r,n}}^{\text{fmaj}}}(t) = \exp\left(\frac{tn(rn + (r-2))}{4} + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k} t^{2k}}{(2k)(2k)!} \sum_{j=1}^n ((rj)^{2k} - 1)\right), \quad (4.3)$$

where \mathcal{B}_n is the n -th Bernoulli number.

PROOF. Using the identity

$$1 - e^{-t} = t \exp\left(\sum_{k=1}^{\infty} \frac{\mathcal{B}_k t^k}{k(k!)}\right),$$

we have for any $j \geq 1$,

$$1 - e^{-rjt} = -rjt \exp\left(\sum_{k=1}^{\infty} \frac{\mathcal{B}_k (-rjt)^k}{k(k!)}\right) = -rjt \exp\left(\frac{rjt}{2} + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k} (rjt)^{2k}}{(2k)(2k)!}\right).$$

since $\mathcal{B}_1 = -\frac{1}{2}$ and $\mathcal{B}_{2k+1} = 0$ for $k \geq 1$. Thus,

$$[rj]_{e^t} = \frac{1 - e^{-rjt}}{1 - e^{-t}} = rj \exp\left(\frac{(rj-1)t}{2} + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k} t^{2k}}{(2k)(2k)!} ((rj)^{2k} - 1)\right),$$

and from which

$$\begin{aligned} [r]_{e^t} [2r]_{e^t} \cdots [nr]_{e^t} &= \prod_{j=1}^n rj \exp\left(\frac{(rj-1)t}{2} + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k} t^{2k}}{(2k)(2k)!} ((rj)^{2k} - 1)\right) \\ &= r^n n! \exp\left(\frac{tn(rn + (r-2))}{4} + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k} t^{2k}}{(2k)(2k)!} \sum_{j=1}^n ((rj)^{2k} - 1)\right) \end{aligned}$$

and (4.3) follows. □

Theorem 4.3 *Let $X_{G_{r,n}}^{\text{fmaj}}$ be the random variable taking values $\text{fmaj}(\pi)$ of random elements π in $G_{r,n}$. Then the standardized random variable $Z_{G_{r,n}}^{\text{fmaj}} := (X_{G_{r,n}}^{\text{fmaj}} - \mu_{G_{r,n}}^{\text{fmaj}})/\sigma_{G_{r,n}}^{\text{fmaj}}$ converges to a random variable having the standard normal distribution $N(0, 1)$ as $n \rightarrow +\infty$.*

PROOF. Here, the mgf of $Z_{G_{r,n}}^{\text{fmaj}}$ is given by

$$\begin{aligned} M_{Z_{G_{r,n}}^{\text{fmaj}}}(t) &= e^{-t\mu_{G_{r,n}}^{\text{fmaj}}/\sigma_{G_{r,n}}^{\text{fmaj}}} M_{X_{G_{r,n}}^{\text{fmaj}}}\left(\frac{t}{\sigma_{G_{r,n}}^{\text{fmaj}}}\right) \\ &= \exp\left(-\frac{t\mu_{G_{r,n}}^{\text{fmaj}}}{\sigma_{G_{r,n}}^{\text{fmaj}}} + \frac{tn(rn + (r-2))}{4\sigma_{G_{r,n}}^{\text{fmaj}}} + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k} t^{2k}}{(2k)(2k)! (\sigma_{G_{r,n}}^{\text{fmaj}})^{2k}} \sum_{j=1}^n ((rj)^{2k} - 1)\right). \end{aligned}$$

By Theorem 4.1, we have

$$\lim_{n \rightarrow \infty} \left(-\frac{t\mu_{G_{r,n}}^{\text{fmaj}}}{\sigma_{G_{r,n}}^{\text{fmaj}}} + \frac{tn(rn + (r-2))}{4\sigma_{G_{r,n}}^{\text{fmaj}}}\right) = \lim_{n \rightarrow \infty} \frac{t}{\sigma_{G_{r,n}}^{\text{fmaj}}} \left(\frac{n(rn + r - 2)}{4} - \mu_{G_{r,n}}^{\text{fmaj}}\right) = 0. \quad (4.4)$$

Since $\sigma_{G_{r,n}}^{\text{fmaj}} \sim rn^{3/2}/6$ and $\sum_{j=1}^n ((rj)^2 - 1) \sim r^2n^3/3$ as $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{(\sigma_{G_{r,n}}^{\text{fmaj}})^2} \sum_{j=1}^n ((rj)^2 - 1) = \lim_{n \rightarrow \infty} \frac{r^2n^3/3}{r^2n^3/36} = 12.$$

By adapting the proof of [10, Lemma 3.4] (i.e., choosing $\alpha > r$ and $\beta > 36/r^2$ such that $r(n+1) < \alpha n$ and $(\sigma_{G_{r,n}}^{\text{fmaj}})^2 - n^3/\beta > 0$ for all $n > N$) and using the fact that $\mathcal{B}_2 = \frac{1}{6}$, we have that

$$\lim_{n \rightarrow \infty} \sum_{k=2}^{\infty} \frac{\mathcal{B}_{2k} t^{2k}}{(2k)(2k)! (\sigma_{G_{r,n}}^{\text{fmaj}})^2} \sum_{j=1}^n ((rj)^{2k} - 1) = 0$$

so that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{\mathcal{B}_{2k} t^{2k}}{(2k)(2k)! (\sigma_{G_{r,n}}^{\text{fmaj}})^{2k}} \sum_{j=1}^n ((rj)^{2k} - 1) = \lim_{n \rightarrow \infty} \frac{\mathcal{B}_2 t^2}{2(2)! (\sigma_{G_{r,n}}^{\text{fmaj}})^2} \sum_{j=1}^n ((rj)^2 - 1) = \frac{t^2}{2}. \quad (4.5)$$

Combining (4.4) and (4.5), we obtain

$$\lim_{n \rightarrow \infty} M_{Z_{G_{r,n}}^{\text{fmaj}}}(t) = e^{t^2/2},$$

which is the mgf of the standard normal distribution. This finishes the proof. \square

5 The distribution of excedance on involutions

Denote by $\mathcal{I}_n^r = \{p \in G_{r,n} : p^2 = e\}$ the set of all involutions in $G_{r,n}$ and $i_n^r = \#\mathcal{I}_n^r$. Let $i_n^r(t) := \sum_{p \in \mathcal{I}_n^r} t^{\text{exc}(p)}$ and $i_r(x, t) := \sum_{n \geq 0} i_n^r(t) x^n / n!$, where $i_0^r(t) := 1$. It is clear that $i_n^r(1) = i_n^r$ so that $i_r(x, 1) = \sum_{n \geq 0} i_n^r x^n / n!$. In [13], Chow and Mansour have proved that

$$i^r(x, t) = \begin{cases} e^{x+rtx^2/2} & \text{if } r \text{ is odd,} \\ e^{(t+1)x+rtx^2/2} & \text{if } r \text{ is even.} \end{cases}$$

In particular, we have

$$i^r(x, 1) = \begin{cases} e^{x+rx^2/2} & \text{if } r \text{ is odd,} \\ e^{2x+rx^2/2} & \text{if } r \text{ is even.} \end{cases} \quad (5.1)$$

Lemma 5.1 *Let $r \geq 1$. We have*

$$i_n^r \sim \begin{cases} \frac{e^{\sqrt{n/r}-1/4r}}{\sqrt{2}} \left(\frac{rn}{e}\right)^{n/2} \left(1 + \frac{1+6r}{24r^{3/2}n^{1/2}} + O(n^{-1})\right) & \text{if } r \text{ is odd,} \\ \frac{e^{2\sqrt{n/r}-1/r}}{\sqrt{2}} \left(\frac{rn}{e}\right)^{n/2} \left(1 + \frac{2+3r}{6r^{3/2}n^{1/2}} + O(n^{-1})\right) & \text{if } r \text{ is even} \end{cases}$$

and

$$\frac{i_{n-1}^r}{i_n^r} \sim \begin{cases} \frac{1}{\sqrt{nre}} - \frac{3}{4nr\sqrt{e}} + \frac{9}{32nr\sqrt{nr}\sqrt{e}} + O(1/n^2) & \text{if } r \text{ is odd,} \\ \frac{1}{\sqrt{nre}} - \frac{1}{2nr\sqrt{e}} + \frac{9}{8nr\sqrt{nr}\sqrt{e}} + O(1/n^2) & \text{if } r \text{ is even.} \end{cases}$$

PROOF. By virtue of (5.1), the exponential generating function of i_n^r 's is of the form $e^{P(x)}$ with $P(x)$ a polynomial in x with non-negative aperiodic coefficients so that [19, Corollary VIII.2] applies giving the leading term of the asymptotic expansion of the coefficients.

When r is odd, [19, Corollary VIII.2] yields

$$\frac{i_n^r}{n!} \sim \frac{e^{\frac{\sqrt{1+4rn}-1}{2r} + \frac{(\sqrt{1+4rn}-1)^2}{8r}}}{\sqrt{\frac{\pi(\sqrt{1+4rn}-1)\sqrt{1+4rn}}{r}} \left(\frac{\sqrt{1+4rn}-1}{2r}\right)^n}.$$

Using the Stirling formula $n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}$ and with the aid of a computer algebra system such as Maple, we obtain

$$i_n^r \sim \frac{e^{\frac{4\sqrt{rn}-1}{4r}}}{\sqrt{2}} \left(\frac{rn}{e}\right)^{n/2} \left(1 + \frac{1+6r}{24r^{3/2}n^{1/2}} + \frac{1+12r-36r^2+96r^3}{1152r^3n} + O(n^{-3/2})\right)$$

and

$$\frac{i_{n-1}^r}{i_n^r} \sim \frac{1}{\sqrt{nre}} - \frac{3}{4nr\sqrt{e}} + \frac{9}{32nr\sqrt{nr}\sqrt{e}} + O(1/n^2).$$

When r is even, the same procedure yields

$$i_n^r \sim \frac{e^{\frac{2\sqrt{rn}-1}{r}}}{\sqrt{2}} \left(\frac{rs}{e} \right)^{n/2} \left(1 + \frac{2+3r}{6r^{3/2}n^{1/2}} + \frac{4+12r-9r^2+6r^3}{72r^3n} + O(n^{-3/2}) \right)$$

and from which we obtain

$$\frac{i_{n-1}^r}{i_n^r} \sim \frac{1}{\sqrt{nre}} - \frac{3}{2nr\sqrt{e}} + \frac{9}{8nr\sqrt{nr}\sqrt{e}} + O(1/(n^2)),$$

as desired. \square

Theorem 5.2 *The mean $\mu_{\mathcal{J}_n^r}^{\text{exc}} = E(X_{\mathcal{J}_n^r}^{\text{exc}})$ and variance $(\sigma_{\mathcal{J}_n^r}^{\text{exc}})^2 = \text{Var}(X_{\mathcal{J}_n^r}^{\text{exc}})$ of the random variable $X_{\mathcal{J}_n^r}^{\text{exc}}$ are given by*

$$\mu_{\mathcal{J}_n^r}^{\text{exc}} = \begin{cases} \frac{n}{2} \left(1 - \frac{i_{n-1}^r}{i_n^r} \right) & \text{if } r \text{ is odd,} \\ \frac{n}{2} & \text{if } r \text{ is even,} \end{cases}$$

and

$$(\sigma_{\mathcal{J}_n^r}^{\text{exc}})^2 = \begin{cases} \frac{n}{4r} + \frac{n(r-1)}{4r} \left(\frac{i_{n-1}^r}{i_n^r} \right) - \frac{n^2}{4} \left(\frac{i_{n-1}^r}{i_n^r} \right)^2 & \text{if } r \text{ is odd,} \\ \frac{n}{2} \left(\frac{i_{n-1}^r}{i_n^r} \right) & \text{if } r \text{ is even.} \end{cases}$$

PROOF. Differentiating $i^r(x, t) = \sum_{n \geq 0} i_n^r(t) x^n / n!$ with respect to t , followed by setting $t = 1$, we have

$$\sum_{n \geq 0} (i_n^r)'(1) x^n / n! = \begin{cases} \frac{rx^2}{2} e^{x+rx^2/2} & \text{if } r \text{ is odd,} \\ \left(x + \frac{rx^2}{2} \right) e^{2x+rx^2/2} & \text{if } r \text{ is even.} \end{cases} \quad (5.2)$$

By virtue of (5.1) and (5.2), upon expanding the exponentials on the right side followed by equating the coefficients of x^n , we obtain

$$(i_n^r)'(1) = \begin{cases} \frac{rn(n-1)i_{n-2}^r}{2} & \text{if } r \text{ is odd,} \\ \frac{2ni_{n-1}^r + rn(n-1)i_{n-2}^r}{2} & \text{if } r \text{ is even.} \end{cases}$$

Thus,

$$\mu_{\mathcal{J}_n^r}^{\text{exc}} = \frac{(i_n^r)'(1)}{i_n^r} = \begin{cases} \frac{rn(n-1)i_{n-2}^r}{2i_n^r} & \text{if } r \text{ is odd,} \\ \frac{2ni_{n-1}^r + rn(n-1)i_{n-2}^r}{2i_n^r} & \text{if } r \text{ is even.} \end{cases}$$

Using the recurrence relation [13, Theorem 9 with $t = 1$], namely,

$$i_n^r = \begin{cases} i_{n-1}^r + r(n-1)i_{n-2}^r & \text{if } r \text{ is odd,} \\ 2i_{n-1}^r + r(n-1)i_{n-2}^r & \text{if } r \text{ is even,} \end{cases} \quad (5.3)$$

we can express i_{n-2}^r in terms of i_n^r and i_{n-1}^r . In doing so, the above expressions for $\mu_{\mathcal{J}_n^r}^{\text{exc}}$ becomes

$$\mu_{\mathcal{J}_n^r}^{\text{exc}} = \begin{cases} \frac{n}{2} \left(1 - \frac{i_{n-1}^r}{i_n^r} \right) & \text{if } r \text{ is odd,} \\ \frac{n}{2} & \text{if } r \text{ is even.} \end{cases}$$

Differentiating $i^r(x, t)$ twice with respect to t , followed by setting $t = 1$, we have

$$\sum_{n \geq 0} (i_n^r)''(1) \frac{x^n}{n!} = \begin{cases} \frac{r^2 x^4}{4} e^{x+rx^2/2} & \text{if } r \text{ is odd,} \\ \left(x + \frac{rx^2}{2} \right)^2 e^{2x+rx^2/2} & \text{if } r \text{ is even.} \end{cases}$$

Expanding the exponentials on the right side according to (5.1) and equating the coefficients of x^n , we have

$$(i_n^r)''(1) = \begin{cases} 6r^2 \binom{n}{4} i_{n-4}^r & \text{if } r \text{ is odd,} \\ 2 \binom{n}{2} i_{n-2}^r + 6r \binom{n}{3} i_{n-3}^r + 6r^2 \binom{n}{4} i_{n-4}^r & \text{if } r \text{ is even.} \end{cases}$$

Using now (5.3) repeatedly, we can express i_{n-4}^r , i_{n-3}^r and i_{n-2}^r in terms of i_n^r and i_{n-1}^r , the end result being

$$(i_n^r)''(1) = \begin{cases} \frac{rn^2 + n(1-2r)i_n^r}{4r} + \frac{(-2rn^2 + n(3r-1))i_{n-1}^r}{4r} & \text{if } r \text{ is odd,} \\ \frac{n(n-2)}{4} i_n^r + \frac{n}{2} i_{n-1}^r & \text{if } r \text{ is even.} \end{cases}$$

Finally, we have

$$\begin{aligned} (\sigma_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}})^2 &= \frac{(i_n^r)''(1)}{i_n^r} + \mu_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}} - (\mu_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}})^2 \\ &= \begin{cases} \frac{n}{4r} + \frac{n(r-1)}{4r} \left(\frac{i_{n-1}^r}{i_n^r} \right) - \frac{n^2}{4} \left(\frac{i_{n-1}^r}{i_n^r} \right)^2 & \text{if } r \text{ is odd,} \\ \frac{n}{2} \left(\frac{i_{n-1}^r}{i_n^r} \right) & \text{if } r \text{ is even.} \end{cases} \end{aligned}$$

□

Theorem 5.3 *Let $X_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}}$ be the random variable taking values $\text{exc}(\pi)$ of random involutions π in $G_{r,n}$. Then the standardized random variable*

$$Z_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}} := \frac{X_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}} - \mu_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}}}{\sigma_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}}}$$

converges to a random variable having the standard normal distribution $N(0, 1)$ as $n \rightarrow +\infty$.

PROOF. It was proved in [13] that $i_n^r(t)$ has only real zeros. By Lemma 5.1 and Theorem 5.2,

$$(\sigma_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}})^2 = \begin{cases} O(n) & \text{if } r \text{ is odd} \\ O(n^{1/2}) & \text{if } r \text{ is even} \end{cases}.$$

Since $(\sigma_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}})^2 \rightarrow +\infty$ as $n \rightarrow +\infty$ in either case, according to Pitman [24], $Z_{\mathcal{J}_n^{\text{exc}}}^{\text{exc}} \rightarrow N(0, 1)$ as $n \rightarrow +\infty$. □

6 The distribution of excedance on derangements

The enumeration of derangements in $G_{r,n}$ has received much attention recently from several authors. Faliharimalala and Zeng [18] counted derangements in $G_{r,n}$ by their flag major indices and obtained

$$d_{r,n}(q) = [r]_q [2r]_q \cdots [nr]_q \sum_{k=0}^n \frac{(-1)^k q^{r \binom{k}{2}}}{[r]_q [2r]_q \cdots [kr]_q},$$

where $d_{r,n}(q) := \sum_{p \in \mathcal{D}_{r,n}} q^{\text{fmaj}(p)}$. Specializing $q = 1$, we get the following formula of the number of r -colored derangements,

$$d_{r,n}(1) = r^n n! \sum_{k=0}^n \frac{(-1)^k}{r^k k!},$$

which also readily follows from the principle of inclusion-exclusion.

Chow and Mansour [13] enumerated derangements in $G_{r,n}$ by their excedances and obtained

$$\sum_{n \geq 0} d_{r,n}(q) \frac{x^n}{n!} = \frac{(1-q)e^{(r-1)qx}}{e^{r qx} - qe^{rx}}, \quad (6.1)$$

where $d_{r,n}(q) := \sum_{p \in \mathcal{D}_{r,n}} q^{\text{exc}(p)}$.

Theorem 6.1 The mean $\mu_{\mathcal{D}_r,n}^{\text{exc}} = E(X_{\mathcal{D}_r,n}^{\text{exc}})$ and variance $(\sigma_{\mathcal{D}_r,n}^{\text{exc}})^2 = \text{Var}(X_{\mathcal{D}_r,n}^{\text{exc}})$ of the random variable $X_{\mathcal{D}_r,n}^{\text{exc}}$ are given by

$$\mu_{\mathcal{D}_r,n}^{\text{exc}} = \frac{n+1}{2} - \frac{1}{2r} + o(1) \quad \text{and} \quad (\sigma_{\mathcal{D}_r,n}^{\text{exc}})^2 = \frac{n+1}{12} - \frac{1}{6r} + o(1)$$

as $n \rightarrow +\infty$.

PROOF. Replacing in (6.1) x by x/r , we have

$$\sum_{n \geq 0} d_{r,n}(q) \frac{x^n}{r^n n!} = \frac{(1-q)e^{(r-1)qx/r}}{e^{qx} - qe^x}. \quad (6.2)$$

Differentiating (6.2) with respect to q , followed by letting $q \rightarrow 1$, we have

$$\sum_{n \geq 0} d'_{r,n}(1) \frac{x^n}{r^n n!} = \frac{x(2r-2-x(r-2))e^{-x/r}}{2r(1-x)^2}.$$

By expanding the right side of the preceding identity as a formal power series in x , we get that

$$\frac{d'_{r,n}(1)}{r^n n!} = \sum_{j=0}^{n-1} \frac{(n+1-j)r-2}{2r} \frac{(-1)^j}{r^j j!} = \sum_{j=0}^{n-1} (-1)^j \frac{n+1+j}{2j! r^j}. \quad (6.3)$$

Hence, the mean of the excedance number in the r -colored derangements is given by

$$\mu_{\mathcal{D}_r,n}^{\text{exc}} = \frac{\sum_{j=0}^{n-1} (-1)^j \frac{n+1+j}{2j! r^j}}{\sum_{j=0}^n \frac{(-1)^j}{r^j j!}}, \quad (6.4)$$

which implies that

$$\mu_{\mathcal{D}_r,n}^{\text{exc}} = \frac{n+1}{2} - \frac{1}{2r} \frac{\sum_{j=0}^{n-2} \frac{(-1)^j}{j! r^j}}{\sum_{j=0}^n \frac{(-1)^j}{r^j j!}} = \frac{n+1}{2} - \frac{1}{2r} + o(1). \quad (6.5)$$

Differentiating (6.2) twice with respect to q , followed by letting $q \rightarrow 1$, we obtain

$$\sum_{n \geq 0} d''_{r,n}(1) \frac{x^n}{r^n n!} = \frac{x^2((6-6r+r^2)x^2 - 2(2r^2-9r+6)x + 6(1-r)^2)e^{-x/r}}{6r^2(1-x)^3}.$$

By expanding the right side of the preceding identity as a formal power series in x , we get that

$$\frac{d''_{r,n}(1)}{r^n n!} = \sum_{j=0}^{n-2} \left(\frac{3(n-j)^2 + n - 2 - j}{12} - \frac{n-j}{r} + \frac{1}{r^2} \right) \frac{(-1)^j}{r^j j!},$$

which is equivalent to

$$\frac{d''_{r,n}(1)}{r^n n!} = n(n-1) \frac{(-1)^{n-1}}{r^{n-1}(n-1)!} + n(n-1) \frac{(-1)^n}{r^n n!} + \sum_{j=0}^n \frac{3n^2 + (6j+1)n + 3j^2 - j - 2}{12} \frac{(-1)^j}{r^j j!}. \quad (6.6)$$

Hence, for all $n \geq 2$ we have that

$$\frac{d''_{r,n}(1)}{d_{r,n}} = \frac{n(n-1) \frac{(-1)^{n-1}}{r^{n-1}(n-1)!} + n(n-1) \frac{(-1)^n}{r^n n!} + \sum_{j=0}^n \frac{3n^2 + (6j+1)n + 3j^2 - j - 2}{12} \frac{(-1)^j}{r^j j!}}{\sum_{j=0}^n \frac{(-1)^j}{r^j j!}},$$

which implies that for n large enough

$$\begin{aligned} \frac{d''_{r,n}(1)}{d_{r,n}} &= \frac{3n^2 + n - 2}{12} - \frac{6n - 1}{12r} - \frac{\frac{3}{12r} \sum_{j=0}^{n-1} \frac{(j+1)(-1)^j}{r^j j!} - n(n-1) \frac{(-1)^{n-1}}{r^{n-1}(n-1)!} (1 - \frac{1}{rn})}{\sum_{j=0}^n \frac{(-1)^j}{r^j j!}} \\ &\approx \frac{3n^2 + n - 2}{12} - \frac{6n - 1}{12r} - \frac{3}{12r} + \frac{3}{12r^2} \\ &= \frac{3n^2 + n - 2}{12} - \frac{3n + 1}{12r} + \frac{1}{4r^2}. \end{aligned} \quad (6.7)$$

Therefore, using (6.5) and (6.7) we obtain

$$(\sigma_{\mathcal{D}_{r,n}}^{\text{exc}})^2 = \frac{d''_{r,n}(1)}{d_{r,n}} + \mu_{\mathcal{D}_{r,n}}^{\text{exc}} - (\mu_{\mathcal{D}_{r,n}}^{\text{exc}})^2 = \frac{n+1}{12} - \frac{1}{6r} + o(1).$$

□

Theorem 6.2 *Let $X_{\mathcal{D}_{r,n}}^{\text{exc}}$ be the random variable taking values $\text{exc}(\pi)$ of random derangements π in $G_{r,n}$. Then the standardized random variable*

$$Z_{\mathcal{D}_{r,n}}^{\text{exc}} := \frac{X_{\mathcal{D}_{r,n}}^{\text{exc}} - \mu_{\mathcal{D}_{r,n}}^{\text{exc}}}{\sigma_{\mathcal{D}_{r,n}}^{\text{exc}}}$$

converges to a random variable having the standard normal distribution $N(0, 1)$ as $n \rightarrow +\infty$.

PROOF. It was proved in [13] that $d_{r,n}(q)$ has only real zeros. Since $(\sigma_{\mathcal{D}_{r,n}}^{\text{exc}})^2 \rightarrow +\infty$ as $n \rightarrow +\infty$, according to Pitman [24], $Z_{\mathcal{D}_{r,n}}^{\text{exc}} \rightarrow N(0, 1)$ as $n \rightarrow +\infty$. □

7 The distribution of the number of fixed points

We consider in this section the distribution of the number of fixed points of $p \in G_{r,n}$. We first establish certain properties of the generating function $F_{r,n}(t) := \sum_{p \in G_{r,n}} t^{\text{fix}(p)}$ of r -colored permutations p by their number of fixed points $\text{fix}(p)$, as follows.

Lemma 7.1 *Let $F_{r,n}(t) = \sum_{j=0}^n \binom{n}{j} d_{r,n-j} t^j$. Then*

- (a) $F'_{r,n}(t) = nF_{r,n-1}(t)$,
- (b) $F_{r,n}(1) = r^n n!$,
- (c) $F'_{r,n}(1) = r^{n-1} n!$, and
- (d) $F''_{r,n}(1) = r^{n-2} n!$.

PROOF. Differentiating $F_{r,n}$ once with respect to t , we have

$$F'_{r,n}(t) = \sum_{j=1}^n j \binom{n}{j} d_{r,n-j} t^{j-1} = n \sum_{j=1}^n \binom{n-1}{j-1} d_{r,(n-1)-(j-1)} t^{j-1} = nF_{r,n-1}(t),$$

which is (a). Since for $j = 0, 1, \dots, n$,

$$\binom{n}{j} d_{r,n-j} = \#\{p \in G_{r,n} : \text{fix}(p) = j\},$$

$F_{r,n}(t)$ is the generating function of the elements of the wreath product $G_{r,n}$ by their number of fixed points so that $F_{r,n}(1) = \#G_{r,n} = r^n n!$, which is (b). Setting $t = 1$ in (a) and using (b), we get $F'_{r,n}(1) = nF_{r,n-1}(1) = nr^{n-1}(n-1)! = r^{n-1} n!$, which is (c). Differentiating $F'_{r,n}(t)$ once again with respect to t and exploiting (a), we obtain

$$F''_{r,n}(t) = nF'_{r,n-1}(t) = n(n-1)F_{r,n-2}(t).$$

Setting $t = 1$ and using (b), we get $F''_{r,n}(1) = n(n-1)r^{n-2}(n-2)! = r^{n-2} n!$, which is (d). □

Denote by $X_{r,n}^{\text{fix}}$ the random variable taking value the number of fixed points of p randomly drawn from $G_{r,n}$. The probability generating function of $X_{r,n}^{\text{fix}}$ is precisely given by $F_{r,n}(t)/F_{r,n}(1)$.

Theorem 7.2 *The mean $\mu_{G_{r,n}}^{\text{fix}}$ and variance $(\sigma_{G_{r,n}}^{\text{fix}})^2$ of $X_{r,n}^{\text{fix}}$ are given by*

$$\mu_{G_{r,n}}^{\text{fix}} = \frac{1}{r} \quad \text{and} \quad (\sigma_{G_{r,n}}^{\text{fix}})^2 = \frac{1}{r}.$$

PROOF. By Lemma 7.1, we have

$$\begin{aligned}\mu_{G_{r,n}}^{\text{fix}} &= \frac{F'_{r,n}(1)}{F_{r,n}(1)} = \frac{r^{n-1}n!}{r^n n!} = \frac{1}{r}, \\ (\sigma_{G_{r,n}}^{\text{fix}})^2 &= \frac{F''_{r,n}(1) + F'_{r,n}(1)}{F_{r,n}(1)} - (\mu_{G_{r,n}}^{\text{fix}})^2 = \frac{r^{n-2}n! + r^{n-1}n!}{r^n n!} - \left(\frac{1}{r}\right)^2 = \frac{1}{r}.\end{aligned}$$

□

Proposition 7.3 For $0 \leq j \leq n$, let $\{p \in G_{r,n} : \text{fix}(p) = j\}$ be the set of elements of $G_{r,n}$ having j fixed points. Then for fixed j ,

$$\lim_{n \rightarrow \infty} \frac{\#\{p \in G_{r,n} : \text{fix}(p) = j\}}{r^n n!} = \frac{e^{-1/r}(1/r)^j}{j!}.$$

PROOF. It is easy to see that $\#\{p \in G_{r,n} : \text{fix}(p) = j\} = \binom{n}{j} d_{r,n-j}$ so that

$$\lim_{n \rightarrow \infty} \frac{\#\{p \in G_{r,n} : \text{fix}(p) = j\}}{r^n n!} = \lim_{n \rightarrow \infty} \frac{\binom{n}{j} d_{r,n-j}}{r^n n!} = \frac{(1/r)^j}{j!} \lim_{n \rightarrow \infty} \sum_{k=0}^{n-j} \frac{(-1)^k}{r^k k!} = \frac{e^{-1/r}(1/r)^j}{j!}.$$

□

Proposition 7.3 states that the number of fixed points of a randomly chosen permutation $p \in G_{r,n}$ follows a Poisson distribution with mean $1/r$ as $n \rightarrow +\infty$. When $r = 1$, $G_{r,n} = \mathfrak{S}_n$ and Proposition 7.3 reduces to the corresponding symmetric group result due to Montmort [23].

Let $X_{G_{r,n}}^{\text{fix}}$ be the random variable which takes value the number of fixed points of an element p randomly drawn from $G_{r,n}$.

Proposition 7.4 Let $k \in \mathbb{N}$. The k th moment of $X_{G_{r,n}}^{\text{fix}}$ is given by

$$E[(X_{G_{r,n}}^{\text{fix}})^k] = \sum_{l=0}^n \frac{S(k, l)}{r^l} = B_n\left(\frac{1}{r}\right),$$

where $S(k, l)$ is a Stirling number of the second kind and $B_n(x)$ is the n th Bell polynomial.

PROOF. Recall that the Stirling number $S(k, l)$ of the second kind can be written as

$$S(k, l) = \frac{1}{l!} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} j^k.$$

We have

$$\begin{aligned}E[(X_{G_{r,n}}^{\text{fix}})^k] &= \sum_{j=0}^n j^k P(\text{fix}(p) = j) = \sum_{j=0}^n \frac{j^k \binom{n}{j} d_{r,n-j}^r}{r^n n!} \\ &= \sum_{j=0}^n \sum_{l=0}^{n-j} \frac{(-1)^l j^k}{r^{k+l} j! l!} = \sum_{j=0}^n \sum_{l=j}^n \frac{(-1)^{l-j} j^k}{r^l j! (l-j)!} \\ &= \sum_{l=0}^n \frac{1}{r^l} \left(\frac{1}{l!} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} j^k \right) \\ &= \sum_{l=0}^n \frac{S(k, l)}{r^l} = B_n\left(\frac{1}{r}\right),\end{aligned}$$

where the last equality follows from the well known identity [3]:

$$\sum_{k=0}^n S(n, k) x^k = B_n(x)$$

and $B_n(x)$ is the n th Bell polynomial. □

When $r = 1$, Proposition 7.4 specializes to the classical result that the k th moment of X is given by

$$E[X^k] = \sum_{l=0}^n S(k, l),$$

where X is the random variable taking value the number of fixed points of a uniformly distributed random permutation $\sigma \in \mathfrak{S}_n$.

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