GENERALIZATION OF A STATISTIC ON LINEAR DOMINO ARRANGEMENTS

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ABSTRACT. In this paper, we generalize an earlier statistic on square-and-domino tilings by considering only those squares covering a multiple of $k$, where $k$ is a fixed positive integer. We consider the distribution of this statistic jointly with the one that records the number of dominos in a tiling. We derive both finite and infinite sum expressions for the corresponding joint distribution polynomials, the first of which reduces when $k = 1$ to a prior result. The cases $q = 0$ and $q = -1$ are noted for general $k$. Finally, the case $k = 2$ is considered specifically, where further results may be given, including a combinatorial proof when $q = -1$.

1. INTRODUCTION

Let $F_n$ be the Fibonacci number defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. See, for example, sequence A000045 in [12]. Let $G_n = G_n(t)$ be the Fibonacci polynomial defined by $G_n = G_{n-1} + tG_{n-2}$ if $n \geq 2$, with $G_0 = 0$ and $G_1 = 1$; note that $G_n(1) = F_n$ for all $n$. See, for example, [10]. Finally, the $q$-binomial coefficient \( \binom{x}{k}_q \) is defined by

\[
\binom{x}{k}_q = \begin{cases} 
\prod_{i=1}^{k} \frac{1-q^{x-i+1}}{1-q^i}, & \text{if } k \geq 0; \\
0, & \text{if } k < 0.
\end{cases}
\]

Polynomial generalizations of $F_n$ have arisen in connection with statistics on binary words [3], Morse code sequences [4], lattice paths [5], and linear domino arrangements [10, 11]. Let us recall now a statistic related to domino arrangements. If $n \geq 1$, then let $F_n$ denote the set of coverings of the numbers 1, 2, …, $n$, arranged in a row by indistinguishable dominos and indistinguishable squares, where pieces do not overlap, a domino is a rectangular piece covering two numbers, and a square is a piece covering a single number. The members of $F_n$ are also called (linear) tilings or domino arrangements. (If $n = 0$, then $F_0$ consists of the empty tiling having length zero.)

Note that members of $F_n$ correspond uniquely to words in the alphabet $\{d, s\}$ comprising $i$ $d$'s and $n-2i$ $s$'s for some $i$, $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. In what follows, we will frequently identify tilings $c$ by such words $c_1c_2\cdots$. For example, if $n = 4$, then $F_4 = \{dd, dss, sds, ssd, sss\}$. Note that $|F_n| = F_{n+1}$ for all $n$. Given $\pi \in F_n$, let $\rho(\pi)$ denote the sum of the numbers covered by squares in $\pi$. For example, if $n = 15$ and $\pi = sds^2d^2sds^2s \in F_{15}$ (see Figure 1 below), then $\rho(\pi) = 1 + 4 + 5 + 10 + 15 = 35$.

![Figure 1. The tiling $\pi = sds^2d^2sds^2s \in F_{15}$ has $\rho(\pi) = 35.$](image)

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The statistic $\rho$ was introduced in [11], where its distribution was studied on $r$-mino arrangements. Let $v(\pi)$ denote the number of dominos in the tiling $\pi$. Then the joint distribution for the $\rho$ and $v$ statistics on $\mathcal{F}_n$ is given by

$$\sum_{\pi \in \mathcal{F}_n} q^{\rho(\pi)} t^{v(\pi)} = \sum_{j=0}^{n} q^{\left(\frac{n-2j+1}{2}\right)} \binom{n-j}{j} t^j, \quad n \geq 0,$$

where $q$ and $t$ are indeterminates. Equation (1) is the $r = 2$ case (corresponding to square-and-domino tilings) of [11, Theorem 2.1], which is a result on more general $r$-mino arrangements. Here, we will provide a different generalization of (1). Note that (1) reduces to the well-known formula $F_{n+1} = \sum_{j=0}^{n} \binom{n-j}{j}$ when $q = t = 1$.

Recently, generalizations of the Fibonacci sequence have been studied which specify the recurrence for each value of the index mod $k$, where $k$ is a fixed positive integer. For example, the recurrence

$$Q_m = a_j Q_{m-1} + b_j Q_{m-2}, \quad m \equiv j \pmod{k},$$

with $Q_0 = 0$ and $Q_1 = 1$, was considered in [8], where a Binet-like formula is derived. See also [6] for the case when $b_j = 1$ for all $j$ and [13] for the case $k = 2$. These generalizations so far have been studied primarily from an algebraic standpoint such as through the use of generating functions [6] or orthogonal polynomials [8]. In [7], a special case of (2) and a closely related sequence are studied from a more combinatorial viewpoint in terms of statistics on linear tilings and new generalizations of $F_n$ are obtained which extend prior ones.

In this paper, we continue this study by considering a generalization of the $\rho$ statistic defined above, where one looks only at squares that cover multiples of $k$. More precisely, let $\rho_k$ record the sum divided by $k$ of all the multiples of $k$ which are covered by squares within a member of $\mathcal{F}_n$. Note that $\rho_k$ reduces to $\rho$ when $k = 1$.

In the next section, we obtain an explicit formula for all $k$ (see Theorem 2.2 below) for the joint distribution

$$a_n^{(k)}(q, t) := \sum_{\pi \in \mathcal{F}_n} q^{\rho_k(\pi)} t^{v(\pi)}.$$ 

This yields an infinite family of $q$-generalizations for the numbers $G_n(t)$ defined above, and setting $q = 1$ yields seemingly new expressions for $G_n(t)$. When $k = 1$ in our formula, we obtain the explicit expression (1) above, but with a different proof than that given in [11]. We also note some special cases of $q$ and provide an infinite expansion for $a_n^{(k)}(q, t)$ (see Theorem 2.7 below). In the third section, we consider specifically the case $k = 2$, where further combinatorial results may be given. In particular, we provide a combinatorial proof explaining the values of $a_n^{(2)}(-1, 1)$ as well as an explicit expression for the sum of the $\rho_2$ values taken over all the members of $\mathcal{F}_n$. Note that $\rho_2$ records half the sum of the even numbers covered by squares within a tiling.

2. General Formulas

Suppose $k$ is a fixed positive integer. Given $\pi \in \mathcal{F}_n$, let $v(\pi)$ denote the number of dominos of $\pi$ and let $\rho_k(\pi)$ denote the sum divided by $k$ of all the multiples of $k$ covered by squares of $\pi$. For example, if $\pi = s^2 d^3 s d s d s d s d^2 d^2 \in \mathcal{F}_{25}$ (see Figure 2 below), then $v(\pi) = 9$ and

$$\rho_3(\pi) = \frac{9 + 12 + 15 + 21}{3} = 19.$$
If \( q \) and \( t \) are indeterminates, then define the distribution polynomial \( a_n^{(k)}(q, t) \) by
\[
a_n^{(k)}(q, t) := \sum_{\pi \in \mathcal{F}_n} q^{\rho_k(\pi)} t^{v(\pi)}, \quad n \geq 1,
\]
with \( a_n^{(0)}(q, t) := 1 \). For example, if \( n = 6 \) and \( k = 3 \), then
\[
a_6^{(3)}(q, t) = 2t^2 + t^3 + q(1 + t)(t + 2qt + q^2 + q^2t).
\]
Note that \( a_n^{(k)}(1, t) = G_{n+1} \) for all \( k \) and \( n \).

![Figure 2](image_url) The tiling \( \pi = s^d d^3 ds ds d^2 s^d \in \mathcal{F}_5 \) has \( \rho_3(\pi) = 19 \).

In what follows, we will often suppress arguments and write \( a_n \) for \( a_n^{(k)}(q, t) \). Considering whether the last piece within a member of \( \mathcal{F}_n \) is a square or a domino yields the recurrence
\[
a_n = q^n a_{n-1} + ta_{n-2}, \quad n \geq 2,
\]
if \( n \) is divisible by \( k \), and the recurrence
\[
a_n = a_{n-1} + ta_{n-2}, \quad n \geq 2,
\]
if \( n \) is not, with the initial conditions \( a_0 = 1 \) and
\[
a_1 = \begin{cases} q, & \text{if } k = 1; \\ 1, & \text{if } k > 1. \end{cases}
\]

To solve recurrences (3) and (4), we first ascertain an explicit formula for the generating function of the numbers \( a_n \).

**Theorem 2.1.** We have
\[
\sum_{n \geq 0} a_n x^n = \left( \frac{\sum_{r=0}^{k-1} x^r G_{r+1} - tx^k \sum_{r=0}^{k-1} (-tx)^r G_{k-1-r}}{\prod_{i=0}^{j} (1 - 2tq^i x^k G_{k-1} - (-t)^k q^{2i} x^{2k})} \right) \sum_{j \geq 0} \frac{G_k q^{j+1} x^j}{x^{jk}}.
\]

**Proof.** It is more convenient to first consider the generating function for the numbers \( a_n' := a_{n-1}^{(k)}(q, t) \). Then the sequence \( a_n' \) has initial values \( a_0' = 0 \) and \( a_1' = 1 \) and satisfies the recurrences
\[
a_{mk+r}' = a_{mk+r-1}' + ta_{mk+r-2}', \quad 2 \leq r \leq k \quad \text{and} \quad m \geq 0,
\]
with
\[
a_{mk+1}' = q^m a_{mk}' + ta_{mk-1}', \quad m \geq 1.
\]
Let
\[
a_r(x) = \sum_{m \geq 0} a_{mk+r}' x^m,
\]
where \( r \in [k] \). Then multiplying the recurrences (6) and (7) by \( x^m \), and summing the first over \( m \geq 0 \) and the second over \( m \geq 1 \), gives
\[
a_r(x) = a_{r-1}(x) + ta_{r-2}(x), \quad 3 \leq r \leq k,
\]
\[
a_2(x) = a_1(x) + txa_k(x),
\]
\[
a_1(x) = 1 + qxa_k(qx) + txa_{k-1}(x).
\]
By induction on $r$, we obtain
\[ a_r(x) = G_{r-1} a_2(x) + t G_{r-2} a_1(x), \quad 2 \leq r \leq k. \]
Therefore,
\[ a_r(x) = G_{r-1} (a_1(x) + t x a_k(x)) + t G_{r-2} a_1(x), \]
which implies
\[ a_r(x) = G_r a_1(x) + t x G_{r-1} a_k(x), \quad 2 \leq r \leq k. \]
Taking $r = k$ in (8) gives
\[ a_1(x) = \frac{1 - txG_{k-1}}{G_k} a_k(x). \]

By induction on $r$, we obtain
\[ a_r(x) = \frac{G_r + (-t)^r x G_{k-r}}{G_k} a_k(x), \quad 1 \leq r \leq k. \]

Since $a_1(x) = 1 + q x a_k(q x) + t x a_{k-1}(x)$, the last relation may be rewritten as
\[ a_k(x) = \frac{G_k}{1 - 2 tx G_{k-1} + (-t)^k x^2} + \frac{qxG_k}{1 - 2 tx G_{k-1} + (-t)^k x^2} a_k(q x). \]

Iterating (9) yields
\[ a_k(x) = \sum_{j=0}^{\infty} \frac{G_k^{j+1} q^{(j+1) \frac{j}{2}} x^j}{\prod_{i=0}^{j} (1 - 2 t q^i x G_{k-1} + (-t)^k q^{2i} x^2)}. \]

Thus, we have
\[ a_r(x) = (G_r + (-t)^r x G_{k-r}) \sum_{j=0}^{\infty} \frac{G_k^j q^{(j+1) \frac{j}{2}} x^j}{\prod_{i=0}^{j} (1 - 2 t q^i x G_{k-1} + (-t)^k q^{2i} x^2)}, \quad 1 \leq r \leq k, \]
which implies
\[
\sum_{n \geq 0} a_n x^n = \sum_{r=1}^{k} \sum_{m \geq 0} a_{m+r} x^{mk+r} = \sum_{r=1}^{k} x^r a_r(x^k) \\
= \left( \sum_{r=1}^{k} x^r G_r + x^k \sum_{r=1}^{k} (-tx)^r G_{k-r} \right) \sum_{j=0}^{\infty} \frac{G_k^j q^{(j+1) \frac{j}{2}} x^j}{\prod_{i=0}^{j} (1 - 2 t q^i x G_{k-1} + (-t)^k q^{2i} x^{2k})}. 
\]

The result now follows since
\[ \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} a'_n x^n = \frac{1}{x} \sum_{n \geq 0} a'_n x^n. \]

We now derive an explicit formula for the polynomials $a_n^{(k)}(q, t)$.

**Theorem 2.2.** If $n = km + r$, where $m \geq 0$ and $0 \leq r \leq k - 1$, then
\[ a_n = G_{r+1} S(m) + (-t)^{r+1} G_{k-1-r} S(m-1), \]
where
\[ S(m) = \sum_{j=0}^{m} (-1)^j G_j^j q^{(j+1) \frac{j}{2}} t^{m-(k-1)j} \sum_{a=j}^{m} a^a d^{m+a-j} \binom{a}{j} \binom{m+j-a}{j} q^j, \quad m \geq 0, \]
with $S(-1) = 0$ and
\[ d_\pm = G_{k-1} \pm \sqrt{G_k G_{k-2}}. \]
Proof. Note first that
\[ d_{\pm} = G_{k-1} \pm \sqrt{G_{k-1}^2 - (-t)^{k-2}}, \]
by the identity \( G_m^2 - G_{m+1}G_{m-1} = (-t)^{m-1} \), which can be shown by induction (see, e.g., [2, Identity 8] for the \( t = 1 \) case). Then
\[ 1 - 2tq^ixG_{k-1} + (-t)^kq^{2i}x^2 = (1 - d_+tq^ix)(1 - d_-tq^ix). \]

Let \( n = mk + r \), where \( m \geq 0 \) and \( 0 \leq r \leq k-1 \). By Theorem 2.1, we have
\[ a_n = G_{r+1} [x^m] (a(x)) + (-t)^{r+1} G_{k-1-r} [x^{m-1}] (a(x)), \]
where
\[ a(x) = \frac{G_k^j q^{(j+1)} x^j}{\prod_{i=0}^j (1 - d_+tq^i x)(1 - d_-tq^i x)}. \]
Using the expansion [1]
\[ \frac{y^j}{\prod_{i=0}^j (1 - q^i y)} = \sum_{a \geq j} \binom{a}{j}_q y^a \]
and the fact \( d_+d_- = (-t)^{k-2} \), we have
\[
[x^m](a(x)) = \sum_{j \geq 0} [x^m] \left( \frac{G_k^j q^{(j+1)} x^j}{\prod_{i=0}^j (1 - d_+tq^i x)(1 - d_-tq^i x)} \right)
= \sum_{j \geq 0} G_k^j q^{(j+1)} \left[ \frac{(d_+tq^j)^j}{\prod_{i=0}^j (1 - d_+tq^i x)} \cdot \frac{(d_-tq^j)^j}{\prod_{i=0}^j (1 - d_-tq^i x)} \right]
= \sum_{j \geq 0} (-1)^{kj} G_k^j q^{(j+1)} t^{m-(k-1)} \sum_{a=j}^m \binom{a}{j}_q (d_+t)^a \cdot \binom{m+j-a}{j}_q (d_-t)^{m+j-a}
= \sum_{j \geq 0} \binom{n}{j}_q (d_+t)^j \cdot \binom{m+j-a}{j}_q,
\]
which completes the proof. \( \square \)

Letting \( k = 1 \) in Theorem 2.2 gives the following expression for \( a_n^{(1)}(q,t) \).

Corollary 2.3. If \( n \geq 0 \), then
\[
a_n^{(1)}(q,t) = \sum_{j=0}^n q^{(n-j+1)} \binom{n-j}{j}_q t^j.
\]
Proof. When \( k = 1 \), we have \( d_\pm = \pm \frac{1}{\sqrt{t}} \) since \( G_0 = 0 \) and \( G_{-1} = \frac{1}{t} \). Taking \( k = 1 \) in (10) then gives

\[
\begin{align*}
a_n^{(1)}(q, t) &= S(n) = \sum_{j=0}^{n} (-1)^j q^{\binom{j+1}{2}} t^n \sum_{a=j}^{n} \binom{a}{j} q^{n-a} \binom{n+j-a}{j} \cdot (-1)^{n-a} q^{n+j-a} \binom{n+j-a}{j} q \\
&= \sum_{j=0}^{n} q^{\binom{j+1}{2}} t^j \sum_{a=j}^{n} (-1)^{n-a} q^{n-j} \binom{a}{j} q^{2n-j-a} \binom{2n-j-a}{n-j} q \\
&= \sum_{j=0}^{n} (-1)^j q^{\binom{n-j+1}{2}} t^j \sum_{a=0}^{j} (-1)^a q^{a+n-j} \binom{a+n-j}{n-j} q^{n-j} \\
&= \sum_{j=0}^{n} (-1)^j q^{\binom{n-j+1}{2}} \binom{n-j}{n-j} q^{j} t^j = \sum_{j=0}^{n} q^{\binom{n-j+1}{2}} q^{j} t^j,
\end{align*}
\]

where we have used the identity

\[
\sum_{a=0}^{n-m} (-1)^a q^{a+m} \binom{n-a}{m} = \begin{cases} \binom{n+m}{m} q^2, & \text{if } n \equiv m \pmod{2}; \\ 0, & \text{otherwise,} \end{cases} \quad (0 \leq m \leq n).
\]

Note that (12) may be obtained by writing

\[
\sum_{a \geq 0} (-1)^a \binom{a+m}{m} x^a \cdot \sum_{a \geq 0} \binom{a}{m} x^a = \frac{1}{\prod_{i=0}^{m} (1 + q^i x)} \cdot \frac{x^m}{\prod_{i=0}^{m} (1 - q^i x)} = \frac{x^m}{\prod_{i=0}^{m} (1 - q^{2i} x^2)}
\]

\[
= \sum_{a \geq 0} \binom{a+m}{m} x^{2a+m} q^2,
\]

and extracting the coefficient of \( x^n \) from both sides. \( \square \)

Remark: Formula (11) corresponds to the \( r = 2 \) case of [11, Theorem 2.1], which is a result on more general \( r \)-mino arrangements where no restriction is placed on the positions of \( r \)-minos or squares. The proof there was combinatorial, though it does not seem that it can be extended to prove Theorem 2.2 above.

Taking \( q = 1 \) and \( r = k - 1 \) in (10), and noting \( a_n^{(k)}(1, t) = G_{n+1} \), yields the following identity.
Corollary 2.4. If \( m \geq 0 \) and \( k \geq 1 \), then

\[
G_{(m+1)k} = G_k \sum_{j=0}^{k} (-1)^j G_j t^{m-(k-1)} j \sum_{a=j}^{m} d_+^a d_-^{m-j-a} \binom{a}{j} \binom{m+j-a}{j},
\]

where \( d_\pm = G_{k-1} \pm \sqrt{G_k G_{k-2}} \).

We have the following explicit formula for the number of members of \( \mathcal{F}_n \) (weighted according to the value of \( v \)) in which no square covers a multiple of \( k \).

Corollary 2.5. If \( n = km + r \), where \( m \geq 0 \) and \( 0 \leq r \leq k-1 \), then

\[
d_n^{(k)}(0, t) = t^m G_{r+1} T(m) + (-1)^{r+1} t^{m+r} G_{k-1-r} T(m-1),
\]

where

\[
T(m) = \sum_{i=0}^{m} \binom{m+1}{2i+1} G_{k-1}^{m-2i} (G_k G_{k-2})^i.
\]

Proof. Setting \( q = 0 \) in (10) implies

\[
a_m^{(k)}(0, t) = t^m G_{r+1} \sum_{a=0}^{m} d_+^a d_-^{m-a} + (-1)^{r+1} t^{m+r} G_{k-1-r} \sum_{a=0}^{m-1} d_+^a d_-^{m-1-a},
\]

with

\[
\frac{\sum_{a=0}^{m} d_+^a d_-^{m-a}}{d_+ - d_-} = \frac{1}{2\sqrt{G_k G_{k-2}}} \sum_{i=0}^{m} \binom{m+1}{2i+1} G_{k-1}^{m-2i} (\sqrt{G_k G_{k-2}})^{2i+1}.
\]

For example, when \( k = 1 \) in (14), we see that \( a_n^{(1)}(0, t) \) equals \( t^2 \) for \( n \) even and zero for \( n \) odd. Taking \( k = 2 \) in (14) gives \( a_n^{(2)}(0, t) = t^m \) and \( a_n^{(2)}(0, t) = (m+1) t^m \) for \( m \geq 0 \). These formulas are readily seen directly.

We next consider the case \( q = -1 \). Recall that for any generating function in \( q \), the evaluation at \( q = -1 \) gives the difference in cardinalities between those members of a structure having an even value for the statistic counted by \( q \) with those having an odd value. Letting \( q = -1 \) and \( t = 1 \) in (5) gives the following formulas, where \( f_i := \sum_{n \geq 0} a_n^{(i)}(-1, 1) x^n \):

\[
f_1 = \frac{(1 - x - x^3 - x^4)(1 - x^6)}{1 - x^{12}},
\]
\[
f_2 = \frac{(1 + x + x^3 + x^4 + 2x^5 - x^6 + x^7 + x^9 - x^{10})(1 - x^{12})}{1 - x^{24}},
\]
\[
f_3 = \frac{1 + x + 2x^2 - x^3 + x^4}{1 - x^6},
\]
\[
f_4 = \frac{(1 + x + 2x^2 + 3x^3 - 2x^4 + x^5 - x^6)(1 + x^4 + x^8)}{1 - 5x^8 + x^{16}}.
\]

The first three generating functions show that the sequences \( a_n^{(i)}(-1, 1), i = 1, 2, 3 \), are periodic with periods 12, 24, and 6, respectively. The sequences \( a_n^{(1)}(-1, 1) \) and \( a_n^{(2)}(-1, 1) \) are seen to satisfy the stronger conditions \( p_{n+6} = -p_n \) and \( p_{n+12} = -p_n \) for all \( n \geq 0 \). From the appearance of the generating function \( f_4 \), it seems that the sequence \( a_n^{(4)}(-1, 1) \) would not be periodic, which is indeed the case. It turns out that there are no other values of \( k \) for which the sequence \( a_n^{(k)}(-1, 1) \) is periodic.
Proposition 2.6. The sequence $a_n^{(k)}(-1, 1)$ is never periodic (or eventually periodic) when $k \geq 4$.

Proof. Substituting $q = -1$ and $t = 1$ into the infinite part of (5) gives

$$\sum_{j=0}^{\infty} \prod_{i=0}^{j} (1 - 2F_{k-1}(-1)^j x^k + (-1)^k x^{2k}) = \sum_{m=0}^{\infty} \frac{F_{2m}^{2m}(-1)^{m} x^{2mk}}{(1 - 2F_{k-1}(-1)^k x^k + (-1)^k x^{2k})(1 + (1 + (-1)^k x^{2k})^2 - 4F_{k-1}^2(-1)^k x^{2k}) m}.$$  

Then the equation above for any polynomial (possibly zero), and $d(x)$ is of the form $d(x) = x^{m+1} e(x)$, with $m$ denoting the degree of $c(x)$ (we take $m$ to be $-1$ if $c(x)$ is the zero polynomial) and $e(x)$ being a polynomial of degree at most $\ell - 1$. Then $(1 - x^\ell)(a(x) - b(x) c(x)) = b(x) d(x)$ implies that the equation $b(x) = 0$ must have at least one root of unity among its roots since $e(x) = \frac{d(x)}{x^{m+1}}$ is of degree at most $\ell - 1$, with $e(x)$ not identically zero.

Then the equation $b(u) = 0$, where $u = x^\frac{1}{2\ell}$, must also have at least one root of unity among its roots, since $r$ a root of unity implies $r^{2k}$ is as well.

The equation $b(u) = 0$ is given by

$$1 + (F_k^2 - 4F_{k-1}^2 + 2(-1)^k) u + u^2 = 0.$$  

If $k \geq 4$, then

$$F_k^2 - 4F_{k-1}^2 + 2(-1)^k \leq -5,$$

since

$$F_k^2 - 4F_{k-1}^2 = (F_k - 2F_{k-1})(F_k + 2F_{k-1}) = -F_{k-3}(F_k + 2F_{k-1}) \leq -7.$$  

Note that an equation of the form

$$1 - au + u^2 = 0, \quad a \geq 5,$$

has (real) roots $\frac{a}{2} \pm \frac{\sqrt{a^2 - 4}}{2}$. So the only possible roots of unity that are also roots to such an equation are $\pm 1$. However, the equations $\frac{a}{2} + \frac{\sqrt{a^2 - 4}}{2} = \pm 1$ and $\frac{a}{2} - \frac{\sqrt{a^2 - 4}}{2} = \pm 1$ have solutions $a = \pm 2$ in each case, but $a \geq 5$. Thus no roots of unity satisfy equation (15) when $k \geq 4$, which implies the result. 

$\square$
Remark: When \( k = 1, 2, 3 \), the equation (15) is satisfied by roots of unity and it works out that the sequences \( a^{(k)}_n(-1, 1) \) are periodic in these cases.

Let \((x : q)_s = \prod_{i=0}^{s-1} (1 - q^i x)\). We conclude this section with the following infinite expansion for the numbers \( a^{(k)}_n(q, t) \) for all \( k \geq 1 \).

**Theorem 2.7.** If \( n = km + r \), where \( m \geq 1 \) and \( 0 \leq r \leq k - 1 \), then

\[
a_n = t^m G_{r+1} \sum_{s=0}^{\infty} q^{sm} \left( d_+^m b_s + d_-^m c_s \right) + (-1)^{r+1} t^{m+r} G_{k-1-r} \sum_{s=0}^{\infty} q^{s(m-1)} \left( d_+^{m-1} b_s + d_-^{m-1} c_s \right),
\]

where

\[
b_s = \sum_{j \geq s} \frac{(-1)^s G^j_k q^{(j+1) \frac{r_1}{2}} + (r_{1}')^{s} d_+}{t^j (q : q)_s(q : q)_{j-s} \prod_{i=0}^j (q^i d_+ - q^i d_-)},
\]

\[
c_s = \sum_{j \geq s} \frac{(-1)^s G^j_k q^{(j+1) \frac{r_1}{2}} + (r_{1}^{1'})^{s} d_-}{t^j (q : q)_s(q : q)_{j-s} \prod_{i=0}^j (q^i d_- - q^i d_+)},
\]

and

\[
d_\pm = G_{k-1} \pm \sqrt{G_{k-1} G_{k-2}}.
\]

**Proof.** Note first that \( d_\pm = G_{k-1} \pm \sqrt{G_{k-1} (-t)^{k-2}} \), as in the proof of Theorem 2.2, and thus

\[1 - 2t q^s x G_{k-1} + (-t)^k q^{2s} x^2 = (1 - \rho_s x)(1 - \theta_s x),\]

where \( \rho_s = d_+ t q^s \) and \( \theta_s = d_- t q^s \).

Let \( n = mk + r \), where \( m \geq 1 \) and \( 0 \leq r \leq k - 1 \). By partial fractions, let us write

\[
\sum_{j \geq s} \frac{G^j_k q^{(j+1) \frac{r_1}{2}} x^j}{\prod_{i=0}^j (1 - 2t q^i x G_{k-1} + (-t)^k q^{2i} x^2)} = \sum_{s=0}^{\infty} \frac{b_s}{1 - \rho_s x} + \sum_{s=0}^{\infty} \frac{c_s}{1 - \theta_s x},
\]

where \( b_s \) and \( c_s \) are constants to be determined. By Theorem 2.1,

\[
a_n = G_{r+1} [x^m] \left( \sum_{j \geq s} \frac{G^j_k q^{(j+1) \frac{r_1}{2}} x^j}{\prod_{i=0}^j (1 - 2t q^i x G_{k-1} + (-t)^k q^{2i} x^2)} \right) + (-t)^{r+1} G_{k-1-r} [x^{m-1}] \left( \sum_{j \geq s} \frac{G^j_k q^{(j+1) \frac{r_1}{2}} x^j}{\prod_{i=0}^j (1 - 2t q^i x G_{k-1} + (-t)^k q^{2i} x^2)} \right)
\]

\[= G_{r+1} [x^m] \left( \sum_{s=0}^{\infty} \frac{b_s}{1 - \rho_s x} + \sum_{s=0}^{\infty} \frac{c_s}{1 - \theta_s x} \right) + (-t)^{r+1} G_{k-1-r} [x^{m-1}] \left( \sum_{s=0}^{\infty} \frac{b_s}{1 - \rho_s x} + \sum_{s=0}^{\infty} \frac{c_s}{1 - \theta_s x} \right)
\]

\[= G_{r+1} \sum_{s=0}^{\infty} (b_s \rho_s^m + c_s \theta_s^m) + (-t)^{r+1} G_{k-1-r} \sum_{s=0}^{\infty} (b_s \rho_s^{m-1} + c_s \theta_s^{m-1}).
\]
We also have
\[
\begin{align*}
b_s &= \sum_{j=s} \frac{G_k^j q^{(i+1)}}{\rho_s^j \prod_{i=0}^{s-1} (1 - \rho_i / \rho_s) \prod_{i=s+1}^j (1 - \rho_i / \rho_s) \prod_{i=0}^j (1 - \theta_i / \rho_s)} \\
&= \sum_{j=s} \frac{(-1)^s G_k^j q^{(i+1)} \rho_s^{j+1}}{t j \prod_{i=0}^{s-1} (q^i - q^s) \prod_{i=s+1}^j (q^s - q^i) \prod_{i=0}^j (d_+ t q^s - d_- t q^i)} \\
&= \sum_{j=s} \frac{(-1)^s G_k^j q^{(i+1) + (\tau_1^s)} d_+}{t^j (q : q)_s (q : q) j - \prod_{i=0}^j (q^s d_+ - q^i d_-)}
\end{align*}
\]
and, similarly,
\[
c_s = \sum_{j=0} (-1)^s G_k^j q^{(i+1) + (\tau_1^s) + s} d_-
\]
which gives (16). \(\square\)

3. The Case \(k = 2\)

In this section, we consider further results concerning the polynomial sequence \(a_n^{(2)} = a_n^{(2)}(q, t)\). Taking \(k = 2\) in (10), and noting \(d_+ = d_- = 1\) in this case, gives the explicit formulas
\[
a^{(2)}_{2m} = \sum_{j=0}^{m} q^{(i+1)} t^{m-j} \sum_{a=j}^{m} \binom{a}{j} \binom{m+j-a}{j} \rho_s^{j+1} \\
- t \sum_{j=0}^{m-1} q^{(i+1)} t^{m-1-j} \sum_{a=j}^{m-1} \binom{a}{j} \binom{m+j-1-a}{j} \rho_s^{j+1}
\]
(17)
\[
a^{(2)}_{2m+1} = \sum_{j=0}^{m} q^{(i+1)} t^{m-j} \sum_{a=j}^{m} \binom{a}{j} \binom{m+j-a}{j} \rho_s^{j+1}
\]
(18)

Though we are unable to give simpler expressions for the polynomials (17) and (18), they are seen to be solutions to the following relatively simple recurrences.

**Proposition 3.1.** If \(m \geq 2\), then
\[
a^{(2)}_{2m} = (q^m + q t + t) a^{(2)}_{2m-2} - q t^2 a^{(2)}_{2m-4},
\]
with \(a^{(2)}_{0} = 1\) and \(a^{(2)}_{2} = q + t\), and
\[
a^{(2)}_{2m+1} = (q^m + 2t) a^{(2)}_{2m-1} - t^2 a^{(2)}_{2m-3},
\]
with \(a^{(2)}_{1} = 1\) and \(a^{(2)}_{3} = q + 2t\).

**Proof.** We provide a combinatorial argument, the initial values being clear. To show (19), first note that if \(m \geq 2\), then the total weight of all the members of \(F_{2m}\) ending in \(ss\) is \(q^m a^{(2)}_{2m-2}\), while the weight of those ending in \(d\) is \(t a^{(2)}_{2m-2}\). To determine the weight of the members of \(F_{2m}\) ending in \(ds\), first insert a domino before the final square within any member of \(F_{2m-2}\) ending in \(s\). By subtraction, the total weight of all the members of
Iterating the last equation gives

\[ f_{n+1} \] for the sequences \( F \).

By similar reasoning, the total weight of all members of \( F \) is

\[ q^m a_{2m-1}^{(2)} + t a_{2m-2}^{(2)} \] and thus

\[ f = \sum_{n \geq 0} a_n^{(2)}(q,t)x^n, \]

which we'll also denote by \( f(x) \).

**Proposition 3.2.** We have

\[ f(x; q, t) = (1 + x - tx^2) \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}} x^{2j}}{\prod_{i=0}^{j} (1 - qt^j x^2)^2}. \]  

**Proof.** This follows from setting \( k = 2 \) in (5) above, but we give an alternative derivation using Proposition 3.1 as follows. Let \( b(x) = \sum_{m \geq 0} a_{2m}^{(2)} x^m \). Multiplying (19) by \( x^m \), and summing over \( m \geq 2 \), implies

\[ b(x) - 1 - (t + q)x = qx(b(qx) - 1) + tx(1 + q)(b(x) - 1) - qt^2 x^2 b(x), \]

or

\[ b(x) = \frac{1}{1 - tx} + \frac{q x}{(1 - tx)(1 - qtx)} b(qx). \]

Iterating the last equation gives

\[ b(x) = \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}} x^j}{(1 - tx) \prod_{i=0}^{j} (1 - qt^i x^2)^2} \]

\[ = (1 - tx) \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j} (1 - qt^i x^2)^2}. \]

Similarly, if \( c(x) = \sum_{m \geq 0} a_{2m+1}^{(2)} x^m \), then we have

\[ c(x) = \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j} (1 - qt^i x^2)^2}, \]

and thus

\[ f(x) = b(x^2) + xc(x^2) \]

\[ = (1 - tx^2) \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}} x^{2j}}{\prod_{i=0}^{j} (1 - qt^i x^2)^2} + x \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}} x^{2j}}{\prod_{i=0}^{j} (1 - qt^i x^2)^2}, \]

as desired.

Substituting \( q = -1 \) in (21) yields the following result.

**Corollary 3.3.** We have

\[ \sum_{n \geq 0} a_n^{(2)}(-1, t)x^n = \frac{(1 + x + tx^2)(1 + x - tx^2)(1 - x + tx^2)}{1 - (2t^2 - 1)x^4 + t^4 x^8}. \]
Corollary 3.4. The sequence \( a_n^{(2)}(-1, 1) \) is determined by the condition

\[
f(n + 12) = -f(n), \quad n \geq 0,
\]

with the values of \( a_n^{(2)}(-1, 1) \) for \( 0 \leq n \leq 11 \) given by \( 1, 1, 0, 1, 2, -1, 1, 0, 1, -1, 0 \).

**Proof.** Letting \( t = 1 \) in (22), we have

\[
\sum_{n \geq 0} a_n^{(2)}(-1, 1) x^n = \frac{(1 + x + x^2)(1 + x - x^2)(1 - x + x^2)}{1 - x^4 + x^8}
\]

\[
= \frac{(1 + x + x^3 + x^5 - x^6)(1 + x^4)(1 - x^{12})}{(1 - x^4 + x^8)(1 + x^4)(1 - x^{12})}
\]

\[
= \frac{(1 + x + x^3 + x^4 + 2x^5 - x^6 + x^7 + x^9 - x^{10})(1 - x^{12})}{1 - x^{24}}
\]

which implies the result. \( \square \)

**Combinatorial proof of Corollary 3.4.**

Let \( \mathcal{F}'_n \) and \( \mathcal{F}_n^0 \) denote the subsets of \( \mathcal{F}_n \) having even and odd \( \rho_2 \) values, respectively. We first define an involution of \( \mathcal{F}_n \) off of a set \( \mathcal{F}'_n \) which pairs members of \( \mathcal{F}_n^e \) and \( \mathcal{F}_n^o \). Let \( \mathcal{F}'_n \subseteq \mathcal{F}_n \) consist of those tilings of the form

\[(23) \quad \pi = d^i (sd^{2i} s)(sd^{2i} s) \cdots (sd^{2i} s), \]

if \( n \) is even, and of the form

\[(24) \quad \pi = d^i (sd^{2i} s)(sd^{2i} s) \cdots (sd^{2i} s)sd^j, \]

if \( n \) is odd, for some \( \ell \) where \( i, j, i_1, i_2, \ldots, i_\ell \geq 0 \). We define an involution of \( \mathcal{F}_n - \mathcal{F}_n' \) as follows. Given \( \lambda \in \mathcal{F}_n - \mathcal{F}_n' \), let \( j_0 \) denote the smallest index \( j \geq 1 \) such that either

(i) an odd number of dominos occurs between the \((2j - 1)\)-st and \((2j)\)-th squares, or

(ii) an even number of dominos occurs between the \((2j - 1)\)-st and \((2j)\)-th squares with at least one domino between the \((2j)\)-th and \((2j + 1)\)-st squares (or between the \((2j)\)-th square and the end of the tiling, if the \((2j)\)-th square is right-most).

Now exchange positions of the \((2j_0)\)-th square and the domino that precedes it if (i) occurs, or exchange the positions of the \((2j_0)\)-th square and the domino that directly follows it if (ii) occurs. Let \( \lambda' \) denote the resulting member of \( \mathcal{F}_n' \). Then \( \lambda \) and \( \lambda' \) have opposite \( \rho_2 \)-parity (since their \( \rho_2 \) values differ by one), and the mapping \( \lambda \mapsto \lambda' \) is an involution of \( \mathcal{F}_n - \mathcal{F}_n' \). For example, if \( n = 28 \) and \( \lambda = d^3 sd^2 s^3 d^3 sd^2 s \in \mathcal{F}_{28} \), then \( j_0 = 3 \) and \( \lambda' = d^3 sd^2 s^3 d^2 sd^2 s \). See Figure 3 below, where the \((2j_0 - 1)\)-st and \((2j_0)\)-th squares are shaded in each tiling.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
|   |   |   | | | | | | | | | | | | | | | | | | | | | | | | | | |

**Figure 3.** The tiling \( \lambda \) has \( \rho_2(\lambda) = 35 \), while \( \rho_2(\lambda') = 34 \).

We now consider the signed sum of members of \( \mathcal{F}'_n \), i.e., \( \sum_{\pi \in \mathcal{F}'_n} (-1)^{\rho_2(\pi)} \). First observe that if \( i \) is even in (23) and (24) above, then one may verify that

\[
\rho_2(\pi) \equiv \left( \frac{\ell + 1}{2} \right) \pmod{2},
\]
whereas if \(i\) is odd, then

\[
\rho_2(\pi) \equiv \left( \frac{\ell}{2} \right) (\mod 2).
\]

For the remainder of the proof, we will assume that \(n\) is even, the proof in the odd case being similar. Assume further that \(n = 2m\), where \(m\) is odd, as the argument for the case of even \(m\) is basically the same.

First suppose that \(\pi \in F'_{n}\) is of the form in (23) above, with \(i\) even. Note that \(m\) odd implies \(\ell\) is odd. Let \(\bar{\pi}\) be the tiling of length \(m\) given by

\[
\bar{\pi} = d^i s d^{i_1} s d^{i_2} \cdots s d^{i_{\ell}};
\]

note that all members of \(F_m\) arise uniquely as \(\pi\) ranges over all members of \(F'_{n}\) for which \(i\) is even. Let \(s(\sigma)\) denote the number of squares in a tiling \(\sigma\). Then we have

\[
\rho_2(\pi) \equiv \left( \frac{\ell + 1}{2} \right) \equiv \frac{\ell + 1}{2} = \frac{s(\bar{\pi}) + 1}{2} (\mod 2).
\]

If \(\pi \in F'_{n}\) is of the form in (23) with \(i\) odd, then \(m\) odd implies \(\ell\) is even. Let \(\pi^*\) be the tiling of length \(m - 1\) given by

\[
\pi^* = d^{i-1} s d^{i_1} s d^{i_2} \cdots s d^{i_{\ell}};
\]

note that all members of \(F_{m-1}\) arise uniquely in this manner. Observe that in this case

\[
\rho_2(\pi) \equiv \left( \frac{\ell}{2} \right) \equiv \frac{\ell}{2} = \frac{s(\pi^*)}{2} (\mod 2).
\]

Therefore, we have

\[
\sum_{\pi \in F'_{n}} (-1)\rho_2(\pi) = \sum_{\pi \in F'_{n}, i \text{ even}} (-1)\rho_2(\pi) + \sum_{\pi \in F'_{n}, i \text{ odd}} (-1)\rho_2(\pi)
\]

\[
= \sum_{\sigma \in F_m} (-1)^{(s(\sigma) + 1)/2} + \sum_{\sigma \in F_{m-1}} (-1)^{s(\sigma)/2}.
\]

(25)

To evaluate the last two sums, we consider the statistic \([s(\sigma)/2]\) on \(F_r\) where \(r \geq 1\) and pair members of \(F_r\) of opposite parity with respect to this statistic. Given \(\sigma = \sigma_1 \sigma_2 \cdots \in F_r\), let \(a_o\) denote the smallest index \(a \geq 1\) such that either

(i) \(\sigma_{2a-1} = d\), or
(ii) \(\sigma_{2a-1} \sigma_{2a} = ss\).

Define an involution of \(F_r\) by replacing \(\sigma_{2a-1} = d\) with \(ss\) if (i) occurs or by replacing \(\sigma_{2a-1} \sigma_{2a} = ss\) with \(d\) if (ii) occurs. Note that this mapping changes the value of \([s(\sigma)/2]\) by one, whence it changes its parity. If \(r \equiv 0 (\mod 3)\), then there is a single unpaired tiling in \(F_r\), namely, \((sd)^{r/3}\), which has sign \((-1)^{[r/6]}\). If \(r \equiv 1 (\mod 3)\), then the single unpaired tiling \((sd)^{(r-1)/3}s\) has sign \((-1)^{[r+2]/6}\). If \(r \equiv 2 (\mod 3)\), then each member of \(F_r\) is paired with another of opposite parity, whence the resulting sum is zero.
Theorem 3.5. The coefficient of $x^n$ for $n \geq 0$ in $\frac{d}{dq} f(x; q, t) |_{q=1}$ is given by

$$
\frac{(i \sqrt{7})^{n+1}}{8(4t+1)} \left( \frac{(2n+1)(4t+1)(-1)^n + 2n(n+1) - 4t - 1}{2i \sqrt{t}} U_n(y) + \frac{4t + 1}{2i \sqrt{t}} U_n(y) \right),
$$

where $y = \frac{1}{2i \sqrt{t}}$ and $i = \sqrt{-1}$.

Proof. Differentiating the generating function $f(x; q, t)$ in (21) with respect to $q$, and substituting $q = 1$, yields

$$
g(x; t) := \left. \frac{d}{dq} f(x; q, t) \right|_{q=1} = \frac{x^2(1 - tx^2)(1 + tx^2)}{(1 - x - tx^2)^3(1 + x - tx^2)^2}.
$$

By partial fractions, we may rewrite this as

$$
g(x; t) = -\frac{3 - 2tx}{16(1 + x - tx^2)} + \frac{2 + x}{8(1 + x - tx^2)^2} - \frac{1 + 2tx}{16(1 - x - tx^2)}
$$

$$
+ \frac{1 - tx}{4t(1 - x - tx^2)} - \frac{1 - 2tx - x}{4t(1 - x - tx^2)^3}.
$$

By the fact that $\sum_{n \geq 0} U_n(t)x^n = \frac{1}{1 - 2tx + x^2}$, we obtain

$$
\sum_{n \geq 1} nU_n(t)x^{n-1} = \frac{2t - 2x}{(1 - 2tx + x^2)^2}
$$

and

$$
\sum_{n \geq 2} n(n - 1)U_n(t)x^{n-2} = \frac{8t^2 - 2 - 12tx + 6x^2}{(1 - 2tx + x^2)^3}.
$$

Applying the preceding to (25) shows that if $m \equiv 0 \pmod{3}$, i.e., if $m = 6p + 3$ for some $p$ (since $m$ was assumed odd) and $n = 12p + 6$, then

$$
ad_n^{(2)}(-1, 1) = \sum_{n \in \mathcal{F}_n'} (-1)^{p_2(x)}
$$

$$
= \sum_{\sigma \in \mathcal{F}_{6p+3}} (-1)^{\lfloor s(\sigma)/2 \rfloor} + \sum_{\sigma \in \mathcal{F}_{6p+2}} (-1)^{\lfloor s(\sigma)/2 \rfloor} = (-1)^{(6p+3)/6} + 0 = (-1)^{b+1}.
$$

Similarly, if $n = 12p + 2$, then $ad_n^{(2)}(-1, 1) = (-1)^{p+1} + (-1)^p = 0$, and if $n = 12p + 10$, then $ad_n^{(3)}(-1, 1) = 0 + (-1)^{p+1} = (-1)^{b+1}$. This yields the values of $ad_n^{(2)}(-1, 1)$ given in Corollary 3.4 above in the case when $n = 2m$, where $m$ is odd. The other cases are obtained similarly. \hfill \square

Remark: Comparable proofs may be given to explain the periodic nature of the $ad_n^{(1)}(-1, 1)$ and $ad_n^{(3)}(-1, 1)$ values witnessed above.

Let $U_n(t)$ denote the $n$-th Chebyshev polynomial of the second kind defined by $U_n+1(t) = 2tU_n(t) - U_{n-1}(t)$, with $U_0(t) = 1$ and $U_1(t) = 2t$ (see, e.g., [9]).

Theorem 3.5. The coefficient of $x^n$ for $n \geq 0$ in $\frac{d}{dq} f(x; q, t) |_{q=1}$ is given by

$$
\frac{(i \sqrt{7})^{n+1}}{8(4t+1)} \left( \frac{(2n+1)(4t+1)(-1)^n + 2n(n+1) - 4t - 1}{2i \sqrt{t}} U_n(y) + (4t + 1)(-1)^n + 4t - 1 - 2n(n+2)U_{n-1}(y) \right),
$$

where $y = \frac{1}{2i \sqrt{t}}$ and $i = \sqrt{-1}$.

Proof. Differentiating the generating function $f(x; q, t)$ in (21) with respect to $q$, and substituting $q = 1$, yields

$$
g(x; t) := \left. \frac{d}{dq} f(x; q, t) \right|_{q=1} = \frac{x^2(1 - tx^2)(1 + tx^2)}{(1 - x - tx^2)^3(1 + x - tx^2)^2}.
$$

By partial fractions, we may rewrite this as

$$
g(x; t) = -\frac{3 - 2tx}{16(1 + x - tx^2)} + \frac{2 + x}{8(1 + x - tx^2)^2} - \frac{1 + 2tx}{16(1 - x - tx^2)}
$$

$$
+ \frac{1 - tx}{4t(1 - x - tx^2)} - \frac{1 - 2tx - x}{4t(1 - x - tx^2)^3}.
$$

By the fact that $\sum_{n \geq 0} U_n(t)x^n = \frac{1}{1 - 2tx + x^2}$, we obtain

$$
\sum_{n \geq 1} nU_n(t)x^{n-1} = \frac{2t - 2x}{(1 - 2tx + x^2)^2}
$$

and

$$
\sum_{n \geq 2} n(n - 1)U_n(t)x^{n-2} = \frac{8t^2 - 2 - 12tx + 6x^2}{(1 - 2tx + x^2)^3}.
$$
Let \( y = \frac{1}{2i\sqrt{t}} \), where \( i = \sqrt{-1} \). Extracting the coefficient of \( x^n \) from each summand then gives

\[
[x^n] \left( -\frac{3 - 2tx}{16(1 + x - tx^2)} \right) = -\frac{(-i\sqrt{t})^n}{16}(3U_n(y) - 2i\sqrt{t}U_{n-1}(y)),
\]

\[
[x^n] \left( \frac{2 + x}{8(1 + x - tx^2)^2} \right) = \frac{(2 + n)(-i\sqrt{t})^n}{8}U_n(y),
\]

\[
[x^n] \left( -\frac{1 + 2tx}{16(1 + x - tx^2)^2} \right) = -\frac{(i\sqrt{t})^n}{16}(U_n(y) - 2i\sqrt{t}U_{n-1}(y)),
\]

\[
[x^n] \left( \frac{1 - tx}{4t(1 + x - tx^2)^2} \right) = \frac{(1 + 4t + (t + 1)n)(i\sqrt{t})^n}{4t(1 + 4t)}U_n(y) - \frac{(1 + n)(2-t-1)(i\sqrt{t})^{n-1}}{4t(1 + 4t)}U_{n-1}(y),
\]

\[
[x^n] \left( -\frac{1 - 2tx - x}{4t(1 + x - tx^2)^2} \right) = \frac{(tn^2 - (t+2)n - 2(1 + 4t))(i\sqrt{t})^n}{8t(1 + 4t)}U_n(y) + \frac{(tn^2 + (4t-1)n - 1 + 3t)(i\sqrt{t})^n}{4(1 + 4t)}U_{n-1}(y).
\]

Adding all of these expressions yields the desired result. \( \square \)

Let \( t_n(\rho_2) \) denote the sum of the \( \rho_2 \) values of all the members of \( \mathcal{F}_n \). Letting \( t = 1 \) in the prior theorem, and noting \( i^nU_n(-i/2) = F_{n+1} \), gives the following expression for \( t_n(\rho_2) \).

**Corollary 3.6.** If \( n \geq 0 \), then

\begin{equation}
(26) \quad t_n(\rho_2) = (-1)^n \frac{(2n+1)F_{n+1} - 2F_n + (2n^2 + 2n - 5)F_{n+1} + (4n^2 + 8n - 6)F_n}{16}.
\end{equation}

**References**


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