ON COMPUTATIONAL COMPLEXITY OF PLANE CURVE INVARIANTS

FEDOR DUZHIN AND TAO BIAOSHUAI

ABSTRACT. The theory of generic smooth closed plane curves initiated by Vladimir Arnold is a beautiful fusion of topology, combinatorics, and analysis. The theory remains fairly undeveloped. We review existing methods to describe generic smooth closed plane curves combinatorially, introduce a new one, and give an algorithm for efficient computation of Arnold's invariants. Our results provide a good source of future research projects that involve computer experiments with plane curves. The reader is not required to have background in topology and even undergraduate students with basic knowledge of differential geometry and graph theory will easily understand our paper.

1. INTRODUCTION

A generic plane curve is an immersion of the circle into the plane having only transversal double points as singularities. In particular, immersions with triple points and self-tangency points (see Figure 1) are not considered generic. Non-generic curves are exceptional cases — a random curve doesn't have complex singularities. It is important, however, to study non-generic curves too because they occur if one deforms one generic curve into another one.

The theory of generic plane curves is somewhat similar to that of knots. However, according to V. Arnold (see [1]),

> The combinatorics of plane curves seems to be far more complicated than that of knot theory (which might be considered as a simplified "commutative" version of the combinatorics of plane curve and which is probably embedded in plane curves theory).

Let K be a generic plane curve. Arnold's invariants J^+, J^-, and St are computed as follows. First, one assigns an orientation to K (in other words, chooses one of the two directions). Then, K is deformed (as in Figure 2) into one of the standard curves $K_0, K_1, K_2, \ldots$ shown in Figure 3. During deformation, singularities shown in Figure 1 occur. Undergoing a direct self-tangency changes J^+ by ± 2, an inverse self-tangency changes J^- by ± 2, and a triple point changes St by ± 1. Values of J^+, J^-, and St on standard curves K_n for $n \geq 0$ are pre-defined. Details of this construction can be found in [1] or in [3].

The theory of plane curves remains fairly unexplored and computer experiments might be useful. For instance, [4] and [5] develop an algorithm that generates all generic plane curves with a given number of double points. Our paper aims to explain how known formulas for Arnold's invariants can be programmed and what the computational complexity is.

2. PRELIMINARIES

Throughout this paper, both the circle S^1 and the plane $\mathbb{R}^2$ are supposed to be oriented.

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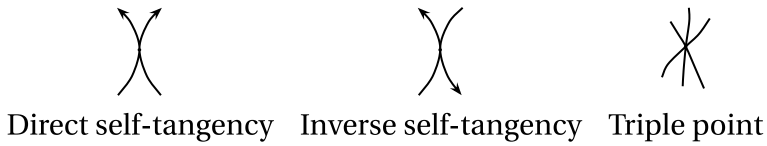


FIGURE 1. Generic singularities.

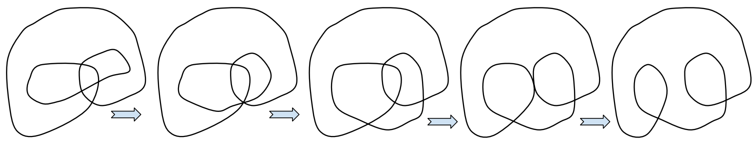


FIGURE 2. A generic deformation.

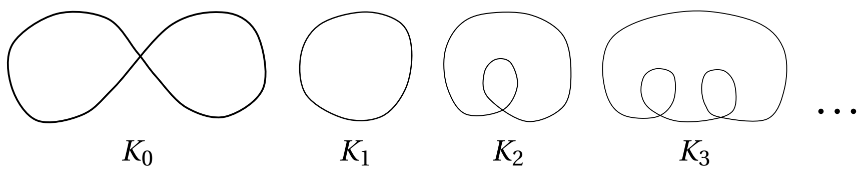


FIGURE 3. Standard curves.

A plane curve is a smooth immersion $\gamma : S^1 \to \mathbb{R}^2$ considered up to a positive change of the variable. A generic plane curve can be identified with the image $\gamma(S^1) = \Gamma \subset \mathbb{R}^2$. In the sequel, Γ denotes both the curve as a map and the curve as a subset of the plane. We hope that such an abuse of notation does not lead to a confusion.

We suppose that the parameter is natural, that is, $\|\gamma'(t)\| = 1$ for all $t \in S^1$. We also assume that the circle has a base point $t_0 \in S^1$ and that $\gamma(t_0)$ is not a double point of the curve. Whenever we refer to 'travelling along the curve', we mean that one starts at $\gamma(t_0)$.

Definition 2.1. Let $\gamma : S^1 \to \mathbb{R}^2$ be a generic plane curve and let $p \in \Gamma = \gamma(S^1)$ be a double point, where $\gamma^{-1}(p) = \{t_1, t_2\}$. The two pre-images t_1, t_2 of p are referred to as *passings* of p. Further, a passing of a double point is called *positive* if the other passing is seen as going from left to right and *negative* otherwise[1]. Finally, the sign of the double point p is the sign of its first passing as one travels along Γ.

Definition 2.2. Let $\Gamma \subset \mathbb{R}^2$ be a generic plane curve. A *region* is a path-connected component of $\mathbb{R}^2 - \Gamma$. The *outer* region is the unbounded path-connected component of $\mathbb{R}^2 - \Gamma$. The *index* $\mathrm{ind}(R)$ of a region R is the degree of the map $\rho : S^1 \to S^1$ given by $\rho(t) = \dfrac{\gamma(t) - a_0}{\|\gamma(t) - a_0\|}$ for any $a_0 \in R$, which makes sense because $\|\rho(t)\| = 1$.

Obviously, $\mathrm{ind}(R)$ doesn't depend on the choice of $a_0 \in R$. It is also obvious that the index of the outer region is 0 and that indices of adjacent regions differ by 1 with the bigger index indicating the region on the left of the curve. It is also easy to show that a generic curve with n double points has $n+2$ regions.

[1]Alternatively, we can describe the sign of a passing as follows. Let t_1 and t_2 be two passings of the same double point. Then the sign of t_1 is the same as the sign of the frame $\gamma'(t_2), \gamma'(t_1)$. You can imagine that you're going along the y-axis and see how the x-axis intersects your trajectory from left to right.

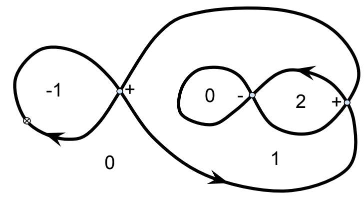


FIGURE 4. Indices of regions and signs of double points.

Definition 2.3. Let $\gamma : S^1 \to \mathbb{R}^2$ be a plane curve (not necessarily generic). Recall that $\gamma'(t)$ is a unit vector and hence can be considered as a point of the unit circle in $\mathbb{R}^2$. The *rotation number* of the curve γ is the degree of the map $\tau : S^1 \to S^1$ given by $\tau(t) = \gamma'(t)$.

We also need to consider non-generic curves, i.e., those that have more complicated singularities than transversal double points. We allow the three complex singularities shown in Figure 1 — a direct self-tangency, an inverse self-tangency, or a triple point. We do not consider anything more complicated in this paper, like quadruple points, triple points with two tangent branches etc.

Obviously, the rotation number doesn't change under isotopies in the class of immersions. H. Whitney proved in [9] that the rotation number is a complete isotopy invariant of a plane curve. In other words, two generic plane curves of the same rotation number can be continuously deformed into one another. Moreover, there is a *generic* deformation (see Figure 2), i.e., one that goes through generic curves, with finitely many exceptions, where each of the exceptions is a non-generic curve with precisely one singularity of one of the types shown in Figure 1. In particular, any curve can be deformed into K_r shown in Figure 3, where r is (the absolute value of) the rotation number.

Example 2.4. Figure 4 shows a curve with 3 double points with signs of double points and indices of regions indicated. The base point marked by the crossed circle. The rotation number of this curve is 0.

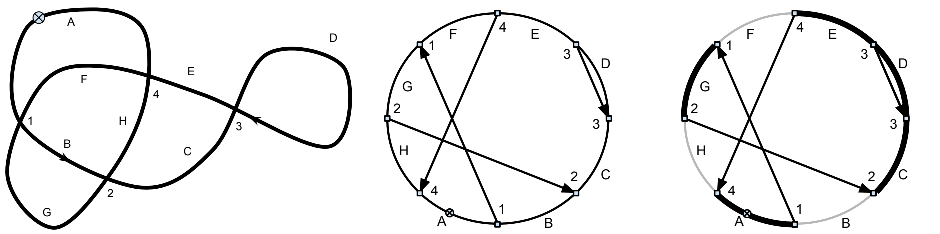
3. PROGRAMMING PLANE CURVES

Since our ultimate goal is to develop computer programmes, we need a way to describe generic curves combinatorially. Here, we introduce three methods — (marked) Gauss diagram, arc diagram, and edge list.

Gauss diagram. An abstract *Gauss diagram* is a circle with a collection of oriented chords. Chords cannot share common endpoints but might intersect inside the circle.

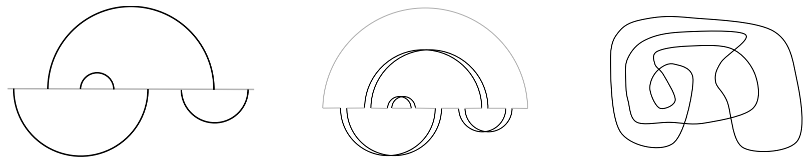
Consider a generic plane curve, i.e., an immersion $\gamma : S^1 \to \mathbb{R}^2$. Each double point p has two passings $t_1, t_2 \in S^1$ introduced in Definition 2.1. The Gauss diagram of the curve Γ is the circle S^1 with a chord joining t_1 and t_2 for each double point p. The chord is oriented from the positive to the negative passing. Figure 5 shows a generic plane curve and its Gauss digram.

There exist Gauss diagrams that represent more than one plane curve and there exist Gauss diagrams that do not represent any plane curve (in fact, Gauss diagrams correspond rather to curves on surfaces than to plane ones). However, if we include an additional information — mark portions of the circle mapped by γ onto the boundary of the outer region, then such a *marked Gauss diagram* (Figure 5, third illustration) is a complete invariant of a generic closed curve. Specifically, given a marked Gauss diagram, if it corresponds to a generic closed curve, then such a closed curve is unique as shown in [2].



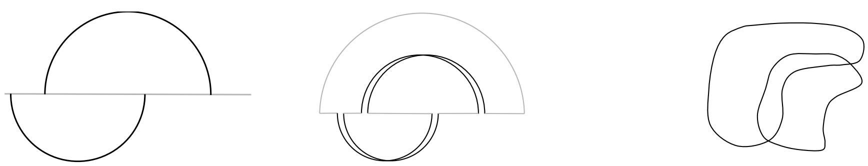
A generic curve Its Gauss diagram Marked Gauss diagram

FIGURE 5. A Gauss diagram.



An arc diagram Conversion to a curve The closed curve

FIGURE 6. Constructing a generic closed curve from an arc diagram.



An arc diagram Conversion to a curve Two intersecting closed curves.

FIGURE 7. An arc diagram might correspond to several intersecting closed curves.

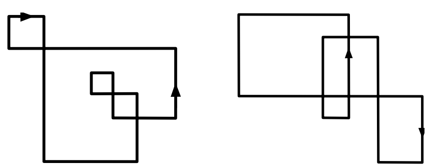


FIGURE 8. Curves from Figures 4 and 5.

Arc diagram. Following [4] and [5], we define *arc diagrams*. An abstract arc diagram is a collection of non-intersecting upper and lower semi-circles aligned along the x-axis. To construct the generic closed curve from an arc diagram, one replaces each arc with a double point as shown in Figure 6 and joins the left and the right endpoints by a semi-circle in the upper half-plane.

It is shown in [4] that every closed curve has at least one arc diagram. Conversely, an arc diagram might not correspond to any closed curve as shown in Figure 7 (sometimes, replacing arcs with double points could produce several closed curves) but whenever it does, the curve is determined uniquely by the diagram.

Edge list. To produce an *edge list*, we first need to draw our generic closed curve as a broken line with segments parallel to the coordinate axes as in Figure 8. Then we treat the curve as a directed graph with n vertices of degree 4 at double points of the curve and $2n$ edges.

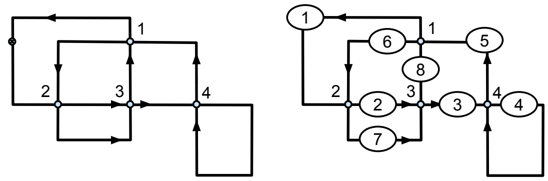


FIGURE 9. Enumeration of edges and vertices.

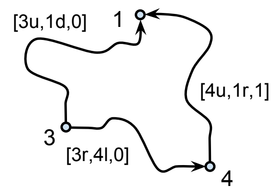


FIGURE 10. The boundary of a region as retrieved from the edge list.

We are going to standardize enumeration of vertices and edges. Again, we assume that there is a base point on the curve and, moreover, we require that the base point belongs to the boundary of the outer region as shown in Figure 9.

Then the edges of the graph are numbered in the order of appearance as one travels along the curve. The first edge joins the first vertex to the second one (or to itself) and contains the base point.

For each edge, we will store the initial and the terminal vertex, the direction of the first and the last segment (up, down, left, or right), and the turning number — the number of left turns minus the number of right turns.

Example 3.1. Consider the curve shown in Figure 9. The edge list of this curve is $[1u,2l,3]$, $[2r,3l,0]$, $[3r,4l,0]$, $[4r,4d,-3]$, $[4u,1r,1]$, $[1l,2u,1]$, $[2d,3d,2]$, $[3u,1d,0]$. This notation has the following meaning. The first edge, for instance, goes from 1 to 2, its initial direction is upward, it arrives to 2 from the left, and turns left three times.

Lemma 3.2. *A generic closed curve is uniquely determined by its edge list.*

Proof. To recover the curve, it's sufficient to know its Gauss diagram and identify the boundary of the outer region. It's easy to get the Gauss diagram — the turning numbers are not even needed.

Now observe that boundaries of regions can be obtained from the edge list. For instance, edges $[3u,1d,0]$, $[3r,4l,0]$, and $[4u,1r,1]$ in Example 3.1 form the boundary of the region shown schematically in Figure 10.

Let B be the boundary of a region. The edge list allows us to find the number of left turns minus the number of right turns as one is moving along B keeping the region on the left hand side. Clearly, we get -4 for the outer region and 4 for all other regions. The proof is completed. $\square$

Remark. The only thing we need the turning numbers for is the ability to determine which of the regions is the outer one. Of course, we could just agree that the first edge belongs to the boundary of the outer region and somehow remember whether the outer region is on its left or its right hand side. However, the idea is that the edge list, being geometric in nature, could be read from an actual curve or an arc diagram and finding turning numbers might be easier than identifying the outer region right away.

Encoding non-generic curves. If two generic curves Γ_1 and Γ_2 have the same rotation number, then Γ_1 can be continuously deformed into Γ_2 as shown in Figure 2. Such a deformation is a continuous path in the space of all curves (we don't define the space of all curves here in order to avoid unnecessary technicalities). By a small perturbation of this path, we can always make sure that it goes all the way through generic curves with finitely many exceptions — non-generic curves with a simplest possible singularity. The simplest possible singularity can be one of the three types — a direct self-tangency, an inverse self tangency, or a triple point.

If we wanted to study non-generic curves specifically, then we could consider a deformation of, say, one curve with a triple point to another curve with a triple point. Then, by analogy with the generic case, we would probably require such a deformation to go through simple non-generic curves, with finitely many exceptional cases of more complicated singularities — curves with a triple point and a self-tangency point, curves with two triple points, curves with a quadruple point etc.

The natural extension of this approach would be do define the complexity of a singularity such that generic curves have complexity 0, curves with one self-tangency or a triple point have complexity 1, and curves with more complicated singularities are of complexity 2 and higher. Then a generic deformation of two curves of complexity n should go almost entirely through singular curves of complexity n with finitely many exceptions of complexity $n+1$ that cannot be removed by a small perturbation of the path.

The invariants J^+, J^-, and St have order 1 in the sense that they are nontrivial for generic curves, i.e., curves of complexity 0. If we extend them to non-generic curves, then they will be very simple for curves of complexity 1 (for instance, St takes value 1 on any curve with a triple point and 0 on a curve with a self-tangency point) and just vanish on curves of higher complexity.

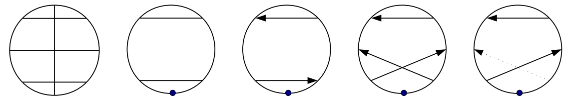
The theory of higher order plane curve invariants is yet to be developed. The reader might refer to the rich literature about Vassiliev knot invariants for the analogy in knot theory. However, the situation with plane curves is more difficult because it's not clear how one should define the complexity of a singularity. What is a more complicated singularity — a triple point where two of the branches touch or a quadruple point? One point with six branches intersecting transversally or four different points of direct self-tangency? Do we need to include cusps too or we should remain in the class of immersions?

Currently we only encode and compute invariants of generic curves. All the three encoding schemes — Gauss diagrams, arc diagrams, and edge lists, are not designed to account for non-generic curves. If the theory of higher order plane curve invariants is ever developed, then one will probably want to encode non-generic curves too. It may be possible to modify Gauss diagrams, arc diagrams, and edge lists to do it. However, since there is no theory of non-generic curves and higher order invariants yet, we are not going to discuss such modifications here, although combinatorial descriptions of non-generic plane curves might be a good topic for future research.

4. Computation of J^+, J^-, and St

Formulas of Polyak. In [6], Polyak gives explicit formulas for Arnold's invariants J^+, J^-, and St in terms of Gauss diagrams.

Definition 4.1. A *chord diagram* is an oriented circle with finitely many chords (unordered pairs of points on it) such that no two endpoints of different chords coincide. Similarly, a *Gauss diagram* is an oriented circle with finitely many oriented chords. Both chord and



A B C D $C \subset D$

FIGURE 11. Gauss and chord diagrams

Gauss diagrams are considered up to an orientation-preserving diffeomorphism of the circle.

A *based* chord or Gauss diagram is a diagram with a distinguished point that is not on any of the endpoints of the chords.

To denote that c is one of the chords of a chord or Gauss diagram D, we write $c \in D$. Given two diagrams A and B of the same type, we say that A is a *subdiagram* of B, denoted $A \subset B$ if A is obtained from B by removing a number of chords.

If a chord diagram C is obtained from a Gauss diagram G by forgetting orientations of chords, we say that G is *of type* C and write $C \leftrightarrow G$.

Example 4.2. In Figure 11, A is a chord diagram, B is a based chord diagram, C and D are based Gauss diagrams. Further, C is a subdiagram of D of type B. Thus we can write $B \leftrightarrow C \subset D$.

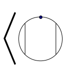
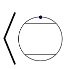
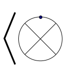
Given a generic closed curve Γ, one can construct its Gauss diagram G. The chords of G represent double points of the curve. The base point on Γ will give us a distinguished point on G, turning G into a based Gauss diagram. The signs of double points can be read from the diagram — in order to find the sign of a double point p, we go along G until the first passing of p. Should it be the endpoint of the arrow, the sign is negative, otherwise it's positive. The same procedure defines signs of chords of an abstract based Gauss diagram.

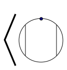
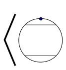
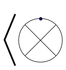
Definition 4.3. Let C be a based chord diagram and let G be a based Gauss diagram. Define

$$\langle C, G\rangle = \sum_{C \leftrightarrow A \subset G} \prod_{c \in A} \operatorname{sign}(c),$$

where the sum is taken over all subdiagrams A in G of type C.

Theorem 4.4 (Polyak, [6])**.** *Let G be the based Gauss diagram of a generic closed curve Γ of rotation number r with n double points. Then we have*

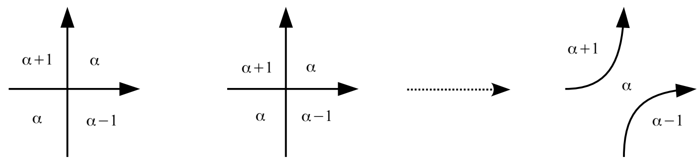
(1) $J^+(\Gamma) = \Big\langle$  $, G\Big\rangle - \Big\langle$  $, G\Big\rangle - 3\Big\langle$  $G\Big\rangle - \dfrac{n + r^2 - 1}{2} = J^-(\Gamma) + n,$

(2) $St(\Gamma) = -\dfrac{1}{2}\Big\langle$  $, G\Big\rangle + \dfrac{1}{2}\Big\langle$  $, G\Big\rangle + \dfrac{1}{2}\Big\langle$  $, G\Big\rangle + \dfrac{n + r^2 - 1}{4}.$

It is straightforward to obtain the Gauss diagram from either an arc diagram or an edge list. Thus computing values of J^+, J^-, and St according to (1) and (2) requires $O(n^2)$ operations.

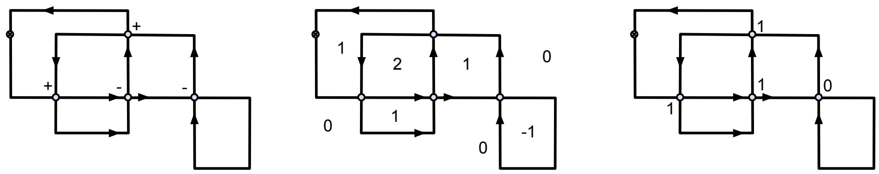
Formulas of Viro and Shumakovich. Recall that a curve separates the plane into regions (Definition 2.2). Moreover, the so-called *Alexander rule* (Figure 12) describes indices of regions in a neighbourhood of a double point.

Definition 4.5. The *smoothing* of an oriented generic closed curve Γ is the operation shown in Figure 12 on the right applied at all double points. Let $\tilde{\Gamma}$ be the result of the smoothing — a disjoint union of oriented circles (analogous to Seifert circles in knot theory). Path-connected components of $\mathbb{R}^2 - \tilde{\Gamma}$ inherit indices of regions of Γ.



The Alexander rule Smoothing of a double point

FIGURE 12. Indices of regions in a neighbourhood of a double point.



Signs of double points Indices of regions Indices of double points

FIGURE 13. Ingredients for Shumakovich's formula.

Observe that while a bounded path-connected component of $\mathbb{R}^2 - \Gamma$ is always homeomorphic to a disk, path-connected components of $\mathbb{R}^2 - \tilde{\Gamma}$ may have the topology of a disk (or a plane) with h holes. If $\tilde{R}$ is a path-connected component of $\mathbb{R}^2 - \tilde{\Gamma}$, then we have $\chi(\tilde{R}) = 1 - h$, where χ denotes the Euler characteristic.

Theorem 4.6 (Viro, [8])**.** *Let $\tilde{\Gamma}$ be the smoothing of a generic plane curve Γ with n double points. Then we have*

$$J^+(\Gamma) = 1 + n - \sum_{\tilde{R}} \operatorname{ind}^2(\tilde{R})\chi(\tilde{R}) = J^-(\Gamma) + n \tag{3}$$

where the sum is taken over all path-connected components $\tilde{R}$ of $\mathbb{R}^2 - \tilde{\Gamma}$.

Definition 4.7. The index of a double point shown in Figure 12 is α.

Theorem 4.8 (Shumakovich, [7])**.** *Suppose that the base point of a generic plane curve Γ belongs to the boundary of the outer region. Then we have*

$$St(\Gamma) = \sum_p \operatorname{sign}(p) \operatorname{ind}(p), \tag{4}$$

where the sum is taken over all double points p of Γ.

Example 4.9. We'll apply (3) and (4) to the curve in Example 3.1. The ingredients for (4) are shown in Figure 13. Thus,

$$St(\Gamma) = +1 + 1 - 1 - 0 = 1.$$

The ingredients for (3) are shown in Figure 14. Thus,

$$J^+(\Gamma) = 1 + 4 - \left(0^2 \cdot (-1) + 1^2 \cdot 0 + 2^2 \cdot 1 + (-1)^2 \cdot 1\right) = 0, \quad J^-(\Gamma) = -4.$$

5. OUR ALGORITHM

If our goal is to implement an algorithm that computes the values of $J^+(\Gamma)$, $J^-(\Gamma)$, and $St(\Gamma)$ for a given generic closed curve Γ with n double points, then an important issue is the method of describing Γ combinatorially. Gauss diagrams work well with Polyak's formulas (1) and and (2), but the computational complexity is $O(n^2)$.

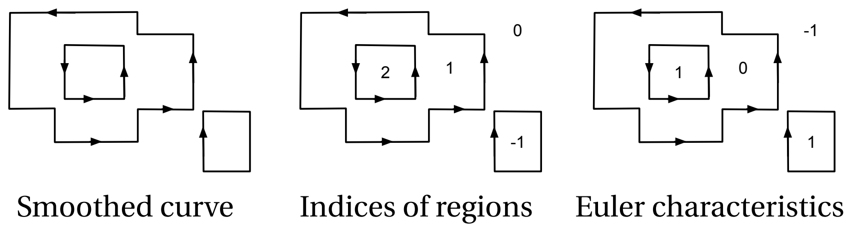


FIGURE 14. Ingredients for Viro's formula.

We will show how the complexity $O(n)$ can be achieved if the curve is stored as an edge list and the formulas of Viro (3) and Shumakovich (4) are applied. Specifically, we'll explain how each of the six ingredients illustrated in Figures 13 and 14 can be computed by going once either through all the double points or through all the edges.

Computing signs of double points. This is easy — we just travel along the curve using the edge list.

Identifying regions of the curve and computing their indices. In order to identify regions of the curve, we need one extra step. We'll construct a $4\times 4n$ table that essentially provides the same information as the edge list, but the order of entries is rearranged. The best way to explain it is by an example.

Example 5.1. We'll again refer to the same curve as the one in Example 3.1 (Figures 9, 13, and 14). The edge list is $[1u,2l,3]$, $[2r,3l,0]$, $[3r,4l,0]$, $[4r,4d,-3]$, $[4u,1r,1]$, $[1l,2u,1]$, $[2d,3d,2]$, $[3u,1d,0]$. The table we are constructing is

$$\begin{bmatrix} 1r & 1u & 1l & 1d & 2r & 2u & 2l & 2d & 3r & 3u & 3l & 3d & \cdots \\ 4u & 2l & 2u & 3u & 3l & 1l & 1u & 3d & 4l & 1d & 2r & 2d & \cdots \\ 5 & 1 & 6 & 8 & 2 & 6 & 1 & 7 & 3 & 8 & 2 & 7 & \cdots \\ - & + & + & - & + & - & - & + & + & + & - & - & \cdots \end{bmatrix} \tag{5}$$

For instance, the 1st column says that if you travel to the right from vertex 1 (first entry), then you come to vertex 4 from top (second entry) along edge 5 (third entry), which is actually oriented in the opposite direction, from vertex 4 to vertex 1 (minus in the fourth entry).

In other words, for each $i = 1,2,\ldots,2n$, the i th element of the edge list, $[a,b,c]$ produces two columns of the table

$$\begin{bmatrix} a \\ b \\ i \\ + \end{bmatrix} \text{ and } \begin{bmatrix} b \\ a \\ i \\ - \end{bmatrix}$$

at positions determined by a and b respectively.

Now we can specify regions by listing the edges that occur in the boundary of each region, with the sign showing whether the direction of the edge is the same or the opposite as the natural orientation of the region's boundary (the convention is that the region is on the left of its boundary).

Example 5.2. We'll again refer to the same curve as the one in Example 3.1 (Figures 9, 13, and 14). The regions are $(-1,-5,4,-3,-7)$ — the outer region, $(1,-6)$, $(-2,7)$, $(2,8,6)$, $(3,5,-8)$, (-4).

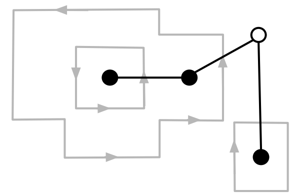


FIGURE 15. The dual graph of a smoothed curve.

At the same time, we can store regions occurring in the neighbourhood of each vertex in the counter-clockwise order, starting from the top-right corner: Vertex 1: $1,2,4,5$; Vertex 2: $4,2,1,3$; Vertex 3: $5,4,3,1$; Vertex 4: $1,5,1,6$.

It is obvious that table (5) allows us to generate the list of regions by checking each edge at most four times. We can always produce the list of regions in the lexicographic order.

The Alexander Rule (Figure 12) allows us to compute the indices of three regions adjacent to a vertex if the index of the fourth region is known. We know that the index of the outer region is 0 and we know that the outer region is adjacent to the 1st vertex. Thus we can compute the indices of all the four regions in the neighbourhood of the first vertex. Now we just need to go through all the vertices, each time finding indices of regions that haven't been found before. Since vertices are enumerated in the order of appearance along the curve, the situation when at some point none of the four indices is known is impossible. Thus eventually we will find indices of all the regions.

Computing indices of double points. It's very easy now — we just apply Definition 4.7.

Computing indices of regions of the smoothed curve. Again, this is easy. Initially we have $n+2$ regions enumerated from 1 to $n+2$. Now we need to go through all double points, each time making note that two regions of the original curve are being merged by assigning them new labels in the list of the regions of the smoothed curve.

Example 5.3. Let's do it again for the curve from Example 5.2. The regions have been found already. The smoothing at vertex 1 merges regions 2 and 5 into one region, that we label as, say, $2'$. From vertex 2, regions 2 and 3 are merged and region 3 is now also a part of $2'$. From vertex 3 and vertex 4, regions 3 and 5 are merged; and from vertex 4, regions 1 and 1 are also merged, so we don't get a new info here. The result can be stored as

Region of Γ	Boundary	Index	Region of $\tilde{\Gamma}$
1	$(-1,-5,4,-3,-7)$	0	$1'$
2	$(1,-6)$	1	$2'$
3	$(-2,7)$	1	$2'$
4	$(2,8,6)$	2	$4'$
5	$(3,5,-8)$	1	$2'$
6	(-4)	-1	$6'$

Euler characteristic of regions of the smoothed curve. This is probably the trickiest thing to compute. We need to construct a graph whose vertices correspond to regions of the smoothed curve and there is an edge joining two vertices whenever the corresponding regions share a common boundary. In other words, it's the dual graph of the smoothed curve.

Example 5.4. Figure 15 shows the dual graph for the curve from Example 3.1.

Obviously, such a dual graph is a rooted tree whose root represents the outer region. The Euler characteristic is $1-h$, where h is the number of holes. The number h can be easily read from the rooted tree — it's the degree of the root (which doesn't actually matter) or the degree minus one for any other vertex.

The rooted tree can be constructed by going through all the double points of the original curve; each of them contributes up to two edges to the rooted tree.

6. Summary

Our algorithm can be used for computer experiments. For instance, one can generate all curves with a given number of double points, compute the three invariants for each curve, and look at the distribution. Since it's a large computation, the efficiency of the algorithm is important. One of the authors tried to do it when he was a year 3 student with not much success because he didn't invent the efficient algorithm introduced in the present paper but used Polyak's formulas instead. We encourage young researchers, especially undergraduate students to implement our algorithm — it'll be a very interesting project.

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Nanyang Technological University
E-mail address: `fduzhin@gmail.com`

Nanyang Technological University
E-mail address: `516481722@qq.com`