# A PROBABILISTIC INTERPRETATION OF A SEQUENCE RELATED TO NARAYANA POLYNOMIALS

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ABSTRACT. A sequence of coefficients appearing in a recurrence for the Narayana polynomials is generalized. The coefficients are given a probabilistic interpretation in terms of beta distributed random variables. The recurrence established by M. Lasalle is then obtained from a classical convolution identity. Some arithmetical properties of the generalized coefficients are also established.

## 1. INTRODUCTION

The Narayana polynomials

$$\mathcal{N}_r(z) = \sum_{k=1}^{r} N(r,k) z^{k-1} \tag{1.1}$$

with the Narayana numbers $N(r,k)$ given by

$$N(r,k) = \frac{1}{r}\binom{r}{k-1}\binom{r}{k} \tag{1.2}$$

have a large number of combinatorial properties. In a recent paper, M. Lasalle [19] established the recurrence

$$(z+1)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n\geq 1} (-z)^n \binom{r-1}{2n-1} A_n \mathcal{N}_{r-2n+1}(z). \tag{1.3}$$

The numbers $A_n$ satisfy the recurrence

$$(-1)^{n-1}A_n = C_n + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} A_j C_{n-j}, \tag{1.4}$$

with $A_1 = 1$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$ the Catalan numbers. This recurrence is taken here as being the definition of $A_n$. The first few values are

$$A_1 = 1,\ A_2 = 1,\ A_3 = 5,\ A_4 = 56,\ A_5 = 1092,\ A_6 = 32670. \tag{1.5}$$

Lasalle [19] shows that $\{A_n : n \in \mathbb{N}\}$ is an increasing sequence of positive integers. In the process of establishing the positivity of this sequence, he contacted D. Zeilberger, who suggested the study of the related sequence

$$a_n = \frac{2A_n}{C_n}, \tag{1.6}$$

with first few values

$$a_1 = 2,\ a_2 = 1,\ a_3 = 2,\ a_4 = 8,\ a_5 = 52,\ a_6 = 495,\ a_7 = 6470. \tag{1.7}$$

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The recurrence (1.4) yields

$$(-1)^{n-1}a_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1}\binom{n+1}{j+1} \frac{a_j}{n-j+1}. \tag{1.8}$$

This may be expressed in terms of the numbers

$$\sigma_{n,r} := \frac{2}{n}\binom{n}{r-1}\binom{n+1}{r+1} \tag{1.9}$$

that appear as entry $A108838$ in $OEIS$ and count Dyck paths by the number of long interior inclines. The fact that $\sigma_{n,r}$ is an integer also follows from

$$\sigma_{n,r} = \binom{n-1}{r-1}\binom{n+1}{r} - \binom{n-1}{r-2}\binom{n+1}{r+1}. \tag{1.10}$$

The relation (1.8) can also be written as

$$a_n = (-1)^{n-1}\left[2 + \frac{1}{2}\sum_{j=1}^{n-1} (-1)^j \sigma_{n,j} a_j\right]. \tag{1.11}$$

The original approach by M. Lasalle [19] is to establish the relation

$$(z+1)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = \sum_{n\geq 1} (-z)^n \binom{r-1}{2n-1} A_n(r)\mathcal{N}_{r-2n+1}(z) \tag{1.12}$$

for some coefficient $A_n(r)$. The expression

$$\mathcal{N}_r(z) = \sum_{m\geq 0} z^m (z+1)^{r-2m-1}\binom{r-1}{2m} C_m \tag{1.13}$$

given in [12], is then employed to show that $A_n(r)$ is independent of $r$. This is the definition of $A_n$ given in [19]. Lasalle mentions in passing that "J. Novak observed, as empirical evidence, that the integers $(-1)^{n-1}A_n$ are precisely the (classical) cumulants of a standard semicircular random variable".

The goal of this paper is to revisit Lasalle's results, provide probabilistic interpretation of the numbers $A_n$ and to consider Zeilberger's suggestion.

The probabilistic interpretation of the numbers $A_n$ starts with the semicircular distribution

$$f_1(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.14}$$

Let $X$ be a random variable with distribution $f_1$. Then $X_* = 2X$ satisfies

$$\mathbb{E}\left[X_*^r\right] = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ C_m & \text{if } r \text{ is even, with } r = 2m, \end{cases} \tag{1.15}$$

where $C_m = \frac{1}{m+1}\binom{2m}{m}$ are the Catalan numbers. The moment generating function

$$\varphi(t) = \sum_{n=0}^{\infty} \mathbb{E}\left[X^n\right]\frac{t^n}{n!} \tag{1.16}$$

is expressed in terms of the modified Bessel function of the first kind $I_\alpha(x)$ and the cumulant generating function

$$\psi(t) = \log\varphi(t) = \sum_{n=1}^{\infty} \kappa_1(n)\frac{t^n}{n!} \tag{1.17}$$

has coefficients $\kappa_1(n)$, known as the cumulants of $X$. The identity

$$A_n = (-1)^{n+1}\kappa_1(2n)2^{2n}, \tag{1.18}$$

is established here. Lasalle's recurrence (1.4) now follows from the convolution identity

$$\kappa(n) = \mathbb{E}\left[X^n\right] - \sum_{j=1}^{n-1}\binom{n-1}{j-1}\kappa(j)\mathbb{E}\left[X^{n-j}\right] \tag{1.19}$$

that holds for any pair of moments and cumulants sequences [24]. The coefficient $a_n$ suggested by D. Zeilberger now takes the form

$$a_n = \frac{2(-1)^{n+1}\kappa_1(2n)}{\mathbb{E}\left[X_*^{2n}\right]}, \tag{1.20}$$

with $X_* = 2X$.

In this paper, these notions are extended to the case of random variables distributed according to the symmetric beta distribution

$$f_\mu(x) = \frac{1}{B(\mu+\frac{1}{2},\frac{1}{2})}(1-x^2)^{\mu-1/2}, \quad \text{for } |x| \le 1,\, \mu > -\tfrac{1}{2} \tag{1.21}$$

and 0 otherwise. The semi-circular distribution is the particular case $\mu = 1$. Here $B(a,b)$ is the classical beta function defined by the integral

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \quad \text{for } a,\, b > 0. \tag{1.22}$$

These ideas lead to introduce a generalization of the Narayana polynomials and these are expressed in terms of the classical Gegenbauer polynomials $C_n^{\mu+\frac{1}{2}}$. The coefficients $a_n$ are also generalized to a family of numbers $\{a_n(\mu)\}$ with parameter $\mu$. The special cases $\mu = 0$ and $\mu = \pm\frac{1}{2}$ are discussed in detail.

Section 2 produces a recurrence for $\{a_n\}$ from which the facts that $a_n$ is increasing and positive are established. The recurrence comes from a relation between $\{a_n\}$ and the Bessel function $I_\alpha(x)$. Section 3 gives an expression for $\{a_n\}$ in terms of the determinant of an upper Hessenberg matrix. The standard procedure to evaluate these determinants gives the original recurrence defining $\{a_n\}$. Section 4 introduces the probabilistic interpretation of the numbers $\{a_n\}$. The cumulants of the associated random variable are expressed in terms of the Bessel zeta function. Section 5 presents the Narayana polynomials as expected values of a simple function of a semicircular random variable. These polynomials are generalized in Section 6 and they are expressed in terms of Gegenbauer polynomials. The corresponding extensions of $\{a_n\}$ are presented in Section 7. The paper concludes with some arithmetical properties of $\{a_n\}$ and its generalization corresponding to the parameter $\mu = 0$. These are described in Section 8.

## 2. THE SEQUENCE $\{a_n\}$ IS POSITIVE AND INCREASING

In this section a direct proof of the positivity of the numbers $a_n$ defined in (1.8) is provided. Naturally this implies $A_n \ge 0$. The analysis employs the *modified Bessel function of the first kind*

$$I_\alpha(z) := \sum_{j=0}^{\infty}\frac{1}{j!\,(j+\alpha)!}\left(\frac{z}{2}\right)^{2j+\alpha}. \tag{2.1}$$

Formulas for this function appear in [16].

**Lemma 2.1.** *The numbers $a_n$ satisfy*

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!}\frac{x^{j-1}}{(j-1)!} = \frac{2}{\sqrt{x}}\frac{I_2(2\sqrt{x})}{I_1(2\sqrt{x})}. \tag{2.2}$$

*Proof.* The statement is equivalent to

$$\sqrt{x}I_1(2\sqrt{x}) \times \sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!}\frac{x^{j-1}}{(j-1)!} = 2I_2(2\sqrt{x}). \tag{2.3}$$

This is established by comparing coefficients of $x^n$ on both sides and using (1.8). □

Now change $x$ to $x^2$ in Lemma 2.1 to write

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!}\frac{x^{2j-2}}{(j-1)!} = \frac{2}{x}\frac{I_2(2x)}{I_1(2x)}. \tag{2.4}$$

The classical relations

$$\frac{d}{dz}\left(z^{-m}I_m(z)\right) = z^{-m}I_{m+1}(z), \text{ and } \frac{d}{dz}\left(z^{m+1}I_{m+1}(z)\right) = z^{m+1}I_m(z) \tag{2.5}$$

give

$$I_1'(z) = I_2(z) + \frac{1}{z}I_1(z). \tag{2.6}$$

Therefore (2.4) may be written as

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}a_j}{(j+1)!}\frac{x^{2j-2}}{(j-1)!} = \frac{1}{x}\frac{d}{dx}\log\left(\frac{I_1(2x)}{2x}\right). \tag{2.7}$$

The relations (2.5) also produce

$$\frac{d}{dz}\left(\frac{z^{m+1}I_{m+1}(z)}{z^{-m}I_m(z)}\right) = z^{2m+1}\frac{I_m^2(z)-I_{m+1}^2(z)}{I_m^2(z)}. \tag{2.8}$$

In particular,

$$\frac{d}{dz}\left(\frac{z^2I_2(z)}{z^{-1}I_1(z)}\right) = z^3 - z^3\frac{I_2^2(z)}{I_1^2(z)}. \tag{2.9}$$

Replacing this relation in (2.7) gives the recurrence stated next.

**Proposition 2.2.** *The numbers $a_n$ satisfy the recurrence*

$$2na_n = \sum_{k=1}^{n-1}\binom{n}{k-1}\binom{n}{k+1}a_ka_{n-k}, \quad \textit{for } n \geq 2, \tag{2.10}$$

*with initial condition $a_1 = 1$.*

**Corollary 2.3.** *The numbers $a_n$ are nonnegative.*

**Proposition 2.4.** *The numbers $a_n$ satisfy*

$$4a_n = \sum_{k=1}^{n-1}\binom{n-1}{k-1}\binom{n-1}{k}a_ka_{n-k} - \sum_{k=2}^{n-2}\binom{n-1}{k-2}\binom{n-1}{k+1}a_ka_{n-k}. \tag{2.11}$$

*Proof.* This follows from (2.10) and the identity

$$\binom{n}{k-1}\binom{n}{k+1} = \frac{n}{2}\left[\binom{n-1}{k-1}\binom{n-1}{k} - \binom{n-1}{k-2}\binom{n-1}{k+1}\right].$$

□

**Corollary 2.5.** *The numbers $a_n$ are nonnegative integers. Moreover $a_n$ is even if $n$ is odd.*

*Proof.* Corollary 2.3 shows $a_n > 0$. It remains to show $a_n \in \mathbb{Z}$ and to verify the parity statement. This is achieved by simultaneous induction on $n$.

Assume first $n = 2m+1$ is odd. Then (1.9) shows that $\frac{1}{2}\sigma_{n,r} \in \mathbb{Z}$ and (1.11), written as

$$a_n = (-1)^{n-1}\left[2 + \sum_{r=1}^{n-1} \frac{\sigma_{n,r}}{2} a_r\right], \tag{2.12}$$

proves that $a_n \in \mathbb{Z}$. Now write (2.10) as

$$2(2m+1)a_{2m+1} = 2\sum_{k=1}^{m} \binom{2m+1}{k-1}\binom{2m+1}{k+1} a_k a_{2m+1-k} \tag{2.13}$$

and observe that either $k$ or $2m+1-k$ is odd. The induction hypothesis shows that either $a_k$ or $a_{2m+1-k}$ is even. This shows $a_{2m+1}$ is even.

Now consider the case $n = 2m$ even. If $r$ is odd, then $a_r$ is even; if $r$ is even then $r-1$ is odd and $\frac{1}{2}\sigma_{n,r} \in \mathbb{Z}$ in view of the identity

$$\sigma_{n,r} = \frac{2}{r-1}\binom{n-1}{r-2}\binom{n+1}{r+1}. \tag{2.14}$$

The result follows again from (2.12). □

**Corollary 2.6.** *The numbers $A_n$ are nonnegative integers.*

The recurrence in Proposition 2.2 is now employed to prove that $\{a_n : n \geq 2\}$ is an increasing sequence. The first few values are 1, 2, 8, 52.

**Theorem 2.7.** *For $n \geq 3$, the inequality $a_n > a_{n-1}$ holds.*

*Proof.* Take the terms $k = 1$ and $k = n-1$ in the sum appearing in the recurrence in Proposition (2.2) and use $a_n > 0$ to obtain

$$a_n \geq \frac{1}{2n}\left[\binom{n}{0}\binom{n}{2} a_1 a_{n-1} + \binom{n}{n-2}\binom{n}{2} a_{n-1} a_1\right]. \tag{2.15}$$

Since $a_1 = 2$ the previous inequality yields

$$a_n \geq (n-1)a_{n-1}. \tag{2.16}$$

Hence, for $n \geq 3$, this gives $a_n - a_{n-1} \geq (n-2)a_{n-1} > 0$. □

## 3. An expression in form of determinants

The recursion relation (1.8) expressed in the form

$$\sum_{j=1}^{m}(-1)^{j-1}\binom{m}{j-1}\binom{m+1}{j+1}a_j = 2m \tag{3.1}$$

is now employed to produce a system of equations for the numbers $a_n$ by varying $m$ through $1, 2, 3, \cdots, n$. The coefficient matrix has determinant $(-1)^{\binom{n}{2}} n!$ and Cramér's rule gives

$$a_n = \frac{(-1)^{n-1}}{n!}\det\begin{pmatrix} \binom{1}{1-1}\binom{1+1}{1+1} & 0 & 0 & \cdots & 0 & 2 \\ \binom{2}{1-1}\binom{2+1}{1+1} & \binom{2}{2-1}\binom{2+1}{2+1} & 0 & \cdots & 0 & 4 \\ \binom{3}{1-1}\binom{3+1}{1+1} & \binom{3}{2-1}\binom{3+1}{2+1} & \binom{3}{3-1}\binom{3+1}{3+1} & \cdots & 0 & 6 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{1-1}\binom{n+1}{1+1} & \binom{n}{2-1}\binom{n+1}{2+1} & \binom{n}{3-1}\binom{n+1}{3+1} & \cdots & \binom{n}{n-2}\binom{n+1}{n} & 2n \end{pmatrix}. \tag{3.2}$$

The power of $-1$ is eliminated by permuting the columns to produce the matrix

$$B_n = \begin{pmatrix} 2 & \binom{1}{1-1}\binom{1+1}{1+1} & 0 & 0 & 0 \\ 4 & \binom{2}{1-1}\binom{2+1}{1+1} & \binom{2}{2-1}\binom{2+1}{2+1} & 0 & \cdots \\ 6 & \binom{3}{1-1}\binom{3+1}{1+1} & \binom{3}{2-1}\binom{3+1}{2+1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2n & \binom{n}{1-1}\binom{n+1}{1+1} & \binom{n}{2-1}\binom{n+1}{2+1} & \binom{n}{3-1}\binom{n+1}{3+1}\cdots & \binom{n}{n-2}\binom{n+1}{n} \end{pmatrix}. \tag{3.3}$$

The representation of $a_n$ in terms of determinants is given in the next result.

**Proposition 3.1.** *The number $a_n$ is given by*

$$a_n = \frac{\det B_n}{n!} \tag{3.4}$$

*where $B_n$ is the matrix in* (3.3).

Recall that an *upper Hessenberg matrix* is one of the form

$$H_n = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \beta_{n-1,n} \\ \beta_{n,1} & \beta_{n,2} & \beta_{n,3} & \beta_{n,4} & \cdots & \cdots & \cdots & \beta_{n,n-1} & \beta_{n,n} \end{pmatrix}. \tag{3.5}$$

The matrix $B$ is of this form with

$$\beta_{i,j} = \begin{cases} 2i & \text{if } 1 \le i \le n \quad \text{and } j = 1, \\ \binom{i}{j-2}\binom{i+1}{j} & \text{if } j-1 \le i \le n \text{ and } j > 1. \end{cases} \tag{3.6}$$

It turns out that the recurrence (1.8) used to define the numbers $a_n$ can be recovered if one employs (3.4).

**Proposition 3.2.** *Define $\alpha_n$ by*

$$\alpha_n = \frac{\det B_n}{n!} \tag{3.7}$$

*where $B$ is the matrix* (3.3)*. Then* $\{\alpha_n\}$ *satisfies the recursion*

$$(-1)^{n-1}\alpha_n = 2 + \sum_{j=1}^{n-1} (-1)^j \binom{n-1}{j-1}\binom{n+1}{j+1}\frac{\alpha_j}{n-j+1} \tag{3.8}$$

*and the initial condition* $\alpha_1 = 1$*. Therefore* $\alpha_n = a_n$*.*

*Proof.* For convenience define $\det H_0 = 1$. The determinant of a Hessenberg matrix satisfies the recurrence

$$\det H_n = \sum_{r=1}^{n} (-1)^{n-r}\beta_{n,r} \det H_{r-1} \prod_{i=r}^{n-1} \beta_{i,i+1}. \tag{3.9}$$

A direct application of (3.9) yields

$$\begin{aligned}
\alpha_n &= \frac{1}{n!}\left\{(-1)^{n-1}(2n)(n-1)! + \sum_{r=2}^{n}(-1)^{n-r}\binom{n}{r-2}\binom{n+1}{r}\det B_{r-1}\prod_{i=r}^{n-1} i\right\} \\
&= 2(-1)^{n-1} + \frac{1}{n!}\sum_{r=2}^{n}(-1)^{n-r}\binom{n}{r-2}\binom{n+1}{r}\alpha_{r-1}\,(n-1)! \\
&= 2(-1)^{n-1} + \sum_{r=2}^{n}(-1)^{n-r}\frac{1}{n}\binom{n}{r-2}\binom{n+1}{r}\alpha_{r-1} \\
&= 2(-1)^{n-1} + \sum_{r=2}^{n}(-1)^{n-r}\binom{n}{r-2}\binom{n+1}{r}\frac{\alpha_{r-1}}{n-r+2} \\
&= 2(-1)^{n-1} + (-1)^{n-1}\sum_{r=1}^{n}(-1)^{j}\binom{n-1}{j-1}\binom{n+1}{j+1}\frac{\alpha_j}{n-j+1}.
\end{aligned}$$

This is (3.8). □

**Corollary 3.3.** *The modified Bessel function of the first kind admits a determinant expression*

$$I_1(x) = x\exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}\det B_j}{(j+1)!\,j!^2}\left(\frac{x}{2}\right)^{2j}\right). \tag{3.10}$$

*Proof.* This follows by integrating the identity

$$\frac{2I_2(2x)}{x\,I_1(2x)} = \frac{1}{x}\frac{d}{dx}\log\frac{I_1(2x)}{2x}. \tag{3.11}$$

□

## 4. THE PROBABILISTIC BACKGROUND: CONJUGATE RANDOM VARIABLES

This section provides the probabilistic tools required for an interpretation of the sequence $A_n$ defined in (1.4). The specific connections are given in Section 5.

Consider a random variable $X$ with the *symmetric beta distribution* given in (1.21). The moments of the symmetric beta distribution, given by

$$\mathbb{E}\left[X^n\right] = \frac{1}{B(\mu+\frac{1}{2},\frac{1}{2})}\int_{-1}^{1} x^n(1-x^2)^{\mu-1/2}\,dx, \tag{4.1}$$

vanish for $n$ odd and for $n = 2m$ they are

$$\mathbb{E}\left[X^{2m}\right] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+m)}\frac{(2m)!}{2^{2m}\,m!}. \tag{4.2}$$

Therefore the moment generating function is

$$\varphi_\mu(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[X^n\right] \frac{t^n}{n!} = \Gamma(\mu+1) \sum_{m=0}^{\infty} \frac{t^{2m}}{2^{2m}\, m!\,\Gamma(\mu+m+1)}. \tag{4.3}$$

The next proposition summarizes properties of $\varphi_\mu(t)$. The first one is to recognize the series in (4.3) from (2.1). The zeros $\{j_{\mu,k}\}$ of the Bessel function of the first kind

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{4.4}$$

appear in the factorization of $\varphi_\mu$ in view of the relation $I_\mu(z) = e^{-\pi i \mu/2} J_\mu(iz)$.

**Proposition 4.1.** *The moment generating function $\varphi_\mu(t)$ of a random variable $X \sim f_\mu$ is given by*

$$\varphi_\mu(t) = \Gamma(\mu+1) \left(\frac{2}{t}\right)^\mu I_\mu(t). \tag{4.5}$$

*Note* 4.2. The Catalan numbers $C_n$ appear as the even-order moments of $f_\mu$ when $\mu = 1$. More precisely, if $X$ is distributed as $f_1$ (written as $X \sim f_1$), then

$$\mathbb{E}\left[(2X)^{2n}\right] = C_n \text{ and } \mathbb{E}\left[(2X)^{2n+1}\right] = 0. \tag{4.6}$$

*Note* 4.3. The moment generating function of $f_\mu$ admits the Weierstrass product representation

$$\varphi_\mu(t) = \prod_{k=1}^{\infty} \left(1 + \frac{t^2}{j_{\mu,k}^2}\right) \tag{4.7}$$

where $\{j_{\mu,k}\}$ are the zeros of the Bessel function of the first kind $J_\mu$.

**Definition 4.4.** The *cumulant generating function* is

$$\begin{aligned} \psi_\mu(t) &= \log \varphi_\mu(t) \\ &= \log\left(\sum_{n=0}^{\infty} \mathbb{E}\left[X^n\right] \frac{t^n}{n!}\right). \end{aligned}$$

The product representation of $\varphi_\mu(t)$ yields

$$\begin{aligned} \log \varphi_\mu(t) &= \sum_{k=1}^{\infty} \log\left(1 + \frac{t^2}{j_{\mu,k}^2}\right) \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{t}{j_{\mu,k}}\right)^{2n} \\ &:= \sum_{n=1}^{\infty} \kappa_\mu(n) \frac{t^n}{n!}. \end{aligned}$$

The series converges for $|t| < j_{\mu,1}$. The first Bessel zero satisfies $j_{\mu,1} > 0$ for all $\mu \geq 0$. It follows that the series has a non-zero radius of convergence.

*Note* 4.5. The coefficient $\kappa_\mu(n)$ is the $n$-th *cumulant* of $X$. An expression that links the moments to the cumulants of $X$ is provided by V. P. Leonov and A. N. Shiryaev [20]:

$$\kappa_\mu(n) = \sum_{\mathcal{V}} (-1)^{k-1} (k-1)! \prod_{i=1}^{k} \mathbb{E}(2X)^{|V_i|} \tag{4.8}$$

where the sum is over all partitions $\mathcal{V} = \{V_1, \cdots, V_k\}$ of the set $\{1, 2, \dots, n\}$.

In the case $\mu = 0$ the moments are Catalan numbers or 0, in the case $\mu = 1$ the moments are central binomial coefficients or 0. Therefore, in both cases, the cumulants $\kappa_\mu(n)$ are integers. An expression for the general value of $\mu$ involves

$$\zeta_\mu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\mu,k}^s} \tag{4.9}$$

the *Bessel zeta function*, sometimes referred as the *Rayleigh function*.

The next result gives an expression for the cumulants of a random variable $X$ with a distribution $f_\mu$. The special case $\mu = 1$, described in the next section, provides the desired probabilistic interpretation of the original sequence $A_n$.

**Theorem 4.6.** *Let $X \sim f_\mu$. Then*

$$\kappa(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2(-1)^{n/2+1}(n-1)!\zeta_\mu(n) & \text{if } n \text{ is even.} \end{cases} \tag{4.10}$$

*Proof.* Rearranging the expansion in Definition 4.4 gives

$$\begin{aligned} \log \varphi_\mu(t) &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{t}{j_{\mu,k}}\right)^{2n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^{2n} \sum_{k=1}^{\infty} \frac{1}{j_{\mu,k}^{2n}}. \end{aligned}$$

Now compare powers of $t$ in this expansion with the definition

$$\log \varphi_\mu(t) = \sum_{n=1}^{\infty} \kappa_\mu(n) \frac{t^n}{n!} \tag{4.11}$$

to obtain the result. □

The next ingredient in the search for an interpretation of the sequence $A_n$ is the notion of conjugate random variables. The properties described below appear in [25]. A complex-valued random variable $Z$ is called a *regular random variable* (rrv for short) if $\mathbb{E}|Z|^n < \infty$ for all $n \in \mathbb{N}$ and

$$\mathbb{E}[h(Z)] = h(\mathbb{E}[Z]) \tag{4.12}$$

for all polynomials $h$. The class of rrv is closed under compositions with polynomials (if $Z$ is rrv and $P$ is a polynomial, then $P(Z)$ is rrv) and it is also closed under addition of independent rrv. The basic definition is stated next.

**Definition 4.7.** Let $X, Y$ be real random variables, not necessarily independent. The pair $(X, Y)$ is called *conjugate random variables* if $Z = X + iY$ is an rrv. The random variable $X$ is called *self-conjugate* if $Y$ has the same distribution as $X$.

The property of rrv's may be expressed in terms of the function

$$\Phi(\alpha,\beta) := \mathbb{E}\left[\exp(i\alpha X + i\beta Y)\right]$$

The next theorem gives a condition for $Z = X + iY$ to be an rrv. The random variables $X$ and $Y$ are not necessarily independent.

**Theorem 4.8.** *Let $Z = X + iY$ be a complex valued random variable with $\mathbb{E}[Z] = 0$ and $\mathbb{E}[Z^n] < \infty$. Then $Z$ is an rrv if and only if $\Phi(\alpha, i\alpha) = 1$ for all $\alpha \in \mathbb{C}$.*

This is now reformulated for real and independent random variables.

**Theorem 4.9.** *Let $X$, $Y$ be independent real valued random variables with finite moments. Define the characteristic functions of $X$ and $Y$ as*

$$\Phi_X(\alpha)=\mathbb{E}\left[e^{i\alpha X}\right]=\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!}\mathbb{E}\left[X^n\right] \text{ and } \Phi_Y(\beta)=\mathbb{E}\left[e^{i\alpha Y}\right]=\sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}\mathbb{E}\left[Y^n\right].$$

*Then $Z=X+iY$ is an rrv with mean zero if and only if $\Phi_X(\alpha)\Phi_Y(i\alpha)=1$ for every $\alpha\in\mathbb{R}$.*

**Example 4.10.** Let $X$ and $Y$ be independent Gaussian variables with zero mean and same variance $\sigma^2$. Then $X$ and $Y$ are conjugate since

$$\varphi_X(t)=\exp\left(\frac{\sigma^2}{2}t^2\right) \text{ and } \varphi_{iY}(t)=\exp\left(-\frac{\sigma^2}{2}t^2\right).$$

*Note* 4.11. Suppose $Z=X+iY$ is a rrv with $\mathbb{E}[Z]=0$ and $z\in\mathbb{C}$. The condition (4.12) becomes

$$\mathbb{E}\left[h(z+X+iY)\right]=h(z). \tag{4.13}$$

Given a sequence of polynomials $\{Q_n(z)\}$ such that $\deg(Q_n)=n$ and with leading coefficient 1, an elementary argument shows that there is a unique sequence of coefficients $\alpha_{j,n}$ such that the relation

$$Q_{n+1}(z)-zQ_n(z)=\sum_{j=0}^{n}\alpha_{j,n}Q_j(z) \tag{4.14}$$

holds. This section discusses this recurrence for the sequence of polynomials

$$P_n(z):=\mathbb{E}(z+X)^n \tag{4.15}$$

associated to a random variable $X$. The polynomial $P_n$ is of degree $n$ and has leading coefficient 1. It is shown that if the cumulants of odd order vanish, then the even order cumulants provide the coefficients $\alpha_{j,n}$ for the recurrence (4.14).

**Theorem 4.12.** *Let $X$ be a random variable with cumulants $\kappa(m)$. Assume the odd-order cumulants vanish and that $X$ has a conjugate random variable $Y$. Define the polynomials*

$$P_n(z)=\mathbb{E}\left[(z+X)^n\right]. \tag{4.16}$$

*Then $P_n$ satisfies the recurrence*

$$P_{n+1}(z)-zP_n(z)=\sum_{m\ge1}\binom{n}{2m-1}\kappa(2m)P_{n-2m+1}(z). \tag{4.17}$$

*Proof.* Let $X_1$, $X_2$ independent copies of $X$. Then

$$\mathbb{E}\left[X_1\left((X_1+iY_1+z+X_2)^n-(z+X_2)^n\right)\right]=$$
$$=\sum_{j=0}^{n}\binom{n}{j}\mathbb{E}\left[X_1(X_1+iY_1)^j(z+X_2)^{n-j}\right]-\mathbb{E}\left[X_1(z+X_2)^n\right].$$

This last expression becomes

$$\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}\left[X_1(X_1+iY_1)^j(z+X_2)^{n-j}\right]=\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}\left[X_1(X_1+iY_1)^j\right]\mathbb{E}\left[(z+X_2)^{n-j}\right].$$

On the other hand

$$\mathbb{E}\left[X_1\left((X_1+z+X_2+iY_1)^n-(z+X_2)^n\right)\right]= \sum_{r=0}^{n}\binom{n}{r}\mathbb{E}\left[X_1(X_1+z)^{n-r}\right]\mathbb{E}\left[(X_2+iY_1)^r\right]-\mathbb{E}\left[X_1(z+X_2)^n\right].$$

The cancellation property (4.28) shows that the only surviving term in the sum corresponds to $r=0$, therefore

$$\mathbb{E}\left[X_1\left((X_1+z+X_2+iY_1)^n-(z+X_2)^n\right)\right]=\mathbb{E}\left[X_1(X_1+z)^n\right]-\mathbb{E}\left[X_1\right]\mathbb{E}\left[(z+X_2)^n\right]$$

and $\mathbb{E}[X_1]=0$ since $\kappa(1)=0$. This shows the identity

$$\sum_{j=1}^{n}\binom{n}{j}\mathbb{E}\left[X_1(X_1+iY_1)^j\right]\mathbb{E}\left[(z+X_2)^{n-j}\right]=\mathbb{E}\left[X_1(X_1+z)^n\right]. \tag{4.18}$$

The cumulants of $X$ satisfy

$$\kappa(m)=\mathbb{E}X(X+iY)^{m-1}, \quad \text{for } m\geq 1, \tag{4.19}$$

(see Theorem 3.3 in [13]), therefore in the current situation

$$\mathbb{E}\left[X_1(X_1+iY_1)^j\right]=\begin{cases}0 & \text{if } j \text{ is even,}\\ \kappa(2m) & \text{if } j=2m+1 \text{ is odd.}\end{cases} \tag{4.20}$$

On the other hand

$$\begin{aligned}\mathbb{E}\left[X_1(X_1+z)^n\right] &= \mathbb{E}\left[(X_1+z)^{n+1}-z(X_1+z)^n\right]\\ &= P_{n+1}(z)-zP_n(z).\end{aligned}$$

Replacing in (4.18) yields the result. □

Recall that a random variable has a Laplace distribution if its probability density function is

$$f_L(x)=\tfrac{1}{2}e^{-|x|}. \tag{4.21}$$

Assume $X_\mu$ has a probability density function $f_\mu$ defined in (1.21) and moment generating function given by (4.7). The next lemma constructs a random variable $Y_\mu$ conjugate to $X_\mu$.

**Lemma 4.13.** *Let $Y_{\mu,n}$ be a random variable defined by*

$$Y_{\mu,n}=\sum_{k=1}^{n}\frac{L_k}{j_{\mu,k}} \tag{4.22}$$

*where $\{L_k : k\in\mathbb{N}\}$ is a sequence of independent, identically distributed Laplace random variables. Then $\lim_{n\to\infty} Y_{\mu,n}=Y_\mu$ exists and is a random variable with continuous probability density. Moreover, the moment generating function of $iY_\mu$ is*

$$\mathbb{E}\left[e^{itY_\mu}\right]=\prod_{k=1}^{\infty}\left(1+\frac{t^2}{j_{\mu,k}^2}\right)^{-1}. \tag{4.23}$$

*the reciprocal of the moment generating function of $f_\mu$ given in* (4.7).

*Proof.* The characteristic function of a Laplace random variable $\dfrac{iL_k}{j_{\mu,k}}$ is

$$\varphi_{iL_k}(t) = \frac{1}{1+\frac{t^2}{j_{\mu,k}^2}}. \tag{4.24}$$

The values

$$\mathbb{E}\left[\frac{L_k}{j_{\mu,k}}\right] = 0, \text{ and } \mathbb{E}\left[\frac{L_k^2}{j_{\mu,k}^2}\right] = \frac{2}{j_{\mu,k}^2}, \tag{4.25}$$

guarantee the convergence of the series

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\frac{L_k}{j_{\mu,k}}\right] \text{ and } \sum_{k=1}^{\infty} \mathbb{E}\left[\frac{L_k^2}{j_{\mu,k}^2}\right]. \tag{4.26}$$

(The last series evaluates to $1/(2\mu+2)$). This ensures the existence of the limit defining $Y_\mu$ (see [17] for details). The continuity of the limiting probability density $Y_\mu$ is ensured by the fact that at least one term (in fact all) in the defining sum has a continuous probability density that is of bounded variation. □

*Note* 4.14. In the case $X_\mu \sim f_\mu$ is independent of $Y_\mu$, then the conjugacy property states that if $h$ is an analytic function in a neighborhood $\mathcal{O}$ of the origin, then

$$\mathbb{E}\left[h(z+X_\mu+iY_\mu)\right] = h(z), \quad \text{for } z \in \mathcal{O}. \tag{4.27}$$

In particular

$$\mathbb{E}\left[(X_\mu+iY_\mu)^n\right] = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

*Note* 4.15. In the special case $\mu = n/2-1$ for $n \in \mathbb{N}$, $n \geq 3$, the function (4.23) has been characterized in [11] as the moment generating function of the total time $T_n$ spent in the sphere $S^{n-1}$ by an $n$-dimensional Brownian motion starting at the origin.

## 5. The Narayana polynomials and the sequence $A_n$

The result of Theorem 4.12 is now applied to a random variable $X \sim f_1$. In this case the polynomials $P_n$ correspond, up to a change of variable, to the Narayana polynomials $\mathcal{N}_n$. The recurrence established by M. Lasalle comes from the results in Section 4. In particular, this provides an interpretation of the sequence $\{A_n\}$ in terms of cumulants and the Bessel zeta function.

Recall the probability density function $f_1$

$$f_1(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & \text{for } |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

**Lemma 5.1.** *Let $X \sim f_1$. The Narayana polynomials appear as the moments*

$$\mathcal{N}_r(z) = \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{r-1}\right], \tag{5.2}$$

*for $r \geq 1$.*

*Proof.* The binomial theorem gives

$$\mathbb{E}\left[(1+z+2\sqrt{z}X)^{r-1}\right] = \sum_{j=0}^{r-1}\binom{r-1}{j}(z+1)^{r-1-j}z^{j/2}\mathbb{E}\left[(2X)^j\right].$$

The result now follows from (4.6) and (1.13). □

In order to apply Theorem 4.12 consider the identities

$$\begin{aligned}\mathcal{N}_r(z) &= \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{r-1}\right] \\ &= (2\sqrt{z})^{r-1}\mathbb{E}\left[(X+z_*)^{r-1}\right] \\ &= (2\sqrt{z})^{r-1}P_{r-1}(z_*),\end{aligned} \tag{5.3}$$

with

$$z_* = \frac{1+z}{2\sqrt{z}}. \tag{5.4}$$

The recurrence (4.17) applied to the polynomial $P_n(z_*)$ yields

$$\frac{\mathcal{N}_{n+2}(z)}{(2\sqrt{z})^{n+1}} - \frac{(1+z)}{2\sqrt{z}}\frac{\mathcal{N}_{n+1}(z)}{(2\sqrt{z})^{n}} = \sum_{m\geq 1}\binom{n}{2m-1}\kappa(2m)\frac{\mathcal{N}_{n-2m+2}(z)}{(2\sqrt{z})^{n-2m+1}} \tag{5.5}$$

that reduces to

$$(1+z)\mathcal{N}_r(z) - \mathcal{N}_{r+1}(z) = -\sum_{m\geq 1}\binom{r-1}{2m-1}\kappa(2m)2^{2m}z^m\mathcal{N}_{r+1-2m}(z), \tag{5.6}$$

by using $r=n+1$. This recurrence has the form of (1.12).

**Theorem 5.2.** *Let $X \sim f_1$. Then the coefficients $A_n$ in Definition 1.3 are given by*

$$A_n = (-1)^{n+1}\kappa(2n)2^{2n}. \tag{5.7}$$

The expression in (4.10) gives the next result.

**Corollary 5.3.** *Let*

$$\zeta_\mu(s) = \sum_{k=1}^{\infty}\frac{1}{j_{\mu,k}^s} \tag{5.8}$$

*be the Bessel zeta function. Then the coefficients $A_n$ are given by*

$$A_n = 2^{2n+1}(2n-1)!\zeta_1(2n).$$

The scaled coefficients $a_n$ are now expressed in terms of the Bessel zeta function.

**Corollary 5.4.** *The coefficients $a_n$ are given by*

$$a_n = 2^{2n+1}(n+1)!(n-1)!\zeta_1(2n). \tag{5.9}$$

*Note* 5.5. This expression for the coefficients and the recurrence

$$(n+\mu)\zeta_\mu(2n) = \sum_{r=1}^{n-1}\zeta_\mu(2r)\zeta_\mu(2n-2r). \tag{5.10}$$

given in [15], provide a new proof of the recurrence in Proposition (2.2).

## 6. The generalized Narayana polynomials

The Narayana polynomials $\mathcal{N}_r(z)$, defined in (1.1), have been expressed as the moments

$$\mathcal{N}_r(z) = \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{r-1}\right], \tag{6.1}$$

for $r \geq 1$. Here $X$ is a random variable with density function $f_1$. This suggests the extension

$$\mathcal{N}_n^{\mu}(z) = \mathbb{E}\left[\left(1+z+2\sqrt{z}X\right)^{n-1}\right], \tag{6.2}$$

with $X \sim f_\mu$. Therefore, $\mathcal{N}_n = \mathcal{N}_n^1$.

*Note* 6.1. The same argument given in (5.6) gives the recurrence

$$(1+z)\mathcal{N}_r^{\mu}(z) - \mathcal{N}_{r+1}^{\mu}(z) = -\sum_{m\geq 1}\binom{r-1}{2m-1}\kappa(2m)2^{2m}z^m\mathcal{N}_{r+1-2m}^{\mu}(z), \tag{6.3}$$

where $\kappa(2n)$ are the cumulants of $X \sim f_\mu$. Theorem 5.2 gives an expression for the generalization of the Lasalle numbers:

$$A_n^{\mu} := (-1)^{n+1}\kappa(2n)2^{2n} \tag{6.4}$$

and the corresponding expression in terms of the Bessel zeta function:

$$A_n^{\mu} := 2^{2n+1}(2n-1)!\zeta_\mu(2n). \tag{6.5}$$

The generalized Narayana polynomials are now expressed in terms of the Gegenbauer polynomials $C_n^{\mu}(x)$ defined by the generating function

$$\sum_{n=0}^{\infty} C_n^{\mu}(x)t^n = (1-2xt+t^2)^{-\mu}. \tag{6.6}$$

These polynomials admit several hypergeometric representations:

$$\begin{aligned} C_n^{\mu}(x) &= \frac{(2\mu)_n}{n!}\,{}_2F_1\left(-n, n+2\mu; \mu+\tfrac{1}{2}; \frac{1-x}{2}\right) \\ &= \frac{2^n(\mu)_n}{n!}(x-1)^n\,{}_2F_1\left(-n, -n-\mu+\tfrac{1}{2}; -2n-2\mu+1; \frac{2}{1-x}\right) \\ &= \frac{(2\mu)_n}{n!}\left(\frac{x+1}{2}\right)^n{}_2F_1\left(-n, -n-\mu+\tfrac{1}{2}; \mu+\tfrac{1}{2}; \frac{x-1}{x+1}\right). \end{aligned} \tag{6.7}$$

The connection between Narayana and Gegenbauer polynomials comes from the expression for $C_n^{\mu}(z)$ given in the next proposition.

**Proposition 6.2.** *The Gegenbauer polynomials are given by*

$$C_n^{\mu}(z) = \frac{(2\mu)_n}{n!}\mathbb{E}\left[\left(z+\sqrt{z^2-1}X_{\mu-1/2}\right)^n\right]. \tag{6.8}$$

*Proof.* The Laplace integral representation

$$C_n^{\mu}(\cos\theta) = \frac{\Gamma(n+2\mu)}{2^{2\mu-1}n!\Gamma^2(\mu)}\int_0^{\pi}\left(\cos\theta + i\sin\theta\cos\phi\right)^n\sin^{2\mu-1}\phi\,d\phi \tag{6.9}$$

appears as Theorem 6.7.4 in [3]. The change of variables $z=\cos\theta$ and $X=\cos\phi$ gives

$$\begin{aligned} C_n^{\mu}(z) &= \frac{\Gamma(n+2\mu)}{2^{2\mu}n!\Gamma^2(\mu)}\int_{-1}^{1}\left(z+\sqrt{z^2-1}X\right)^n(1-X^2)^{\mu-1}\,dX \\ &= \frac{(2\mu)_n}{n!}\mathbb{E}\left[\left(z+\sqrt{z^2-1}X_{\mu-1/2}\right)^n\right], \end{aligned}$$

as claimed. Since this is a polynomial identity in $z$, it can be extended to all $z \in \mathbb{C}$. □

**Theorem 6.3.** *The Gegenbauer polynomial $C_n^{\mu}$ and the generalized polynomial $\mathcal{N}_n^{\mu}$ satisfy the relation*

$$\mathcal{N}_{n+1}^{\mu}(z) = \frac{n!}{(2\mu+1)_n}(1-z)^n C_n^{\mu+\frac{1}{2}}\left(\frac{1+z}{1-z}\right). \tag{6.10}$$

*Proof.* Introduce the variable

$$Z = \frac{1+z}{1-z} \tag{6.11}$$

so that

$$z = \frac{Z-1}{Z+1} \text{ and } \frac{Z}{\sqrt{Z^2-1}} = \frac{1+z}{2\sqrt{z}}. \tag{6.12}$$

Then

$$\begin{aligned} C_n^{\mu+\frac{1}{2}}\left(\frac{1+z}{1-z}\right) &= \frac{(2\mu+1)_n}{n!}\left(\frac{2\sqrt{z}}{1-z}\right)^n \mathbb{E}\left[\left(\frac{1+z}{2\sqrt{z}} + X_{\mu}\right)^n\right] \\ &= \frac{(2\mu+1)_n}{n!\,(1-z)^n}\mathbb{E}\left[\left(1+z+2\sqrt{z}X_{\mu}\right)^n\right] \\ &= \frac{(2\mu+1)_n}{n!\,(1-z)^n}\mathcal{N}_{n+1}^{\mu}(z), \end{aligned}$$

using $Z^2-1 = 4z/(1-z)^2$. □

The expression (6.7) now provides hypergeometric expressions for the original Narayana polynomials

$$\mathcal{N}_{n+1}(z) = \frac{2(1-z)^n}{(n+2)(n+1)}C_n^{3/2}\left(\frac{1+z}{1-z}\right). \tag{6.13}$$

**Corollary 6.4.** *The Narayana polynomials are given by*

$$\begin{aligned} \mathcal{N}_{n+1}(z) &= (1-z)^n {}_2F_1\left(-n, n+3; 2; \frac{z}{z-1}\right) \\ &= \frac{(2n+2)!}{(n+2)!\,(n+1)!}z^n {}_2F_1\left(-n, -n-1; -2n-2; \frac{z-1}{z}\right) \\ &= {}_2F_1(-n, -n-1; 2; z). \end{aligned} \tag{6.14}$$

*Each one of these identities yields a corresponding representation as a finite sum:*

$$\begin{aligned} \mathcal{N}_{n+1}(z) &= \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n+k+2}{k}z^k(1-z)^{n-k} \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\binom{2n+2-k}{n-k}z^{n-k}(1-z)^k \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1}{k}z^k. \end{aligned} \tag{6.15}$$

Note that the first two expressions coincide up to the change of summation variable $k \to n-k$ while the third identity is nothing but (1.1).

*Note* 6.5. The representation

$$C_n^{\mu}(z) = \frac{(\mu)_n}{n!}(2x)^n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n-\mu; \frac{1}{x^2}\right) \tag{6.16}$$

that appears in as 6.4.12 in [3], gives the expression

$$\mathcal{N}_{n+1}(z) = \frac{(2n+2)!}{(n+1)!\,(n+2)!}\left(\frac{1+z}{2}\right)^n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; -n-\frac{1}{2}; \left(\frac{1-z}{1+z}\right)^2\right) \tag{6.17}$$

that yields the finite sum representation

$$\mathcal{N}_{n+1}(z) = \frac{1}{2^{n-1}(n+2)} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k}\binom{2n+1-2k}{n-2k}(1-z)^{2k}(1+z)^{n-2k}. \tag{6.18}$$

*Note* 6.6. The polynomials $S_n(z) = z\mathcal{N}_n^1(z)$ satisfy the symmetry identity

$$S_n(z) = z^{n+1} S_n(z^{-1}). \tag{6.19}$$

These polynomials were expressed in [21] as

$$S_n(z) = (z-1)^{n+1} \int_0^{z/(z-1)} P_n(2x-1)\, dx \tag{6.20}$$

where $P_n(x) = C_n^{1/2}(x)$ are the Legendre polynomials. An equivalent formulation is provided next.

**Theorem 6.7.** *The polynomials* $S_n(z)$ *are given by*

$$S_n(z) = \frac{1}{2^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n+1-k}\binom{2n-2k}{n-k}\binom{n+1-k}{k}(1-z)^{2k}(1+z)^{n+1-2k}.$$

*Proof.* The integration rule

$$\int C_n^{\mu}(x)\, dx = \frac{1}{2(\mu-1)} C_{n+1}^{\mu-1}(x) \tag{6.21}$$

implies

$$\int_0^{z/(z-1)} C_n^{1/2}(2x-1)\, dx = -\frac{1}{2} C_{n+1}^{-1/2}\left(\frac{z+1}{z-1}\right), \tag{6.22}$$

since the generating function

$$\sum_{n=0}^{\infty} t^n C_n^{-1/2}(z) = (1-2zt+t^2)^{1/2} \tag{6.23}$$

gives $C_{n+1}^{-1/2}(-1) = 0$ for $n > 1$. Then (6.20) yields

$$S_n(z) = -\frac{1}{2}(z-1)^{n+1} C_{n+1}^{-1/2}\left(\frac{z+1}{z-1}\right). \tag{6.24}$$

A classical formula for the Gegenbauer polynomials states

$$C_n^{\mu}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!} \frac{(\mu)_{n-k}}{(n-2k)!}(2z)^{n-2k} \tag{6.25}$$

and the identity

$$\left(-\frac{1}{2}\right)_k = -\frac{1}{2^{2k-1}} \frac{(2k-2)!}{(k-1)!}$$

produce

$$C_n^{-1/2}(z) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{k+1}}{n-k}\binom{2n-2k-2}{n-k-1}\binom{n-k}{k} z^{n-2k}. \tag{6.26}$$

The result now follows from (6.24). ∎

## 7. THE GENERALIZATION OF THE NUMBERS $a_n$

The terms forming the original suggestion of Zeilberger

$$a_n = \frac{2A_n}{C_n} \tag{7.1}$$

have been given a probabilistic interpretation: let $X$ be a random variable with a symmetric probability distribution function. The numerator $A_n$ is

$$A_n = (-1)^{n+1}\kappa(2n)2^{2n} \tag{7.2}$$

where $\kappa(2n)$ is the even-order cumulant of the scaled random variable $X_* = 2X$. The denominator $C_n$ is interpreted as the even-order moment of $X_*$:

$$C_n = \mathbb{E}\left[X_*^{2n}\right]. \tag{7.3}$$

These notions are used now to define an extension of the coefficients $a_n$.

**Definition 7.1.** Let $X$ be a random variable with vanishing odd cumulants. The numbers $a_n$ are defined by

$$a_n = \frac{2(-1)^{n+1}\kappa(2n)}{\mathbb{E}\left[X_*^{2n}\right]}. \tag{7.4}$$

In the special case $X$ has a symmetric beta distribution with parameter $\mu$, that is $X \sim f_\mu$ and $X_* = 2X$, these numbers are computed using the cumulants

$$\kappa_\mu(2n) = (-1)^{n+1}2^{2n+1}(2n-1)!\zeta_\mu(2n) \tag{7.5}$$

and the even order moments

$$\mathbb{E}\left[X_*^{2n}\right] = \frac{(2n)!}{n!}\frac{1}{(\mu+1)_n} \tag{7.6}$$

to produce

$$a_n = a_n(\mu) = 2^{2n+1}\,(n-1)!\,(\mu+1)_n\,\zeta_\mu(2n). \tag{7.7}$$

The value

$$\zeta_\mu(2) = \frac{1}{4(\mu+1)} \tag{7.8}$$

yields the initial condition $a_1(\mu) = 2$.

The recurrence (5.10) now provides the next result. Recall that when $x$ is not necessarily a positive integer, the binomial coefficient is given by

$$\binom{x}{k} = \frac{\Gamma(x+1)}{\Gamma(x-k+1)\,k!}. \tag{7.9}$$

**Proposition 7.2.** *The coefficients $a_n(\mu)$ satisfy the recurrence*

$$a_n(\mu) = \frac{1}{2\binom{n+\mu-1}{n-1}}\sum_{k=1}^{n-1}\binom{n+\mu-1}{n-k-1}\binom{n+\mu-1}{k-1}a_k(\mu)a_{n-k}(\mu), \tag{7.10}$$

*with initial condition $a_1(\mu) = 2$.*

*Proof.* Start with the convolution identity for Bessel zeta functions (5.10) and replace each zeta function by its expression in terms of $a_n(\mu)$ from (7.7), which gives

$$(n+\mu)\frac{a_n(\mu)}{2^{2n+1}(n-1)!(\mu+1)_n} = \sum_{k=1}^{n-1} \frac{a_k(\mu)}{2^{2k+1}(k-1)!(\mu+1)_k}\,\frac{a_{n-k}(\mu)}{2^{2n-2k+1}(n-k-1)!(\mu+1)_{n-k}}$$

and after simplification

$$a_n(\mu) = \frac{1}{2(n+\mu)}\sum_{k=1}^{n-1}\frac{(n-1)!}{(k-1)!(n-k-1)!}\,\frac{(\mu+1)_n}{(\mu+1)_k(\mu+1)_{n-k}}a_k(\mu)a_{n-k}(\mu).$$

The result now follows by elementary algebra. □

*Note* 7.3. In the case $\mu = 1$, the recurrence (7.10) becomes (2.10) and the coefficients $a_n(1)$ are the original numbers $a_n$.

*Note* 7.4. The recurrence (7.10) can be written as

$$a_n(\mu) = \frac{1}{2}\sum_{k=1}^{n-1}\frac{\Gamma(n)\Gamma(\mu+1)\Gamma(n+\mu)}{\Gamma(\mu+k+1)\Gamma(n+\mu-k+1)\Gamma(n-k)\Gamma(k)}a_k(\mu)a_{n-k}(\mu).$$

**Theorem 7.5.** *The coefficients $a_n(\mu)$ are positive and increasing for $n \geq \lfloor \frac{\mu+3}{2} \rfloor$.*

*Proof.* The positivity is clear from (7.7). Now take the terms corresponding to $k = 1$ and $k = n-1$ in (7.10) to obtain

$$a_n(\mu) \geq \frac{n-1}{\mu+1}a_1(\mu)a_{n-1}(\mu) = \frac{2(n-1)}{\mu+1}a_{n-1}(\mu). \tag{7.11}$$

This yields

$$a_n(\mu) - a_{n-1}(\mu) \geq \frac{2n-3-\mu}{\mu+1}a_{n-1}(\mu) \tag{7.12}$$

and the result follows. □

Some other special cases are considered next.

**The case** $\mu = 0$. In this situation one obtains the arcsine probability distribution function given by

$$f_0(x) = \begin{cases} \frac{1}{\pi}\,\frac{1}{\sqrt{1-x^2}}, & \text{for } |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{7.13}$$

By the recurrence on the $\zeta_0$ function, the coefficients

$$a_n(0) = 2^{2n}(n-1)!\,n!\,\zeta_0(2n) \tag{7.14}$$

satisfy

$$a_n(0) = \frac{1}{2}\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n-1}{k-1}a_k(0)a_{n-k}(0) \tag{7.15}$$

with $a_1(0) = 2$. Now define as Lasalle $b_n = \frac{1}{2}a_n(0)$. Then (7.15) becomes

$$\begin{aligned} b_n &= \sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n-1}{k-1}b_kb_{n-k}, \\ b_1 &= 1. \end{aligned} \tag{7.16}$$

In particular $b_n$ is a positive integer.

The following comments are obtained by an analysis similar to that for $a_n$.

*Note* 7.6. The recurrence

$$\sum_{j=1}^{n} (-1)^{j-1}\binom{n}{j}\binom{n-1}{j-1} b_j = 1$$

gives the generating function

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1} b_j}{j!} \frac{x^{2j-2}}{(j-1)!} = \frac{I_1(2x)}{x\, I_0(2x)} = \frac{1}{2x}\frac{d}{dx}\log I_0(2x).$$

*Note* 7.7. The sequence $b_n$ admits a determinant representation $b_n = \det(M_n)$, where

$$M_n = \begin{pmatrix} 1 & \binom{1}{1}\binom{1-1}{1-1} & 0 & 0 & \cdots & 0 \\ 1 & \binom{2}{1}\binom{2-1}{1-1} & \binom{2}{2}\binom{2-1}{2-1} & 0 & \cdots & 0 \\ 1 & \binom{3}{1}\binom{3-1}{1-1} & \binom{3}{2}\binom{3-1}{2-1} & \binom{3}{3}\binom{3-1}{3-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{n}{1}\binom{n-1}{1-1} & \binom{n}{2}\binom{n-1}{2-1} & \binom{n}{3}\binom{n-1}{3-1} & \cdots & \binom{n}{n-1}\binom{n-1}{n-2} \end{pmatrix}. \tag{7.17}$$

*Note* 7.8. The identity $I_2(x) = I_0(x) - \frac{2}{x} I_1(x)$ expressed as

$$\frac{I_1(2x)}{xI_0(2x)}\left[1 + \frac{1}{2}x^2\frac{2I_2(2x)}{xI_1(2x)}\right] = 1 \tag{7.18}$$

provides the relation

$$b_n = \frac{1}{2}\sum_{j=1}^{n-1}\binom{n-1}{j}\binom{n}{j-1} b_j a_{n-j}. \tag{7.19}$$

**The case** $\mu = \frac{1}{2}$. Now one obtains the uniform distribution on $[-1,1]$ with even moments

$$\mathbb{E}(2X)^{2n} = \mathbb{E}X_*^{2n} = \frac{2^{2n}}{2n+1} \tag{7.20}$$

and vanishing odd moments. The sequence of cumulants is

$$\kappa_{1/2}(2n) = 2(-1)^{n+1}(2n-1)!\zeta_{1/2}(2n) \tag{7.21}$$

where the Bessel zeta function is

$$\zeta_{1/2}(2n) = \sum_{k=1}^{\infty}\frac{1}{\pi^{2n}k^{2n}} = \frac{1}{\pi^{2n}}\zeta(2n) = \frac{2^{2n-1}}{(2n)!}|B_{2n}|, \tag{7.22}$$

with $B_n$ the Bernoulli numbers. This follows from the identity

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \tag{7.23}$$

and it yields

$$\kappa_{1/2}(2n) = 2^{2n}\frac{B_{2n}}{2n} \text{ and } \kappa_{1/2}(2n+1) = 0, \tag{7.24}$$

with $\kappa_{1/2}(0) = 0$. These are the coefficients of $u^n/n!$ in the cumulant moment generating function

$$\log\varphi_{1/2}(u) = \log\frac{\sinh u}{u} = \frac{1}{6}u^2 - \frac{1}{180}u^4 + \frac{1}{2835}u^6 + \cdots. \tag{7.25}$$

Finally, the corresponding sequence

$$a_n\left(\tfrac{1}{2}\right) = \frac{2(-1)^{n+1}\kappa(2n)}{\mathbb{E}\left[X_*^{2n}\right]} \tag{7.26}$$

is given by

$$a_n\left(\tfrac{1}{2}\right) = 2^{2n}\frac{2n+1}{n}|B_{2n}|. \tag{7.27}$$

The first few terms are

$$a_1\left(\tfrac{1}{2}\right) = 2,\; a_2\left(\tfrac{1}{2}\right) = \frac{4}{3},\; a_3\left(\tfrac{1}{2}\right) = \frac{32}{9},\; a_4\left(\tfrac{1}{2}\right) = \frac{96}{5},\; a_5\left(\tfrac{1}{2}\right) = \frac{512}{3}, \tag{7.28}$$

and, as expected, this is an increasing sequence for $n \geq 3$. The convolution identity (5.10) for Bessel zeta functions gives the well-known quadratic relation for the Bernoulli numbers

$$\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k} = -(2n+1)B_{2n}, \quad \text{for } n > 1. \tag{7.29}$$

Moreover, the moment-cumulants relation (1.19) gives, replacing $n$ by $2n$ and after simplification, the other well-known identity for the Bernoulli numbers

$$\sum_{j=1}^{n}\binom{2n+1}{2j}2^{2j}B_{2j} = 2n, \text{ for } n \geq 1. \tag{7.30}$$

*Note* 7.9. A generating function of the sequence $a_n\left(\tfrac{1}{2}\right)$ is given by

$$\frac{I_{3/2}(x)}{xI_{1/2}(x)} = \frac{x\tanh x - 1}{x^2} = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}2a_j\left(\tfrac{1}{2}\right)}{(2j+1)(2j-1)!}x^{2j-2}.$$

**The limiting case** $\mu = -\frac{1}{2}$ produces

$$f_{-1/2}(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1) \tag{7.31}$$

(the discrete Rademacher distribution). For a random variable $X$ with this distribution, the odd moments of $X_* = 2X$ vanish while the even order moments are

$$\mathbb{E}\left[X_*^{2n}\right] = 2^{2n}. \tag{7.32}$$

Therefore

$$\kappa_{-1/2}(2n) = (-1)^{n+1}2^{2n+1}(2n-1)!\zeta_{-1/2}(2n). \tag{7.33}$$

The identity

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \tag{7.34}$$

shows that the zeros are $j_{k,-1/2} = (2k-1)\pi/2$ and therefore

$$\zeta_{-1/2}(2n) = \sum_{k=1}^{\infty}\frac{2^{2n}}{\pi^{2n}(2k-1)^{2n}} = \frac{2^{2n}-1}{\pi^{2n}}\zeta(2n). \tag{7.35}$$

The expression for $\kappa_{-1/2}(2n)$ may be simplified by the relation

$$E_n = -\frac{2}{n+1}(2^{n+1}-1)B_{n+1} \tag{7.36}$$

between the Euler numbers $E_n$ and the Bernoulli numbers. It follows that

$$\kappa_{-1/2}(2n) = -2^{4n-1}E_{2n-1}. \tag{7.37}$$

The corresponding sequence $a_n\left(-\tfrac{1}{2}\right)$ is now given by

$$a_n\left(-\tfrac{1}{2}\right) = (-1)^n2^{2n}E_{2n-1} \tag{7.38}$$

and its first few values are

$$a_1\left(-\tfrac{1}{2}\right)=2,\ a_2\left(-\tfrac{1}{2}\right)=4,\ a_3\left(-\tfrac{1}{2}\right)=32,\ a_4\left(-\tfrac{1}{2}\right)=544,\ a_5\left(-\tfrac{1}{2}\right)=15872.$$

*Note* 7.10. The generating function of the sequence $a_n\left(-\tfrac{1}{2}\right)$ is given by

$$\frac{I_{1/2}(x)}{xI_{-1/2}(x)}=\frac{\tanh x}{x}=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}2a_j\left(-\tfrac{1}{2}\right)}{(2j-1)!}x^{2j-2}.$$

*Note* 7.11. The convolution identity (5.10) yields the well-known quadratic recurrence relation

$$\sum_{k=1}^{n-1}\binom{2n-2}{2k-1}E_{2k-1}E_{2n-2k-1}=2E_{2n-1},\ \text{for } n>1, \tag{7.39}$$

and the moment-cumulant relation (1.19) gives the other well-known identity

$$\sum_{k=1}^{n}\binom{2n-1}{2k-1}2^{2k-1}E_{2k-1}=1,\ \text{for } n\geq 1. \tag{7.40}$$

## 8. Some arithmetic properties of the sequences $a_n$ and $b_n$

Given a sequence of integers $\{x_n\}$ it is often interesting to examine its arithmetic properties. For instance, given a prime $p$, one may consider the $p$-adic valuation $\nu_p(x_n)$, defined as the largest power of $p$ that divides $x_n$. Examples of this process appear in [2] for the Stirling numbers and in [1, 22] for a sequence of coefficients arising from a definite integral.

The statements described below give information about $\nu_p(a_n)$. These results will be presented in a future publication. M. Lasalle [19] established the next theorem by showing that $A_n$ and $C_n$ have the same parity. The fact that the Catalan numbers are odd if and only if $n=2^r-1$ for some $r\geq 2$ provides the proof. This result appears in [14, 18].

**Theorem 8.1.** *The integer $a_n$ is odd if and only if $n=2(2^m-1)$.*

The previous statement may be expressed in terms of the sequence of binary digits of $n$.

**Fact 8.2.** *Let $\{B(n)\}$ be the sequence of binary digits of $n$ and denote $\bar{x}$ a sequence of a arbitrary length consisting of the repetitions of the symbol $x$. The following statements hold (experimentally)*

*1) $\nu_2(a_n)=0$ if and only if $B(n)=\{\bar{1},0\}$.*

*2) $\nu_2(a_n)=1$ if and only if $B(n)=\{\bar{1}\}$ or $\{1,\bar{0}\}$.*

*3) $\nu_2(a_n)=2$ if and only if $B(n)=\{1,0,\bar{1},0\}$.*

The experimental findings for the prime $p=3$ are described next.

**Fact 8.3.** *Suppose $n$ is not of the form $3^m-1$. Then*

$$\nu_3(a_{3n-2})=\nu_3(a_{3n-1})=\nu_3(a_{3n}). \tag{8.1}$$

*Define $w_j=3^j-1$. Suppose $n$ lies in the interval $w_j+1\leq n\leq w_{j+1}-1$. Then*

$$\nu_3(a_{3n+2})=j-\nu_3(n+1). \tag{8.2}$$

*If $n=w_j$, then $\nu_3(a_{3n})=0$.*

*Now assume that $n=3^m-1$. Then*

$$\nu_3(a_{3n})=\nu_3(a_{3n-1})-1=\nu_3(a_{3n-2})-1=m. \tag{8.3}$$

**Fact 8.4.** *The last observation deals with the sequence* $\{a_n(\mu)\}$. *Consider it now as defined by the recurrence* (7.10). *The initial condition* $a_1(\mu) = 2$, *motivated by the origin of the sequence, in general does not provide integer entries. For example, if* $\mu = 2$, *the sequence is*

$$\left\{2, \frac{2}{3}, \frac{8}{9}, \frac{7}{3}, \frac{88}{9}, \frac{1594}{27}, \frac{1448}{3}\right\},$$

*and for* $\mu = 3$

$$\left\{2, \frac{1}{2}, \frac{1}{2}, \frac{39}{40}, 3, \frac{263}{20}, \frac{309}{4}\right\}.$$

*Observe that the denominators of the sequence for* $\mu = 2$ *are always powers of* 3, *but for* $\mu = 3$ *the arithmetic nature of the denominators is harder to predict. On the other hand if in the case* $\mu = 3$ *the initial condition is replaced by* $a_1(3) = 4$, *then the resulting sequence has denominators that are powers of* 5. *This motivates the next definition.*

**Definition 8.5.** Let $x_n$ be a sequence of rational numbers and $p$ be a prime. The sequence is called $p$-integral if the denominators of $x_n$ are powers of $p$.

*Therefore if* $a_1(3) = 4$, *then the sequence* $a_n(3)$ *is* 5*-integral. The same phenomena appears for other values of* $\mu$, *the data is summarized in the next table.*

| $\mu$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $a_1(\mu)$ | 2 | 4 | 10 | 12 | 84 | 264 | 990 | 2860 |
| $p$ | 3 | 5 | 7 | 7 | 11 | 11 | 13 | 17 |

*Note* 8.6. The sequence {2, 4, 10, 12, 84, 264, 990, 2860} does not appear in Sloane's database OEIS.

*This suggests the next conjecture.*

**Conjecture 8.7.** Let $\mu \in \mathbb{N}$. Then there exists an initial condition $a_1(\mu)$ and a prime $p$ such that the sequence $a_n(\mu)$ is $p$-integral.

Some elementary arithmetical properties of $a_n$ are discussed next. A classical result of E. Lucas states that a prime $p$ divides the binomial coefficient $\binom{a}{b}$ if and only if at least one of the base $p$ digits of $b$ is greater than the corresponding digit of $a$.

**Proposition 8.8.** *Assume* $n$ *is odd. Then* $a_n$ *is even.*

*Proof.* Let $n = 2m+1$. The recurrence (2.10) gives

$$\begin{aligned} 2(2m+1)a_{2m+1} &= \sum_{k=1}^{2m} \binom{2m+1}{k-1}\binom{2m+1}{k+1} a_k a_{2m+1-k} \\ &= 2\sum_{k=1}^{m} \binom{2m+1}{k-1}\binom{2m+1}{k+1} a_k a_{2m+1-k}. \end{aligned}$$

For $k$ in the range $1 \le k \le m$, one of the indices $k$ or $2m+1-k$ is odd. The induction argument shows that for each such $k$, either $a_k$ or $a_{2m+1-k}$ is an even integer. This completes the argument. □

**Lemma 8.9.** *Assume* $n = 2^m - 1$. *Then* $\frac{1}{2}a_n$ *is an odd integer.*

*Proof.* Proposition 8.8 shows that $\frac{1}{2}a_n$ is an integer. The relation (1.8) may be written as

$$(-1)^{n-1}a_n = 2 + \frac{1}{n}\sum_{j=1}^{n-1}(-1)^j \binom{n}{j-1}\binom{n+1}{j+1}a_j. \tag{8.4}$$

This implies

$$n\left[(-1)^{n-1}\tfrac{1}{2}a_n - 1\right] = \frac{1}{2}\sum_{j=1}^{n-1}(-1)^j \binom{n}{j-1}\binom{n+1}{j+1}a_j. \tag{8.5}$$

Observe that if $j$ is odd, then $a_j$ is even and $\binom{n+1}{j+1}$ is also even. Therefore the corresponding term in the sum is divisible by 4. If $j$ is even, then Lucas' theorem shows that 4 divides $\binom{n+1}{j+1}$. It follows that the right hand side is an even number. This implies that $\frac{1}{2}a_n$ is odd, as claimed. □

The next statement, which provides the easier part of Theorem 8.1, describes the indices that produce odd values of $a_n$.

**Theorem 8.10.** *If $n = 2(2^m - 1)$, then $a_n$ is odd.*

*Proof.* Isolate the term $j = n/2$ in the identity (8.4) to produce

$$\begin{aligned}\left[(-1)^n a_n + 2\right](2^m - 1) &= \binom{2^{m+1}-2}{2^m-2}\binom{2^{m+1}-1}{2^m}\tfrac{1}{2}a_{n/2} \\ &+ \frac{1}{2}\sum_{j\neq n/2}(-1)^j\binom{n}{j-1}\binom{n+1}{j+1}a_j.\end{aligned}$$

Lemma 8.9 shows that $\frac{1}{2}a_{n/2}$ is odd and the binomial coefficients on the first term of the right-hand side are also odd by Lucas' theorem. Each term of the sum is even because $a_j$ is even if $j$ is odd and for $j$ even $\binom{n}{j-1}$ is even. Therefore the entire right-hand side is even which forces $a_n$ to be odd. □

The final result discussed here deals with the parity of the sequence $b_n$. The main tool is the recurrence

$$b_n = \sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n-1}{k-1}b_k b_{n-k} \tag{8.6}$$

with $b_1 = 1$. Observe that the binomial coefficients appearing in this recurrence are related to the Narayana numbers $N(n,k)$ (1.2) by

$$\binom{n-1}{k}\binom{n-1}{k-1} = (n-1)N(n-1,k-1). \tag{8.7}$$

Arithmetic properties of the Narayana numbers have been discussed by M. Bona and B. Sagan [5]. It is established that if $n = 2^m - 1$ then $N(n,k)$ is odd for $0 \le k \le n-1$; while if $n = 2^m$ then $N(n,k)$ is even for $1 \le k \le n-2$.

The next theorem is the analog of M. Lasalle's result for the sequence $b_n$.

**Theorem 8.11.** *The coefficient $b_n$ is an odd integer if and only if $n = 2^m$, for some $m \ge 0$.*

*Proof.* The first few terms $b_1 = 1$, $b_2 = 1$, $b_3 = 4$ support the base case of an inductive proof. If $n$ is odd, then

$$b_n = (n-1)\sum_{k=1}^{n-1}N(n-1,k-1)b_k b_{n-k} \tag{8.8}$$

shows that $b_n$ is even.

Consider now the case $n=2^m$. Then Lucas' theorem shows that $\binom{2^m-1}{k}\binom{2^m-1}{k-1}$ is odd for all $k$. The inductive step states that $b_k$ is even if $k \neq 2^r$. In the case $k=2^r$, then $b_{n-k}$ is odd if and only if $k=2^{m-1}$, in which case all the terms in (8.8) are even with the single exception $\binom{2^m-1}{2^{m-1}}\binom{2^m-1}{2^{m-1}-1}b_{2^{m-1}}^2$. This shows that $b_n$ is odd.

Finally, if $n=2j$ is even with $j \neq 2^r$, then

$$b_n = \binom{2j-1}{j}\binom{2j-1}{j-1}b_j^2 + 2\sum_{k=1}^{j-1}\binom{n-1}{k}\binom{n-1}{k-1}b_k b_{n-k}. \tag{8.9}$$

Now simply observe that $j \neq 2^r$, therefore $b_j$ is even by induction. It follows that $b_n$ itself is even.

This completes the proof. □

## 9. One final question

Sequences of combinatorial origin often turn out to be unimodal or logconcave. Recall that a sequence $\{x_j : 1 \leq j \leq n\}$ is called *unimodal* if there is an index $m_*$ such that $x_1 \leq x_2 \leq \cdots \leq x_{m_*}$ and $x_{m_*+1} \geq x_{m_*+2} \geq \cdots \geq x_n$. The sequence is called *logconcave* if $x_{n+1}x_{n-1} \leq x_n^2$ and *logconvex* if $x_{n+1}x_{n-1} \geq x_n^2$. An elementary argument shows that a logconcave sequence is always unimodal. The reader will find in [4, 6, 7, 8, 9, 10, 23, 26] a variety of examples of these type of sequences.

**Conjecture 9.1.** The sequences $\{a_n\}$ and $\{b_n\}$ are logconvex.

**Postscript**. After submission of the present work, the authors learned through the preprint [27] a discussion on monotonicity of sequences which usually arise in conjunction with the ratio $\{z_{n+1}/z_n\}$ and root $\{z_n^{1/n}\}$ tests for infinite series. Appealing to such results, the authors of [27] have produced a proof of Conjecture 9.1.

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