LEBESGUE CONSTANTS OF MULTIPLE FOURIER SERIES

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Dedicated to the memory of E. Belinsky, my friend and colleague

Abstract

This is an attempt of a comprehensive treatment of the results concerning estimates of the L^1-norms of linear means of multiple Fourier series, the Lebesgue constants. Most of them are obtained by estimating the Fourier transform of a function generating such a method. Frequently the properties of the support of this function affects distinctive features in behavior of these norms. By this geometry enters and works hand-in-hand with analysis; moreover, the results are classified mostly in accordance with their geometrical nature. Not rarely Number Theory tools are brought in. We deal only with the trigonometric case - no generalizations for other orthogonal systems are discussed nor are applications to approximation. Several open problems are posed.

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Contents

Preface

The purpose of these notes is to give a comprehensive treatment of results and methods for estimation of the L^1-norms of linear means of multidimensional Fourier series, the so-called Lebesgue constants.

My teacher R. M. Trigub upon seeing me return again and again to these problems, once suggested that now is a proper time to assemble a comprehensive survey on the Lebesgue constants of multiple Fourier series and that I should try to fill this gap. I am very grateful to him for "infecting" me with this tempting idea as well as for repeatedly encouraging me to continue with the work.

Of course, there already were several related surveys in the literature (see [Zh, AIN, AAP, Go, Dy3]), as well as several important books on different aspects of multidimensional Fourier Analysis; those by E. M. Stein (see [SW, S3]) should be mentioned first (for the sake of completeness, let us mention also the book [Ya] devoted just to multiple trigonometric series; unfortunately, it is badly written and moreover has many gaps and mistakes). Much can be found in the recent monograph by Trigub and Belinsky [TB]. But in no one of these works is a real attempt to draw the entire picture of the L^1 aspects of multiple Fourier series. As for myself, I continued to be uncertain as to the justification for such a work, and for a long period I did no more than to collect all known references on the subject.

An impetus has been given to me by E. M. Stein - once he asked me whether I can outline just the results on the Lebesgue constants in which the Fourier transform methods are involved. A brief survey of such results became the first version of these notes. I would like to use this opportunity to express my gratitude to E. M. Stein.

The up-to-date situation in this topic may be characterized by the words "a topic now in disrepute due to its difficulty", the words said about a related field in the marvellous book of Davis and Chang [DC]. Indeed, important problems continue to be open for decades, while interesting publications have seldom appeared during this period. Nevertheless, the continuing vitality of questions related to the topic and efforts which are being made by several enthusiasts, reluctant to give up this circle of ideas, instills hope for new growth of popularity of this part of Fourier Analysis. A brief survey of the main results on Lebesgue constants of multiple Fourier partial sums is given in author's article [L15] in Math. Encyclopedia.

One may see already from the table of contents that the structure of the work is determined by geometrical features rather than analytical ones. It is partly a matter of taste, since both aspects are strongly connected and work hand-in-hand. Numerous appearances of Number Theory tools also should not be underestimated.

The outline of these notes is as follows. After some preliminaries (a brief survey of one-dimensional results among them), in Section 1 we first of all prove a two-sided estimate of the Lebesgue constants of spherical partial sums. The upper estimate is due to V. A. Yudin, while the lower one is essentially based on an idea due to V. A. Ilyin. From this it progresses to certain generalization. In Section 2 we give very general results mostly due to E. Belinsky. Section 3 is devoted to various connections between Fourier series and Fourier integrals; all these "equiconvergence" results are related to Lebesgue constants. In Section 4 we investigate various generalizations of the Bochner-Riesz means. The next Section 5 is also devoted to generalizations of the Bochner-Riesz means but of a different nature. What is preserved is the spherical way of summation. In Section 6 a collection of "polyhedral" results is given. Some of them are very subtle; actually the number of interesting problems in this case is much larger than one would probably expect. In Section 7 results are considered in which partial sums or more general linear means are taken with respect to "hyperbolic crosses". Then, in Section 8, we present some cases in which the operator of taking partial sums turns out to be unbounded. In Section 9 we give some results on integrability of multiple trigonometric series. In the next section we consider results concerning the Nikolskii type problem for Lebesgue constants. It is intimately connected with the previous section. In the last section we give some more results which are not proved by means of the Fourier transform methods. I have tried to cover the literature on the subject completely as well as to discuss all essential results on the topic - some of them are very recent.

My friends and colleagues E. Belinsky, M. Skopina, and especially A. Podkorytov were the readers of earlier variants of this work. Many improvements are due to their precise remarks and stimulating discussions.

Unfortunately, E. Belinsky untimely passed away in 2004. His influence, inspiration and support were very important to me during all my life. These notes are dedicated to his memory.

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Finally I wish to mention that only one person deserves "acknowledgements" for possible mistakes, misprints and all poor passages. "It's Me, O Lord...", as R. Kent entitled his book.

0 Preliminaries

First we give a brief survey of one-dimensional results, then some notation and preliminary discussions for several dimensions will follow.

0.1. Let us draw the state of affairs in the one-dimensional case. The well-known asymptotic result would apparently come back to one's mind at the sight of the words "Lebesgue constants":

$$\pi^{-1}\int\limits_{\mathbf{T}}\left|1/2+\sum_{k=1}^{N}\cos kx\right|dx=4\pi^{-2}\ln N+O(1),$$

where $\mathbf{T}=(-\pi,\pi]$. This relation, whose left-hand side is traditionally called the Nth Lebesgue constant, can be found in any textbook or monograph (see, e.g., [Br], [Zg]) dealing with the Fourier series

$$\sum_{k}\hat{f}(k)e^{ikx},$$

where

$$\hat{f}(k)=(2\pi)^{-1}\int\limits_{\mathbf{T}}f(t)e^{-ikt}dt$$

is the kth Fourier coefficient. Since the Lebesgue constants are the norms of partial sums of the Fourier series, the relation itself expresses the fact that the partial sums of the Fourier series of a continuous function fail to converge to the function at some points. Of course, in view of Carleson's celebrated theorem [Cl], it converges almost everywhere, but the Fourier series of an integrable function may diverge already at each point; the latter fact is Kolmogorov's famous result [K2]. It is known long ago that considering the sequence of the arithmetic (Fejér, Cesáro) means

$$(N+1)^{-1}\sum_{k=0}^{N}S_k(f;x)$$

instead of the sequence of the partial sums

$$S_N(f;x)=\sum_{k=-N}^{N}\hat{f}(k)e^{ikx}$$

improves the situation in the sense that convergence and approximation properties of such means are better. The same is true for many other linear means

$$L_N^{\lambda}(f;x)=\sum_{k}\lambda_{N,k}\hat{f}(k)e^{ikx}.$$

In view of the Banach-Steinhaus theorem, the question whether these means define a regular method of summability (convergent, to the same sum, for convergent series) or not can be derived from the behavior of the sequence of norms of operators L_N^{λ} taking $L^1(\mathbf{T}^n)$ into $L^1(\mathbf{T}^n)$, or $C(\mathbf{T}^n)$ into $C(\mathbf{T}^n)$ which is the same. We will call these norms the Lebesgue constants as well. The uniform boundedness of this sequence yields the regularity of the method, otherwise some information on the growth in N of this sequence may be helpful.

Usual Lebesgue constants (see above) appear when taking $\lambda_{N,k} = 1$ for $|k| \le N$ and $\lambda_{N,k} = 0$ otherwise.

General necessary and sufficient conditions for the regularity of the method of summability $L_N^\lambda(f;x)$ were obtained by J. Karamata and M. Tomič [KT]. These are:

1) $\lim\limits_{N\to\infty} \lambda_{N,k} = 1$ for all k.

2) For every N there exists a number M_N, such that for all m

$$\int_0^\pi \left| \lambda_{N,0}/2 + \sum_{k=1}^m \lambda_{N,k} \cos kx \right| dx \le M_N.$$

3) The total variation of the functions

$$x\lambda_{N,0}/2 + \sum_{k=1}^m k^{-1} \lambda_{N,k} \sin kx$$

is uniformly bounded.

In the case of triangular matrices, that is, those with $\lambda_{N,k} = 0$ for $k \ge N$, only two conditions remain to be valid, namely, the first one and the second with $m = N-1$ and absolute constant instead of M_N - this was established already by Lebesgue.

As it frequently happens to conditions that are simultaneously necessary and sufficient, at best, their verification can be a cumbersome work, and in a sense, impossible in many cases. The problem arises of finding verifiable sufficient conditions in terms of multipliers $\lambda_{N,k}$. For triangular matrices this problem has been posed by S. M. Nikolskii [N]; he also succeeded to solve the problem for $\lambda_{N,k}$ being convex or concave. Actually, some results of such type incidentally appeared earlier (see, e.g., the papers by Hille and Tamarkin [HT] or Sz.-Nagy [SN]), but real interest to the problem and prgress has been visible just after the paper [N]. These results are partly surveyed in [Br, Ch.VII] and [Ti, Ch.VIII]. One can find a kind of survey in [Te2]. A detailed survey is given in [T1]. The following result by Telyakovskii [Te] is one of the most developed.

Theorem 0.1. *If a triangular method of summability L_N^λ satisfy*

$$\sum_{k=0}^{N-1} |\Delta\lambda_{N,k}| \le C,$$

where $\Delta\lambda_{N,k} = \lambda_{N,k} - \lambda_{N,k+1}$, and

$$\sum_{k=2}^{N-2} \left| \sum_{l=1}^{q(k.l)} \frac{\Delta\lambda_{N,k-l} - \Delta\lambda_{N,k+l}}{l} \right| \le C,$$

then for the regularity of the method there necessary and sufficient **1)** *above and*

$$\left| \sum_{k=1}^{N-1} \frac{\lambda_{N,k}}{N-k} \right| \le C.$$

Here and in what follows by C we denote various constants, generally speaking, different.

The last condition in Theorem 0.1 has already appeared in Nikolskii's paper [N]; the point is that the assumptions in [N] were much more restrictive.

These problems are intimately related to the well-known problem of integrability of trigonometric series: given a trigonometric series

$$a_0/2 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

under which assumptions on its coefficients this series is the Fourier series of an integrable function. The assumptions considered are such that the series converges to a function continuous everywhere except one point. Thus, the problem whether this series is the Fourier series is reduced to that of integrability of the sum of the series, or as it is used to say, integrability of the series. We will say in this case that a sequence belongs to $\widehat{L^1}$.

Frequently, the cosine series

$$a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$$

and the sine series

$$\sum_{k=1}^{\infty} b_k \sin kx$$

are investigated separately, since there is a difference in their behavior. Usually, integrability of the latter requires additional assumptions.

Of course, considered were series of the form

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

(see, e.g., [Mo4]) as well, but this does not add anything essential to our consideration.

To the best of our knowledge, there exists no convenient description of $\widehat{L^1}$ in terms of a given sequence alone. Actually, there are some characterizations, e.g., [E, Ro, Ry, Sa], but they are too complicated to be applied to concrete problems and they involve properties of functions. Hence, certain subspaces of $\widehat{L^1}$ are studied so that they are both as wide as possible and described in terms convenient for applications.

Let us be more precise. First of all, in view of the Riemann-Lebesgue theorem $\widehat{L^1}$ itself is a subspace of c_0, the space of null sequences.

In 1922, Sidon [Si1] (see also [B, vol.I]) gave an example of an even monotone null sequence which is not in $\widehat{L^1}$. This means that also the space of sequences of bounded variation

$$bv = \left\{ d = \{d_k\} : \|d\|_{bv} = \sum_{k=0}^{\infty} |\Delta d_k| < \infty \right\}$$

is not a subspace of $\widehat{L^1}$. Here $\Delta d_k = d_k - d_{k+1}$.

It is well-known (see, e.g., [B,Zy]) that possessing a null sequence of bounded variation as its Fourier coefficients, the cosine series converges for every $x \neq 0 (\operatorname{mod} 2\pi)$, while the sine series converges everywhere.

In 1913 W.H. Young [Yo] proved that if $\{a_k\}$ is a convex null sequence, that is,

$$\Delta^2 a_k = \Delta(\Delta a_k) \geq 0$$

for $k = 0, 1, 2, \dots,$ then the cosine series is the Fourier series of an integrable function. In 1923 Kolmogorov [K] extended this result to the class of quasi-convex sequences $\{a_k\}$, namely those satisfying

$$\sum_{k=0}^{\infty} (k+1)|\Delta^2 a_k| < \infty.$$

In 1934 Pfleger [P] proved that like every real sequence of bounded variation is a difference of two monotone sequences, every real quasi-convex sequence is a difference of two convex sequences.

The first period of investigation was over, in a sense, in 1956 when R.P. Boas generalized all previous results [Bo1].

Let us give a list of spaces χ all of which are subspaces of $\widehat{L^1}$ and ensure the integrability of corresponding trigonometric series. This list does not pretend to be comprehensive. Though most of the strongest known conditions are in this list, the selection is partly a matter of taste.

1) The so-called Boas-Telyakovskii condition (see, e.g., [Te2]). Let

$$s_d = \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n/2} \frac{\Delta d_{n-k} - \Delta d_{n+k}}{k} \right|,$$

then $d = \{d_k\} \in bt$ if

$$\|d\|_{bt} = \|d\|_{bv} + s_d < \infty.$$

This was a generalization of Boas' result in the way that in [Bo1] the sign of absolute value in the representation for s_d was inside the second sum.

2) A.G. Fomin's condition [Fo2] (see also [BTM1,2], [GM], [GM2]):

$$\|d\|_{a_p} = \sum_{n=0}^{\infty} 2^{n/p'} \left\{ \sum_{k=2^n}^{2^{n+1}-1} |\Delta d_k|^p \right\}^{1/p} < \infty, \qquad 1 < p < \infty, \quad 1/p + 1/p' = 1.$$

3) The Sidon-Telyakovskii condition [T6]:

$$A_k \downarrow 0 \ (k \to \infty), \quad \sum_{k=0}^{\infty} A_k < \infty \qquad \text{and} \qquad |\Delta d_k| < A_k.$$

4) The Buntinas–Tanovic-Miller condition (see, e.g., [BTM2]).

Let $\{k_n\}$ be an increasing sequence and $\{m_n\}$, $1 \leq m_n \leq k_{n+1}$, a non-decreasing sequence. Then $d \in hv^p$ if

$$\sum_{n=0}^{\infty} m_n^{1/p'} \left\{ \sum_{k=k_n}^{k_{n+1}-1} |\Delta d_k|^p \right\}^{1/p} + \sum_{n=0}^{\infty} \ln\left(\frac{k_{n+1}}{m_n}\right) \sum_{k=k_n}^{k_{n+1}-1} |\Delta d_k| < \infty.$$

If $k_n = 2^n$ and $m_n = 2^{n+1}$, we get a_p. If $m_n = 1$, then $\sum_{n=1}^{\infty} |\Delta d_n| \ln n < \infty$. They also introduced a scale of HV^p spaces each of which is a linearization of hv^p.

Recently amalgam spaces (see [AF1, BTM3]), in which the condition for a sequence d is

$$\sum_{n=0}^{\infty}\left\{\sum_{m=1}^{\infty}\left[\sum_{k=m2^n}^{(m+1)2^n-1}|\Delta d_k|\right]^2\right\}^{1/2}<\infty,$$

were used in these problems; for further analysis see [Fi].

For various reasons we would like to pay more attention to Telyakovskii's results. First of all, some other results simply follow from his results. In addition, besides direct answers to the question which has arisen, estimates of the integral

$$I=\int\left|a_0/2+\sum_{k=1}^{\infty}(a_k coskx+b_k\sin kx)\right|dx$$

were obtained over either the entire period or some smaller interval.

Just such estimates give rise to important applications: in summability (more precisely, in estimates of Lebesgue constants), in estimating deviations of functions from the means of their Fourier series, and in best approximation of infinitely differentiable functions.

A typical strong result due to Telyakovskii is the following

Theorem 0.2. *Let* $\{a_n\}$, $\{b_n\}$ *be null sequences. Then*

$$\int_0^{\pi}\left|a_0/2+\sum_{k=1}^{\infty}a_k\cos kx\right|dx=O(||a||_{bv}+s_a),$$

and uniformly with respect to $p=1,2,...$

$$\int_{\pi/(2p+1)}^{\pi}\left|\sum_{k=1}^{\infty}b_k\sin kx\right|dx=\sum_{k=1}^{p}\frac{|b_k|}{k}+O(||b||_{bv}+s_b),$$

and trigonometric series are the Fourier series of integrable functions.

A traditional way to prove such results were the so-called Sidon type inequalities. Those are various inequalities related to the one obtained by Sidon [Si2]

$$(N+1)^{-1}||\sum_{k=0}^{N}c_kD_k||_{L^1}\le\max_{0\le k\le N}|c_k|,$$

where D_k are the Dirichlet kernels of order k and c_k are arbitrary numbers. For generalizations and applications, see, e.g., a survey by Fridli [Fi].

In [L5], a new approach to these problems was introduced.

First, let us consider a locally absolutely continuous function f on $[0,\infty)$ such that

$$\lim_{x\to\infty}f(x)=0$$

and $f\in X$, where X is a subspace of the space of functions of bounded variation; in addition denote

$$\|f\|_{BV}=\int_0^{\infty}|f'(x)|dx<\infty.$$

The spaces X that are already investigated within this scope generalize the known spaces of sequences (see **1)-3)** above).

1) Set

$$S_f = \int_0^\infty \left| \int_0^{u/2} \frac{f'(u-x) - f'(u+x)}{x} dx \right| du.$$

Then $f \in BT$ if $||f||_{BV} + S_f < \infty$.

It is very interesting that the finiteness of the right-hand side means that the even continuation of f has a derivative in the real H^1 space, that is, this derivative itself and its Hilbert transform are both integrable. Similar relations take place also for sequences and are investigated by Fridli (see [Fi] and especially [Fi1]) .

2) The following spaces first were used in a paper by D. Borwein [Bo]. For integrability problems, they were thoroughly investigated in various papers, first of all in those by Móricz and Giang [Mo, GM0,1, Mo2,3]. Thus $f \in A_p$, $1 < p < \infty$, if

$$\int_0^\infty \left(u^{-1} \int_u^\infty |f'(x)|^p \, dx \right)^{1/p} du < \infty.$$

3) The case of $p = \infty$ is of special interest. We have $f \in A_\infty$ if

$$\int_0^\infty \text{ess sup}_{|x|>u} |f'(x)| \, du < \infty.$$

For this, see [T11, BLT]; for various relations between these spaces and history, see [L11].

Let us go on to the Fourier transform approach. We consider

$$\hat{f}_c(y) = \int_0^\infty f(x) \cos xy \, dx$$

and

$$\hat{f}_s(y) = \int_0^\infty f(x) \sin xy \, dx,$$

the cosine and sine Fourier transforms of f, respectively.

Theorem 0.3. *Let f be as above. Then for any $y > 0$ we have*

$$\hat{f}_c(y) = \theta\gamma(y),$$

and

$$\hat{f}_s(y) = y^{-1} f\left(\pi/(2y)\right) + \theta\gamma(y),$$

where $|\theta| \le C$ and $\int_0^\infty |\gamma(y)| dy \le \|f\|_X$.

Such results are of great interest by themselves. Not many of them are known. We mention Trigub's result on asymptotics of the Fourier transform of a convex function (see [T3,4]); this was a generalization and strengthening of Shilov's result [Shi] on the behavior of the Fourier coefficients of a convex function). Next results were due to Trigub [T11] and Giang-Móricz [GM2]. These cover the cases **2)** and **3)** of the space X. More general result was recently obtained in [L13].

Now we are able to strengthen the known integrability results. Given cosine and sine series with the null sequence of coefficients in χ. Set for $x \in [k, k+1]$

$$A(x) = a_k + (k - x)\Delta a_k, \qquad a_0 = 0.$$

Passing now from Fourier integrals to trigonometric series (and vice versa; see Theorem 0.7) we arrive at

Corollary 0.1. *For each y, where $0 < y \le \pi$, we have*

$$\sum_{k=1}^{\infty} a_k \cos ky = \theta\gamma(y)$$

$$\sum_{k=1}^{\infty} a_k \sin ky = y^{-1}A(\pi/(2y)) + \theta\gamma(y)$$

where $|\theta| \le C$ and $\int\limits_0^{\pi} |\gamma(y)|\,dy \le ||a||_{bv} + s_a$.

Theorem 0.2 follows now immediately, merely by integrating the formulas obtained, but one can imagine more sophisticated use of these formulas, say, integration over some other sets.

A brief survey on the integrability of trigonometric series is given in author's article [L14] in Math. Encyclopedia.

Let us return to estimates of Lebesgue constants. We are going to give some results due to Trigub. In the 60s he made a further step by obtaining a series of results connecting summability to problems of absolute convergence of Fourier series and Fourier integrals. If the Lebesgue constants $||L_N^\lambda||$ increase infinitely as N increases, then there is no regularity of the means $L_N^\lambda(f)$. To provide the convergence of $L_N^\lambda(f)$ to f, some smoothness should be added to f related to the rate of growth of L_N^λ.

If to replace the Fourier coefficients $\hat f(k)$ (integrals) by $\hat f_N(k)$ via the rectangle formula for the uniform partition $x_p = \frac{2p\pi}{2N+1}$, with $|p| \le N$, we get

$$\tilde L_N^\lambda(f; x) = \sum_{k=-N}^{N} \lambda_{N,k} \hat f_N(k) e^{ikx},$$

where

$$\hat f_N(k) = \frac{1}{2N+1} \sum_{p=-N}^{N} f(x_p) e^{-ikx_p}.$$

Coefficients $\hat f_N(k)$ are called the Fourier-Lagrange coefficients. When $\lambda_{N,k} = 1$ for all $k \in [-N, N]$ we have that $\tilde L_N^\lambda(f)$ is the interpolation polynomial defined by the values $f(x_p)$ for $p \in [-N, N]$.

The convergence of $L_N^\lambda(f)$ to f at a point x for every $f \in C(\mathbf{T})$ is reduced to the boundedness in N of the norms of functionals, the Lebesgue functions.

Theorem 0.4. *We have*

$$\sup_{||f||_C\le 1} |\tilde{L}_N^\lambda(f;x)| = (\pi/2)|\sin(N+1/2)x|||L_N^\lambda||$$
$$+\,\theta \sup_{||f||_C\le 1} |\tilde{L}_N^\lambda(f;0)|,$$

where $|\theta|\le C$.

In the following theorem a trigonometric polynomial T_N of order not greater than N is replaced by piece-wise sinusoidal function (see **a)**), which allows one to calculate the asymptotics of integral norms of T_N (see **b)**). For this, let us introduce a sequence of functions φ_N corresponding to $\{\lambda_{N,k}\}_{k=-N}^N$ and satisfying the only condition $\varphi_N(x_k)=\lambda_{N,k}$ for $k\in[-N,N]$. For instance, φ_N may be a polynomial or piece-wise linear function. Denote also $\psi_N(x)=x\varphi_N(x)$.

Theorem 0.5. *Two assertions are true.*

1) *For every* $x\in[-\pi,\pi]$ *and* $p=\left[\frac12+\frac{2N+1}{2\pi}x\right]$ *the following inequality holds*

$$\left| T_N(x) - \frac{(-1)^p}{2N+1} T_N'(x_p)\sin(N+1/2)\,x \right|$$
$$\le C\sum_{k=-N}^{N} |T_N(x_k)| \frac{(2N+1)^2}{(2|p-k|+1)^2(4N+1-2|p-k|)^2}.$$

2) *The following asymptotic equality holds*

$$\int_{\mathbf{T}} \left| \sum_{k=-N}^{N} \lambda_{N,k} e^{ikx} \right| dx = (4/\pi)\sum_{k=-N}^{N} |\widehat{\psi_N}(k)| + \theta \sum_{k=-N}^{N} |\widehat{\varphi_N}(k)|$$

with $|\theta|\le C$.

As an application, one can derive asymptotics for the Lebesgue constants for the general sums of Bernstein-Rogosinski type $\sum_{k=-N}^{N}\mu_k S_N(f;x+x_k)$ with the remainder term $\theta\sum|\mu_k|$.

What is of special interest is the case when $\lambda_{N,k}=\lambda(k/N)$ for some fixed function λ (for instance, it contains the case of partial sums λ being the indicator function of the interval $[-1,1]$). It was Trigub who started a systematic study of this case. The main point is that there exists a close connection between summability problems and behavior of the Fourier transform $\hat{\lambda}$ of the function λ generating the method.

Theorem 0.6. *The following assertions hold.*

1) *If* $\lambda\in C(\mathbf{T})$ *and* $\lambda(\pi)=\lambda(0)=0$, *then*

$$\sup_N ||L_N^\lambda||_M =\le C\sum_{k=-\infty}^{\infty} |\widehat{\lambda}(k)|\ln(|k|+1).$$

And if, in addition, λ *is a continuous real even function with alternating, with respect to sign, Fourier coefficients in a cosine series, starting with the first one, then the opposite inequality is valid too.*

2) *If* $\lambda \in C(\mathbf{T})$ *and* $\lambda(\pi) = \lambda(-\pi) = \lambda(0) = 0$, *then denoting* $\lambda_0(x) = \lambda(x) \operatorname{sign} x$ *we have*

$$\sup_N ||L_N^\lambda||_M \le C \sum_{k=-\infty}^{\infty} |\widehat{\lambda}(k)| \ln(|k|+1),$$

under the assumption that the series on the right-hand side converges.

Many details and references may be found in [T1,5,8,13]. Some results are due to Belinsky but mainly he extended this approach to the multidimensional case.

Theorem 0.7. *Let* φ *be a function of bounded variation and* $\lim_{|x|\to\infty} \varphi(x) = 0$. *Then for any* $\varepsilon > 0$

$$\sup_{0<|y|\le\pi} \left| \int_{-\infty}^{+\infty} \varphi(x) e^{-ixy/\varepsilon}\, dx - \varepsilon \sum_{-\infty}^{+\infty} \varphi(\varepsilon k) e^{-iky} \right| \le 2\varepsilon ||\varphi||_{BV}.$$

This result is due to Trigub [T11, Th.4]. In [T12] this is Lemma 1 given as a partial case of more general result; an earlier version for functions with compact support is due to Belinsky [Be0]).

This brief survey of one-dimensional results by no means pretends to be comprehensive. The idea was to introduce the reader to the subject. Mostly the results are emphasized that will be extended to the multidimensional case. For this, we are going to give some preliminaries as well as some general remarks.

0.2. Let f be an integrable function on $\mathbf{T}^n$, $n = 2, 3, ...$, 2π-periodic in each variable. Consider the Fourier series of this function

$$\sum_k \widehat{f}(k) e^{ikx} \tag{0.1}$$

where $x = (x_1, ..., x_n)$ is a point in the real n-dimensional Euclidean space $\mathbf{R}^n$, $k = (k_1, ..., k_n) \in \mathbf{Z}^n$, the lattice of points in $\mathbf{R}^n$ with integer coordinates, $kx = k_1x_1 + ... + k_nx_n$ is the scalar product, and

$$\widehat{f}(k) = (2\pi)^{-n} \int_{\mathbf{T}^n} f(x) e^{-ikx} dx$$

is the kth Fourier coefficient of f.

Kolmogorov's theorem [K2] applied in each variable asserts that the series (0.1) may be divergent at each point of $\mathbf{T}^n$. Thus, it is quite natural to consider the sequence of linear operators

$$L_N^\lambda : \qquad f(x) \mapsto L_N^\lambda(f;x) = \sum_k \lambda_{N,k} \widehat{f}(k) e^{ikx},$$

where $\lambda_{N,k}$ is a sequence of numbers (multipliers), or an important special case when $\lambda_{N,k} = \lambda(k/N))$

$$L_N^\lambda : \qquad f(x) \mapsto L_N^\lambda(f;x) = \sum_k \lambda(k/N) \widehat{f}(k) e^{ikx}, \tag{0.2}$$

where λ is a bounded measurable function (of course, it should be defined at the points of type k/N), and study properties of this sequence in order to derive some information about the function f, or some space of such functions. One receives much information from the behavior of the norms of these operators. When the operators map $C(\mathbf{T}^n)$ into $C(\mathbf{T}^n)$, or $L^1(\mathbf{T}^n)$ into $L^1(\mathbf{T}^n)$, and $\lambda \equiv 1$ on some set and vanishes outside (in the other words, λ is the indicator function of this set) the norms are traditionally called the Lebesgue constants. This term is frequently saved for the general situation of the norms of multipliers defined by a sequence $\{\lambda_{N,k}\}$ or $\{\lambda(k/N)\}$.

0.3. It is well-known [SW, Ch.VII,Th.3.4] that the operator L_N^λ is bounded if and only if the series

$$\sum_k \lambda(k/N)e^{ikx} \tag{0.3}$$

is the Fourier series of some measure μ_N, and $\|L_N^\lambda\| = \|\mu_N\|$. And if this series is the Fourier series of an integrable function, the following relation takes place:

$$\|L_N^\lambda\| = (2\pi)^{-n} \int\limits_{\mathbf{T}^n} \Big|\sum_k \lambda(k/N)e^{ikx}\Big| dx. \tag{0.4}$$

This occurs, for instance, when λ has compact support.

If to take formally an integral instead of the sum on the right-hand side of (0.4), and to fulfil some simple computations (also formal for a moment), considering k as a continuous parameter, one can expect something like $(2\pi)^{-n}\int_{N\mathbf{T}^n} |\widehat{\lambda}(x)|dx$ in place of the right-hand side of (0.4). Here

$$\widehat{\lambda}(x) = \int\limits_{\mathbf{R}^n} \lambda(u)e^{-iux}\, du$$

is the Fourier transform of λ. This process looks so natural and so attractive (and so short!) but this simplicity is deceptive. It may be very subtle and sometimes very cumbersome in reality. We will see below that sometimes it is invalid in some sense!

It is written in [DC] that "the best trick will be to transfer problems" (of convergence of Fourier series) "from Fourier series to Fourier integrals. This is good because it is easier to compute an integral explicitly than to sum a series in a closed form. On the other hand, this is bad because the integrals defining Fourier transforms do not converge absolutely".

We try to consider some problems in which "this is good" (or at least we think so). Even in the cases when "this is bad" (in the mentioned sense), we obtain certain information studying the order of "badness".

It is clear that problems of behavior of the Fourier transform are hardly of the same interest and value as those we are going to consider. Observe that in principle two different types of results might be needed. First, these are the conditions of integrability of the Fourier transform. Though in a recent paper [SK] no one result either due to Trigub or to the author is mentioned (and we dare to think these are of certain interest and importance), it may be considered as more or less comprehensive survey of the cases and conditions of integrability of multidimensional Fourier transform. But we mostly are interested in more subtle results on asymptotic behavior of the Fourier transform. Actually, there are not so many papers devoted to such precise results, even in the one-dimensional case. Nevertheless, neither a (sub)survey on these results nor even a systematic collection of those is given here. Penetrating deeply into details might lead us far away from the initial problems of summability of the Fourier series. Instead, we are giving the results needed in proper places of the text (see Theorem 1.1, Theorem 4.3, Theorems 5.1, 5.2 and 5.3, Theorem 7.1, and Theorem 9.2 with corollaries).

0.4. Just the definition of a partial sum of the multiple Fourier series reveals many problems and points of interest that are of geometric nature. Very often in these notes we run into geometric ideas and techniques. Some geometric argument appears already in simple calculations in the proof of Theorem 1.1 and then throughout the whole subsection 1.2. Geometric effects are essential for generalizations of the Wiener's result in Section 3. In Section 4, geometry appears naturally when a wide class of sets is considered instead of the ball. Here is where geometric ideas related to curvature enter (see also Section 1).

As Ch. Fefferman writes in [F3]: "One of the most fascinating themes in Fourier analysis in the last two decades has been the connection between the Fourier transform and curvature. Stein has been the most important contributor to this set of ideas". For this, see, e.g., the paper [S2] as a representative example in a series of Stein's works devoted to the role of the Gaussian curvature in Fourier Analysis; mention also his recent book [S3].

Peculiar geometric problems appear in Sections 6 and 8.

0.5. Since the behavior of Lebesgue constants strongly depends on the number and order of location of lattice points in sets investigated, close relations of the problem considered to delicate results in Number Theory do not seem surprising. A classical example of such possible relation is Siegel's proof of Minkowski's theorem via the Fourier transform and Poisson summation formula (see, e.g., [SW, Ch.VII]). But in concrete situations involving Number Theory results may nevertheless seem impressive. Given in formulations or hidden in proofs, such connections are quite frequent in our considerations. First, mention that K. I. Babenko has systematically exploited the Riemann Zeta Function as well as sharp estimates of the number of lattice points in balls and on spheres. Number Theory is a permanent tool in Sections 6 and 8. It is very probable that further development in the problems considered in these sections are waiting for advances in related problems of Number Theory.

0.6. There still exist some challenging problems concerning the Lebesgue constants. The main open problem is the asymptotic behavior of the Lebesgue constants of the Bochner-Riesz means of order below critical (see discussion after Theorem 2.2); the case of spherical partial sums is included as well (see Section 1). It was posed almost 30 years ago and till now defies all the assaults. Of course, the same problem is still open for various generalizations (see, e.g., Section 7). A problem of finding minimal conditions on the set defining partial sums such that the corresponding Lebesgue constants are of logarithmic order is posed in Section 6 (see also Theorem 6.2 and below). Section 1 is concluded with an open problem as is Section 8. Theorem 6.3$'$ is important step, highlighting how much is unclear rather than solving the appropriate problem. Some problems appear in related areas when tools from these areas are applied to solving the problems for Lebesgue constants. In our opinion, there is still enough room for interested researches.

0.7. We do not touch the related problems in more general setting, say, for spherical harmonics or even for compact Lie groups. Mention only that the problem of behavior of Lebesgue constants for Lie groups started in 70s by Dreseler and then continued by G. Travaglini and his colleagues (see, e.g., [GT]) is of considerable interest and is very specific. Further, it turned out that the above mentioned main open problem for trigonometric case, for Cesàro means of spherical harmonic expansions was affirmatively solved in a recent paper [L10].

We also do not touch applications of the results considered to approximation, though some of them are known. Moreover, in many cases estimates of the Lebesgue constants immediately imply corresponding estimates of the rate of approximation on related classes of functions.

Among other applications let us just mention that of estimating one-dimensional trigonometric sums by means of multi-dimensional polyhedral Lebesgue constants (see [KS]) and relations between positive definiteness and Lebesgue constants (see, e.g., [Gn] and [Zas]).

1 Spherical partial sums and some generalizations

Main problems arise in multidimensional Fourier Analysis because of various possibilities to define partial sums of the Fourier series. Different geometry implies very important differences in the behavior of such partial sums and, as a consequence, very different convergence and approximate properties of the function considered. It is known that in many respects spherical partial sums are characteristic in a wide range of problems.

1.1. Let us begin with a very exemplary case. Consider the spherical partial sums S_N of the Fourier series of a function f

$$S_N(f;x) = \sum_{|k|\le N} \widehat{f}(k)e^{ikx},$$

where $|k|^2 = k_1^2 + \cdots + k_n^2$. It is well-known (see, e.g., [SW, Ch.VII, Thm.3.4]) that the norm of the operator

$$S_N : \qquad f(x) \mapsto S_N(f;x)$$

taking $C(\mathbf{T}^n)$ into $C(\mathbf{T}^n)$ (or $L^1(\mathbf{T}^n)$ into $L^1(\mathbf{T}^n)$ that is the same) equal the L^1-norm of the corresponding Dirichlet kernel

$$\|S_N\| = (2\pi)^{-n} \int_{\mathbf{T}^n} \left| \sum_{|k|\le N} e^{ikx} \right| dx. \tag{1.1}$$

The following ordinal estimate holds.

Theorem 1.1. *There exist positive constants C_1, and C_2, where $C_1 < C_2$, depending only on n, such that*

$$C_1 N^{(n-1)/2} \le \|S_N\| \le C_2 N^{(n-1)/2}. \tag{1.2}$$

Proof. We give an outline of the proof which illustrates rather general method. It turned out that first the estimate from below was obtained by V.A. Ilyin [I] (even for more general expansions corresponding to the Laplace operator), and after that two-sided estimates were obtained in [Ba2] and [IA]. All those proofs were rather complicated. We will follow a very simple proof of the upper estimate proposed by V. Yudin [Y1] in more general setting; the earlier proof of the upper estimate in (1.2) due to H. Shapiro [Sh] essentially is almost the same. Then, using a trick proposed in [I], we will adjust this proof to the estimate from below as well.

If I_k is the cube with the edge of length 1 and the center at the point k , and $A_N = \bigcup_{k:|k|\le N} I_k$, then

$$\int_{A_N} e^{iux} du = \sum_{k:|k|\le N} \int_{I_k} e^{iux} du$$

$$= \sum_{k:|k|\le N} e^{ikx} \prod_{j=1}^{n} \frac{2\sin(x_j/2)}{x_j} = \prod_{j=1}^{n} \frac{2\sin(x_j/2)}{x_j} \sum_{k:|k|\le N} e^{ikx}.$$

Thus, we obtain

$$\begin{aligned}\|S_N\| &\le (2\pi)^{-n}(\pi/2)^n \int\limits_{\mathbf{T}^n}\Bigg| \int\limits_{A_N} e^{iux}\,du\Bigg|\,dx \\ &\le 4^{-n}\int\limits_{\mathbf{T}^n}\Bigg| \int\limits_{|u|\le N} e^{iux}\,du\Bigg|\,dx + 4^{-n}\int\limits_{\mathbf{T}^n}\Bigg| \int\limits_{W_N} \pm e^{iux}\,du\Bigg|\,dx,\end{aligned} \tag{1.3}$$

where W_N is symmetric difference of the set A_N and the ball $B_N = \{u : |u| \le N\}$. Taking into account that $\operatorname{mes} W_N \le CN^{n-1}$ and applying the Cauchy-Schwarz inequality to the last summand on the right-hand side of (1.3), we obtain, by virtue of Parseval's identity, that

$$\begin{aligned}\|S_N\| &\le 4^{-n}\int\limits_{\mathbf{T}^n}\Bigg| \int\limits_{|u|\le N} e^{iux}du\Bigg|\,dx + O(N^{(n-1)/2}) \\ &= 4^{-n}\int\limits_{N\mathbf{T}^n}\Bigg| \int\limits_{\mathbf{R}^n} \chi_1(u)e^{iux}du\Bigg|\,dx + O(N^{(n-1)/2}) \\ &= 4^{-n}\int\limits_{N\mathbf{T}^n} |\hat{\chi}_1(x)|\,dx + O(N^{(n-1)/2}),\end{aligned} \tag{1.4}$$

where χ_1 is the indicator function of the unit ball $B_1 = \{u : |u| \le 1\}$. Now standard computations of $\hat{\chi}_1$ via the Bessel functions (see, e.g., [SW, **?**]) and consequent integration complete the upper estimate in (1.2).

In order to obtain the lower estimate, let us introduce a small parameter ε, $0 < \varepsilon < 1$, as it was done in [I]. Instead of the equality (1.1), we obtain here an obvious inequality from below

$$\|S_N\| \ge (2\pi)^{-n}\int\limits_{\varepsilon\mathbf{T}^n}\Bigg|\sum_{|k|\le N} e^{ikx}\Bigg|\,dx.$$

Now we can proceed as in the obtaining (1.3) and (1.4), but with estimates from below instead of those from above and signs "$-$" instead of "$+$" on appropriate places. This gives

$$\|S_N\| \ge (2\pi)^{-n}\int\limits_{\varepsilon N\mathbf{T}^n} |\hat{\chi}_1(x)|\,dx - C_3\varepsilon^{n/2}N^{(n-1)/2}. \tag{1.5}$$

The afore-mentioned computations involving Bessel functions yield here the following estimate from below

$$\|S_N\| \ge \left(C_4\varepsilon^{(n-1)/2} - C_3\varepsilon^{n/2}\right)N^{(n-1)/2}.$$

It remains only to choose ε such that $C_1 = C_4\varepsilon^{(n-1)/2} - C_3\varepsilon^{n/2} > 0$. The proof is complete. □

In the less known paper by Podkorytov [P0] similar technique was elaborated independently. This allowed to obtain the following interesting result. To indicate that partial sums correspond to certain set B, we will denote them by

$$S_B(f;x) = \sum_{k\in B}\hat{f}(k)e^{ikx}. \tag{1.6}$$

Theorem 1.2. *Assume that a set $B \subset \mathbb{R}^n$ satisfies the following conditions:*

1) *For some $N_1 \geq N_2 \geq ... \geq N_n \geq 0$ we have*

$$B \subset [-N_1, N_1; -N_2, N_2; ...; -N_n, N_n].$$

2) *For all $j = 1, ..., n$ and all $x_1, ..., x_{j-1}, x_{j+1}, ..., x_n$ the set*

$$B_j(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$$
$$=\{x_j : (x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_n) \in B\}$$

is either empty or is an interval.

Then

$$||S_B|| = O\left(\sqrt{N_2 \cdots N_n}\left(1 + \ln(N_1/N_2)\right)\right).$$

1.2. The proof of Theorem 1.1 is very typical for obtaining estimates of the Lebesgue constants of either partial sums or linear means of Fourier seriesẆe have already mentioned that such was V. Yudin's estimate from above [Y1] for very general sets B, generating the corresponding partial sums, namely, those which are balanced (with each point x the whole set δx, $|\delta| \leq 1$, belongs to B), and having the finite upper Minkowski measure:

$$\limsup_{\varepsilon \to 0} (1/\varepsilon)\, \mathrm{mes}\{x : \rho(x, \partial B) < \varepsilon\} < \infty,$$

where $\rho(x, y)$ is the distance between two points $x, y \in \mathbf{R}^n$, and

$$\rho(x, \partial B) = \inf_{y \in \partial B} \rho(x, y).$$

A similar method was applied to obtain the estimate from below in [L2], where conditions are less restrictive than in the earlier paper [CaS] and the later papers [Br1,2]; above all, they are local.

Theorem 1.3. ([L2,3]) *Let the boundary of a domain B contain a simple (non-intersecting) piece of a surface of smoothness $[(n + 2)/2]$ in which there is at least one point with non-vanishing principal curvatures. Then there exists a positive constant C depending only on B such that*

$$\int_{\mathbf{T}^n} \left| \sum_{k \in NB \cap \mathbf{Z}^n} e^{ikx} \right| dx \geq C N^{(n-1)/2}$$

for large N.

More details will be given below when considering a generalization of this theorem (see Theorem 4.2). And now let us give one related two-dimensional result.

Theorem 1.4. ([Gu]) *Assume that a convex set B is included into $\mathbf{T}^2$. Then for sufficiently large N the inequality*

$$||S_{NB}|| \geq C N^{1/2} \left(\int_{\mathbf{T}} \sqrt{\rho(\varphi)}\, d\varphi \right)^2$$

holds, where $\rho(\varphi)$ *is the curvature radius of* ∂B *at the point where*

$$\max\{x_1 \cos\varphi + x_2 \sin\varphi : x \in B\}$$

is attained.

If

$$\liminf ||S_{NB}||/N^{1/2} = 0,$$

then the boundary of the set B *is degenerate, i.e. for almost all directions* φ *its curvature radius is equal to zero.*

In this theorem as well as in [P7] no assumptions on smoothness of ∂B are involved. Of course, convexity itself yields some minimal smoothness. Since only two-dimensional statements are obtained in [P7] and [Gu], this gives rise to the question: what are the minimal smoothness assumptions for such estimates in case of arbitrary dimension.

2 General estimates

Certainly, in the previous section the Fourier transform is applied to estimates of Lebesgue constants in a comparatively simple situation, while in more general and more complicate situations some steps may be really tedious and entailed with considerable technical difficulties. For example, even for the Fejér type means generated by convex sets, the proof of the boundedness of the norms of corresponding operators in [P1] is rather complicate even in the two-dimensional case, and non-trivial estimates of Fourier transforms are continuously used.

2.1. Let us give now one Belinsky result in which the Fourier transform method is realized in a very general setting. Belinsky was apparently the first to begin a *systematic* study of connections between summability of multi-dimensional Fourier series and integrability of the Fourier transform of a function, generating the method of summability.

Theorem 2.1. ([Be2]) *Let λ be a bounded measurable function with compact support. Then for the norms of a sequence of linear operators* (0.2) *we have the estimate from above*

$$\begin{aligned}\|L_N^\lambda\|_{L^1(\mathbf{T}^n)\to L^1(\mathbf{T}^n)} &\le (2\pi)^{-n}\int\limits_{N\mathbf{T}^n}\prod_{j=1}^n\frac{x_j}{2N\sin(x_j/(2N))}|\widehat{\lambda}(x)|\,dx\\ &+\sum_{j=1}^{m-1}(\pi/2)^{(j+1)n}\int\limits_{N\mathbf{T}^n}|\widehat{\lambda}(x)|\,|x/N|^j\,dx\\ &+\omega_1\int\limits_{(2\pi)^{-1}\mathbf{T}^n}\cdots\int\limits_{(2\pi)^{-1}\mathbf{T}^n}\left(\sum_k\left|\Delta_{k/N}^m(\lambda;u_1/N,...,u_m/N)\right|^2\right)^{1/2}du_1...du_m,\end{aligned}\tag{2.1}$$

and the estimate from below

$$\begin{aligned}C_p\|L_N^\lambda\|_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)} &\ge \left\{(2\pi)^{-n}\int\limits_{\varepsilon N\mathbf{T}^n}\left|\prod_{j=1}^n\frac{x_j}{2N\sin(x_j/(2N))}\widehat{\lambda}(x)\right|^p dx\right\}^{1/p}\\ &-\sum_{j=1}^{m-1}(\pi/2)^{(j+1)n}\left\{\int\limits_{\varepsilon N\mathbf{T}^n}|\widehat{\lambda}(x)|^p\,|x/N|^{jp}\,dx\right\}^{1/p}\\ &-\omega_p\varepsilon^{n/p-n/2}N^{n/p-n}\\ &\times\int\limits_{(2\pi)^{-1}\mathbf{T}^n}\cdots\int\limits_{(2\pi)^{-1}\mathbf{T}^n}\left(\sum_k\left|\Delta_{k/N}^m(\lambda;u_1/N,...,u_m/N)\right|^2\right)^{1/2}du_1...du_m,\end{aligned}\tag{2.2}$$

where $\omega_p=\pi^{nm+n/p-n/2}2^{-nm+n/p-n/2}$.

Here ε, $0<\varepsilon\le 1$, is an arbitrary real number, m is integer, and $1\le p\le 2$. The mth difference $\Delta_z^m(\lambda;h_1,...,h_m)$ is defined recursively by the formulas

$$\Delta_z^1(\lambda;h_1)=\lambda(z+h_1)-\lambda(z)$$

and
$$\Delta_z^m(\lambda; h_1, ..., h_m) = \Delta_{z+h_m}^{m-1}(\lambda; h_1, ..., h_{m-1}) - \Delta_z^{m-1}(\lambda; h_1, ..., h_{m-1}),$$
with $h_j, z \in \mathbf{R}^n$. When $p > 2$, in view of duality (see, e.g., [SW, Ch.I,Th.3.20]) the estimate (2.2) is still valid with $p' = p/(p-1)$ instead of p.

Proof. We follow the argument from [Be2]. We have by definition

$$\|L_N^\lambda\|_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)}$$
$$= \sup_{\|f\|_{L^p(\mathbf{T}^n)}\le 1} \left\{ (2\pi)^{-2n} \int_{\mathbf{T}^n} \left| \int_{\mathbf{T}^n} f(x-u) \sum_k \lambda(k/N) e^{iku}\, du \right|^p dx \right\}^{1/p}.$$

For $p = 1$, we have
$$||L_N^\lambda|| = (2\pi)^{-n} \int_{\mathbf{T}^n} \left| \sum_k \lambda(k/N) e^{ikx} \right| dx. \tag{0.4}$$

For $p > 1$, assume, without loss of generality, that $\lambda = 0$ when $|x_j| > 1$, $j = 1, 2, ..., n$, and set
$$f(x_1, ..., x_n) = C_p^{-1} N^{n/p-n} \prod_{j=1}^n D_N(x_j),$$

where $D_N(x_j)$ is the Dirichlet kernel. A constant C_p is chosen to guarantee the function f to be in the unit ball, that is, $||f||_{L^p(\mathbf{T}^n)} \le 1$. We obtain

$$C_p\|L_N^\lambda\|_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)} \ge N^{n/p-n} \left\{ (2\pi)^{-n} \int_{\mathbf{T}^n} \left| \sum_k \lambda(k/N) e^{ikx} \right|^p dx \right\}^{1/p}.$$

Using the obvious equality

$$\int_{k_j-1/2}^{k_j+1/2} e^{ix_jv_j}\, dv_j = e^{ik_jx_j} 2\sin(x_j/2)/x_j, \qquad j = 1, 2, ..., n,$$

replace the sum by the Fourier transform of λ as in [Y1] (see also [Zg, Ch.V, Th.2.29]; cf. the proof of Theorem 1.1):

$$C_p\|L_N^\lambda\|_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)}$$
$$= N^{n/p-n} \left\{ (2\pi)^{-n} \int_{\mathbf{T}^n} \left| \prod_{j=1}^n \frac{x_j}{2\sin(x_j/2)} \right.\right.$$
$$\left.\left. \times \sum_k \int_{k+(2\pi)^{-1}\mathbf{T}^n} \lambda(k/N) e^{ixv}\, dv \right|^p dx \right\}^{1/p}.$$

Obviously, for $0 < \varepsilon < 1$ we have

$$C_p \|L_N^\lambda\|_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)} \geq N^{n/p-n}\left\{(2\pi)^{-n}\int\limits_{\varepsilon\mathbf{T}^n}\left|\prod_{j=1}^n \frac{x_j}{2\sin(x_j/2)} \times \sum_k \int\limits_{k+(2\pi)^{-1}\mathbf{T}^n} \lambda(k/N)e^{ixv}\,dv\right|^p dx\right\}^{1/p}.$$

The following simple inequality $2/\pi t \leq \sin t$ true for $0 \leq t \leq \pi/2$, and Minkowski's inequality yield the estimate from below

$$\begin{aligned} &C_p N^{n-n/p}(2\pi)^n \|L_N^\lambda\|_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)} \\ &\geq \left\{\int\limits_{\varepsilon\mathbf{T}^n}\left|\prod_{j=1}^n \frac{x_j}{2\sin(x_j/2)}\sum_k \int\limits_{k+(2\pi)^{-1}\mathbf{T}^n} \lambda(v/N)e^{ixv}\,dv\right|^p dx\right\}^{1/p} \\ &- \left\{(\pi/2)^{np}\int\limits_{\varepsilon\mathbf{T}^n}\left|\sum_k \int\limits_{k+(2\pi)^{-1}\mathbf{T}^n}\Big[\lambda(k/N)-\lambda(v/N)\Big]e^{ixv}\,dv\right|^p dx\right\}^{1/p}. \end{aligned}$$

Summing the integrals over $\{k + (2\pi)^{-1}\mathbf{T}^n\}$ and changing the variables, we reduce the first term on the right-hand side to the form claimed. Let us estimate the second term. The change of variables $v_j \to v_j + k_j$, for $j = 1, ..., n$, and the generalized Minkowski's inequality yield

$$\begin{aligned} &\left\{\int\limits_{\varepsilon\mathbf{T}^n}\left|\sum_k \int\limits_{k+(2\pi)^{-1}\mathbf{T}^n}\Big[\lambda(k/N)-\lambda(v/N)\Big]e^{ixv}\,dv\right|^p dx\right\}^{1/p} \\ &\leq \int\limits_{(2\pi)^{-1}\mathbf{T}^n}\left\{\int\limits_{\varepsilon\mathbf{T}^n}\left|\sum_k \Big[\lambda(k/N)-\lambda(k+v/N)\Big]e^{ikx}\right|^p dx\right\}^{1/p} dv \\ &= \int\limits_{(2\pi)^{-1}\mathbf{T}^n}\left\{\int\limits_{\varepsilon\mathbf{T}^n}\left|\sum_k \Delta^1_{k/N}(\lambda;u_1/N)e^{ikx}\right|^p dx\right\}^{1/p} du_1. \end{aligned}$$

Note that the inner integral on the right-hand side is of the same form as that in the beginning of the proof, so the same argument is applicable to it. In what follows we need only estimates from above. Taking into account that

$$\int\limits_{\mathbf{R}^n} \Big[\lambda(u_2) - \lambda(u_2 + u_1/N)\Big] e^{-iu_2 x}\, du_2$$
$$= \int\limits_{\mathbf{R}^n} \lambda(u_2)[e^{-iu_2 x} - e^{-i(u_2 - u_1/N)x}]\, du_2$$
$$= (1 - e^{iu_1 x/N}) \int\limits_{\mathbf{R}^n} \lambda(u_2) e^{-iu_2 x}\, du_2 = (1 - e^{iu_1 x/N})\, \widehat{\lambda}(x),$$

we obtain

$$\int\limits_{(2\pi)^{-1}\mathbf{T}^n} \left\{ \int\limits_{\varepsilon \mathbf{T}^n} \Big| \sum_k \Delta^1_{k/N}(\lambda; u_1/N) e^{ikx} \Big|^p dx \right\}^{1/p} du_1$$
$$\le (\pi/2)^n \int\limits_{(2\pi)^{-1}\mathbf{T}^n} \left\{ \int\limits_{\varepsilon N \mathbf{T}^n} \Big| [1 - e^{iu_1 x/N}]\, \widehat{\lambda}(x) \Big|^p dx \right\}^{1/p} du_1$$
$$+ (\pi/2)^n \int\limits_{(2\pi)^{-1}\mathbf{T}^n} \int\limits_{(2\pi)^{-1}\mathbf{T}^n}$$
$$\left\{ \int\limits_{\varepsilon \mathbf{T}^n} \Big| \sum_k \Delta^2_{k/N}(\lambda; u_1/N, u_2/N) e^{ikx} \Big|^p dx \right\}^{1/p} du_1\, du_2.$$

Using the simple inequality

$$|1 - e^{iu_1 x/N}| \le |u_1||x|/N,$$

we obtain

$$\int\limits_{(2\pi)^{-1}\mathbf{T}^n} \left\{ \int\limits_{\varepsilon N \mathbf{T}^n} \Big| (1 - e^{iu_1 x/N}) \widehat{\lambda}(x) \Big|^p dx \right\}^{1/p} du_1$$
$$\le \left\{ \int\limits_{\varepsilon N \mathbf{T}^n} |x/N|^p |\widehat{\lambda}(x)|^p\, dx \right\}^{1/p}.$$

Repeating the same computations $m-2$ times yields (2.2) with the remainder term

$$\int\limits_{(2\pi)^{-1}\mathbf{T}^n} \cdots \int\limits_{(2\pi)^{-1}\mathbf{T}^n} \left\{ \int\limits_{\varepsilon \mathbf{T}^n} \Big| \sum_k \Delta^m_{k/N}(\lambda; u_1/N, ..., u_m/N) e^{ikx} \Big|^p dx \right\}^{1/p} du_1 ... du_m$$

times $N^{n/p-n}(2\pi)^{-n/p}(\pi/2)^{nm}$. In order to complete the proof, apply Hölder's inequality, with the power $2/p$, to the inner integral, and then Parseval's identity. The converse inequality $(p = 1)$ can be obtained similarly. □

2.2. We mention that Theorem 1.1 follows from Theorem 2.1 as a technical corollary. But to demonstrate the strength of this theorem the following more general result seems to be more impressive.

Let

$$\lambda(x) = R_\alpha = (1 - |x|^2)_+^\alpha.$$

Corresponding linear means generated by this function are called the *Bochner-Riesz means*. An unbreakable interest in these means goes back to Bochner's celebrated paper [Bc].

Theorem 2.2. ([I, Ba2, IA]) *There exist positive constants C_1 and C_2, $C_1 < C_2$, depending only on n and α, such that for* $0 \le \alpha < (n-1)/2$

$$C_1 N^{(n-1)/2-\alpha} \le \left\| L_N^{R_\alpha} \right\|_{L^1(\mathbf{T}^n) \to L^1(\mathbf{T}^n)} \le C_2 N^{(n-1)/2-\alpha}. \tag{2.3}$$

This is the best known result for the Lebesgue constants of the Bochner-Riesz means of order less than critical (and consequently for the partial sums). Some rough estimates can be found in the old papers by J. Mitchell [M1-3].

The problem of the asymptotic behavior of $||L_N^{R_\alpha}||$, that is, the existence of the limit of $||L_N^{R_\alpha}|| N^{\alpha-(n-1)/2}$ as $N \to \infty$ (see [Ba2, Sh]), is still open and may be considered, in a sense, as the **main open problem** in this subject.

The following estimate obtained earlier by Babenko (see [Ba2, Ba3]) is also a comparatively simple consequence of Theorem 2.1.

Corollary 2.1. *The following estimate holds:*

$$\left\| L_N^{R_\alpha} \right\|_{L^p(\mathbf{T}^n) \to L^p(\mathbf{T}^n)} \ge C \left(\frac{N^{p(\alpha_p - \alpha)} - 1}{p(\alpha_p - \alpha)} \right)^{1/p}, \tag{2.4}$$

where $1 \le p \le 2n/(n+1)$, and $0 \le \alpha < \alpha_p = n/p - (n+1)/2$. For $\alpha = \alpha_p$ this estimate is understood to be the limit as $\alpha \to \alpha_p$, which gives a logarithmic order of growth.

Proof of Theorem 2.2 and Corollary 2.1. The Fourier transform of R_α is very well known (see, e.g., [SW, Ch.4]):

$$\widehat{R}^\alpha(x) = 2^{-n/2+\alpha} \pi^{-n/2} \Gamma(\alpha+1) |x|^{-n/2-\alpha} J_{n/2+\alpha}(|x|), \tag{2.5}$$

where J_ν is the Bessel function of the first kind and order ν. Let us estimate the terms with the Fourier transform in (2.1). Extending in each one the domain of integration to the ball of radius $\sqrt{n}\pi N$ and then passing to spherical coordinates, we obtain

$$||L_N^{R_\alpha}|| \le C \int_0^{\sqrt{n}\pi N} \left| \frac{J_{n/2+\alpha}(r)}{r^{n/2+\alpha}} \right| r^{n-1}\, dr + R,$$

where R denotes the last summand in (2.1) or (2.2). We are going to estimate it separately.

Let us use the following asymptotic formulas for Bessel functions (see, e.g., [BE, 7.12(8),7.13.1(3)]):

$$J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(\nu+1)} + O(|t|^{\nu+2}), \tag{2.6}$$

as $t \to 0$, and

$$J_\nu(t) = \sqrt{2}(\pi t)^{-1/2} \cos(t - \nu\pi/4 - \pi/4) + O(t^{-3/2}), \tag{2.7}$$

as $t \to \infty$.

These yield

$$\int\limits_0^{\sqrt{n}\pi N} \left| \frac{J_{n/2+\alpha}(r)}{r^{n/2+\alpha}} \right| r^{n-1}\, dr \le C \frac{N^{(n-1)/2-\alpha} - 1}{(n-1)/2-\alpha} + O(N^{(n-3)/2}).$$

Let us estimate now the remainder R. Split the sum into two ones:

$$\sup_{u_1,\dots,u_m \in (2\pi)^{-1}\mathbf{T}^n} \left\{ \sum_k \left| \Delta^m_{k/N} \left(R_\alpha; u_1/N, \dots, u_m/N \right) \right|^2 \right\}^{1/2}$$
$$= \sup_{u_1,\dots,u_m \in (2\pi)^{-1}\mathbf{T}^n} \left\{ \sum_{|k|<N-m-2} + \sum_{N-m-2 \le |k| \le N} \right\}^{1/2}.$$

Estimate each summand in the second sum by the maximal value of the function. Since R_α is monotone increasing near the origin, we have

$$\sum_{N-m-2 \le |k| \le N} \le CN^{-2\alpha} \sum_{N-m-2 \le |k| \le N} 1 \le CN^{n-1-2\alpha}.$$

The latter value follows from the well-known estimates of the number of points of $\mathbf{Z}^n$ in the n-dimensional ball of radius N. The mean-value theorem for the directional derivative yields that the first sum is

$$N^{-2m} \sum_{|k|<N-m-2} \left| \frac{\partial^m R_\alpha}{\partial u_1 \dots \partial u_m} (k/N + \theta_1 u_1/N + \dots + \theta_m u_m/N) \right|^2$$

with $0 < \theta_j < 1$, $j = 1, 2, \dots, m$. Choose m such that $m \ge \alpha + 1$. If α is integer then the derivative is bounded and

$$\sum_{|k|<N-m-2} \le CN^{n-2-2\alpha}.$$

If α is fractional then estimating the derivative by its maximal value on the interval, we have

$$\sum_{|k|<N-m-2} \le CN^{-2m} \sum_{|k|<N-m-2} \left(1 - (|k| + m + 2)^2 N^{-2} \right)^{2(\alpha-m)}.$$

In view of monotonicity of the function it is possible to make use of integrals in place of sums. This yields

$$\sum_{|k|<N-m-2} \le CN^{n-1-2\alpha},$$

and finally

$$R \le CN^{(n-1)/2-\alpha}. \tag{2.8}$$

This gives the right-hand side of (2.3). Use now (2.6), (2.7), and (2.8). The formula (2.2) yields

$$\begin{aligned}||L_N^{R_\alpha}||_{L^p(\mathbf{T}^n)\to L^p(\mathbf{T}^n)} \ge C_1 &\left[\frac{N^{p(\alpha_p-\alpha)}-1}{p(\alpha_p-\alpha)}\right]^{1/p} \varepsilon^{\alpha_p-\alpha} \\ &- C_2 N^{\alpha_p-\alpha}\varepsilon^{\alpha_p+1/2}.\end{aligned}$$

Choose ε so that

$$C_1\varepsilon^{\alpha_p-\alpha} - C_2\varepsilon^{\alpha_p+1/2} \ge C > 0,$$

and this completes the proof of the corollary. By this we obtain the left-hand side of (2.3) as well. □

The original proofs are much more tedious. For instance, properties of Riemann's Zeta Function is the main tool in [Ba2].

For the sake of convenience, we will give Corollary 2.2 in one of the next sections, Section 4.

2.3. Let us give some other results which are due to Belinsky [Be1]. These results are based on the Poisson summation formula and a certain technique of estimating trigonometric sums and integrals via the Fourier transform.

Consider a function $\lambda(x)$ bounded on $\mathbf{R}^n$ and continuous at the points of $\mathbf{Z}^n$, and construct the formal trigonometric series for any $f \in C(\mathbf{T}^n)$

$$\sum_{k\in\mathbf{Z}^n} \lambda(k)\widehat{f}(k)e^{ikx}. \tag{2.9}$$

Denote

$$U_N(f;x) = (2\pi)^{-n}\int\limits_{\mathbf{T}^n} f(x+u)\sum_m \lambda(m/N)e^{-imu}\,du.$$

The following two propositions are very useful in many applications. They are the corollaries to Theorem 1 in [Be1]. We do not formulate the theorem itself because just these propositions have proved to concentrate its main benefits.

Proposition 2.1. *Suppose $U_N(f;x)$ is defined as above. If $\lambda \in C(\mathbf{R}^n)$ and $\widehat{\lambda} \in L^1(\mathbf{R}^n)$, then*

$$\|U_N\| = (2\pi)^{-n}\int\limits_{\mathbf{R}^n} |\widehat{\lambda}(u)|\,du + \theta(2\pi)^{-n}\int\limits_{\mathbf{R}^n\setminus(N\mathbf{T}^n)} |\widehat{\lambda}(u)|\,du,$$

where $-2 \le \theta \le 0$, and, in general, the constant θ depends on N.

Proposition 2.2. *If λ is compactly supported and continuous, then*

$$\sup_N \|U_N\| = (2\pi)^{-n}\int\limits_{\mathbf{R}^n} |\widehat{\lambda}(u)|\,du. \tag{2.10}$$

All the norms here are the uniform norms. Note, that a similar upper estimate may be found in [SW, Ch.VII, §2] provided both λ and $\widehat{\lambda}$ are integrable:

$$\int\limits_{\mathbf{T}^n}\left|\sum_k \lambda(k/N)e^{ikx}\right|dx \le \int\limits_{\mathbf{R}^n} |\widehat{\lambda}(x)|\,dx. \tag{2.11}$$

2.4. In a very precise form such relations are given in [T10]. It is to these results that we now turn.

Let $B = B(\mathbf{R}^n)$ be the algebra of functions representable in the form

$$\lambda(x) = \int\limits_{\mathbf{R}^n} e^{-ixu}\, d\mu(u)$$

with $||\lambda||_B = \inf \operatorname{var} \mu$, where μ is a finite Borel measure on $\mathbf{R}^n$.

Let $A = A(\mathbf{R}^n)$ be the algebra of functions which expand in an absolutely convergent Fourier integral

$$\lambda(x) = \hat{g}(x) = \int\limits_{\mathbf{R}^n} g(u) e^{-ixu}\, du$$

with

$$||\lambda||_A = ||g||_{L^1} = \int\limits_{\mathbf{R}^n} |g(u)|\, du < \infty.$$

Denote by $||L_N^\lambda||_M$ the norm of the operator L_N^λ taking $L^\infty(\mathbf{T}^n)$ into $L^\infty(\mathbf{T}^n)$ instead of earlier considered cases $L^1(\mathbf{T}^n) \to L^1(\mathbf{T}^n)$ or $C(\mathbf{T}^n) \to C(\mathbf{T}^n)$, where λ is a locally Riemann-integrable function.

Theorem 2.3. *The following relations hold.*

1) *If $\lambda \in B(\mathbf{R}^n)$, then $\lambda \in M(\mathbf{R}^n)$ and $||\lambda||_M \le ||\lambda||_B$.*

2) *If $\lambda \in M(\mathbf{R}^n)$, the function λ can be adjusted on its set of discontinuities so that it belongs to $B(\mathbf{R}^n)$ and $||\lambda||_B \le ||\lambda||_M$.*

Corollary 2.2. *A continuous function on $\mathbf{R}^n$ belongs to M and B simultaneously, and $||\lambda||_M = ||\lambda||_B$.*

This assertion was known earlier - see, e.g., [SW], namely, Ch.I, Theorem 3.19 and Ch.VII, §3, especially Theorem 3.4.

Theorem 2.4. *If $\lambda \in B$ and outside some neighborhood of zero λ has bounded Vitali variation (see the next section), while $\lim\limits_{|x|\to\infty} \lambda(x) = 0$, then $\lambda \in A$ and*

$$||\lambda||_B = (2\pi)^{-n} \int\limits_{\mathbf{R}^n} |\hat{\lambda}(x)|\, dx,$$

where for $\prod\limits_{j=1}^{n} x_j \ne 0$ we have

$$\hat{\lambda}(x) = \lim_{\substack{\min N_j \to \infty,\\ j=1,2,\dots,n}} \int\limits_{-N_1}^{N_1} \dots \int\limits_{-N_n}^{N_n} \lambda(u) e^{-ixu}\, du.$$

3 Fourier series and Fourier integrals

The above estimates were given in terms of absolute integrability of the Fourier transform of the function λ generating a method of summability. Suppose that this function is compactly supported. For its periodical continuation preserving the same notation will not result in any confusion. Knowing already that the Lebesgue constants are estimated via the Fourier transform $\widehat{\lambda}$, let us ask ourselves the following question:

Is it possible to estimate the Lebesgue constants in terms of absolute convergence of the Fourier series of the generating function?

The following result of N. Wiener gives a hint at this (see [W, §12]):

If the support of λ is of diameter $2\pi - \varepsilon$, $\varepsilon > 0$, then $\widehat{\lambda}$ is integrable over $\mathbf{R}$ if and only if λ has an absolutely convergent Fourier series.

Absolute convergence of Fourier series is well studied; see the monograph by Kahane [Kh] and corresponding chapters in general monographs [Br] or [Zg].

3.1 In the general situation, that is, where nothing is known except $\operatorname{supp}\lambda \subset \mathbf{T}$, Wiener's result is no longer the case. Denoting by $\lambda_1(x) = x\lambda(x)$, we have the following result which is due to Trigub (see [T2,3]; a very simple proof can be found in [BLT]):

We have $\widehat{\lambda} \in L^1(\mathbf{R})$ if and only if λ and λ_1 have absolutely convergent Fourier series; the two conditions are independent.

It is to be noted that for $1 < p < \infty$, Plancherel and Polya [PP] showed that when $\operatorname{supp}\lambda \subset \mathbf{T}$ we have $\widehat{\lambda} \in L^p(\mathbf{R})$ if and only if there holds

$$\sum_m |\widehat{\lambda}(m)|^p < \infty.$$

Let us first give a generalization of Trigub's result (see [L]). Set $\lambda_\nu(x) = x^\nu\lambda(x)$ where $\nu = (\nu_1, ..., \nu_n)$ and $x^\nu = x_1^{\nu_1}...x_n^{\nu_n}$. If $\nu = (0, ..., 0, 1, 0, ..., 0)$ with $\nu_j = 1$, let us denote $\lambda_\nu := \lambda_j$.

Proposition 3.1. *Let $\operatorname{supp}\lambda \subset \mathbf{T}^n$. Then $\widehat{\lambda} \in L^1(\mathbf{R}^n)$ if and only if for every $\nu = (\nu_1, ..., \nu_n)$ with $\nu_j = 0$ or 1, the function λ_ν after 2π-periodic continuation has absolutely convergent Fourier series, written $\lambda_\nu \in A(\mathbf{T}^n)$.*

Proof. The necessity of these conditions is a very simple generalization of the corresponding one-dimensional result and can be found in [Be1, Lemma]. Let us prove the converse part of the proposition. We use an inductive argument similar to that used in [PP] for L^p with $p > 1$.

We already have the one-dimensional result (see above). Assuming its validity for the dimension $n-1$, let us prove it for the n-dimensional case. Denote $k' = (k_1, ..., k_{n-1})$ and $x' = (x_1, ..., x_{n-1})$; obviously $k = (k', k_n)$ and $x = (x', x_n)$. Since for any $u_n \in \mathbf{R}$

$$\widehat{\lambda}(k', u_n) = \int_{\mathbf{R}} \left[\int_{\mathbf{R}^{n-1}} \lambda(x) e^{-ik'x'}\, dx' \right] e^{-iu_n x_n}\, dx_n$$

is the one-dimensional Fourier transform in the x_n-variable of the function

$$\int_{\mathbf{R}^{n-1}} \lambda(x) e^{-ik'x'}\, dx'$$

with compact support, Trigub's result yields for any $k' \in \mathbf{Z}^{n-1}$

$$\int\limits_{\mathbf{R}} |\widehat{\lambda}(k', u_n)|\, du_n \le C \sum_{k_n} \Big\{ |\widehat{\lambda}(k', k_n)| + |\widehat{\lambda}_{(0,\dots,0,1)}(k', k_n)| \Big\}.$$

Summing in k', we obtain

$$\sum_{k'} \int\limits_{\mathbf{R}} |\widehat{\lambda}(k', u_n)|\, du_n \le C \sum_{k} \Big\{ |\widehat{\lambda}(k)| + |\widehat{\lambda}_{(0,\dots,0,1)}(k)| \Big\}.$$

In view of assumptions of the proposition, the right-hand side is bounded, and consequently the series on the left-hand side is convergent. By the B. Levi theorem the series

$$\sum_{k'} |\widehat{\lambda}(k', u_n)|$$

converges almost everywhere, and hence

$$\int\limits_{\mathbf{R}} \sum_{k'} |\widehat{\lambda}(k', u_n)|\, du_n = \sum_{k'} \int\limits_{\mathbf{R}} |\widehat{\lambda}(k', u_n)|\, du_n.$$

Further, using the inductive assumption, we obtain

$$\begin{aligned} \int\limits_{\mathbf{R}^n} |\widehat{\lambda}(u)|\, du &= \int\limits_{\mathbf{R}} \Bigg[\int\limits_{\mathbf{R}^{n-1}} |\widehat{\lambda}(u)|\, du' \Bigg] du_n \\ &\le C \int\limits_{\mathbf{R}} \sum_{k'} \sum_{\nu:\nu_n=0} |\widehat{\lambda}_\nu(k', u_n)|\, du_n \\ &= C \sum_{\nu} \sum_{k'} \int\limits_{\mathbf{R}}^{n} |\widehat{\lambda}_\nu(k', u_n)|\, du_n \le C \sum_{\nu} \sum_{k} |\widehat{\lambda}_\nu(k)|, \end{aligned}$$

and the proposition is proved. □

To confirm that this result is substantial, we also have to prove the independence of these conditions. It suffices to give an example of a function λ such that $\lambda_\nu \in A(\mathbf{T}^n)$ for all $\nu \ne (1, ..., 1)$ from the proposition and $\lambda_{(1,\dots,1)} \notin A(\mathbf{T}^n)$. Indeed, since for every $\lambda \in A(\mathbf{T}^n)$ we have

$$\sum_{m} |\widehat{\lambda}_{(0,\dots,0,2,0,\dots,0)}(m)| \le C \sum_{m} |\widehat{\lambda}(m)|,$$

to verify the other combinations one has to choose λ_ν on place of λ successively for each ν.

The following function delivers the desired counterexample. Set

$$\lambda(x) = \ln^{-1}\Bigg(e\pi^n \prod_{j=1}^{n} (\pi - x_j)^{-1} \Bigg)$$

for $x \in \mathbf{T}^n_+$ and let λ be continued to all $\mathbf{T}^n$ to be even in each variable. A standard integration by part argument shows that $\lambda_\nu \in A(\mathbf{T}^n)$ for $\nu \ne (1, ..., 1)$. Since $\lambda_{(1,\dots,1)}$ is odd in each variable only for this function we have an additional term

$$\prod_{j=1}^{n}(1-m_j^{-1})m_j^{-1}\ln^{-1}(em_1...m_n)$$

when calculating the Fourier coefficients. This yields $\lambda_{(1,...,1)} \notin A(\mathbf{T}^n)$. The property of being even at least in one variable removes such a term.

3.2. We are in a position now to give a non-trivial extension of Wiener's result to several dimensions (see [L0]). For this we need some additional notation and definitions.

A closed subset S of $\mathbf{T}^1$ is called a W-set, or a W_1-set, if S is of diameter less than 2π.

Let us consider the sets

$$\mathbf{T}_k^{n-1}(a) = \mathbf{T}^n \cap \{x_k = a\}$$

with $-\pi \le a \le \pi$, $k = 1,2,...,n$, and identify $\mathbf{T}_k^{n-1}(-\pi)$ and $\mathbf{T}_k^{n-1}(\pi)$. A closed subset S of $\mathbf{T}^n$, $n = 2,3,...$, is called a W_j-set, $j = 1,2,...,n$, if for every $k = 1,2,...,n$, $k \ne j$, the sets $\mathbf{T}_k^{n-1}(a)$, $-\pi \le a \le \pi$, are W_j-sets. A set is called a $W_{\alpha_1...\alpha_m}$-set, $m \le n$ and $\{\alpha_1,...,\alpha_m\} \subseteq \{1,2,...,n\}$, if it is a W_{α_k}-set for every $k = 1,2,...,m$. We will denote a $W_{1...n}$-set as a W-set.

We need the following auxiliary result.

Lemma 3.1. *If for any $j = 1,2,...,n$ the set $S = \operatorname{supp}\lambda$ is a W_j-set, then $\lambda \in A(\mathbf{T}^n)$ implies $\lambda_j \in A(\mathbf{T}^n)$.*

Proof. Under the assumptions of the lemma we have:

(i) for any point

$$A_0^j = (x_1,...,x_{j-1},x_{x+1},...,x_n) \in \mathbf{T}^{n-1}$$

the points

$$A_\pi^j = (x_1,...,x_{j-1},\pi,x_{x+1},...,x_n)$$

and

$$A_{-\pi}^j = (x_1,...,x_{j-1},-\pi,x_{x+1},...,x_n)$$

cannot belong to S simultaneously;

(ii) for any four points

$$(x_1,...,x_{j-1},\pm\pi,x_{j+1},...,x_{k-1},\pm\pi,x_{k+1},...,x_n),$$

$k = 1,2,...,n$, $k \ne j$, only those can belong to the support which lie on the same side of the hyperplane $x_j = 0$.

Observe, that $\mathbf{T}^n \setminus S$ is open with respect to $\mathbf{T}^n$. Hence if $A_{-\pi}^j \in S$, then A_π^j is an interior point for $\mathbf{T}^n \setminus S$, and there exists a neighborhood $U(A_\pi^j) \subset \mathbf{T}^n \setminus S$ of A_π^j (with respect to $\mathbf{T}^n$). If $A_\pi^j \in S$ a similar neighborhood $U(A_{-\pi}^j)$ can be found. The properties **(i)** and **(ii)** are still valid for the set

$$\widetilde{S} = (\mathbf{T}^n \setminus U(A_{-\pi}^j)) \setminus U(A_\pi^j).$$

Nothing is known about the number of neighborhoods removed from $\mathbf{T}^n$ to form $\widetilde{S}$. Ensure, by a standard compactness argument, that $\widetilde{S}$ can be built by removing only a finite number of such neighborhoods. Indeed, otherwise cutting $\mathbf{T}^n$ by hyper-planes parallel to the coordinate hyper-planes so that the faces of $\mathbf{T}^n$ corresponding to $x_j = \pm\pi$ are divided into 2^{n-1} equal parts, we get that the impossibility of removing only a finite number of neighborhoods is valid at least for one of these parts. Applying to it the same argument and continuing this process to infinity, we arrive at the contradiction with **(i)**.

Now we are going to build a function $\varphi \in A(\mathbf{T}^n)$ so that $\varphi = x_j$ on $\widetilde{S}$. The above construction has given us a set $\widetilde{S}$ with the same properties as S but of simpler structure. First we continue φ from S to $\widetilde{S}$ by $\varphi = x_j$ on $\widetilde{S}$ as well. Then we continue (see [S1, Ch.VI]) this φ to the whole of $\mathbf{T}^n$ as a k-smooth function with $k > n/2$ so that $\varphi(A^j_\pi) = \varphi(A^j_{-\pi})$ for all $A^j_{\pm\pi}$, $j = 1, 2, ..., n$. This is possible since the number of removed neighborhoods is finite, their structure is arbitrary (their boundaries can be chosen smooth enough), and conditions **(i)** and **(ii)** are satisfied. Hence $\varphi \in A(\mathbf{T}^n)$ (see, e.g., [SW, Ch.VII, Cor.1.9]). Since $A(\mathbf{T}^n)$ is the Banach algebra, $\lambda \in A(\mathbf{T}^n)$ implies $\lambda_j = \varphi\lambda \in A(\mathbf{T}^n)$. □

We are now in a position to prove the following generalization of Wiener's result.

Proposition 3.2. *If the set $S = \operatorname{supp}\lambda$ is a W-set, then $\widehat{\lambda} \in L^1(\mathbf{R}^n)$ if and only if λ, being continued 2π-periodically in each variable, has an absolutely convergent Fourier series.*

Proof. Indeed, in view of Proposition 3.1 we have to check $\lambda_\nu \in A(\mathbf{T}^n)$ for all appropriate ν. This is done by using the lemma in each variable. □

Theorem 3.1. *If $S = \operatorname{supp}\lambda$ is a W-set, then there exist two positive constants C_1 and C_2, $C_1 < C_2$, such that*

$$C_1 \sum_m |\widehat{\lambda}(m)| \le \sup_N ||L_N^\lambda|| \le C_2 \sum_m |\widehat{\lambda}(m)|.$$

This follows immediately from Propositions 2.2 and 3.2. A simple version of this theorem ($\operatorname{supp}\lambda \subset \mathbf{T}^n$ and λ vanishes on $\mathbf{T}^n_+$) can be found in [Be1, Prop.3].

3.3. In various questions of summability, certain assumptions on λ connected to bounded variation are rather natural; see in the one-dimensional case, e.g., [HT, Te, Be0]. Let us give one result of Trigub which is quite general and seems to be very useful for passage from Fourier series to Fourier integrals. We recall some well-known notions. The Vitali variation is defined as follows (see, e.g., [CA, AC]). Let λ be a complex-valued function and

$$\Delta_u\lambda(x) = \left(\prod_{j=1}^n \Delta_{u_j}\right)\lambda(x),$$

$$\Delta_{u_j}\lambda(x) = \lambda(x) - \lambda(x_1, ..., x_{j-1}, x_j + u_j, x_{j+1}, ..., x_n),$$

be a "mixed" difference with respect to parallelepiped $[x, x+u]$. Let us take an arbitrary number of non-overlapping parallelepipeds, and form a mixed difference with respect to each of them. Then the Vitali variation is

$$V(\lambda) = \sup \sum |\Delta_u\lambda(x)|,$$

where the least upper bound is taken over all the sets of such parallelepipeds. For smooth functions λ, the Vitali variation is expressed as the following integral

$$V(\lambda) = \int_{\mathbf{R}^n} \left|\frac{\partial^n\lambda(x)}{\partial x_1 ... \partial x_n}\right| dx.$$

The Tonelli variation is a different thing [To]. Roughly speaking, a function is of bounded Tonelli variation if it has a bounded variation in each variable, and these variations are integrable as functions of the remained variables. For a smooth function λ it is equal to

$$\int\limits_{\mathbf{R}^n} \sum_{j=1}^{n} \left| \frac{\partial \lambda(x)}{\partial x_j} \right| dx.$$

Let us write $\lambda \in V_0$ if its Vitali variation is bounded and $\lim\limits_{|x|\to\infty} \lambda(x) = 0$. In this case the function is of bounded variation with respect to any smaller number of variables, that is, belongs to the class of functions of bounded Hardy variation (cf. [AC]). In other words, functions depending only on a smaller number of variables than n are excluded.

Theorem 3.2. ([T7,8]) *The following relations hold:*

1) *For each $\lambda \in V_0$, and for every $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $\varepsilon_j > 0$, $j = 1, ..., n$,*

$$\sup_{0<|u_j|\le\pi/\varepsilon_j} \left| \int\limits_{\mathbf{R}^n} \lambda(x) e^{-iux}\, dx - \prod_{j=1}^{n} \varepsilon_j \sum_{k} \lambda(\varepsilon_1 k_1, ..., \varepsilon_n k_n) e^{-i(\varepsilon_1 k_1 u_1 + ... + \varepsilon_n k_n u_n)} \right| \le CV(\lambda) \sum_{j=1}^{n} \varepsilon_j \prod_{q\ne j} |u_q|^{-1}. \tag{3.1}$$

2) *If, moreover, λ has also a bounded Tonelli variation dominated by $V(\lambda)$, then $|u_q|^{-1}$ is replaced by $(1 + |u_q|)^{-1}$ in* (3.1)*.*

3) *If λ satisfies* **1)** *and* **2)** *then for $N = (N_1, ..., N_n)$ and*

$$k/N = (k_1/N_1, ..., k_n/N_n)$$

we have

$$\|L_N^\lambda\| = (2\pi)^{-n} \int\limits_{|x_j|\le\pi N_j} |\widehat{\lambda}(x)|\, dx + \theta V(\lambda) \sum_{j=1}^{n} \prod_{q\ne j} \ln(N_q + 1), \tag{3.2}$$

with $|\theta| \le C$.

In this theorem constants C depend only on n; the integrals and sums are treated in the Cauchy sense. We shall be concerned with functions of bounded variation later on, while considering the radial case as well as problems of integrability of trigonometric series.

Proof. First, let us check that the integral (the Fourier transform $\widehat{\lambda}$) exists in the improper sense for $\prod_{j=1}^{n} u_j \neq 0$. For this, it suffices to prove the following Cauchy type inequality

$$\left| \int\limits_{\substack{N \le |x_j| \le N+\delta, \\ j=1,2,...,n}} \lambda(x) e^{-iux}\, dx \right| \le C V_N(\lambda) \prod_{j=1}^{n} |u_j|^{-1}, \tag{3.3}$$

where $V_N(\lambda)$ is the total Vitali variation of λ restricted to the set

$$\{x : |x_j| \ge N, \quad j = 1, 2, ..., n\}.$$

For smooth functions this inequality is established by n-tuple integration by parts. In the general case, a function may be replaced by its Steklov type function which is smoother than the given function (see, e.g., [Bc, §44 and Appendix])

$$\lambda_h(x) = h^{-n} \int\limits_{\substack{0 \le y_j \le h, \\ j=1,2,...,n}} \lambda(x+y)\, dy;$$

we can do this smoothing for several times repeatedly, if needed. Since we have $V_N(\lambda_h) \le V_N(\lambda)$, integration by parts is applicable here, letting then $h \to 0$.

The series in question is an analog of the Riemann integral sum for $\widehat{\lambda}$, and thus converges for $\prod_{j=1}^{n} u_j \neq 0$.

1) Let us prove this part of the theorem for $u_j > 0$, $j = 1, 2, ..., n$. At each point, there exists the limit

$$\lambda(x+0) = \lim_{y \to x, y \ge x} \lambda(y).$$

Without loss of generality, one may take $\lambda(x+0) = \lambda(x)$ everywhere, since the number of points where $\lambda(x+0) \neq \lambda(x)$ is at most countable, while

$$|\lambda(x+0) - \lambda(x)| = \lim_{u \to +0} |\Delta_u \lambda(x)| \le V_{[x,x+\varepsilon]}(\lambda).$$

Replacing the function by its Steklov type function as above, we can treat λ as smooth enough in what follows.

Denote by $h_\varepsilon = h_\varepsilon(t)$ an increasing step-wise function on $\mathbf{R}$ with the jumps $\varepsilon > 0$ at the points $\{k\varepsilon\}$, $k \in \mathbf{Z}$. Then

$$|t - h_\varepsilon(t)| \le \varepsilon$$

and

$$t - h_\varepsilon(t) \sim \sum_{\nu=-\infty}^{\infty} \alpha_\nu e^{2\pi\nu t/\varepsilon}$$

with $|\alpha_\nu| \le C_\varepsilon (|\nu|+1)^{-1}$. The last inequality follows, e.g., from the known estimate of the Fourier coefficients of a function of bounded variation (see, e.g., [Br, Ch.II, §2] or [Zg, Ch.II, Th.4.12]). Now the series can be written as the following repeated Stieltjes integral

$$\int\limits_{\mathbf{R}^n} \lambda(x_1, \ldots, x_n) e^{iux} dh_{\varepsilon_1}(x_1) \ldots dh_{\varepsilon_n}(x_n).$$

Let us continue the calculations for $n = 2$, since the case $n > 2$ is treated completely in the same way. We have

$$\int_{\mathbf{R}}\int_{\mathbf{R}} \lambda(x_1, x_2) e^{-i(u_1x_1+u_2x_2)} [dx_1 d(x_2 - h_{\varepsilon_2}(x_1)) + dh_{\varepsilon_2}(x_2) d(x_1 - h_{\varepsilon_1}(x_1))]$$
$$= \int_{\mathbf{R}}\int_{\mathbf{R}} \lambda(x_1, x_2) e^{-i(u_1x_1+u_2x_2)}\, dx_1 d(x_2 - h_{\varepsilon_2}(x_1))$$
$$+ \int_{\mathbf{R}}\int_{\mathbf{R}} \lambda(x_1, x_2) e^{-i(u_1x_1+u_2x_2)}\, dh_{\varepsilon_2}(x_2) d(x_1 - h_{\varepsilon_1}(x_1))]$$
$$= J_1 + J_2.$$

After integrating by parts in x_2, we obtain

$$J_1 = \int_{\mathbf{R}} e^{-iu_1x_1}\, dx_1 \int_{\mathbf{R}} [h_{\varepsilon_2}(x_2) - x_2] \lambda'_{x_2}(x_1, x_2) e^{-iu_2x_2}\, dx_2$$
$$+ \int_{\mathbf{R}} e^{iu_1x_1} dx_1 \int_{\mathbf{R}} iu_2 [x_2 - h_{\varepsilon_2}(x_2)] \lambda(x_1, x_2) e^{-iu_2x_2}\, dx_2.$$

Integrating the first summand by parts once more, now in x_1, and taking into account that

$$\lim_{|x_1|\to\infty} \int_{\mathbf{R}} |\lambda'_{x_2}(x_1, x_2)|\, dx_2$$
$$\leq \lim_{|x_1|\to\infty} \int_{|v|\geq|x_1|} dv \int_{\mathbf{R}} |\lambda''_{x_1,x_2}(v, x_2)|\, dx_2 = 0,$$

and $|x_2 - h_{\varepsilon_2}(x_2)| < \varepsilon_2$, we get the bound

$$\theta V(\lambda) \varepsilon_2 |u_1|^{-1}, \quad |\theta| \leq 1.$$

Insert the series for $x_2 - h_{\varepsilon_2}(x_2)$ into the second summand for J_1. Since its partial sums are bounded and (3.3) holds, one may change the order of summation and integration. We arrive at the following value

$$iu_2 \sum \alpha_\nu \int_{\mathbf{R}}\int_{\mathbf{R}} \lambda(x_1, x_2) e^{-i(u_1x_1+u_2x_2-2\pi i\nu x_2/\varepsilon_2)}\, dx_1 dx_2.$$

Proceeding as for the first summand, we obtain the bound

$$O\left(|u_2| \varepsilon_2 V(\lambda) \sum_\nu (|\nu| + 1)^{-1} |u_1|^{-1} |u_2 - 2\pi\nu/\varepsilon_2|^{-1} \right).$$

It remains now to take into account that $|u_2| \leq \pi/\varepsilon_2$, and for $\nu \neq 0$ we have

$$|u_2 - 2\pi\nu/\varepsilon_2| \geq \pi(2|\nu| - 1)/\varepsilon_2.$$

This yields the bound $O(\varepsilon_2|u_1|^{-1})$.

The integral J_2 is estimated similarly.

2) For the first summand in J_1, we have the immediate bound $O(\varepsilon_2)$. The second one is estimated similarly by applying the inequality

$$\left|\int_{\mathbf{R}^n} \lambda(x)e^{iux}dx\right| \le C|u_j|^{-1}$$

instead of (3.3); the inequality is obtained by integration once by parts.

3) To prove this, use the previous statement with $\varepsilon_j = N_j^{-1}$, $j = 1, 2, \ldots, n$. Since

$$\begin{aligned}&\int_{\mathbf{T}^n}\left|\sum \lambda(k/N)e^{ikx}\right| dx\\ &= \int_{\substack{|y_j|\le\pi N_j,\\ j=1,2,\ldots,n}} \left|\sum \lambda(k/N)e^{-iy(k/N)}\right| dy \prod_{j=1}^n N_j^{-1},\end{aligned}$$

we have

$$\begin{aligned}&\left|\int_{\substack{|y_j|\le\pi N_j,\\ j=1,2,\ldots,n}} |\hat{\lambda}(y)|dy - \int_{\mathbf{T}^n}\left|\sum \lambda(k/N)e^{ikx}\right| dx\right|\\ &\le \int_{\substack{|y_j|\le\pi N_j,\\ j=1,2,\ldots,n}} \left|\hat{\lambda}(y) - \sum \lambda(k/N)e^{-iy(k/N)}\prod_{j=1}^n N_j^{-1}\right| dy\\ &\le CV(\lambda)\sum_{j=1}^n \prod_{q\ne j} \ln(1+\pi N_j).\end{aligned}$$

The theorem is proved. □

4 Generalizations of the Bochner-Riesz means

In this section, we are going to clarify for which linear means the (2.3) type estimates hold. These are, in a sense, certain generalizations of the Bochner-Riesz means. The main point is the geometric properties the support of the function generating linear means.

4.1. The estimates from above were obtained by Colzani and Soardi [CoS]. Their method is the direct generalization of that used by V. Yudin [Y1] for partial sums.

Suppose $S \subset \mathbf{R}^n$ is an open bounded set whose boundary ∂S has finite upper Minkowski measure. Let us consider complex-valued bounded functions λ on $\mathbf{R}^n$ satisfying the following assumptions:

$$\lambda(x) = 0 \quad \text{if } x \text{ does not belong to} \quad S; \tag{4.1}$$

there exist an integer $m \geq 0$ and real numbers $\alpha > -1/2$ and $\beta > -3/2$ such that

$$\lambda \in C^{m+1}(S); \tag{4.2}$$

$$|D^\xi \lambda(x)| \leq C\rho(x, \partial S)^\alpha \tag{4.3}$$

if $\xi_1 + ... + \xi_n = m$ and $x \in S$;

$$|D^\xi \lambda(x)| \leq C\rho(x, \partial S)^\beta \tag{4.4}$$

if $\xi_1 + ... + \xi_n = m + 1$ and $x \in S$.

If (4.1)–(4.4) are satisfied with $m \geq 1$, the function λ must also satisfy the following condition

$$\lambda \in C^{m-1}(\mathbf{R}^n). \tag{4.5}$$

Since λ is supposed to be bounded, we may assume $\alpha \geq 0$ whenever $m = 0$. Let us set

$$\gamma = \min\Big(1, \alpha + 1/2, \beta + 3/2\Big). \tag{4.6}$$

If $\beta = -1/2$ and $\alpha \geq 1/2$, let

$$\lambda \in C^{m+2}(S); \tag{4.7}$$

and

$$|D^\xi \lambda(x)| \leq C\rho(x, \partial S)^{-3/2} \tag{4.8}$$

if $\xi_1 + ... + \xi_n = m + 2$ and $x \in S$.

Theorem 4.1. ([CoS]) *Let S be as above and λ satisfies* (4.1)–(4.5)*, and, in addition,* (4.7), (4.8) *when $\beta = -1/2$ and $\alpha \geq 1/2$. Let $p_c = 2n(n + 2(m + \gamma))^{-1}$. Then for all $N > 2$:*

1) *If $m + \gamma \leq n/2$*

$$\|L_N^\lambda\|_p \leq C_p N^{n/2-(m+\gamma)} \qquad \text{if} \quad 1 \leq p < p_c,$$
$$\|L_N^\lambda\|_p \leq C_p N^{n-n/p} \ln^{1/p} N \qquad \text{if} \quad p = p_c,$$
$$\|L_N^\lambda\|_p \leq C_p N^{n-n/p} \qquad \text{if} \quad p_c < p \leq 2.$$

2) *If $m + \gamma > n/2$*

$$\|L_N^\lambda\|_1 \leq C.$$

Vignati [V] generalized these results to the case of non-isotropic metrics in $\mathbf{R}^n$. V. Yudin [Y3] showed that these estimates cannot be asymptotically improved for $N \to \infty$ in the class of the sets considered.

4.2. Special examination of general conditions for lower estimates was begun in [Y2], where the lower bound $\ln^n N$ for the order of growth of the Lebesgue constants of "all reasonable" partial sums is established, namely, for those generated by sets, assumed to be convex, closed, bounded, and containing a certain ball. Nazarov brought our attention to the fact that these assumptions seem to be unnatural and restrictive; his conjecture is that the only, in a sense, assumption should be the one that a ball is contained in a generating set. This conjecture is a natural extension of the Littlewood conjecture. This problem is still open.

Investigation of estimates of the Lebesgue constants from below was continued, as it was mentioned above, in [CaS], and then in [L2,3] (see Theorem 1.3). The recent result from [LRZ] generalizes the left-hand inequality in (2.3) in the spirit of Theorem 1.3.

Let $S = \operatorname{supp} \lambda$ be the support of a function $\lambda(x)$, where S is not necessarily a compactum. In what follows we shall be interested in functions $\lambda(x) = \lambda_{r,\alpha}(x)$, which are r-smooth inside S and may be represented in a certain neighborhood of ∂S as follows:

$$\lambda_{r,\alpha}(x) = f(x)(\rho(x))^\alpha, \tag{4.9}$$

where $f \in C^r(\mathbf{R}^n)$ and does not vanish on ∂S, while $\rho(x) = 0$ if $x \notin S$, and $\rho(x) = \rho(x, \partial S)$ if $x \in S$ (see the notion of regularized distance in [S1, Ch.6]). Notice, that $\rho(x)$ is a smooth function in a neighborhood of ∂S when $x \in S$ (see, e.g., [Gi, Appendix B]). It should be mentioned that in [CoS] the following obvious consequence of (4.1)–(4.5) is proved.

Lemma 4.1. *Suppose $S \subset \mathbf{R}^n$ is a bounded open set such that S has finite upper Minkowski measure and λ is a bounded complex-valued function on $\mathbf{R}^n$ satisfying* (4.1)-(4.5). *Then there exists a constant $C > 0$ such that*

$$|\lambda(x)| \le A\rho(x, \partial S)^{\alpha+m}$$

for all $x \in S$.

The following theorem shows that for the norms of the above generalizations of the Bochner-Riesz means the upper estimates match the lower ones, obtained under very similar assumptions.

Theorem 4.2. ([LRZ]) *Suppose that there exist an open set U and a hyper-surface V of smoothness*

$$r > \max(1, (n-1)/2 + \alpha),$$

where $0 \le \alpha < (n-1)/2$, with non-vanishing principal curvatures, such that $\partial S \cap U = V$. Suppose, further, that in $U \cap S$ we have $\lambda(x) = \lambda_{r,\alpha}(x)$. Then there exists a positive constant $C_{S,\lambda}$ depending only on S and λ such that

$$\|L_{NS}^\lambda\| \ge C_{S,\lambda} N^{(n-1)/2-\alpha}$$

for large N.

4.3. We want to indicate three focal points on which the proof of Theorem 4.2 is based. The first one as well as the idea of the proof was suggested by Belinsky.

Lemma 4.2. ([L2, LRZ]) *Let K be a set in $\mathbf{R}^n$ and ψ be a bounded measurable function with support in K. Then for every point $x_0 \in \mathbf{R}^n$, for every ball $B_\delta(x_0)$ of radius δ centered at x_0, and for every function φ supported in $B_\delta(x_0)$ and having the Fourier transform integrable over all $\mathbf{R}^n$, there exists a constant C, depending only on φ, such that*

$$\|L_K^\psi\| \geq C\|L_{K\cap B_\delta(x_0)}^{\psi\varphi}\|.$$

Proof. We have

$$\|L_K^\psi\| = \sup\nolimits_{\|f\|\leq 1} \|L_K^\psi(f;\cdot)\| \geq \sup\nolimits_{\|T_{B_\delta(x_0)}\|\leq 1} \|L_K^\psi(T_{B_\delta(x_0)};\cdot)\|, \tag{4.10}$$

where $T_{B_\delta(x_0)}$ denotes a trigonometric polynomial from the set of all those with spectrum in $B_\delta(x_0)$. According to [SW, Ch.VII, §2] (see (2.11)), the following inequality holds for every $f \in C(\mathbf{T}^n)$:

$$\left\|L_{B_\delta(x_0)}^\phi(f;\cdot)\right\| \leq (2\pi)^{-n}\|\hat{\phi}\|_{L_1(\mathbf{R}^n)}\|f\|.$$

Since the image of $L_{B_\delta(x_0)}^\phi$ is only a part of all polynomials $T_{B_\delta(x_0)}$, it follows from (4.10) that

$$\|L_K^\psi\| \geq \sup_{\left\|L_{B_\delta(x_0)}^\phi(f;\cdot)\right\|\leq 1} \left\|L_K^\psi\left(L_{B_\delta(x_0)}^\phi(f;\cdot);\cdot\right)\right\|$$

$$\geq C \sup_{\|f\|\leq 1} \left\|L_{K\bigcap B_\delta(x_0)}^{\psi\phi}(f;\cdot)\right\| = C\left\|L_{K\bigcap B_\delta(x_0)}^{\psi\phi}\right\|.$$

The lemma is proved. □

This lemma is of certain interest by itself, but mainly as a tool for estimates from below. A similar way to make "global from local" may be found in [Se].

The next step of the proof is the application of Theorem 2.1, more precisely the lower estimate for $p = 1$. After that we need appropriate asymptotic estimates of the Fourier transform of the functions considered. The following result is strongly based on the estimates of singularities of the Radon transform obtained by Ramm and Zaslavsky (see [RZ1, RZ2]).

Theorem 4.3. ([LRZ], see also [RZ] and [RK]) *Let S be the compact support of a function $\lambda(x) = \lambda_{r,\alpha}(x)$ with $\alpha \geq 0$ and*

$$r > \max(1, (n-1)/2 + \alpha).$$

Let S be convex, with the r-smooth boundary ∂S, and suppose the principal curvatures of ∂S never vanish. Let $\theta \in \mathbf{R}^n$ be a vector on the unit sphere, $x^+(\theta)$ and $x^-(\theta)$ be the (uniquely defined) points of ∂S at which the function $\theta_1 x_1 + ... + \theta_n x_n$ attains maximum and minimum on ∂S, respectively. Then for $t \to +\infty$

$$\widehat{\lambda}(t\theta) = t^{-\alpha-(n+1)/2}\left(\Xi^+(\theta)e^{itx^+(\theta)\theta} + \Xi^-(\theta)e^{itx^-(\theta)\theta} + o(1)\right),$$

$$\Xi^\pm(\theta) = (2\pi)^{(n-1)/2}\Gamma(\alpha+1)e^{\pm i\pi(2\alpha+n+1)/4}f(x^\pm(\theta))(\varkappa^\pm(\theta))^{-1/2},$$

where the remainder term is small uniformly in θ, and $\varkappa^\pm(\theta)$ is the Gaussian curvature of ∂S at the points $x^\pm(\theta)$, respectively.

This result continues and develops the well-known asymptotic estimate for the indicator function of a convex set [GGV]. There is an "almost all" gap between Theorem 4.3 and the result in [P7] in the two-dimensional case. We must mention that many authors use one result of Herz [Hz] to estimate the Fourier transform of the indicator function of a convex set with smooth boundary. But smoothness assumptions in this work are essentially more restrictive than those in [GGV] (and, of course, in Theorem 4.3) since the author was interested in sharp estimate for the remainder term. This explains, for example, the excess smoothness conditions in [CaS] or [Br1, Br2].

4.4. Special attention must be given to the following circumstance. One can see that in many results cited the value $(n-1)/2$ is of special meaning and importance. It is not accidentally, and this number is called "critical order" for the Bochner-Riesz means. Let us compare Theorem 2.2 with the following well-known result of Stein.

Theorem 4.4. ([S0]) *The following asymptotic formula holds:*

$$\|L_N^{R_{(n-1)/2}}\|_{L_1(\mathbf{T}^n)\to L_1(\mathbf{T}^n)} = \omega_n \ln N + O(1). \tag{4.11}$$

This asymptotics was obtained as a corollary to some general estimates of the difference between the corresponding kernel

$$\sum_{|k|\le N} R_{(n-1)/2}(k/N)\, e^{ikx}$$

and its integral analog. The constant ω_n was not indicated explicitly. Here the Lebesgue constants of the Bochner-Riesz means lose their power rate of growth, and behave as the Lebesgue constants of one-dimensional partial sums (cf. **0.1**). This likeness is not casual. Before formulating one recent generalization of Theorem 4.4, we wish to derive (4.11) from Theorem 2.1 as a simple consequence (the promised Corollary 2.2).

Corollary 4.1. *The following asymptotic formula holds:*

$$||L_N^{R_{(n-1)/2}}||_{L_1(\mathbf{T}^n)\to L_1(\mathbf{T}^n)} = \frac{4\Gamma((n+1)/2)}{\pi^{3/2}\Gamma(n/2)} \ln N + O(1).$$

Proof. ([Be2]) Theorem 2.1, (2.5), (2.6) and (2.7) yield

$$||L_N^{R_{(n-1)/2}}|| = \pi^{-(n+1)/2}\Gamma((n+1)/2) \times \int\limits_{1\le|x|\le N} \left|\prod_{j=1}^{n} \frac{x_j}{2N\sin(x_j/(2N))} \frac{\cos(|x|-\pi n/2)}{|x|^n}\right| dx + O(1).$$

Using the relation

$$\frac{x_j}{2N\sin(x_j/(2N))} - 1 = O(|x_j/N|^2)$$

and proceeding as in the proof of Corollary 2.1, we obtain

$$||L_N^{R_{(n-1)/2}}|| = \pi^{-(n+1)/2}\Gamma((n+1)/2) \times \int\limits_{1\le|x|\le N} |x|^{-n}|\cos(|x|-\pi n/2)|\, dx + O(1).$$

After passage to spherical coordinates, the right-hand side equals

$$||L_N^{R_{(n-1)/2}}|| = \pi^{-(n+1)/2}\Gamma((n+1)/2)\frac{2\pi^{n/2}}{\Gamma(n/2)}$$
$$\times \int_1^N |t^{-1}\cos(t-\pi n/2)|\,dt + O(1).$$

It is well-known that the last integral is

$$(2/\pi)\ln N + O(1) \tag{4.12}$$

(see, e.g., [Zg, Vol.1, Ch.2]), and this completes the proof. □

Applying again Theorem 2.1 and certain technique similar to that in the proof of Theorem 4.3 allows us to obtain such logarithmic asymptotics in a more general setting.

Theorem 4.5. ([L8]) *Let S be the compact support of a function $\lambda = \lambda_{n,(n-1)/2}$, with the n-smooth boundary ∂S. Assume that S is convex and the principal curvatures of ∂S never vanish. Then there exists a positive constant $C_{S,\lambda}$ depending only on S and λ such that*

$$\|L_N^\lambda\|_{L_1(\mathbf{T}^n)\to L_1(\mathbf{T}^n)} = C_{S,\lambda}\ln N + o(\ln N) \tag{4.13}$$

for large N.

Remark 4.1. The following formula is given in [L8] to calculate $C_{S,\lambda}$

$$C_{S,\lambda} = (2\pi)^{(n+3)/2}\Gamma((n+1)/2)\int_{|\theta|=1} d\theta \int_0^{2\pi} |(-1)^n\phi^+(\theta)e^{it} + \phi^-(\theta)|\,dt,$$

where $\phi^\pm(\theta) = f(x^\pm(\theta))(\varkappa^\pm(\theta))^{-1/2}$ (cf. Theorem 4.3). For the Lebesgue constants of the usual Bochner-Riesz means, simple computations yield the same constant as in Corollary 2.2.

Remark 4.2. It is obvious that taking $\lambda = \lambda_{r,\alpha}$, with $r > n$ and $\alpha > (n-1)/2$ in Theorem 4.5, we will obtain $\|L_N^\lambda\| = O(1)$ (cf. **2)** in Theorem 4.1).

5 "Radial" results

Considerable study has been given to the Bochner-Riesz means and certain of their generalizations. But we have not discussed yet one more peculiarity of the Bochner-Riesz means: the function R_α generating these means is *radial*, that is, depends only on $|x|$. Such functions play a special role in Fourier Analysis, and there are many ways to exploit the radiality.

For example, the Fourier transform of an (integrable) radial function $f(x) = f_0(|x|)$ is also radial and is represented by the formula (see, e.g., [Bc, Th.56] or [SW, Ch.IV])

$$\hat{f}(x) = \hat{f}_0(|x|) = (2\pi)^{n/2} \int_0^\infty f_0(t)(|x|t)^{-n/2+1} J_{n/2-1}(|x|t) t^{n-1}\, dt,$$

which is sometimes attributed to Cauchy and Poisson.

5.1. A situation is more complicated when functions are allowed to be non-integrable. Let us start with one special class of radial functions, written MV. Let $\lambda(x) = \lambda_0(|x|)$ be a radial function satisfying the following conditions:

$$\lambda_0, \lambda_0', ..., \lambda_0^{[(n-2)/2]} \quad \text{are locally absolutely continuous on} \quad (0, \infty). \tag{5.1}$$

$$\lim_{t\to\infty} \lambda_0(t) = 0. \tag{5.2}$$

Set $\Lambda(t) = t^{(n-1)/2} \lambda_0^{((n-1)/2)}(t)$. Further assumptions are

$$\lim_{t\to\infty} \Lambda(t) = 0. \tag{5.3}$$

$$||\lambda||_{MV} = \sup_{t>0} |\lambda_0| + \int_0^\infty |d\Lambda(t)| < \infty. \tag{5.4}$$

Here the fractional derivative is understood in the Weyl sense (see, e.g., [BE, Co]); the definition needs to be specified. For $0 < \delta < 1$ and a locally integrable function g on $(0, \infty)$,

$$W_\omega^\delta(g; t) = \begin{cases} \frac{1}{\Gamma(\delta)} \int\limits_t^\omega g(u)(u-t)^{\delta-1}\, du, & 0 < t < \omega \\ 0, & t \ge \omega \end{cases}$$

is the fractional (Weyl type) integral of order δ and, following Cossar [Co], define a fractional Weyl derivative of order α by

$$g^{(\alpha)}(t) = \lim_{\omega\to\infty} -\frac{d}{dt} W_\omega^{1-\alpha}(g; t)$$

when $0 < \alpha < 1$. For $\alpha = r + \delta$, $r = 1, 2, ...,$ and $0 < \delta < 1$,

$$g^{(\alpha)}(t) = \frac{d^r}{dt^r} g^{(\delta)}(t)$$

is the fractional derivative of order α.

Theorem 5.1 ([BL1, BL2]). *Let $\lambda(x) = \lambda_0(|x|)$ be a radial function satisfying* (5.1)–(5.4). *Suppose, in addition, that λ_0 is continuous at zero. Then*

$$\|L_N^\lambda\|_{L^1(\mathbf{T}^n)\to L^1(\mathbf{T}^n)} = (2\pi)^{-n} \int\limits_{|x|\le\pi N} |\widehat{\lambda}(x)|\,dx \\ +O(||\lambda_0||_{MV}). \tag{5.5}$$

Proof. We are going to prove that the series

$$\sum_k \lambda(k/N)e^{ikx} \tag{0.3}$$

is the Fourier series of an integrable function; by this the norm in question satisfies (0.4). Estimate

$$R_N = \left| ||L_N^\lambda|| - (2\pi)^{-n} \int\limits_{|x|\le\pi N} |\widehat{\lambda}(x)|\,dx - (2\pi)^{-n}|\lambda(0)| \right| \\ \le (2\pi)^{-n} \int\limits_{\mathbf{T}^n} \left| \sum_{k\ne 0} \lambda(k/N)e^{ikx} - \Phi_N(x) \right| dx,$$

where

$$\Phi_N(x) = \begin{cases} N^n\widehat{\lambda}(Nx), & |x|\le\pi, \\ 0, & x\in\mathbf{T}^n\setminus\{x:|x|\le\pi\}. \end{cases}$$

Consider the periodic, in each variable, continuation of this function and calculate its kth Fourier coefficient. No confusion will be resulted by saving the same notation. In [BL1, BL2] the following results were obtained: if a function λ satisfies conditions (5.1)–(5.4) its Fourier transform can be calculated as follows:

$$\widehat{\lambda}(x) = \frac{(2\pi)^{n/2}(-1)^{[n/2]}}{\Gamma((n-1)/2)} |x|^{1-n/2} \int\limits_0^\infty \Lambda(t)t^{n/2}Q(|x|t)\,dt, \tag{5.6}$$

where

$$Q(r) = \int\limits_0^1 (1-s)^{(n-3)/2} s^{n/2} J_{n/2-1}(rs)\,ds,$$

and the inverse formula holds for $|x|>0$:

$$\lambda(x) = \lim_{A\to\infty} (2\pi)^{-n} \int\limits_{|u|\le A} \widehat{\lambda}(u)e^{ixu}\,du. \tag{5.7}$$

For generalizations of this result, see [L6, L7]; they were obtained as well as those in [BL1, BL2] under additional assumptions which are removed in [LT]. For $|k|>0$, (5.6) and (5.7) yield

$$\widehat{\Phi}_N(k) = (2\pi)^{-n} \int\limits_{|u|\le\pi} N^n\widehat{\lambda}(Nu)e^{-iku}\,du$$
$$= (2\pi)^{-n} \int\limits_{|u|\le\pi N} \widehat{\lambda}(u)e^{-ik/Nu}\,du$$
$$= \lambda(-k/N) - (2\pi)^{-n} \int\limits_{|u|>\pi N} \widehat{\lambda}(u)e^{-iuk/N}\,du.$$

For $k = 0$, the passage to the spherical coordinates and (5.6) yield the following equality

$$\widehat{\Phi}_N(0) = (2\pi)^{-n} \int\limits_{|u|\le\pi N} \widehat{\lambda}(u)\,du$$
$$= \frac{2^{n-1}(-1)^{[n/2]}}{\Gamma(n/2)\Gamma((n-1)/2)} \int\limits_0^{\pi N} r^{n/2}\,dr \int\limits_0^{\infty} \Lambda(t)t^{n/2}Q(rt)\,dt.$$

Denote

$$q(r) = \int\limits_0^1 (1-s)^{(n-3)/2}s^{n/2-1}J_{n/2}(rs)\,ds.$$

For q, the following asymptotic relation was obtained in [BL2] (see also [L6, L7]):

$$q(r) = \alpha_1 r^{-(n-1)/2}J_{n-1/2}(r) + \alpha_2 r^{-n/2} + O(r^{-(n+2)/2}) \tag{5.8}$$

as $r \to \infty$, where $\alpha_1 = \Gamma((n-1)/2)$ and α_2 is some constant depending only on n. Integrating by parts and using one of the two versions of the well-known formula (see, e.g., [BE, 7.2.8(50),(51)])

$$\frac{d}{dt}t^{\pm\nu}J_\nu(t) = \pm t^{\pm\nu}J_{\nu\mp1}(t), \tag{5.9$\pm$}$$

we obtain

$$\widehat{\Phi}_N(0) = \frac{2^{n-1}(-1)^{[n/2]}}{\Gamma(n/2)\Gamma((n-1)/2)} \int\limits_0^{\pi N} r^{n/2-1}\,dr \Big\{ \Lambda(t)t^{n/2}q(rt)\,\Big|_0^\infty$$
$$- \int\limits_0^\infty t^{n/2}q(rt)\,d\Lambda(t) \Big\} = -\int\limits_0^\infty t^{n/2} \int\limits_0^{\pi N} r^{n/2-1}q(rt)\,dr\,d\Lambda(t).$$

The right-hand side equals

$$
\begin{aligned}
&-\int_0^\infty \int_0^{\pi N t} r^{n/2-1} q(r)\, dr\, d\Lambda(t) \\
&= -\int_{1/(\pi N)}^\infty \int_1^{\pi N} r^{n/2-1} q(rt)\, dr\, d\Lambda(t) + O(||\lambda||_{MV}) \\
&= -\alpha_2 \int_{1/(\pi N)}^\infty \int_1^{\pi N} r^{-1}\, dr\, d\Lambda(t) \\
&- \int_{1/(\pi N)}^\infty \int_1^{\pi N} r^{n/2-1}[q(rt) - \alpha_2 r^{-n/2}]\, dr\, d\Lambda(t) + O(||\lambda||_{MV}).
\end{aligned} \tag{5.9}
$$

Let us estimate the first integral on the right-hand side. We get after the obvious substitutions

$$
\begin{aligned}
\int_{1/(\pi N)}^\infty \int_1^{\pi N} r^{-1}\, dr\, d\Lambda(t) &= \int_1^\infty r^{-1}\Lambda(r/(\pi N))\, dr \\
&= \int_{1/(\pi N)}^\infty r^{(n-3)/2} \lambda_0^{((n-1)/2)}(r)\, dr.
\end{aligned}
$$

Integration by parts yields the bound $O(||\lambda||_{MV})$. This is obvious for n odd but the same is true also for n even - this needs some additional calculations, simple in fact, with fractional derivatives. In order to estimate the second integral and thus to get $|\widehat{\Phi}_N(0)| \le O(||\lambda||_{MV})$ it suffices, taking into account (5.4), to prove the boundedness of the value

$$
\sup_{N,t:\pi N t>1} \left| \int_1^{\pi N t} r^{n/2-1}[q(r) - \alpha_2 r^{-n/2}]\, dr \right|.
$$

It follows from (5.8) that the right-hand side is equal to

$$
\begin{aligned}
&\sup_{N,t:\pi N t>1} \left| \alpha_1 \int_1^{\pi N t} r^{-1/2} J_{n-1/2}(r)\, dr + O\left(\int_1^{\pi N t} r^{-2} dr \right) \right| \\
&= \sup_{N,t:\pi N t>1} \left| \alpha_1 \int_1^{\pi N t} r^{-1/2} J_{n-1/2}(r)\, dr \right| + O(1).
\end{aligned}
$$

The asymptotic formula (2.7) makes the claimed estimate obvious. Therefore we get the bound

$$
R_N \le (2\pi)^{-n} \int_{\mathbf{T}^n} \left| \sum_{k\neq 0} \left[\int_{|u|>\pi N} \widehat{\lambda}(u) e^{-iku/N}\, du \right] e^{ikx} \right| dx + C||\lambda||_{MV}.
$$

Applying the Cauchy-Schwarz inequality to the outer integral and then Parseval's identity, we obtain

$$R_N \le C\Big\{\sum_{k\ne 0}\Big|\int\limits_{|u|>\pi N}\widehat{\lambda}(u)e^{-iku/N}\,du\Big|^2\Big\}^{1/2} + C||\lambda||_{MV}.$$

The Cauchy-Poisson formula for the Fourier transform (see above) and (5.6) yield the following estimate

$$R_N \le CN^{n/2-1}\Big\{\sum_{k\ne 0}|k|^{2-n}\Big|\int\limits_{\pi N}^{\infty}J_{n/2-1}(|k|r/N)r\,dr\int\limits_0^{\infty}\Lambda(t)t^{n/2}Q(rt)\,dt\Big|^2\Big\}^{1/2}$$
$$+ C||\lambda||_{MV}.$$

Integration by parts in t implies

$$R_N \le CN^{n/2-1}\Big\{\sum_{k\ne 0}|k|^{2-n}\Big|\int\limits_{\pi N}^{\infty}J_{n/2-1}(|k|r/N)\,dr\,\Big[\Lambda(t)t^{n/2}q(rt)\Big]_0^{\infty}$$
$$-\int\limits_{\pi N}^{\infty}J_{n/2-1}(|k|r/N)\,dr\int\limits_0^{\infty}t^{n/2}q(rt)\,d\Lambda(t)\Big|^2\Big\}^{1/2} + C||\lambda||_{MV}.$$

After applying generalized Minkowski's inequality and (5.4) we get, as above, that it suffices to prove the boundedness of the value

$$\sup_{N,t}N^{n/2-1}\left\{\sum_{k\ne 0}|k|^{2-n}\Big|t^{n/2}\int\limits_{\pi N}^{\infty}J_{n/2-1}(|k|r/N)q(rt)\,dr\,\Big|^2\right\}^{1/2}$$
$$=\sup_{N,t}N^{n/2-1}\Big\{\sum_{k\ne 0}|k|^{2-n}\Big|t^{n/2}\int\limits_{\pi N}^{\infty}r^{n/2}J_{n/2-1}(|k|r/N)\,dr$$
$$\times\int\limits_0^1(1-s)^{(n-3)/2}s^{n/2-1}J_{n/2}(rts)r^{n/2}\,ds\,\Big|^2\Big\}^{1/2}.$$

Integrating by parts and using (5.9-), we obtain

$$\sup_{N,t}N^{n/2}\Big\{\sum_{k\ne 0}|k|^{-n}\,\Big|\,t^{n/2}J_{n/2}(|k|r/N)q(rt)\,\Big|_{\pi N}^{\infty}$$
$$+\,t^{n/2+1}\int\limits_{\pi N}^{\infty}J_{n/2}(|k|r/N)\,dr\int\limits_0^1(1-s)^{(n-3)/2}s^{n/2}J_{n/2+1}(rts)\,ds\Big|^2\Big\}^{1/2}.$$

Relations (5.8) and (5.9±) as well as the fact that the series $\sum\limits_{k\ne 0}|k|^{-n-1}$ converges imply the boundedness of the integrated terms. Further, integrating by parts as in the proof of (5.8), we obtain the following asymptotic relation

$$\int_0^1 (1-s)^{(n-3)/2} s^{n/2} J_{n/2+1}(rts)\,ds$$
$$= \alpha_3 r^{-n/2}\sin(r-\pi n/2) + O(r^{-(n+2)/2}).$$

No additional term appears here, unlike in (5.8), a special connection between the powers and the order of the Bessel function resulted a somewhat unusual asymptotics in the latter. Estimates using the remainder term are now trivial. Let us estimate

$$\sup_{N,t} N^{n/2}\Big\{\sum_{k\neq 0}|k|^{-n}\Big|\, t\int_{\pi N}^{\infty} r^{-n/2}J_{n/2}(|k|r/N)\sin(rt-\pi n/2)\,dr\,\Big|^2\Big\}^{1/2}.$$

Again integration by parts yields

$$\begin{aligned}\sup_{N,t} N^{n/2}\Big\{\sum_{k\neq 0}|k|^{-n}\Big|\, & r^{-n/2}J_{n/2}(|k|r/N)\cos(rt-\pi n/2)\,\Big|_{\pi N}^{\infty} \\ -(|k|/N)\int_{\pi N}^{\infty} & r^{-n/2}J_{n/2+1}(|k|r/N)\cos(rt-\pi n/2)\,dr\,\Big|^2\Big\}^{1/2}.\end{aligned} \tag{5.10}$$

The integrated terms in (5.10) are easily estimated. Apply now (2.7) to the last integral in (5.10). Estimates for the remainder term are obvious. Using also simple trigonometric identities, we get that the estimate is in order

$$\sup_{N,t} N^{(n-1)/2}\left\{\sum_{k\neq 0}|k|^{1-n}\left|\int_{\pi N}^{\infty} r^{-(n+1)/2}\sin r(|k|/N-t)\,dr\right|^2\right\}^{1/2}. \tag{5.11}$$

Observe that the estimates for similar values with $\sin r(|k|/N+t)$ or $\cos r(|k|/N\pm t)$ on place of $\sin r(|k|/N-t)$ are the same. Assume that Nt is large enough. Split the sum in (5.11) into three ones: over $\{k : 1\le |k| < Nt-1\}$, over $\{k : Nt-1\le |k|\le Nt+1\}$, and over $\{k : Nt+1 < |k| < \infty\}$. For the integral in (5.11), integration by parts implies the following bound, up to some constant,

$$N^{-(n+1)/2}||k|/N-t|^{-1} = N^{-(n-1)/2}|\,|k|-Nt|^{-1}.$$

Therefore the boundedness of the following sums

$$\sum_{1\le|k|<Nt-1}|k|^{1-n}(Nt-|k|)^{-2}$$

and

$$\sum_{Nt+1<|k|<\infty}|k|^{1-n}(|k|-Nt)^{-2}$$

has to be established when estimating over the first and the third domains, respectively. This is easily demonstrated by passing to integrals instead of the sums. For the second one, we obtain

$$\sup_{N,t} N^{(n-1)/2}\Bigg\{\sum_{Nt-1\le|k|\le Nt+1}|k|^{1-n}\Bigg|\int\limits_{\pi N}^{\infty} r^{-(n+1)/2}\sin r(|k|/N-t)\,dr\Bigg|^2\Bigg\}^{1/2}$$

$$\le N^{(n-1)/2}\Bigg\{\sum_{Nt-1\le|k|\le Nt+1}|k|^{1-n}\Bigg|\int\limits_{\pi N}^{\infty} r^{-(n+1)/2}\,dr\Bigg|^2\Bigg\}^{1/2}$$

$$\le C\Bigg\{\sum_{Nt-1\le|k|\le Nt+1}|k|^{1-n}\Bigg\}^{1/2}\le C.$$

When Nt is small similar estimates are valid after splitting the sum in (5.11) into two ones: over $\{k : 1 \le |k| \le 3\}$ and over $\{k : 3 < |k| < \infty\}$. The proof is complete. □

Remark 5.1. In fact, it is obtained in the proof that under the assumptions considered, (0.3) is the Fourier series of a function which belongs not only to $L^1(\mathbf{T}^n)$ but to $L^2(\mathbf{T}^n)$ as well.

Remark 5.2. Observe that besides other applications, say, to approximation on the class of functions with bounded poly-harmonic operator, Theorem 5.1 allows to obtain once again (4.11) as a simple corollary. Indeed, conditions (5.1)–(5.4) are verified easily. Then the estimates are similar to those in the proof of Corollary 2.2, and, of course, the same constant is obtained in calculations.

It is interesting that we have several different results which proved to be sharp by obtaining (4.11) as a simple consequence.

5.2. One of the features of radial functions is combining, in a certain sense, some multi-dimensional properties and those typical for the one-dimensional case. One of the ways to express this is the following

Theorem 5.2. ([LT]) *For $\lambda \in MV$ there holds*

$$\begin{aligned}\widehat{\lambda}(x) = |x|^{-(n-1)/2-\alpha}\Bigg\{&C_1\int\limits_0^{\infty}\Lambda(t)\sin(|x|t-\pi n/2)\,dt\\ &+C_2|x|^{-1}\Lambda(\pi/(2|x|))\\ &+O\Bigg(|x|^{-1}\int\limits_0^{\infty}\min\big(2|x|t/\pi,\pi/(2|x|t)\big)|d\Lambda(t)|\Bigg)\Bigg\}.\end{aligned}\tag{5.12}$$

Remark 5.3. In a weaker form this theorem can be found in [BL1, L6, L7]. The question of integrability of the Fourier transform, say over $|x| > 1$, is reduced to the integrability of the one-dimensional Fourier transform (when the first term in (5.12) is handled) and to the condition

$$\int\limits_0^1 t^{-1}|\Lambda(t)|\,dt<\infty\tag{5.13}$$

when the second term is integrated, since the remainder term is always integrable. The condition (5.13) is sharp and cannot be removed; indeed, there exist functions in MV which do not satisfy it (see [L6, L7, LT]).

Hence, we can apply to functions $\lambda \in MV$ many one-dimensional results, in which the behavior of Fourier transform is involved. Let us give such an example (cf. [LN] and [U]).

Proposition 5.1 ([BL1])**.** *Let $\lambda \in MV$ be supported in $|x| \le \pi$. Assume further that Λ satisfies* (5.13) *and has at least one point of discontinuity. Then*

$$\|L_N^\lambda\|_{L^1(\mathbf{T}^n)\to L^1(\mathbf{T}^n)} = M(\Lambda)\ln N + o(\ln N),$$

where $M(\Lambda)$ is an average of some almost periodic function built in accordance with Λ.

Proof. Applying Theorems 5.1 and 5.2 and taking into account (2.2) and (5.12), we arrive at the integral

$$\int_1^N \left| \int_0^\infty g(t)e^{-ixt}\,dt \right| dx$$

to be estimated, where g is either an odd or even continuation of Λ in accordance with n. The following argument is due to Belinsky. Let φ be the so-called "jump" function defined as follows:

$$\varphi(t) = \begin{cases} 1, & 0 < t < \pi \\ 1/2, & t = 0, t = \pi \\ 0, & -\pi < t < 0. \end{cases}$$

We can write

$$g(t) = g_1(t) + g_2(t) = g_1(t) + \sum_{k=1}^\infty a_k\varphi(t - t_k),$$

where g_1 is a (continuous) function with integrable Fourier transform, $\sum\limits_{k=1}^\infty |a_k| < \infty$, and t_k are the points of jump discontinuity of g. We have

$$\int_{-\pi}^{\pi} g_2(t)e^{-ixt}\,dt = (-i/x)\sum_{k=1}^\infty a_k(e^{-it_kx} - e^{-i\pi x}).$$

Now, the following integral

$$\int_{-N}^{N} |x|^{-1}\left|\sum_{k=1}^\infty a_k(e^{-it_kx} - e^{-i\pi x})\right| dx$$

has to be estimated. Setting

$$\psi(x) = \sum_{k=1}^\infty a_k(e^{-it_kx} - e^{-i\pi x}),$$

let us show that the limit

$$\lim_{N\to\infty} \ln^{-1} N \int\limits_{-N}^{N} |x|^{-1}|\psi(x)|\,dx$$

exists. Indeed, this limit coincides with

$$\lim_{N\to\infty} \ln^{-1} N \sum_{m=1}^{N-1} m^{-1}\Big[\int\limits_{m}^{m+1} |\psi(x)|\,dx + \int\limits_{m}^{m+1} |\psi(-x)|\,dx\Big],$$

therefore the sequence

$$\int\limits_{m}^{m+1} |\psi(x)|\,dx + \int\limits_{m}^{m+1} |\psi(-x)|\,dx$$

is summed by the method of logarithmic means. Since this method being regular is stronger than that of arithmetic means, we obtain

$$\lim_{N\to\infty} \ln^{-1} N \int\limits_{-N}^{N} |x|^{-1}|\psi(x)|\,dx = \lim_{N\to\infty} (2N)^{-1} \int\limits_{-N}^{N} |x|^{-1}|\psi(x)|\,dx$$

provided the limit on the right-hand side does exist. But it is really so, since ψ is a uniformly almost periodic function (in the Bohr sense; see [Le, Ch.1]). Just this limit on the right-hand side can be taken as a definition of $M(\Lambda)$. Proposition is proved. □

This result as well as Theorem 5.1 are generalizations of one-dimensional results in [Be0].

Under different assumptions, a similar connection between the radial Fourier transform and the one-dimensional one was obtained by Podkorytov. Let us give the precise formulation.

Theorem 5.3 ([P4])**.** *Let $\lambda_0 \in C[0,\infty)$ and $\lambda_0(t) = 0$ for $t \ge 1$. Then the following integrals converge simultaneously:*

$$\int\limits_{\mathbf{R}^n} |\widehat{\lambda}(x)|\,dx$$

and

$$\int\limits_{0}^{\infty} s^{(n-1)/2} \left| \int\limits_{0}^{1} t^{(n-1)/2}\lambda_0(t)\cos(2\pi st - \pi(n-1)/4)\,dt \right| ds.$$

To compare the latter two theorems, one can see that (5.1)–(5.4) together with (5.13) is the price one pays for λ not being obliged to possess an integrable Fourier transformṪhe assumption that λ has compact support is by no means important and can be removed by positing some smoothness conditions at infinity.

5.3. Let us describe one more "radial" result due to Trigub. It generalizes his own one-dimensional result Theorem 0.6.

Consider a function $\lambda_0(t)$ in $[0,\pi]$ and expand it in a cosine series:

$$\lambda_0(t) \sim \sum_{j=0}^{\infty} a_j \cos jx. \tag{5.14}$$

Theorem 5.4 ([T6]). *Let* $\lambda_0 \in C^{[(n-1)/2]}[0,\pi]$, *and* $\lambda_0^{(r)}(\pi) = 0$ *for* $0 \le r \le [(n-1)/2]$. *Then*

$$\sup_N \int_{\mathbf{T}^n} \left| \sum_{|k|\le N} \lambda_0(|k|\pi/N) e^{ikx} \right| dx \le C \sum_{j=0}^{\infty} j^{(n-1)/2} |a_j| \ln(j+1). \tag{5.15}$$

It is supposed that the series on the right-hand side converges and C depends only on n. For the summability on the whole class of periodic continuous functions, it suffices to supplement (5.15) with the condition $\lambda_0(0) = 1$.

It may be shown that for a_j, $j \ge 1$, with alternating signs, the opposite inequality holds provided $n = 1(\mathrm{mod}\, 4)$. It is possible to consider the sine expansion of λ_0 as well.

Laying some smoothness conditions on λ_0 which provide the convergence of the series on the right-hand side of (5.15), one can get convenient sufficient conditions for summability.

Proof. First, integrate the series (5.14) term by term $[(n-1)/2]$ times. Then, integrating the series obtained $[(n-1)/2]$ times over $[t,\pi]$ and taking into account the boundary condition at π, we obtain

$$\lambda_0(t) = \sum_{j=0}^{\infty} \alpha_j \psi_j(t),$$

where ψ_j are defined by

$$\psi_j^{([(n-1)/2])}(t) = \begin{cases} j^{[(n-1)/2]}(-1)^{[(n-1)/2]/2}(\cos jt + (-1)^{j+1}), & [(n-1)/2] \text{ is even} \\ j^{[(n-1)/2]}(-1)^{([(n-1)/2]+1)/2} \sin jt, & [(n-1)/2] \text{ is odd}, \end{cases}$$

and $\pi s_j^{(r)}(\pi) = 0$ for $0 \le r \le [(n-1)/2]$. Actually, ψ_j is the difference between $\cos jt$ and its Taylor polynomial of the corresponding degree at π.

This yields

$$\sum_{|k|\le N} \lambda_0(|k|\pi/N) e^{ikx} = \sum_{j=0}^{\infty} \alpha_j \sum_{|k|\le N} \psi_j(|k|\pi/N) e^{ikx},$$

and hence

$$\int_{\mathbf{T}^n} \left| \sum_{|k|\le N} \lambda_0(|k|\pi/N) e^{ikx} \right| dx \le \sum_{j=0}^{\infty} |\alpha_j| \int_{\mathbf{T}^n} \left| \sum_{|k|\le N} \psi_j(|k|\pi/N) e^{ikx} \right| dx.$$

What remains to prove is the following

Lemma 5.1. *For any integer* $j \ge 0$

$$\sup_N \int_{\mathbf{T}^n} \left| \sum_{|k|\le N} \psi_j(|k|\pi/N)^{ikx} \right| dx \le C j^{(n-1)/2} \ln(j+1).$$

Proof of Lemma 5.1. Applying (2.11), we obtain

$$\sup_N \int_{\mathbf{T}^n} \left| \sum_{|k|\le N} \psi_j(|k|\pi/N) e^{ikx} \right| dx \le \int_{\mathbf{R}^n} |\hat{\psi}_j(y)| dy,$$

here ψ_j is radial and vanishes for $|x| > \pi$.

The Cauchy-Poisson formula yields

$$\int_{\mathbf{R}^n} |\hat{\psi}_j(y)| dy = C \int_0^\infty t^{n/2} dt \left| \int_0^\pi \psi_j(u) u^{n/2} J_{n/2-1}(ut)\, du \right| .$$

As for C, it suffices that C depends only on n. Applying the Taylor formula yields

$$\psi_j(u) = \frac{1}{[(n-1)/2]!} \int_\pi^u \psi_j^{([(n-1)/2]+1)}(t)(t-u)^{[(n-1)/2]} dt$$

for $u \in [0,\pi]$. Now change the order of integration and substitute $\psi_j^{([(n-1)/2]+1)}$ (see above). We obtain

$$\int_{\mathbf{R}^n} |\hat{\psi}_j(y)|\, dy \le j^{[(n-1)/2]+1} \frac{C}{[(n-1)/2]!} \times \int_0^\infty t^{n/2} dt \left| \int_0^\pi e^{iju} du \int_0^u (u-z)^{[(n-1)/2]} z^{n/2} J_{n/2-1}(tz)\, dz \right| . \tag{5.16}$$

To continue, we need to know certain properties of the function

$$i(\mu,\eta,t) = \int_0^1 z^\mu J_\eta(tz)\, dz,$$

where $\mu + \eta - 1$.

Lemma 5.2. *Assuming that $t > 0$ and $\mu + \eta > -1$, we have*

(1) $i(\mu,\eta,t) = t^{-1} J_{\eta+1}(t) + (\eta + 1 - \mu) t^{-1} i(\mu - 1, \eta + 1, t)$.

(2) $i'(\mu,\eta,t) + (\mu+1)t^{-1} i(\mu,\eta,t) = t^{-1} J_\eta(t)$.

(3) The function $i(\mu,\eta,t)$ behaves as $O(t^\eta)$ for $t \to 0$, while for $t \to \infty$ it behaves as $O(t^{-3/2})$ for $\mu > 1/2$, and as $O(t^{-1-\mu})$ for $\mu \le 1/2$.

For the proof of (1) and (3), see, e.g., [L6, L7]; as for (2), it is proved by differentiation of i and integration by parts.

The inner integral on the right-hand side of (5.16), the one over $[0,u]$, is equal to

$$\sum_{p=0}^{[(n-1)/2]} \binom{[(n-1)/2]}{p} (-u)^{[(n-1)/2]-p} \int_0^u z^{p+n/2} J_{n/2-1}(tz)\, dz$$
$$= u^{[(n-1)/2]+n/2+1} \sum_{p=0}^{[(n-1)/2]} \binom{[(n-1)/2]}{p} \times (-1)^{[(n-1)/2]-p} i(p+n/2, n/2-1, ut).$$

Indeed, the binomial formula is used and linear substitution in the integral is fulfilled. Use now (1) in Lemma 5.2. After $[(n-1)/2]$ times, we have

$$\begin{aligned} i(p+n/2, n/2-1, r) &= r^{-1} J_{n/2}(r) - p r^{-2} J_{n/2+1}(r) \\ &\quad + \ldots + (-p)(-p+2)\ldots(-p+2([(n-1)/2]-2)) \\ &\qquad \times r^{-[(n-1)/2]} J_{n/2+[(n-1)/2]-1}(r) \\ &\qquad + (-p)(-p+2)\ldots(-p+2([(n-1)/2]-1)) \\ &\qquad \times r^{-[(n-1)/2]} i(p+n/2 \\ &\qquad - [(n-1)/2], k/2+[(n-1)/2]-1, r). \end{aligned}$$

Since

$$\sum_{p=0}^{[(n-1)/2]} \binom{[(n-1)/2]}{p} (-1)^p p^q = 0$$

for $0 \leq q \leq [(n-1)/2]-1$, for some β_p, depending only on p and n, the same integral is equal to

$$\begin{aligned} u^{[(n-1)/2]+n/2+1} &\sum_{p=0}^{[(n-1)/2]} \beta_p \, (ut)^{-[(n-1)/2]} \\ &\times i(p+n/2-[(n-1)/2], n/2+[(n-1)/2]-1, ut), \end{aligned}$$

where $\beta_1 = 0$ if $[(n-1)/2] \geq 1$.

Changing the order of summation and integration, we have to prove the following estimate

$$\begin{aligned} \int_0^\infty t^{n/2-[(n-1)/2]}\, dt \left| \int_0^\pi e^{iju} u^{n/2+1} i(\mu,\eta,ut)\, du \right| \\ = O\Big(j^{(n-1)/2-[(n-1)/2]-1} \ln(j+1) \Big), \end{aligned} \tag{5.17}$$

where $\mu = p+n/2-[(n-1)/2]$, $\eta = n/2+[(n-1)/2]-1$, and $1 \leq p \leq [(n-1)/2]$. If $[(n-1)/2] = 0$, that is, $n = 2$, we have the only integral

$$\int_0^\infty t\, dt \left| \int_0^\pi e^{iju} u^2 i(1,0,ut)\, du \right| = \int_0^\infty dt \left| \int_0^\pi e^{iju} u J_1(ut)\, du \right|. \tag{5.18}$$

When $j \geq 1$ and $t \in [0,1]$, integration by parts yields

$$\begin{aligned} \int_0^\pi e^{iju} u^{n/2+1} i(\mu,\eta,ut)\, du &= (ij)^{-1} e^{ij\pi} \pi^{n/2+1} i(\mu,\eta,\pi t) \\ &\quad - (ij)^{-1} \int_0^\pi e^{iju} (n/2+1) u^{n/2} i(\mu,\eta,ut)\, du \\ &\quad - t(ij)^{-1} \int_0^\pi e^{iju} u^{n/2+1} i(\mu,\eta,ut)\, du. \end{aligned}$$

By (3) in Lemma 5.2 we have $i(\mu,\eta,r)=O(r^\eta)$, and by (2)

$$|i'(\mu,\eta,r)|=O(r^{\eta-1})$$

for $0\le r\le\pi$. This estimates the inner integral in (5.17) over $[0,1]$ by $O(j^{-1})$. To estimate the integral in t over $[1,\infty)$, split the inner one in two: over $[0,\pi/t]$ and over $[\pi/t,\pi]$.

As for the integral over $[0,\pi/t]$, integrate, as above, by parts and obtain

$$(ij)^{-1}e^{ij\pi/t}(\pi/t)^{n/2+1}i(\mu,\eta,\pi)-(ij)^{-1}\int_0^{\pi/t}e^{iju}(n/2+1)u^{n/2}i(\mu,\eta,ut)\,du$$
$$-\,t(ij)^{-1}\int_0^{\pi/t}e^{iju}u^{n/2+1}i'(\mu,\eta,ut)\,du.$$

Applying similar estimates, we have

$$j^{-1}\int_1^\infty t^{n/2-[(n-1)/2]}O(t^{-n/2-1})dt=O(j^{-1})$$

for $[(n-1)/2\ge 1$. For $u\ge\pi/t$, apply the asymptotics of i and i':

$$\rho(t)=i(\mu,\eta,t)-\sqrt{(2/\pi)}t^{-3/2}\cos(t-\pi(\eta+1)/2-\pi/4)=O(t^{-5/2})$$

and

$$\rho'(t)=O(t^{-5/2})$$

(the latter for $\mu\ge 3/2$). Both estimates follow from (2.7) and Lemma 5.2. More precisely,the first one follows from (1), (3) and (2.7); while the second one from (2), (3), (2.7), and the estimate already proved. Now substitute the sum

$$\sqrt{(2/\pi)}t^{-3/2}\cos(t-\theta)+\rho(t)$$

for $i(\mu,\eta,t)$, where $\theta=\pi(\eta+1)/2+\pi/4$. For the first summand, we have when $j\ne t$

$$\left|\int_{\pi/t}^{\pi}e^{iju}u^{n/2+1}(ut)^{-3/2}e^{\pm iut}du\right|$$
$$=\left|\int_{\pi/t}^{\pi}e^{iu(j\pm t)}u^{(n-1)/2}\,du\right|t^{-3/2}$$
$$=t^{-3/2}\Bigg|i^{-1}(j\pm t)^{-1}e^{iu(j\pm t)}u^{(n-1)/2}\Big|_{\pi/t}^{\pi}$$
$$-\,(2i)^{-1}(j\pm t)^{-1}(n-1)\int_{\pi/t}^{\pi}e^{iu(j\pm t)}u^{(n-3)/2}\,du\Bigg|$$
$$\le Ct^{-3/2}(|j-t|+1)^{-1}.$$

In this form, the inequality holds to be true for $j = t$ as well. Similarly, for the second summand

$$\left| \int_{\pi/t}^{\pi} e^{iju} u^{n/2+1} \rho(ut) du \right|$$

$$= \left| \int_{\pi/t}^{\pi} e^{iu(j-t)} u^{n/2+1} e^{iut} \rho(ut)\, du \right|$$

$$= \Bigg| i^{-1}(j-t)^{-1} e^{iu(j-t)} u^{n/2+1} e^{iut} \rho(ut) \Big|_{\pi/t}^{\pi}$$

$$- i^{-1}(j-t)^{-1} \int_{\pi/t}^{\pi} e^{iu(j-t)} \Big[(n/2+1) u^{n/2} e^{iut} \rho(ut)$$

$$+ itu^{n/2+1} e^{iut} \rho(ut) + tu^{n/2+1} e^{iut} \rho'(ut) \Big]\, du \Bigg|$$

$$\leq C|j-t|^{-1} |\rho(\pi t)| + C|j-t|^{-1} t^{-n/2-1}$$

$$+ C|j-t|^{-1} \int_{\pi/t}^{\pi} \Big[u|\rho(ut)| + u^2 t |\rho(ut)| + u^2 t |\rho'(ut)| \Big]\, du.$$

Applying the above indicated estimates for ρ and ρ', we obtain

$$\left| \int_{\pi/t}^{\pi} e^{iju} u^{n/2+1} \rho(ut)\, du \right| \leq C t^{-3/2} (|j-t|+1)^{-1}.$$

It remains to estimate the integral (see (5.17))

$$\int_{1}^{\infty} t^{n/2-[(n-1)/2]} t^{-3/2} (|j-t|+1)^{-1} dt.$$

For n odd, this integral equals

$$\int_{1}^{\infty} t^{-1}(|j-t|+1)^{-1} dt = \int_{1}^{j} (t^{-1} + (j-t+1)^{-1}(j+1)^{-1} dt$$

$$+ \int_{j}^{\infty} (-t^{-1} + (t-j+1)^{-1})(j-1)^{-1}\, dt$$

$$= O(j^{-1} \ln\ j).$$

For n even, we have

$$\int_1^\infty t^{-1/2}(|j-t|+1)^{-1}\,dt = 2\int_1^{\sqrt{j}}(j+1-s^2)^{-1}ds + 2\int_{\sqrt{j}}^\infty (s^2-j+1)^{-1}\,ds$$
$$= O(j^{-1/2}\ln\ j).$$

Combining all the estimates obtained, we obtain that the integral in (5.17) is estimated by

$$O(j^{-1}) + O(j^{-[(n-1)/2]-1+(n-1)/2}\ln\ j),$$

and (5.17) is established for $[(n-1)/2] \geq 1$ $(n \geq 3)$.

For $n = 2$, it remains to estimate the integral (cf. (5.18))

$$\int_1^\infty dt\left|\int_0^{\pi/t} e^{iju}uJ_1(ut)\,du\right| = \int_1^\infty dt\left|\int_0^{\pi/t} e^{i(j-t)u}ue^{iut}J_1(ut)\,du\right|.$$

The estimates are continued by integration by parts as above for the integral over $[\pi/t,\pi]$. The bound will be $O(j^{-1}\ln j)$. Lemma 5.1 is proved. □

Obviously, this lemma completes the proof of the theorem. □

6 "Polyhedral" results

Let us again refer to the book [DC]: "Concentric polygons are an obvious thing to try, but this turns out to be no more interesting than repeating several one-dimensional results. It doesn't give any new mathematics, and it avoids having to think deeply about Fefferman's result.[1] To avoid thinking about a subject is almost always a mistake; at best you are in for some big surprises later on".

This passage is a moot point even if one speaks about parallelepipeds with the sides parallel to coordinate planes. What is anticipated here is nothing more than the product of one-dimensional estimates. But even in this case there exists Fefferman's other bright result [F1], which gives an example of a continuous function with everywhere rectangularly divergent partial sums. And considering more general objects within the scope of "polyhedral" case, one can meet with many non-trivial problems. We will touch those closely connected to our topic.

6.1. We must say that, in general, this case has a "logarithmic" nature. More precisely, there exist two positive constants C_1 and C_2, $C_1 < C_2$, such that for each polyhedron E we have

$$C_1 \ln^n N \leq \int\limits_{\mathbf{T}^n} \left| \sum_{k \in NE} e^{ikx} \right| dx \leq C_2 \ln^n N. \tag{6.1}$$

Actually this was proved by Belinsky [Be2]; nothing new was added in later publications [P3, Bb1]. Thus, we see an essential difference between this case and the spherical case characterized in (1.2). In the latter case, the Lebesgue constants are of power growth, the worst possible, in a sense, while (6.1) is the best possible estimate one can achieve for partial sums generated by a non-trivial set. We are going to concentrate on two important problems which are essentially of "polyhedral" nature.

6.2. One of them touches quite a natural question stated as follows.

Can partial sums be defined by sets for which the norms of the corresponding operators (1.1) have an intermediate - between (6.1) *and* (1.2) *- rate of growth with respect to N-dilations of these sets?*

Some trivial solutions were suggested in [Y2], where an intermediate growth is achieved by Cartesian product of the two mentioned situations. Of course, this is possible only for dimension three and greater. Thus a real solution might be that for the two-dimensional case. It was done by Podkorytov (similar but weaker results were given in [YY2]). It is clear (see Theorem 1.3) that the boundary can possess no point of non-vanishing curvature - this readily results in the maximal (power) order $N^{(n-1)/2}$. On the other side, any polyhedron matches (6.1). Thus, the only chance might be delivered by a "polyhedron" with an infinite number of specially organized sides.

Let C_1 and C_2 denote, as above, positive constants such that $C_1 < C_2$.

Theorem 6.1 ([P5]). *The following assertions hold.*

1) *For any $p > 2$ there exists a compact, convex set E for which*

$$C_1 \ln^p N \leq \int\limits_{\mathbf{T}^2} \left| \sum_{k \in NE} e^{ikx} \right| dx \leq C_2 \ln^p N, \qquad N \geq 2. \tag{6.2}$$

[1] The famous solution of the multiplier problem for the ball in [F2].

2) *For any* $p \in (0, 1/2)$ *and* $\alpha > 1$ *there exists a compact, convex set* E *for which*

$$C_1 N^p \ln^{-\alpha p} N \le \int\limits_{\mathbf{T}^2} \Big| \sum_{k \in NE} e^{ikx} \Big| \, dx \le C_2 N^p \ln^{2-2p} N, \qquad N \ge 2. \tag{6.3}$$

The proof is obtained by proceeding, in a sense, as in the proof of Theorem 1.1, using very delicate technique where the behavior of the sequence of lengths and slopes of the sides of E is treated carefully.

6.3. The next question also seems to be very natural.

Is it possible to write a certain asymptotic relation instead of the ordinal estimate (6.1)*?*

Some partial cases were investigated by Daugavet [D], Kuznetsova [Ku1, Ku2, Ku4], Skopina [Sk0, Sk2]. For example, Kuznetsova generalized Daugavet's result as follows.

Theorem 6.2 ([Ku1, Ku2, Ku4])**.** *Let*

$$B_{N_1,N_2} = \{(k_1, k_2) : |k_1|/N_1 + |k_2|/N_2 \le 1\}.$$

The asymptotic equality

$$||S_{B_{N_1,N_2}}|| = 32\pi^{-4} \ln N_1 \ln N_2 - 16\pi^{-4} \ln^2 N_1 + O(\ln N_2)$$

holds uniformly with respect to all natural N_1, N_2, *and* $l = N_2/N_1$.

The case $l = 1$ is the mentioned result of Daugavet. What differentiates both these results from many others is that not dilations of certain fixed domain are taken. This is a source of additional difficulties, and nothing is known for l other than integer as well as for the case of more dimensions. Let us also mention a recent paper [Bak].

As for the "regular" situation, an unexpected result was obtained again by Podkorytov [P6]. He has shown that there are two main cases. The first one, the afore-mentioned asymptotic results may be referred to, deals with the polygons (we are speaking about two-dimensional results) with integer, or rational slopes of sides. In this case one can show that the estimates change insignificantly if one considers the corresponding integrals instead of sums, that is, the Fourier transform $\hat{\chi}_{NE}$ of the indicator function of the N-dilation of the corresponding set E. In other words, the Dirichlet kernel is well approximated by $\hat{\chi}_{NE}$. This circumstance allows to obtain the logarithmic asymptotics, namely, $\int_{\mathbf{T}^2} |\sum_{k \in NE} e^{ikx}| \, dx$ is equivalent to $\ln^2 N$ and both to $\int_{\mathbf{T}^2} |\hat{\chi}_{NE}(x)| \, dx$.

In the second case, that is, when at least one slope is irrational, the situation changes qualitatively: the upper limit and the lower limit of the ratio of $\int_{\mathbf{T}^2} |\sum_{k \in NE} e^{ikx}| \, dx$ and $\ln^2 N$, as $N \to \infty$, may be different. In other words, in this case the behavior of the Fourier transform of the indicator function of NE is not representative of the behavior of the corresponding partial sums. In [P6] the quantitative estimate of this phenomenon is given at once. Namely, for the triangles

$$E = E_\alpha = \{(u, v) : \qquad 0 \le u \le 1, \qquad 0 \le v \le \alpha u\}$$

the following theorem is true.

Theorem 6.3. *There hold two assertions.*

1) *We have*

$$\int_{\mathbf{T}^2} \left| \sum_{k \in NE_\alpha} e^{ikx} \right| dx = \int_{\mathbf{T}^2} |\hat{\chi}_{NE_\alpha}(x)|\, dx$$
$$+ \int_0^{2\pi} \left| \sum_{j=0}^{N} \{\alpha j\} e^{ijt} \right| dt + O(\ln N \ln\ln N),$$

where $\{...\}$ *denotes the fractional part of the corresponding number.*

2) *There exists irrational* α *such that*

$$\overline{\lim_{N\to\infty}} \ln^{-2} N \int_0^{2\pi} \left| \sum_{j=0}^{N} \{\alpha j\} e^{ijt} \right| dt > 0. \tag{6.4}$$

The main defect of this theorem is that it is true only for α from very scarce set, and nothing is known about other α. In a recent paper by Nazarov and Podkorytov [NP] this uncertainty is partly removed. Namely, the following is true. Denote by $I_N(\alpha)$ the integral in (6.4).

Theorem 6.3'. *Let* α *be irrational.*

1) *We have*

$$0 < C_1 \leq \overline{\lim_{N\to\infty}} I_N(\alpha) \ln^{-2} N \leq C_2.$$

2) *We have*

$$\underline{\lim}_{N\to\infty} I_N(\alpha) \ln^{-2} N = 0$$

if and only if α *is a Liouville number, that is, if and only if for each* $M > 0$ *there exist fractions* p/q $(q \geq 2)$ *such that*

$$|\alpha - p/q| \leq q^{-M}.$$

3) *If* $|\alpha - p/q| \leq q^{-M}$ *for some* $M > 2$ *and infinitely many fractions* p/q $(q \geq 2)$*, then the fraction* $I_N(\alpha)\ln^{-2} N$ *has no limit as* $N \to \infty$.

4) *The integral* $I_N(\alpha)$ *is concentrated on a set of small measure, namely, for all* $N \geq 2$ *and* α *irrational there exists a set* $E = E(N, \alpha) \subset \mathbf{T}$ *such that*

$$\text{mes}(E) \leq e^{-\sqrt{\ln N}}$$

while

$$\int_{\mathbf{T}\setminus E} \left| \sum_{0 \leq j \leq N} \{\alpha j\} e^{ijt} \right| dt \leq C \ln^{3/2} N.$$

5) *There exist numbers* $0 < \omega \leq \Omega < \infty$ *such that for almost all* α

$$\omega = \underline{\lim}_{N\to\infty} I_N(\alpha) \ln^{-2} N$$

and

$$\Omega = \overline{\lim_{N\to\infty}} I_N(\alpha) \ln^{-2} N.$$

6.4. Observe that Podkorytov in [P2] and Skopina in [Sk1, Sk2] gave some asymptotic estimates for more general linear means in the cases which we may treat as "polyhedral" as well. Let

$$\rho(x) = \rho_E(x) = \inf\{\alpha > 0 : x/\alpha \in E\}$$

be the Minkowski functional of a set E and

$$L_N^\lambda(f;x) = L_N^{\lambda_E}(f;x) = \sum_{k \in NE} \lambda(\rho(k)/N)\hat{f}(k)e^{ikx}.$$

Theorem 6.4 ([P2]). *Let E be a polyhedron star-shaped with respect to the origin, which is an interior point of it, and $\lambda \in C[0,\infty)$ be supported on $[0,1]$.*

1) *If the extension of at least one of the faces of the polyhedron E passes through the origin, then*

$$\sup_N ||L_N^{\lambda_E}|| = \infty$$

and consequently there exists an $f \in C(\mathbf{T}^n)$ such that

$$\overline{\lim_{N\to\infty}} |L_N^\lambda(f;0)| = \infty.$$

2) *If no extension of a face pass through the origin, then the convergence of the integral*

$$F_n(\lambda) = \int_{\mathbf{R}} |d\hat{\lambda}(r)| \frac{\ln^{n-1}(2+|r|)}{1+|r|}\, dr,$$

where

$$d\hat{\lambda}(r) = \int_0^1 e^{-irt}\, d\lambda(t),$$

is sufficient for the norms $||L_N^\lambda||$ to be bounded, and consequently $L_N^\lambda(f;\cdot)$ converge uniformly to f as $N \to \infty$ for all $f \in C(\mathbf{T}^n)$.

Some results for "polyhedral" functions λ are obtained in [Sk2, Sk3] in the form similar to that given in Theorem 5.1. In particular, the following asymptotic relation holds.

Theorem 6.5 ([Sk3]). *Let E be an n-dimensional polyhedron with vertices having all coordinates rational, star-shaped with respect to the origin, and the origin does not lie on the extension of any face of the polyhedron. Let $\lambda(x) = \lambda_E(x)$. Then*

$$\|L_N^\lambda\|_{L_1(\mathbf{T}^n)\to L_1(\mathbf{T}^n)} = (2\pi)^{-n} \int_{N\mathbf{T}^n} |\widehat{\lambda}(x)|\, dx + O(V_{\lambda_0} + |\lambda_0(0)|) \ln^{n-1} N.$$

On the base of this theorem, it is possible to find the main term of

$$\|L_N^\lambda\|_{L_1(\mathbf{T}^n)\to L_1(\mathbf{T}^n)}$$

in a form suitable for calculations. We mention some special cases. For instance, it is shown in [Sk3], that the following statement holds.

Theorem 6.6. *If E is a convex symmetric $2l$-polygon, and $\lambda_0 \in C[0,1] \cap C^1[0,1)$ is such that $\lambda_0(t) \ge 0$, $\lambda_0(1) = 0$, and both $\lambda_0'(t)$ and $(t-1)\lambda_0'(t)$ are monotone decreasing, then*

$$\|L_N^\lambda\|_{L_1(\mathbf{T}^2)\to L_1(\mathbf{T}^2)}$$

$$= 16\,l\pi^{-4} \int_1^N x^{-1}\lambda_0(1-1/x)\,\ln x\,dx$$

$$+ O\Big(\int_1^N x^{-1}\lambda_0(1-1/x)\,dx + \lambda_0(0)\Big).$$

This allows us to obtain the logarithmic asymptotics provided some better bounds are valid for the remainder terms. The constant in the main term depends on geometric properties of the polyhedron. It is shown in [Sk1] that the Lebesgue constants grow as $(2/\pi)^{2n}\ln^n N$ for parallelepipeds, and as $2(n+1)\pi^{-n-1}\ln^n N$ for simplices. More precisely, let for $N = 0, 1, 2, \ldots$, and $0 \le p \le N$ the means L_N^λ be defined by means of

$$\lambda(x) = \begin{cases} 1, & \text{for} \quad x \in (N-p)E, \\ (N+1-\rho(x))(p+1)^{-1}, & \text{for} \quad x \in NE \setminus (N-p)E, \\ 0, & \text{for} \quad x \notin NE, \end{cases}$$

where E is the same as in Theorem 6.5; then the norms of such operators are equal to

$$||L_N^{\lambda_E}|| = (2\pi)^{-n} \int_{\mathbf{T}^n} |\hat{\lambda}(x)|\,dx + \Sigma,$$

where

$$|\Sigma| \le C_{P,n}(p+1)^{-1}(\ln(N+2))^{n-1}.$$

7 "Hyperbolic" results

Since the appearance of Babenko's paper [Ba1] interest has continued in various questions of Approximation Theory and Fourier Analysis in $\mathbf{R}^n$ connected with the study of linear means with harmonics in "hyperbolic crosses"

$$\Gamma(N,\gamma) = \{k \in \mathbf{Z}^n : h(N,k,\gamma) = \prod_{j=1}^{n} (|k_j|/N)^{\gamma_j} \le 1, \quad \gamma_j \ge 1, j = 1, ..., n\}.$$

We are interested in the hyperbolic means of Bochner-Riesz type of order $\alpha \ge 0$

$$L^{\alpha}_{\Gamma(N,\gamma)} : f(x) \mapsto \sum_{k\in\Gamma(N,\gamma)} (1 - h(N,k,\gamma))^{\alpha}_{+} \hat{f}(k)e^{ikx}.$$

Hyperbolic Bochner-Riesz means (for the two-dimensional Fourier integrals with $\gamma_1 = \gamma_2 = 2$) appeared for the first time in the paper of El-Kohen [EK] in connection with the study of their L^p-norms. His result was not sharp, and shortly after was strengthened by Carbery [C].

The case $\alpha = 0$ - hyperbolic partial sums $L_{\Gamma(N,\gamma)} = L^0_{\Gamma(N,\gamma)}$ - is investigated separately earlier. The exact degree of growth for them $\|L_{\Gamma(N,\gamma)}\| \asymp N^{(n-1)/2}$ (cf. Theorem 1.1) was established in the two-dimensional case independently by Belinsky [Be2] and by A. and V. Yudins [YY1], and afterwards was generalized to the case of arbitrary dimension in [L1]. Recently these results were applied to problems of uniform convergence in [Dy4].

For $\alpha > 0$, the estimates are given in the following

Theorem 7.1. ([L4]) *The following assertions hold.* **1)** *For $\alpha < (n-1)/2$, we have*

$$\|L^{\alpha}_{\Gamma(N,\gamma)}\| \asymp N^{(n-1)/2-\alpha}.$$

2) *For $\alpha = (n-1)/2$, we have*

$$\|L^{(n-1)/2}_{\Gamma(N,\gamma)}\| = \omega_{n,\gamma} \ln^n N + O(\ln^{n-1} N).$$

3) *For $\alpha > (n-1)/2$, we have*

$$\|L^{\alpha}_{\Gamma(N,\gamma)}\| = \omega_{n,\gamma,\alpha} \ln^{n-1} N + O(\ln^{n-2} N).$$

Here and below ω with subscripts denotes, generally speaking, different constants depending only on the indicated indices.

Observe that the critical order $(n-1)/2$ is the same as in the spherical case. But if for the values lower than the critical one the orders of growth of the Lebesgue constants coincide (this is clear in view of Theorem 4.2), the difference between (4.11) and **2)** in Theorem 7.1 is obvious as well as for orders greater than $(n-1)/2$: in the latter case the Lebesgue constants of the usual Bochner-Riesz spherical means are bounded. In order to establish Theorem 7.1, especially **2)** and **3)**, we need the following

Theorem 7.2. ([L4,9, LS]) *For the norms of operators*

$$\bar{L}^{\alpha}_{\Gamma(N,\gamma)} : \qquad f(x) \mapsto \sum_{|k_j|\le N, j=1,...,n} (1 - h(N,k,\gamma))^{\alpha}_{+} \hat{f}(k)e^{ikx}$$

the following asymptotic equality is true

$$\|\bar{L}^{\alpha}_{\Gamma(N,\gamma)}\| = \omega_{n,\gamma,\alpha} \ln^{n-1} N + O(\ln^{n-2} N).$$

This is a strengthening of Kivinukk's result [Ki], where two-sided ordinal inequalities were obtained; by this it was shown for the first time the influence of smoothness at the corner points on the order drop of a logarithmic growth, as compared with the Lebesgue constants of cubic partial sums.

It should be mentioned that these theorems are proved by step by step passage from sums to corresponding integrals. This leads to the Fourier transform of a function generating the method of summability under consideration.

Proof of Theorem 7.1. The proof is inductive. To estimate the passage from the trigonometric sum to the Fourier transform, first the two-dimensional case is considered. Then a geometric argument allows one to make estimates for higher dimensions which are either easier or similar to those for dimension two. Hence we present the two-dimensional proof of the passage from sums to integrals, while the inductive argument for higher dimensions is the same as that for hyperbolic partial sums ($\alpha = 0$) in [L1]. Then the stationary phase method is applied to estimate the Fourier transform. Some ideas from [Be2] are used here.

Since the norms of the operators

$$f \to \int_{\mathbf{T}} f(x_1, x_2) dx_1 \quad \text{and} \quad f \to \int_{\mathbf{T}} f(x_1, x_2) dx_2,$$

taking $C(\mathbf{T}^2)$ into $C(\mathbf{T}^2)$, are bounded, it suffices to estimate the norm of the operator

$$f \to \sum_{\substack{1 \le |m_1|^{\gamma_1} |m_2|^{\gamma_2} \\ \le N^{\gamma_1+\gamma_2}}} (1 - |m_1|^{\gamma_1} |m_2|^{\gamma_2} / N^{\gamma_1+\gamma_2})^{\alpha} \hat{f}(m) e^{imx}.$$

This norm is equal to (cf. (0.4))

$$\begin{aligned}
&\int_{\mathbf{T}^2} \Bigg| \sum_{\substack{1 \le |m_1|^{\gamma_1} |m_2|^{\gamma_2} \\ \le N^{\gamma_1+\gamma_2}}} (1 - |m_1|^{\gamma_1} |m_2|^{\gamma_2} / N^{\gamma_1+\gamma_2})^{\alpha} e^{imx} \Bigg| dx \\
&= \int_{\mathbf{T}^2} \Bigg| \sum_{1 \le |m_1| \le N} e^{im_1 x_1} \sum_{\substack{1 \le |m_2|^{\gamma_2} \\ \le N^{\gamma_1+\gamma_2} |m_1|^{-\gamma_1}}} (1 - |m_1|^{\gamma_1} |m_2|^{\gamma_2} / N^{\gamma_1+\gamma_2})^{\alpha} e^{im_2 x_2} \\
&+ \sum_{1 \le |m_2| \le N} e^{im_2 x_2} \sum_{\substack{1 \le |m_1|^{\gamma_1} \\ \le N^{\gamma_1+\gamma_2} |m_2|^{-\gamma_2}}} (1 - |m_1|^{\gamma_1} |m_2|^{\gamma_2} / N^{\gamma_1+\gamma_2})^{\alpha} e^{im_1 x_1} \\
&+ \le \sum_{1 \le |m_1|, |m_2| \le N} (1 - |m_1|^{\gamma_1} |m_2|^{\gamma_2} / N^{\gamma_1+\gamma_2})^{\alpha} e^{imx} \Bigg| dx
\end{aligned}$$

times $(2\pi)^{-2}$. The estimate for the last sum is given in Theorem 7.2. The first two sums are similar, so we will handle only one of them. We have

$$\begin{aligned}
&\int\limits_{\mathbf{T}^2} \Bigg| \sum_{1\le|m_1|\le N} e^{im_1x_1} \Bigg\{ \sum_{\substack{1\le|m_2|^{\gamma_2}\\ \le N^{\gamma_1+\gamma_2}|m_1|^{-\gamma_1}}} (1-|m_1|^{\gamma_1}|m_2|^{\gamma_2}/N^{\gamma_1+\gamma_2})^{\alpha} e^{im_2x_2} \\
&- \int\limits_{|y_2|^{\gamma_2}\le N^{\gamma_1+\gamma_2}|m_1|^{-\gamma_1}} (1-|m_1|^{\gamma_1}|y_2|^{\gamma_2}/N^{\gamma_1+\gamma_2})^{\alpha} e^{iy_2x_x} dy_2 \Bigg\} \Bigg| \, dx \\
&= \int\limits_{\mathbf{T}^2} \Bigg| \sum_{1\le|m_1|\le N} e^{im_1x_1} \Bigg\{ \sum_{\substack{1\le|m_2|^{\gamma_2}\\ \le N^{\gamma_1+\gamma_2}|m_1|^{-\gamma_1}}} (1-|m_1|^{\gamma_1}|m_2|^{\gamma_2}/N^{\gamma_1+\gamma_2})^{\alpha} e^{im_2x_2} \\
&- N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2} \int\limits_{|z|\le 1} (1-|z|^{\gamma_2})^{\alpha} e^{ix_2zN^{\gamma_1/\gamma_2+1}/|m_1|^{\gamma_1/\gamma_2}} dz \Bigg\} \Bigg| \, dx,
\end{aligned}$$

where z is substituted for $y_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}$. The right-hand side may be rewritten as

$$\begin{aligned}
&\int\limits_{\mathbf{T}^2} \Bigg| \sum_{1\le|m_1|\le N} e^{im_1x_1} \Bigg\{ \sum_{\substack{1\le|m_2|^{\gamma_2}\\ \le N^{\gamma_1+\gamma_2}|m_1|^{-\gamma_1}}} (1-|m_1|^{\gamma_1}|m_2|^{\gamma_2}/N^{\gamma_1+\gamma_2})^{\alpha} e^{im_2x_2} \\
&\qquad - N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2} \Lambda(x_2N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}) \Bigg\} \Bigg| \, dx,
\end{aligned} \tag{7.1}$$

where Λ is the one-dimensional inverse Fourier transform, times 2π, of the function $(1-|z|^{\gamma_2})_+^{\alpha}$. The same argument as that when proving Theorem 5.1 yields the following relation

$$\begin{aligned}
&(2\pi)^{-1} \int\limits_{\mathbf{T}} N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2} \Lambda(x_2N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}) e^{-im_2x_2} \, dx_2 \\
&\quad = (1-|m_1|^{\gamma_2}|m_2|^{\gamma_2}/N^{\gamma_1+\gamma_2})_+^{\alpha} \\
&\quad - (2\pi)^{-1} \int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} \Lambda(t) e^{-itm_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}} \, dt.
\end{aligned} \tag{7.2}$$

Observe that the left-hand side of (7.2) is simply the m_2th Fourier coefficient of the function between the sign of the integral and $e^{-im_2x_2}$. Applying successively the Cauchy-Schwarz inequality and Parseval's identity to (7.1), and then using (7.2), we arrive at the following value which is to be estimated:

$$\Bigg\{ \sum_{1\le|m_1|\le N} \sum_{m_2\ne 0} \Bigg| \int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} \Lambda(t) e^{-itm_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}} \, dt \Bigg|^2 \Bigg\}^{1/2}.$$

We will consider later on the case $m_2 = 0$. For this as well as for all other m_2, we have to know the behavior of $\Lambda(t)$. We have

$$\Lambda(t) = \int_{-1}^{1} (1 - |z|^{\gamma_2})^\alpha e^{itz} dz = 2 \int_0^1 (1 - z^{\gamma_2})^\alpha \cos tz \, dz$$
$$= 2 \int_0^1 (\gamma_2/2)(1 - z^2)^\alpha \cos tz \, dz$$
$$+ 2 \int_0^1 [(1 - z^{\gamma_2})^\alpha - (\gamma_2/2)(1 - z^2)^\alpha] \cos tz \, dz.$$

Using (2.5), we obtain

$$\Lambda(t) = 2^{\alpha-1/2}\Gamma_2\sqrt{\pi}\Gamma(\alpha+1)J_{\alpha+1/2}(t)t^{-\alpha-1/2}$$
$$+ \int_0^1 \varphi(z) \cos tz \, dz,$$

and it is completely clear what is taken as $\varphi(z)$. Integration by parts yields (integrated terms obviously vanish)

$$\Lambda(t) = 2^{\alpha-1/2}\gamma_2\sqrt{\pi}\Gamma(\alpha+1)J_{\alpha+1/2}(t)t^{-\alpha-1/2}$$
$$+ t^{-1} \int_0^1 \Phi(z) \sin tz \, dz,$$

where the function $\Phi(z) = -\varphi'(z)$ behaves as $z^{\min(\gamma_2-1,1)}$ at zero, and as $(1-z)^\alpha$ at 1. Applying (2.7) to the Bessel function and elementary estimates to the integral, we obtain

$$\Lambda(t) = 2^\alpha \gamma_2 \Gamma(\alpha+1) t^{-\alpha-1} \cos(t+\theta) + O(t^{-2-\varepsilon}) \tag{7.3}$$

with some numbers θ and $\varepsilon \geq 0$. Denote

$$M_3 = \{m_2 : 1 \leq |m_2| \leq N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2} + 2\}.$$

First we obtain

$$\left\{ \sum_{1\leq|m_1|\leq N} \sum_{M_3} \left| \int_{|t|>\pi N^{\gamma_1/\gamma_2+1}/|m_1|^{\gamma_1/\gamma_2}} t^{-2-\varepsilon} \, dt \right|^2 \right\}^{1/2}$$

$$\leq \left\{ \sum_{1\leq|m_1|\leq N} \sum_{M_3} (|m_1|^{\gamma_1} N^{-\gamma_1-\gamma_2})^{2(1+\varepsilon)/\gamma_2} \right\}^{1/2} \leq CN^{-\varepsilon}.$$

Denote now

$$M_1 = \{m_2 : 1 \leq |m_2| \leq N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2} - 2\}.$$

To handle the main term in (6.3), let us estimate

$$\Big\{\sum_{1\le|m_1|\le N}\sum_{M_1}\Big|\int_{|t|>\pi N^{\gamma_1/\gamma_2+1}/|m_1|^{\gamma_1/\gamma_2}} t^{-\alpha-1}e^{\pm i(t+\theta)}e^{-itm_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}}\,dt\Big|^2\Big\}^{1/2}$$

$$=\Big\{\sum_{1\le|m_1|\le N}\sum_{M_1}\Big|\int_{|t|>\pi N^{\gamma_1/\gamma+1}/|m_1|^{\gamma_1/\gamma_2}} t^{-\alpha-1}e^{-it(m_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}\pm 1)}\,dt\Big|^2\Big\}^{1/2}.$$

Integrating by parts, we obtain the following bound for the integral on the right-hand side

$$(|m_2|\ |m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}\pm 1)^{-1}(\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^{-1-\alpha}.$$

Hence the bound for the whole right-hand side is

$$\left\{\sum_{1\le|m_1|\le N}\sum_{M_1}(|m_2|\ |m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}\pm 1)^{-2}(N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^{-2-2\alpha}\right\}^{1/2}$$

$$\le CN^{-(1+\alpha)(\gamma_1/\gamma_2+1)}\Big\{\sum_{1\le|m_1|\le N}|m_1|^{2\gamma_1(1+\alpha)/\gamma_2}N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}$$

$$\times(1-|m_1|^{\gamma_1\gamma_2}(N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1\gamma_2}-2)N^{-\gamma_1/\gamma_2-1})^{-1}\Big\}^{1/2}$$

$$\le CN^{-\alpha(\gamma_1/\gamma_2+1)-(\gamma_1/\gamma_2+1)/2}$$

$$\times\left\{\sum_{1\le|m_1|\le N}|m_1|^{\gamma_1/\gamma_2+2\alpha\gamma_1/\gamma_2}N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}\right\}^{1/2}$$

$$=CN^{-\alpha(\gamma_1/\gamma_2+1)}\Big\{\sum_{1\le|m_1|\le N}|m_1|^{2\alpha\gamma_1/\gamma_2}\Big\}^{1/2}\le CN^{1/2-\alpha}.$$

For

$$\Big|\,|m_2|-N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}\Big|\le 2,$$

estimates are straightforward and comparatively simple. Indeed,

$$\Big\{\sum_{1\le|m_1|\le N}\Big|\int_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} t^{-1-\alpha}\,dt\Big|^2\Big\}^{1/2}$$

$$\le C\Big\{\sum_{1\le|m_1|\le N}(|m_1|^{\gamma_1/\gamma_2}N^{-\gamma_1/\gamma_2-1})^{2\alpha}\Big\}^{1/2}\le CN^{1/2-\alpha}.$$

Of course, the latter estimate in this form is true only for $\alpha > 0$. For $\alpha = 0$, some additional though simple estimates are needed, but these have been proved earlier (see [Be2], [L1]).

Let us consider now estimates for

$$m_2 \in M_2 = \{m_2 : |m_2| > N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2} + 2\}.$$

Since M_2 is infinite, more delicate consideration is needed. First, let us integrate by parts

$$\begin{aligned}
&\left\{ \sum_{1\le|m_1|\le N} \sum_{M_2} \left| \int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} \Lambda(t) e^{-itm_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}}\, dt \right|^2 \right\}^{1/2} \\
&\le \left\{ \sum_{1\le|m_1|\le N} \sum_{M_2} (N^{\gamma_1/\gamma_2+1}|m_2|^{-1}|m_1|^{-\gamma_1/\gamma_2})^2 \Lambda^2(\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}) \right\}^{1/2} \\
&+ \left\{ \sum_{1\le|m_1|\le N} \sum_{M_2} (N^{\gamma_1/\gamma_2+1}|m_2|^{-1}|m_1|^{-\gamma_1/\gamma_2})^2 \right. \\
&\times \left. \left| \int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} \Lambda'(t) e^{-itm_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1}}\, dt \right| \right\}^{1/2} \\
&= I' + I''.
\end{aligned}$$

For I', the estimates are very similar to those above. Since for $\alpha \le 1$, which is more than enough for us, $\Lambda(t) = O(t^{-1-\alpha})$, we have

$$\begin{aligned}
I' &\le C\left\{ \sum_{1\le|m_1|\le N} \sum_{M_2} |m_2|^{-2}(N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^{2-2\alpha} \right\}^{1/2} \\
&\le C\left\{ \sum_{1\le|m_1|\le N} (N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1\gamma_2})^{-1-2\alpha} \right\}^{1/2} \\
&= N^{-\gamma_1/\gamma_2+1)/2-\alpha)\gamma_1/\gamma_2+1)} \left\{ \sum_{1\le|m_1|\le N} |m_1|^{\gamma_1/\gamma_2+2\alpha\gamma_1/\gamma_2} \right\}^{1/2} \\
&\le CN^{-\alpha}.
\end{aligned} \tag{7.4}$$

To estimate I'', observe that

$$\begin{aligned}
\Lambda'(t) &= i\int_{-1}^{1} (1-|z|^{\gamma_2})^\alpha z e^{itz} dz \\
&= i\int_{-1}^{1} (1-|z|^{\gamma_2})^\alpha e^{itz} dz + i\int_{-1}^{1} (1-|z|^{\gamma_2})^\alpha (z-1) e^{itz} dz \\
&= 2^\alpha i\gamma_2 \Gamma(\alpha+1) t^{-\alpha-1}\cos(t+\theta) + O(t^{-2-\varepsilon})
\end{aligned}$$

as above. Since $\sum l_{|m_2|>Q}|m_2|^{-2} \le 2Q^{-1}$, estimates for the remainder term are exactly as above. Hence, we again are concerned with the main term in the asymptotic representation for $\Lambda'(t)$. We have

$$\Big\{ \sum_{1\le|m_1|\le N} \sum_{M_2} m_2^{-2} (N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^2$$
$$\times \Big| \int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} t^{-1-\alpha} e^{it(1-m_2|m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1})}\, dt \Big| \Big\}^{1/2}$$
$$\le C\Big\{ \sum_{1\le|m_1|\le N} \sum_{M_2} m_2^{-2} (N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^2$$
$$\times (1-|m_2|\ |m_1|^{\gamma_1/\gamma_2}/N^{\gamma_1/\gamma_2+1})^{-2} (N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^{-2-2\alpha} \Big\}^{1/2}$$
$$\le C\Big\{ \sum_{1\le|m_1|\le N} (N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^{-2\alpha}$$
$$\times \sum_{M_2} (|m_2| - N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2})^{-2} \Big\}^{1/2}$$
$$\le CN^{-\alpha(\gamma_1/\gamma_2+1)} \Big\{ \sum_{1\le|m_1|\le N} |m_1|^{2\alpha\gamma_1/\gamma_2} \Big\}^{1/2} \le CN^{1/2-\alpha}.$$

Let us now come back to the case $m_2=0$. Of course, it may cause no serious problem. We have to estimate

$$\int\limits_{\mathbf{T}^2} \Big| \sum_{1\le|m_1|\le N} e^{im_1x_1} \Big(\int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma/\gamma_2}} \Lambda(t)\, dt - 1 \Big) \Big|\, dx$$
$$= \omega_2 \log N + O\Big(\sum_{1\le|m_1|\le N} \Big| \int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} \Lambda(t)\, dt \Big| \Big).$$

It suffices to use a simple bound $\Lambda(t)=O(t^{-1-\alpha})$. This yields the estimate $O(N^{-\alpha})$ for the remainder term on the right-hand side. Actually, this is the case for any individual m_2, or a finite number of m_2s. So the same good estimate is valid when estimating over $M_3 \setminus M_1$.

Let us sum up previous work. We have succeeded in passing from the trigonometric sum in m_1 to the correspondent integral with an appropriate estimate of the difference in (7.1). Now we go on with

$$\int\limits_{\mathbf{T}^2} \Big| \sum_{1\le|m_1|\le N} e^{im_1x_1} N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2} \Lambda(x_2^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}) \Big|\, dx.$$

Using, as when proving Theorem 1.1, the relation

$$e^{im_1x_1} = \frac{x_1}{2\sin(x_1/2)} \int\limits_{m_1-1/2}^{m_1+1/2} e^{ix_1u}\, du,$$

we have to estimate the difference

$$\int_{\mathbf{T}^2} \Bigg| \sum_{1\le|m_1|\le N} \int_{m_1-1/2}^{m_1+1/2} e^{ix_1u}\,du$$
$$\Bigg\{ \int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} (1-|m_1|^{\gamma_1}|t|^{\gamma_1}N^{-\gamma_1-\gamma_2})^{\alpha} e^{ix_2t}\,dt$$
$$- \int_{|t|\le N^{\gamma_1/\gamma_2+1}|u|^{-\gamma_1/\gamma_2}} (1-|u|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^{\alpha} e^{ix_2t} dt \Bigg\} \Bigg|\, dx.$$

Substituting $u \to m_1 + u$ and applying simple inequalities, and then the Cauchy-Schwarz inequality, Parseval's identity and mean-value theorem, we estimate this difference via

$$\int_{\mathbf{T}} dx_2 \int_{-1/2}^{1/2} du \int_{\mathbf{T}} \Bigg| \sum_{1\le|m_1|\le N} e^{ix_1(m_1+u)}$$
$$\Bigg\{ \int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} (1-|m_1|^{\gamma_1}|t|^{\gamma_1-\gamma_2})^{\alpha} e^{ix_2t}\,dt$$
$$- \int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1+u|^{-\gamma_1/\gamma_2}} (1-|m_1+u|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2}) e^{ix_2t} dt \Bigg\} \Bigg|$$
$$\le \int_{\mathbf{T}} \Bigg\{ \sum_{1\le|m_1|\le N} \Bigg| \int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} (1-|m_1|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^{\alpha} e^{ix_2t}\,dt$$
$$- \int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1+s|^{-\gamma_1/\gamma_2}} (1-|m_1+s|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^{\alpha} e^{ix_2t}\,dt \Bigg|^2 \Bigg\}^{1/2} dx_2,$$

where s is some number in $(-1/2, 1/2)$.

After substitutions reducing the inner integrals to those over $[-1,1]$, the right-hand side can be rewritten as

$$\int_{\mathbf{T}} \left\{ \sum_{1\le|m_1|\le N} \left| N|m_1|^{-\gamma_1/\gamma_2} \int_{-1}^{1} (1-|t|^{\gamma_2})^\alpha e^{ix_2tN^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}}\,dt \right.\right.$$

$$\left.\left. - N|m_1+s|^{-\gamma_1/\gamma_2} \int_{-1}^{1} (1-|t|^{\gamma_2})^\alpha e^{ix_2tN^{\gamma_1/\gamma_2+1}}|m_1+s|^{-\gamma_1/\gamma_2}dt \right|^2 \right\}^{1/2} dx_2$$

$$= 4\int_0^{\pi N} \left\{ \sum_{1\le|m_1|\le N} \left| \, |m_1|^{-\gamma_1/\gamma_2} \int_0^1 (1-t^{\gamma_2})^\alpha \cos(vtN^{\gamma_1/\gamma_2}|m_1|^{-\gamma_1/\gamma_2})\,dt \right.\right.$$

$$\left.\left. - |m_1+s|^{-\gamma_1/\gamma_2} \int_0^1 (1-t^{\gamma_2})^\alpha \cos(vtN^{\gamma_1/\gamma_2}|m_1+s|^{-\gamma_1/\gamma_2})\,dt \right|^2 \right\}^{1/2} dv$$

times N^{γ_1/γ_2}.

To estimate this, split the sum over m_1 into two: $1 \le |m_1| \le v^q N^p$ and $|m_1| > v^q N^p$, where p and q will be specified later on. The first part is simpler. Applying the above rough estimate to the inner integrals, we obtain

$$N^{\gamma_1/\gamma_1} \int_0^{\pi N} v^{-1-\alpha} \left\{ \sum_{1\le|m_1|\le v^qN^p} \left| \, |m_1|^{-\gamma_1/\gamma_2}|m_1|^{(1+\alpha)\gamma_1/\gamma_2} N^{-(1+\alpha)\gamma_1/\gamma_2} \right|^2 \right\}^{1/2} dv$$

$$= N^{-\alpha\gamma_1/\gamma_2} \int_0^{\pi N} v^{-1-\alpha} \left\{ \sum_{1\le|m_1|\le v^qN^p} |m_1|^{2\alpha\gamma_1/\gamma_2} \right\}^{1/2} dv$$

$$= N^{-\alpha\gamma_1\gamma_2} N^{p\alpha\gamma_1/\gamma_2+p/2} \int_0^{\pi N} v^{-1-\alpha} v^{q\alpha\gamma_1/\gamma_2+q/2}\,dv.$$

Two things should now be achieved by choice of p and q. First, integration must be guaranteed. Observe that here and in what follows non-integrability at zero might become a real problem. It was possible earlier to separate the integral, say, over $[0,1]$ which results in no confusion; hence the integral can be always understood as that over $[1,\pi N]$. Thus we wish that

$$q/2 + q\alpha\gamma_1/\gamma_2 - \alpha > 0,$$

which leads to the estimate

$$O(N^{-\alpha\gamma_1/\gamma_2} N^{p\alpha\gamma_1/\gamma_2+p/2} N^{q/2+q\alpha\gamma_1/\gamma_2-\alpha}).$$

Analyzing this, we arrive at the restrictions $p+q=1$ and

$$q > \alpha/(1/2+\alpha\gamma_1/\gamma_2).$$

The latter gives, for $\alpha \le 1/2$, that $q<1$. Hence for $\alpha < 1/2$ the bound is $O(N^{1/2-\alpha})$, just the one which is needed. For $\alpha = 1/2$, the precise estimate is $O(\log N)$ which is also good for us. For $\alpha > 1/2$, the estimate is even better than needed; the only point is that sometimes (this depends on γ_1 and γ_2) one should take $q > 1$ and $p < 0$.

Let us proceed now with the case $|m_1| > v^q N^p$, in which more delicate consideration is needed for

$$\int_0^{\pi N} \Bigg\{ \sum_{|m_1|>v^q N^p} \Bigg| v|m_1|^{-\gamma_1/\gamma_2} \int_0^1 (1-t^{\gamma_2})^\alpha \cos(tvN^{\gamma_1/\gamma_2}|m_1|^{-\gamma_1/\gamma_2})\,dt$$
$$- v|m_1+s|^{-\gamma_1/\gamma_2} \int_0^1 (1-t^{\gamma_2})^\alpha \cos(tvN^{\gamma_1/\gamma_2}|m_1+s|^{-\gamma_1/\gamma_2})\,dt \Bigg|^2 \Bigg\}^{1/2} \frac{dv}{v}$$

times N^{γ_1/γ_2}.

We need some more information on the behavior of the inner integrals. First, denoting $(1-t^{\gamma_2})^\alpha = \Phi(t)$ and integrating by parts, we obviously obtain

$$M \int_0^1 \Phi(t) \cos Mt\,dt = -\int_0^1 \Phi'(t) \sin Mt\,dt,$$

and the value we have to estimate is

$$\int_0^{\pi N} \Bigg\{ \sum_{|m_1|>v^q N^p} \Bigg| \int_0^1 \Phi'(t) \sin(tvN^{\gamma_1/\gamma_2}|m_1+s|^{-\gamma_1/\gamma_2})\,dt$$
$$- \int_0^1 \Phi'(t) \sin(tvN^{\gamma_1/\gamma_2}|m_1|^{-\gamma_1/\gamma_2})\,dt \Bigg|^2 \Bigg\}^{1/2} \frac{dv}{v}.$$

The difference of the two integrals is equal to

$$vN^{\gamma_1/\gamma_2}(|m_1+s|^{-\gamma_1/\gamma_2} - |m_1|^{-\gamma_1/\gamma_2})$$
$$\times \int_0^1 \Phi'(t) t \cos(tvN^{\gamma_1/\gamma_2}|m_1+s_{m_1}|^{-\gamma_1/\gamma_2})\,dt,$$

where $|m_1| < |m_1+s_{m_1}| < |m_1+s|$, and the entire value under estimation is bounded by

$$CN^{\gamma_1/\gamma_2} \int_0^{\pi N} \Bigg\{ \sum_{|m_1|>v^q N^p} \Big|\, |m_1|^{-\gamma_1/\gamma_2-1} (vN^{\gamma_1/\gamma_2}|m_1|^{-\gamma_1/\gamma_2})^{-\alpha} \Big|^2 \Bigg\}^{1/2} dv$$
$$= CN^{\gamma_1\gamma_2-\alpha\gamma_1/\gamma_2} \int_0^{\pi N} \Bigg\{ \sum_{|m_1|>v^q N^p} |m_1|^{-2\gamma_1/\gamma_2-2+2\alpha\gamma_1/\gamma_2} \Bigg\}^{1/2} v^{-\alpha} dv$$
$$= CN^{\gamma_1/\gamma_2-\alpha\gamma_1/\gamma_2-p\gamma_1/\gamma_2+p\alpha\gamma_1/\gamma_2-p/2}$$
$$\times \int_0^{\pi N} v^{-\alpha-q\gamma_1/\gamma_2-q/2+q\alpha\gamma_1/\gamma_2} dv.$$

The procedure of the choice of p and q is exactly the same as above. We arrive at the same condition $p+q=1$ as well as the bound

$$q<(1-\alpha)/(\gamma_1/\gamma_2+1/2-\alpha\gamma_1/\gamma_2),$$

the latter at least for $\alpha\le 1/2$. Finally, all this means that the passage from the trigonometric sum to the Fourier transform is properly estimated. What has to be estimated now is the following integral

$$\int\limits_{\mathbf{T}^2}\Bigg|\sum_{1/2\le|u|\le N+1/2} e^{ix_1u}\int\limits_{|t|\le N^{\gamma_1/\gamma_2+1}|u|^{-\gamma_1/\gamma_2}}(1-|u|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_2t}\,dt$$
$$+\int\limits_{1/2\le|t|\le N+1/2} e^{ix_2t}\,dt\int\limits_{|u|\le N^{\gamma_2/\gamma_1+1}|t|^{-\gamma_2/\gamma_1}}(1-|u|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_1u}\,du\Bigg|\,dx.$$

As mentioned above, the case of more dimensions can be handled inductively similarly to that for the case of partial sums, $\alpha=0$, in [L1]. But it can be seen already from the two-dimensional case itself the type of problems one encounters with passing from trigonometric sums to the Fourier transform in the hyperbolic case; compare this with the proof of Theorem 1.1.

Thus we have to estimate now

$$\int\limits_{N\mathbf{T}^n}\Bigg|\int\limits_{\substack{|x_1|^{\gamma_1}\dots|x_n|^{\gamma_n}\le 1,\\ |x_1|,\dots,|x_{n-1}|\ge 1/2}}(1-|x_1|^{\gamma_1}\dots|x_n|^{\gamma_n})^\alpha e^{iux}\,dx\Bigg|\,du.$$

Of course, all the other combinations of $n-1$ variables separated from zero should be considered as well; they are treated similarly.

Denote the inner integral by $\Psi(u)$; we are interested in the behavior of $\Psi(u)$ for u large. Further, it suffices to consider

$$\Psi(u)=\int\limits_{\substack{x_1^{\gamma_1}\dots x_n^{\gamma_n}\le 1,\\ x_1,\dots,x_{n-1}\ge 1/2,\ x_n\ge 0}}(1-x_1^{\gamma_1}\dots x_n^{\gamma_n})^\alpha\cos(ux)\,dx.$$

Introduce a new variable

$$t=x_1^{\gamma_1/\gamma_n}\dots x_{n-1}^{\gamma_{n-1}/\gamma_n}x_n.$$

We obtain

$$\Psi(u)=\int_0^1(1-t^{\gamma_n})^\alpha dt\int\limits_G x_1^{-\gamma_1/\gamma_n}\dots x_{n-1}^{-\gamma_{n-1}/\gamma_n}$$
$$\times\cos(u_1x_1+\dots+u_{n-1}x_{n-1}+tu_nx_1^{-\gamma_1/\gamma_n}\dots x_{n-1}^{-\gamma_{n-1}/\gamma_n})dx_1\dots dx_{n-1}.$$

Here and in further estimates we denote by the same letter G corresponding domains in $\mathbf{R}_+^{n-1}$. The only thing of importance to us is that the variables are separated from zero and infinity.

Let us introduce new variables

$$v_j = x_j u_j (|u_1|^{\gamma_2} \dots |u_n|^{\gamma_n} |t|^{\gamma_n})^{-1/\gamma}, \quad j = 1, 2, \dots, n-1,$$

with $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. We have

$$\begin{aligned} &t u_n x_1^{-\gamma_1/\gamma_2} \dots x_{n-1}^{-\gamma_{n-1}/\gamma_n} \\ &= v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n} (t^{\gamma_n} |u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{1/\gamma} \end{aligned}$$

and

$$\begin{aligned} &x_1^{-\gamma_1/\gamma_n} \dots x_{n-1}^{-\gamma_{n-1}/\gamma_n} \, dx_1 \dots dx_{n-1} \\ &= t^{n\gamma_n/\gamma - 1} (|u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{n/\gamma} |u_1 \dots u_n|^{-1} v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n} \, dv_1 \dots dv_{n-1}. \end{aligned}$$

By this we obtain

$$\begin{aligned} \Psi(u) &= |u_1 \dots u_n|^{-1} (|u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{n/\gamma} \\ &\times \int_0^1 (1 - t^{\gamma_n})^{\alpha} \, t^{n\gamma_n/\gamma - 1} \, dt \\ &\int_G \cos\Big((t^{\gamma_n} |u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{1/\gamma} (v_1 + \cdots + v_{n-1} + v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n}) \Big) \\ &\quad \times v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n} \, dv_1 \dots dv_{n-1}. \end{aligned}$$

It is convenient to perform one more substitution $t^{\gamma_n/\gamma} \to t$. By this we represent Ψ in the following form

$$\begin{aligned} \Psi(u) &= (\gamma/\gamma_n) |u_1 \dots u_n|^{-1} (|u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{n/\gamma} \\ &\times \int_0^1 (1 - t^{\gamma_n})^{\alpha} \, t^{n-1} \, dt \\ &\int_G \cos\Big(t (|u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{1/\gamma} (v_1 + \cdots + v_{n-1} + v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n}) \Big) \\ &\quad \times v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n} \, dv_1 \dots dv_{n-1}. \end{aligned}$$

Our next task is to consider the inner integral. It can be rewritten in the following form

$$\int_{R_+^{n-1}} \varphi(v_1, \dots, v_{n-1}) e^{itM(v_1 + \cdots + v_{n-1} + v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n})} \, dv_1 \dots dv_{n-1},$$

where φ is an infinitely differentiable function supported on G, and

$$M = (|u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{1/\gamma}.$$

Denoting also $v = (v_1, \dots, v_{n-1})$ and

$$\Phi(v) = v_1 + \cdots + v_{n-1} + v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n},$$

we have to investigate the behavior of the integral

$$Q_n(tM) = \int\limits_{R_+^{n-1}} \varphi(v) e^{itM\Phi(v)} dv.$$

Let us estimate this integral by means of the stationary phase method, more precisely, its multi-dimensional version (see, e.g., [Fr]); namely, for the integer $k \geq 1$ the following asymptotic formula is valid

$$\begin{aligned} Q_n(R) &= (2\pi)^{(n-1)/2} R^{-(n-1)/2} e^{i(R\Phi(v_0)+\theta(v_0))} \\ &\quad \times |\det \Phi''(v_0)|^{-1/2} (\varphi(v_0) + O(R^{-1})) \\ &\quad + R^{-(n-1)/2} e^{iR\Phi(v_0)} \sum_{j=1}^{k-1} a_j R^{-j} + O(R^{-(n-1)/2-k}), \end{aligned} \tag{7.5}$$

where $v_0 = (v_1^0, v_2^0, \ldots, v_{n-1}^0)$ is a stationary point of Φ; Φ'' is the matrix of the second derivatives of Φ; $\theta(v_0)$ is a real number depending on $\det \Phi''(v_0)$; and a_j are some (complex) numbers. Now we have to find stationary points, if existent, and calculate all the parameters in (7.4).

We obtain for $j = 1, 2, \ldots, n-1$

$$\frac{\partial \Phi}{\partial v_j} = 1 - (\gamma_j/\gamma_n) v_1^{-\gamma_1/\gamma_n} \ldots v_{n-1}^{-\gamma_{n-1}/\gamma_n} v_j^{-1},$$

and solving the system of $n-1$ equations $\frac{\partial \Phi}{\partial v_j} = 0$, $j = 1, 2, ..., n-1$, we get a solution

$$v_j^0 = \gamma_j (\gamma_1^{\gamma_1} \ldots \gamma_n^{\gamma_n})^{-1/\gamma}.$$

Let us prove that just this is the unique stationary point. For this we find the value of the determinant of the second derivatives at this point and prove that it is non-zero. We have

$$\frac{\partial^2 \Phi}{\partial v_j \partial v_k} = \begin{cases} \gamma_j \gamma_k \gamma_n^{-2} v_1^{-\gamma_1/\gamma_n} \ldots v_{n-1}^{-\gamma_{n-1}/\gamma_n} v_j^{-1} v_k^{-1}, & j \neq k \\ (\gamma_j/\gamma_n)(\gamma_j/\gamma_n + 1) v_1^{-\gamma_1/\gamma_n} \ldots v_{n-1}^{-\gamma_{n-1}/\gamma_n} v_j^{-2}, & j = k. \end{cases}$$

By this we obtain

$$\begin{aligned} \det \Phi''(v) &= \gamma_n^{-2(n-1)} (v_1^{-\gamma_1/\gamma_n} \ldots v_{n-1}^{-\gamma_{n-1}/\gamma_n})^{n-1} \\ &\times \begin{vmatrix} \gamma_1(\gamma_1+\gamma_n) v_1^{-1} v_1^{-1} & \gamma_1\gamma_2 v_1^{-1} v_2^{-1} & \ldots & \gamma_1 \gamma_{n-1} v_1^{-1} v_{n-1}^{-1} \\ \gamma_2 \gamma_1 v_2^{-1} v_1^{-1} & \gamma_2(\gamma_2+\gamma_n) v_2^{-1} v_2^{-1} & \ldots & \gamma_2 \gamma_{n-1} v_2^{-1} v_{n-1}^{-1} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{n-1}\gamma_1 v_{n-1}^{-1} v_1^{-1} & \gamma_{n-1}\gamma_2 v_{n-1}^{-1} v_2^{-1} & \ldots & \gamma_{n-1}(\gamma_{n-1}+\gamma_n) v_{n-1}^{-1} v_{n-1}^{-1} \end{vmatrix} \\ &= (v_1^{-\gamma_1/\gamma_n} \ldots v_{n-1}^{-\gamma_{n-1}/\gamma_n})^{n-1} \gamma_n^{-2(n-1)} v_1^{-2} \ldots v_{n-1}^{-2} \gamma_1 \ldots \gamma_{n-1} \Delta, \end{aligned}$$

where

$$\Delta = \begin{vmatrix} \gamma_1+\gamma_n & \gamma_2 & \dots & \gamma_{n-1} \\ \gamma_1 & \gamma_2+\gamma_n & \dots & \gamma_{n-1} \\ \dots & \dots & \dots & \dots \\ \gamma_1 & \gamma_2 & \dots & \gamma_{n-1}+\gamma_n \end{vmatrix}.$$

Standard inductive argument shows that

$$\Delta = (\gamma_1 + \dots + \gamma_n)\gamma_n^{n-2} = \gamma\gamma_n^{n-2}.$$

Hence

$$\det \Phi''(v) = \gamma\gamma_1 \dots \gamma_{n-1}\gamma_n^{-n}(v_1^{-\gamma_1/\gamma_n} \dots v_{n-1}^{-\gamma_{n-1}/\gamma_n})^{n-1} v_1^{-2} \dots v_{n-1}^{-2}.$$

We have

$$\det \Phi''(v_0) = \gamma\gamma_1^{-1} \dots \gamma_n^{-1}(\gamma_1^{\gamma_1/\gamma_n} \dots \gamma_{n-1}^{\gamma_{n-1}/\gamma_n}\gamma_n)^{n-1} > 0;$$

therefore v_0 is a stationary point.

Now we get

$$\begin{aligned}\Phi(v_0) &= (\gamma_1^{\gamma_1} \dots \gamma_n^{\gamma_n})^{-1/\gamma}(\gamma_1 + \dots + \gamma_{n-1}) \\ &\quad + (\gamma_1^{\gamma_1/\gamma_n} \dots \gamma_{n-1}^{\gamma_{n-1}/\gamma_n})^{-1}(\gamma_1^{\gamma_1} \dots \gamma_{n-1}^{\gamma_{n-1}}\gamma_n^{\gamma_n})^{(\gamma_1/\gamma_n+\dots+\gamma_{n-1}/\gamma_n)/\gamma} \\ &= \gamma(\gamma_1^{\gamma_1} \dots \gamma_n^{\gamma_n})^{-1/\gamma}.\end{aligned}$$

Observe that

$$\varphi(v_0) = (v_1^0)^{-\gamma_1/\gamma_n} \dots (v_{n-1}^0)^{-\gamma_{n-1}/\gamma_n}$$

$$= \gamma_n(\gamma_1^{\gamma_1} \dots \gamma_n^{\gamma_n})^{-1/\gamma}.$$

Obviously,

$$\int\limits_{\substack{|u_j|\le 1,\\ j=1,2,\dots,n}} |\Psi(u)|\, du = O(1).$$

It remains to estimate

$$\int\limits_{\substack{1\le|u_j|\le \pi N,\\ j=1,2,\dots,n}} |\Psi(u)|\, du.$$

For the leading term in (7.4), we have

$$\begin{aligned}&\int\limits_{\substack{1\le|u_j|\le \pi N,\\ j=1,2,\dots,n}} |u_1 \dots u_n|^{-1}(|u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{n/\gamma}(|u_1|^{\gamma_1} \dots |u_n|^{\gamma_n})^{(n-1)/(2\gamma)} \\ &\qquad \times \left| \int_0^1 (1-t^{\gamma_n})^{\alpha} t^{(n-1)/2} e^{it(|\gamma_1|^{\gamma_1} \dots |\gamma_n|^{\gamma_n} j)^{-1/\gamma}\Phi(v_0)}\, dt \right| du.\end{aligned}$$

The inner integral is estimated as above. For $\alpha < (n-1)/2$, we obtain

$$\begin{aligned}
&\int\limits_{\substack{1\le|u_j|\le\pi N,\\ j=1,2,\dots,n-1}} |u_1\dots u_n|^{-1}(|u_1|^{\gamma_1}\dots|u_n|^{\gamma_n})^{n/\gamma} \\
&\qquad\qquad \times (|u_1|^{\gamma_1}\dots|u_n|^{\gamma_n})^{(n-1)/(2\gamma)}(|u_1|^{\gamma_1}\dots|u_n|^{\gamma_n})^{-(\alpha+1)/\gamma}du \\
&= 2^n\prod_{j=1}^{n}\int\limits_1^{\pi N} u_j^{-1}u_j^{\gamma_j((n-1)/2-\alpha)/\gamma}\,du_j \\
&\qquad\qquad = O\left(\prod_{j=1}^{n} N^{\gamma_j((n-1)/2-\alpha)/\gamma}\right) = O(N^{(n-1)/2-\alpha}).
\end{aligned}$$

Now we see how to handle the other terms in (7.4), including the remainder one. Also we see that for $\alpha > (n-1)/2$, the bounds are better than the leading term of the asymptotics in Theorem 7.2. It remains to mention that for $\alpha = (n-1)/2$, the leading term in (7.4) gives the product of n integrals estimated by (4.12) each, that is,

$$\omega_n \ln^n N + O(\ln^{n-1} N),$$

and the other terms give better bounds. The theorem is proved. □

8 When is the partial sums operator unbounded?

In previous considerations we studied a sequence of operators of taking partial sums depending on some parameter, either scalar or vector. For any individual value of parameter this operator was bounded even in the case when infinitely many harmonics were involved (cf. Section 7), while the sequence of norms grew infinitely. But some situations when even the norm of an individual operator turns out to be infinite are currently known. In both theorems the second parts are proved by using the following result from [Ru1] (see also [Ru2, Th.3.1.3]):

If the operator of taking partial sums with respect to some dilation of a given set is bounded, then this set may be represented as a finite union of co-sets of discrete subgroups of the lattice $\mathbf{Z}^n$.

To get a contradiction with this statement some theorems from Geometric Number Theory are used (see [Ca1, Ca2]).

8.1. The first result is due to Belinsky [Be5] (see also [MP]). Let $l_1, l_2, ..., l_k$, where $k < n$, be a family of linearly independent vectors in $\mathbf{R}^n$. Consider the sets

$$P_j = \{m \in \mathbf{Z}^n : |l_j m| \leq 1\},$$

and set

$$P = \bigcap_{j=1}^{k} P_j$$

and

$$P_0 = \bigcap_{j=1}^{k} \{x \in \mathbf{R}^n : l_j x = 0\}.$$

Theorem 8.1. *The following two statements hold.*

1) *If in* P_0 *there exists a sublattice of* $\mathbf{Z}^n$ *of dimension* $n-k$, *then*

$$||S_{NP}|| \asymp \ln^k N \qquad as \quad N \to \infty.$$

2) *If no one such sublattice exists in* P_0, *then the operator* S_{NP} *is unbounded for each* $N > 0$.

Proof. Let $\alpha_1, ..., \alpha_{n-k}$ be a basis of the sublattice consisting of all the points in P_0 with integer coordinates. Then by [Ca1, Ch.1] there exist points $\alpha_{n-k+1}, ..., \alpha_n$ such that the system $\alpha_1, ..., \alpha_n$ forms a basis of the lattice $\mathbf{Z}^n$. Let us pass from the canonical basis in $\mathbf{Z}^n$ to the basis $\alpha_1, ..., \alpha_n$. Since this map, written φ, is an automorphism of the group $\mathbf{Z}^n$, the function $f(\varphi^{-1})$ also belongs to $C(\mathbf{T}^n)$. By this, the set P is mapped to the set

$$P_\varphi = \bigcap_{j=1}^{k} \{m \in \mathbf{Z}^n : \quad |\gamma_j m| \leq 1\},$$

where vectors $\gamma_j = (\gamma_{j1}, ..., \gamma_{jn})$ are defined by relations $\gamma_{js} = \alpha_s l_j$ for $s = 1, 2, ..., n$. Since for $1 \leq s \leq n-k$ we have $\alpha_s l_j = 0$, the equalities $\gamma_{js} = 0$ are valid for $1 \leq s \leq n-k$. Hence

$$||S_{NP}|| = ||S_{NP_\varphi}|| \asymp \ln^k N.$$

The last relation is (6.1) indeed.

Let us go on to the second assertion of this theorem. Suppose the contrary, that is, S_{NP} is a bounded operator for some N. Then, by the above mentioned Rudin's result, the set NP can be represented as a finite union of co-sets of discrete subgroups of the lattice $\mathbf{Z}^n$. Observe, that if two subgroups are embedded into NP, then their direct sum is also a group embedded into NP. Therefore, NP is a finite union of co-sets of a certain subgroup Λ. Suppose, further, that $\dim \Lambda < n-k$. Then there exists a vector α orthogonal to the vectors $l_1, ..., l_k$ as well as to the basis of Λ. It follows from [Ca2, Ch.1] that there exists an infinite number of points with integer coordinates belonging to NP so that they are concentrated in a neighborhood of the line $x = at$. This contradicts the fact that NP is representable as a finite union of co-sets of Λ. Hence $\dim \Lambda = n-k$, and the linear span of Λ coincides with the orthogonal complement to the subspace spanned by the vectors $l_1, ..., l_k$, that is, with P. Therefore the sublattice Λ is contained in P, and this contradiction completes the proof of the theorem. □

Remark 8.1. If in the first case of Theorem 8.1 an asymptotics is already proved for the set considered (see Section 6), then one gets the asymptotics in Theorem 8.1 as well.

8.2. The next theorem due to Belinsky and Liflyand [BL3], is of the same nature but deals with hyperbolic partial sums (see also Section 7). Let

$$L_j(x) = l_{j1}x_1 + ... + l_{jn}x_n, \qquad j = 1, 2, ..., n,$$

be linear forms with nonsingular coefficient matrix

$$\Lambda = \{l_{jk}\}, \quad 1 \le j, k \le n, \quad \det \Lambda \ne 0,$$

and

$$H = \{x \in \mathbf{R}^n : \prod_{j=1}^{n} |L_j(x)| \le 1\}.$$

We call the matrix Λ *rational* if each row of this matrix consists of integers, possibly up to a common factor. In the contrary case, the matrix is said to be *irrational.*

Theorem 8.2. *The following two statements hold.*

1) *If the matrix Λ is rational, then*

$$||S_{NH}|| \asymp N^{(n-1)/2}.$$

2) *If Λ is irrational, then there exists an integer N_0 such that the operator S_{NH} is unbounded for all $N > N_0$.*

For $n = 2$, Theorem 8.2 was earlier obtained by Belinsky (see [Be3, Be4]). The proof was not at all like that of Theorem 8.2 in [BL3] but essentially two-dimensional and relied on some other results in Number Theory, in particular, an exact value N_0 was indicated. More precisely, in this case NH may be represented as

$$\{(m_1, m_2) : \quad |m_1^2 - \gamma^2 m_2^2| \le \gamma N\}.$$

If γ is irrational, then the operator is unbounded provided $N > N_0 = 2/\sqrt{5}$. There exists an irrational γ such that the operator is bounded as soon as $N < 2/\sqrt{5}$.

Nevertheless, even in this case the second part of Theorem 8.2 cannot be established for sets

$$H_\gamma = \left\{ \prod |L_j|^{\gamma_j} \leq 1 \right\},$$

since no corresponding results in Number Theory are available to treat this more general case. Any new number theory results for H_γ, that is, some special versions of Minkowski's theorem, will immediately lead to certain extensions of Theorem 8.2.

9 Integrability of multiple trigonometric series

A comparatively detailed survey of one-dimensional results on integrability of trigonometric series was given in Section 0. Corresponding multidimensional extensions are closely related to estimates of Lebesgue constants in the same way as for dimension one. The number of such extensions may be compared with the number of the most important one-dimensional results. It is natural, in some sense, since such extensions are proved mostly by repeating the corresponding one-dimensional argument. Nevertheless, it is not always so simple as it may seem, and sometimes peculiarities of the multidimensional case are displayed. The following authors started and then continued such generalizations: Bugrov [Bu1], Telyakovskii [Te1, Te3], Nosenko [N1-N4], Zaderei [Z1,Z3-Z5], Giang and Móricz [Mo,Mo0-Mo3,Mo5-Mo8,GM0,GM1], Marzuk [Mz], C.-P. Chen [Ch1, Ch2, CH], Papp [Pa], and Ram and Bhatia [RB].

We will give more details for the recent result due to Aubertin and Fournier [AF2] which generalizes their own one-dimensional result [AF1]. It is of special interest not only because of its strength but also because of involving some notions not so usual in this context. Then much attention will be paid to the author's results. Here a new approach connected to the Fourier transform is used to the same extent as for simple trigonometric series (see Section 0).

9.1. The result in [AF2] as well as its prototype in [AF1] (see also [BTM3]) is of special interest because of two reasons. First, it involves amalgam spaces introduced by N. Wiener (see, e.g., [Fe] or [FS]). But, moreover, this leads to special strong conditions apparently incomparable with the other ones, since no bounded variation is assumed.

Let

$$\sum_{m\in\mathbf{Z}^n} c(m)e^{imx} \tag{9.1}$$

be a trigonometric series with coefficients $c(m)$ tending to zero at infinity. It is well-known that this is not sufficient for integrability. A natural restriction is that the differences of the coefficients are also small enough. In what follows mixtures of differences in all directions are involved. For each index j with $1 \le j \le n$, let

$$\Delta_j c(m) = c(m) - c(m_1, ..., m_{j-1}, m_j + 1, m_{j+1}, ..., m_n)$$

be the usual forward difference with respect to the jth component. The operators Δ_j commute; let $\Delta = \prod_{j=1}^{n} \Delta_j$. The condition on the sizes of $\Delta c(m)$ involves amalgams of l^1-norms and l^2-norms. For each positive integer j, let $J(j)$ be the set of integers in the interval $[-2^{j-1}, 2^{j-1})$. Given a multi-index k in $\mathbf{Z}^n_+$, let $J(k)$ be the cartesian product of the sets $J(k_j)$. Also let 2^k be the multi-index with components 2^{k_j}. Given two multi-indices k and m, denote the multi-index with components $k_j m_j$ by $\overline{km}$. For each k in $\mathbf{Z}^n_+$, the sets $J(k)+\overline{m2^k}$ with $m \in \mathbf{Z}^n$ are disjoint and cover $\mathbf{Z}^n$.

Given a function d on $\mathbf{Z}^n$, let $||d||_{1,2,2^k}$ be the quantity obtained by combining norms as follows. First compute the l^1-norm of the restriction of d to each set $J(k)+\overline{m2^k}$. This norm depends on the choice of m, and hence defines a function on $\mathbf{Z}^n$. Then compute the l^2-norm of that function. Further, call a set $J(k)+\overline{m2^k}$ a *middle translate* if some component of m is equal to 0; this is equivalent to the set $J(k)+\overline{m2^k}$ having a member with some component equal to 0. Let $||d||'_{1,2,2^k}$ denote the quantity obtained by proceeding as in the definition of $||d||_{1,2,2^k}$ but omitting the middle translates. Then let

$$I = \{1, 2, ..., n\}$$

and

$$||c||_\Delta = \sum_{k:k_j \ge 0, j \in I} ||\Delta c||'_{1,2,2^k}.$$

Given integers p and q with q in the set I, consider the restriction of the function c to the set of multi-indices m with $m_q = p$. Identify this function in the obvious way with a function, $c_{(q,p)}$ say, on $\mathbf{Z}^{n-1}$, and use the complement of $\{q\}$ in I to index $\mathbf{Z}^{n-1}$ in this case. Then form the difference

$$\sigma_{(q,p)} c = c_{(q,p)} - c_{(q,-p)}.$$

Products of these operators $\sigma_{(q,p)}$ with distinct indices q are well defined, and the operators commute. Given a nonempty subset S of I and a multi-index m in $\mathbf{Z}^{|S|}$, let

$$\sigma_{[S,m]} c = \left[\prod_{j \in S} \sigma_{(j,m_j)} \right] c,$$

where $|S|$ denotes the cardinality of S; then $\sigma_{[S,m]}c$ has $n - |S|$ components. When $|S| < n$, define the functional $||\cdot||_\Delta$ on sequences on $\mathbf{Z}^{n-|S|}$ as before, except for replacing n by $n-|S|$. If $|S| = n$, then $\sigma_{[S,m]}c$ is a constant; in this case define $||\sigma_{[S,m]}c||_\Delta$ to be the absolute value of $\sigma_{[S,m]}c$.

Definition 9.1. Call a function, c say, on $\mathbf{Z}^n$ sufficiently symmetric if

$$||c||_\Sigma = \sum_{|S|>0} \sum_{k \in \mathbf{Z}_+^{|S|}} ||\sigma_{[S,2^k]} c||_\Delta < \infty.$$

If c is *fully* even in the sense that $c(m)$ is not affected if m_j is replaced by $-m_j$ for any j, then c automatically satisfies this definition.

Theorem 9.1. *If the coefficients $c(m)$ of a multiple trigonometric series* (9.1) *tend to zero, have a property that $||c||_\Delta < \infty$, and are sufficiently symmetric, then the series represents an integrable function.*

It is shown in [AF2] that multiple cosine series automatically satisfy the symmetry condition while for multiple sine series additional assumptions have to be imposed on - just like in the one-dimensional case.

9.2. Let us recall what is the idea of the approach introduced in [L5] (see Section 0). First, some known conditions for integrability of trigonometric series which were imposed on the sequence of coefficients of a trigonometric series are adapted to the case of integrals. For the classes of functions satisfying these generalized conditions, an asymptotic behavior of the Fourier transform of a function in such class is established. Then, using Theorem 3.2, even somewhat stronger results are derived for trigonometric series. Let us give the precise formulations for several dimensions.

Let

$$I(q,p), \quad 0 \le q \le n, \quad 1 \le p \le \binom{n}{q},$$

be the pth subset from all possible different subsets of I consisting of $n-q$ elements; let

$$I(q,p;s,r), \quad 0 \le s \le n-q, \quad 1 \le r \le \binom{n-q}{s},$$

be the rth one from all possible different subsets of $I(q,p)$ consisting of $n-q-s$ elements. We denote by $\partial_{q,p} f$ the partial derivative of a function f taken with respect to every variable with index from $I(q,p)$. Given a function φ defined on $\mathbf{R}^n_+$, let $\varphi_{s,r}$ denote the odd continuation of φ in each variable with index in $I(q,p;s,r)$.

Theorem 9.2. *Let f be defined on $\mathbf{R}^n_+$; let for each q,p, $1 \le q \le n$, $1 \le p \le \binom{n}{q}$, the functions $\partial_{q,p} f$ be locally absolutely continuous with respect to every variable with index from $I \setminus I(q,p)$ and*

$$\lim_{x_1+\ldots+x_n \to \infty} \partial_{q,p} f = 0.$$

Then for any $y_1, ..., y_n > 0$ and for any set of numbers

$$\{a_j : a_j = 0 \text{ or } \pi/2, j \in I\}$$

we have

$$\int\limits_{\mathbf{R}^n_+} f(x) \prod_{j=1}^{n} \sin(x_j y_j + a_j)\, dx_j$$

$$= (-1)^n f(\pi/(2y_1), ..., \pi/(2y_n)) \prod_{j=1}^{n} (1 - 2a_j/\pi) y_j^{-1} + \theta\gamma(y), \tag{9.2}$$

where $|\theta| \le C$ and

$$\int\limits_{\mathbf{R}^n_+} |\gamma(y)|\, dy$$

$$\le \sum_{q=0}^{n-1} \sum_{p=1}^{\binom{n}{k}} \sum_{s=0}^{n-q} \sum_{r=1}^{\binom{n-q}{s}} \int\limits_{\mathbf{R}^n_+} \Bigg| \int\limits_{\mathbf{R}^{n-q-s}} (\partial_{q,p} f)_{s,r}(x^y_{q,p}) \prod_{j \in I(q,p;s,r)} dx_j/(y_j - x_j) \Bigg|$$

$$\times \prod_{j \in I \setminus I(q,p)} y_j^{-1} \cos a_j \, dy,$$

provided the right-hand side of the last inequality is finite ($x^y_{q,p}$ means that y_j occur on the places corresponding to the indices $j \in I \setminus I(q,p)$).

Corollary 9.1. *Under the assumptions of Theorem 9.2, the asymptotic relation* (9.2) *holds* *provided*

$$\int\limits_{\mathbf{R}_+^n} |\gamma(y)|\,dy$$

$$\le C_b \sum_{q=0}^{n-1}\sum_{p=1}^{\binom{n}{k}} \int\limits_{\mathbf{R}_+^n} \Bigg(\int\limits_{\substack{y_j\le x_j,\\ j\in I(q,p)}} |(\partial_{q,p}f)_{s,r}(x_{q,p}^y)|^b \prod_{j\in I(q,p)} dx_j/y_j \Bigg)^{1/b}$$

$$\times \prod_{j\in I\setminus I(q,p)} y_j^{-1}\cos y_j\,dy$$

is finite for some $b>1$.

Corollary 9.2. *Under the assumptions of Theorem 9.2, the asymptotic relation* (9.2) *holds provided*

$$\int\limits_{\mathbf{R}_+^n} |\gamma(y)|\,dy$$

$$\le \sum_{q=0}^{n-1} \sum_{p=1^{\binom{n}{k}}\int\limits_{\mathbf{R}_+^n}} \operatorname*{ess\,sup}_{\substack{y_j\le x_j,\\ j\in I(q,p)}} |(\partial_{q,p}f)_{s,r}(x_{q,p}^y)|$$

$$\times \prod_{j\in I\setminus I(q,p)} y_j^{-1}\cos a_j\,dy<\infty.$$

Now we are in a position to obtain estimates for trigonometric series. Set

$$C(x)=\sum_{q=0}^{n-1}\sum_{p=1}^{\binom{n}{k}} \Delta_{I(q,p)}c(m)\prod_{j\in I(q,p)}(m_j-x_j)$$

on $[m_1-1,m_1]\times ...\times[m_n-1,m_n]$, where $\Delta_{I(q,p)}c(m)$ denotes the mixed difference of step 1 with respect to every variable with index from $I(q,p)$.

Theorem 9.3. *Let* $m\in\mathbf{Z}_+^n$ *and*

$$\lim_{m_1+...+m_n\to\infty} c(m)=0.$$

Then for any $0<y_1,...,y_n\le\pi$ *and for any set of numbers*

$$\{a_j : a_j=0 \text{ or } \pi/2, j\in I\}$$

we have

$$\sum_{m\in\mathbf{Z}_+^n} c(m)\prod_{j=1}^n \sin(m_jy_j+a_j)$$
$$=(-1)^n C(\pi/(2y_1),...,\pi/(2y_n))\prod_{j=1}^n(1-2a_j/\pi)y_j^{-1}+\theta\gamma(y), \tag{9.3}$$

where $|\theta| \le C$ *and*

$$\int_{\mathbf{T}^n_+} |\gamma(y)|\, dy$$

$$\le \sum_{q=0}^{n-1} \sum_{p=1}^{\binom{n}{k}} \sum_{s=0}^{n-q} \sum_{r=1}^{\binom{n-q}{s}} \sum_{k:k_j>0} \left| \sum_{\mathbf{Z}^{n-q-s}} (\Delta_{q,p}c)_{s,r}(m^k_{q,p}) \prod_{j\in I(q,p;s,r)} (k_j - m_j)^{-1} \right|$$

$$\times \prod_{j\in I\setminus I(q,p)} k_j^{-1} \cos a_j,$$

provided the right-hand side of the last inequality is finite ($m^k_{q,p}$ *means that* k_j *occur on the places corresponding to the indices* $j \in I \setminus I(q,p)$*).*

Here the procedure of odd continuation is applied to sequences.

Corollary 9.3. *Under the assumptions of Theorem 9.3, the asymptotic relation* (9.3) *holds provided*

$$\int_{\mathbf{T}^n_+} |\gamma(y)|\, dy$$

$$\le C_b \sum_{q=0}^{n-1} \sum_{p=1}^{\binom{n}{k}} \sum_{k:k_j>0} \Bigg(\sum_{\substack{k_j \le m_j,\\ j\in I(q,p)}} |(\Delta_{q,p}c)_{s,r}(m^k_{q,p})|^b \prod_{j\in I(q,p)} k_j^{-1} \Bigg)^{1/b}$$

$$\times \prod_{j\in I\setminus I(q,p)} k_j^{-1} \cos a_j$$

is finite for some $b > 1$.

Corollary 9.4. *Under the assumptions of Theorem 9.3, the asymptotic relation* (9.3) *holds provided*

$$\int_{\mathbf{T}^n_+} |\gamma(y)|\, dy$$

$$\le \sum_{q=0}^{n-1} \sum_{p=1}^{\binom{n}{k}} \sum_{k:k_j>0} \sup_{\substack{y_j \le x_j,\\ j\in I(q,p)}} |(\Delta_{q,p}c)_{s,r}(m^k_{q,p})|$$

$$\times \prod_{j\in I\setminus I(q,p)} k_j^{-1} \cos a_j \, dy < \infty.$$

The results in **9.2** are given in a stronger form than the results referred to above; indeed, they can be derived either from Theorem 9.3 or from one of the corollaries by integrating the obtained point-wise estimates.

9.3. In [Ku3], a result similar to that in Corollary 9.4 is given in a very special setting (both are generalizations of the Sidon-Telyakovskii condition). The series (9.1) is considered to be of the form

$$c(0)+\sum_{l=1}^{\infty}c(l)\sum_{m\in(lV\setminus(l-1)V)\cap\mathbf{Z}^n}e^{imx}, \tag{9.4}$$

where V is a polyhedron with rational vertices, star-shaped with respect to the origin, which is an interior point of it, and such that the extensions of all its faces miss the origin.

Theorem 9.4. *If* $\lim\limits_{l\to\infty}c(l)=0$ *and there exists a decreasing numerical sequence* A_l *such that*

$$|c(l)-c(l+1)|\le A_l$$

for all l, *and* $\sum\limits_{l=0}^{\infty}A_l<\infty$, *then* (9.4) *converges almost everywhere on* $\mathbf{T}^n$ *and is a Fourier series, and*

$$\int\limits_{\mathbf{T}^n}\left|c(0)+\sum_{l=1}^{\infty}c(l)\sum_{m\in(lV\setminus(l-1)V)\cap\mathbf{Z}}e^{imx}\right|dx\le C\sum_{l=1}^{\infty}A_l.$$

We mention also recent results by Kuznetsova [Ku5-Ku7]; in [Ku6] integrability conditions for (9.4) are given in the terms of Orlicz spaces.

Let us mention the following multidimensional result; it can be found in [SW, Ch.VII, 6.5(d)]. Suppose that $\sum\limits_k a_ke^{ikx}$ and

$$\sum_{k\ne0}a_k(k_j/|k|)e^{ikx},\qquad j=1,2,...,n,$$

are the Fourier-Stieltjes series of finite measures, then for each homogeneous (of degree r) harmonic polynomial $P_r(x)$, $r=0,1,2,...$, the series

$$\sum_{k\ne0}(P_r(k)/|k|^r)a_ke^{ikx}$$

is the Fourier series of a function in $L^1(\mathbb{T}^n)$. The techniques used to prove this may be found in [S1, Ch.VII].

10 Nikolskii type results

The problem posed by S. M. Nikolskii (estimating the norms of linear means via multipliers; see Section 0) led not only to numerous one-dimensional results but also to various multi-dimensional generalizations. Among them are those due to I. Matveev [Ma], Grishin [Gr], Lebed' [Lb1], Bugrov [Bu1, Bu2], Trigub [T5, T9], Zaderei [Z2].

10.1. Let us start with one result due to Trigub [T9] in which estimates of Lebesgue constants via multipliers differ from those obtained earlier.

Let $\{e_j\}_{j=1}^n$ be the standard basis in $\mathbf{R}^n$, $I = \{1, ..., n\}$, and $q = \sum q_j e_j$ where the q_j are natural numbers ($j \in I$); analogously $h = \sum h_j e_j$ where the h_j are also natural numbers. Set

$$\Delta_{h_j} \lambda_k = \lambda_k - \lambda_{k+h_j e_j}$$

(the difference operator with step-size h_j in the direction e_j) and

$$\Delta_h^q \lambda_k = \Bigl(\prod_{j \in M_0} \Delta_{h_j}^{q_j} \Bigr) \lambda_k$$

(the "mixed" difference in the direction of all axes).

Theorem 10.1. *For every $p \in [1, 2)$ and q, there exists a constant C, depending only on p, q and n, such that*

$$\begin{aligned} &\int_{\mathbf{T}^n} \Bigl| \sum_{-N_j \le k_j \le N_j} \lambda_k e^{ikx} \Bigr|^p dx \\ &\le C \prod_j (N_j + 1)^{(p-2)/2} \sum_{0 \le s_j \le [\log_2 (N_j+1)]} 2^{(1-p/2) \sum_j s_j} \Bigl(\sum_k |\Delta_h^q \lambda_k|^2 \Bigr)^{p/2}, \end{aligned} \tag{10.1}$$

where λ_k is taken to equal 0 for $k_j \notin [-N_j, N_j]$ at least for one j in the sum $\sum_k$, while $h = h(s, N)$ is defined by the following conditions

$$\frac{N_j + 1}{3 \cdot 2^{s_j}} \le h_j \le \frac{5(N_j + 1)}{6 \cdot 2^{s_j}}$$

and

$$\frac{N_j + 1}{3 \cdot 2^{s_j}} \le h_j \le \frac{N_j + 1}{2^{s_j}},$$

for $j = 1, 2, ..., n$, according as $s_j < [\log_2 (N_j + 1)]$ or $s_j = [\log_2 (N_j + 1)]$.

In several corollaries sufficient conditions are given to ensure that the Lebesgue constants have a given rate of growth. This is done in terms of smoothness of a function generating the sequence $\{\lambda_k\}$, namely

$$\lambda_k = \lambda_{N,k} = \lambda(k_1/N_1, ..., k_n/N_n).$$

For one more application of Theorem 10.1, see the proof of Theorem 11.2.

10.2. The following result is a generalization of Theorem 0.1. It was obtained by Zaderei [Z2].

To present it, some additional notation should be introduced. Recall that

$$I = \{1, 2, ..., n\}.$$

Denote by $a_{k_B, l_{I\setminus B}}$ and $a_{l_B - k_B, k_{I\setminus B}}$ the elements of an n-dimensional sequence

$$\{a_k\} = \{a_{k_1, k_2, ..., k_n}\}$$

with indices k_j and $l_j - k_j$, respectively, for $j \in B \subset I$, while for $j \in I \setminus B$ those with indices l_j and k_j, respectively. Let P_k^n denote the set of all trigonometric polynomials of n variables of degree not greater than k. By

$$\sum_{k_j = l_j, j \in B}^{p_j} a_k, \qquad B = \{j_1, j_2, ..., j_s\},$$

denote the sum

$$\sum_{k_{j_1} = l_{j_1}}^{p_{j_1}} \sum_{k_{j_2} = l_{j_2}}^{p_{j_2}} \cdots \sum_{k_{j_s} = l_{j_s}}^{p_{j_s}} a_{k_B, k_{I\setminus B}};$$

obviously, we denote $a_{k_B, k_{I\setminus B}} = a_k$ for the sake of convenience. If any of the upper limits in this sum is smaller than the corresponding lower one, consider it to be zero; for $B = I$ denote this sum simply by $\sum\limits_{k=l}^{p} a_k$.

Let for $j = 1, 2, ..., n$

$$\Delta_1^j a_k = a_k - a_{k_{I\setminus\{j\}}, k_j + 1}$$

and

$$\Delta_{l_j}^j a_k = a_{k_{I\setminus\{j\}}, k_j - l_j} - a_{k_{I\setminus\{j\}}, k_j + l_j}$$

be the first difference and the first symmetric difference, both with respect to k_j, with step 1 and $2l_j$, respectively.

Let further

$$\Delta_1^{i,j} a_k = \Delta_1^i(\Delta_1^j a_k), \qquad \Delta_1^B a_k = \Delta_1^{B\setminus\{j\}}(\Delta_1^j a_k),$$
$$\Delta_1^I a_k = \Delta_1^{I\setminus B}(\Delta_1^B a_k), \qquad \Delta_{l_i, l_j}^{i,j} a_k = \Delta_{l_i}^i(\Delta_{l_j}^j a_k),$$
$$\Delta_{l_B}^B a_k = \Delta_{l_{B\setminus\{j\}}}^{B\setminus\{j\}}(\Delta_{l_j}^j a_k), \qquad \Delta_l^I = \Delta_{l_{I\setminus B}}^{I\setminus B}(\Delta_{l_B}^B a_k),$$

and $\Delta_1^{\emptyset} a_k = \Delta_{l_\emptyset}^{\emptyset} a_k = a_k$. Obviously, operators Δ_1^i and Δ_1^j as well as $\Delta_{l_i}^i$ and $\Delta_{l_j}^j$ are transitive.

Denote by $\prod\limits_{j \in B} x_j$ the product $x_{j_1} x_{j_2} \cdots x_{j_s}$, and set $\prod\limits_{j \in \emptyset} x_j = 1$.

Let in what follows

$$\mu_j = [k_j/3], \quad \nu_j = k_j - \mu_j, \quad j \in I,$$

and

$$h_k(l) = \begin{cases} 1, & l \in P_\mu^n \\ \prod\limits_{j \in G} \frac{\nu_j - l_j}{\nu_j - \mu_j}, & 0 \le l_{I\setminus G} \le \mu_{I\setminus G}, \\ & \mu_G \le l_G \le \nu_G, \quad G \subseteq I \\ 0, & l \in P_k^n \setminus P_\nu^n. \end{cases}$$

The inequality $\mu_G \le l_G \le \nu_G$ naturally denotes the fact that $\mu_j \le l_j \le \nu_j$ for all $j \in G$.

Starting with a sequence a_k, $k \in \mathbf{Z}_+^n$, define 2^n sequences $a^{(B)} = \{a_k^{(B)}\}$ as follows

$$a_k^{(B)} = a_{l_B - k_B, k_{I\setminus B}} h_l(k), \quad B \subset I.$$

In accordance with the introduced notation

$$a_k = \sum_{B\subset I} a^{(B)}_{l_B - k_B, k_{I\setminus B}} = \sum_{B\subset I} a_k h_l(l_B - k_B, k_{I\setminus B}).$$

Set for $B \subset I$ and $G \subset I$, $B \cap G = \emptyset$,

$$\begin{aligned}&\delta^k_{B;G}(a) = \delta^{(k_1,k_2,\dots,k_n)}_{B;G}(a)\\ &:= \sum_{\substack{l_i=1,\\ i\in B}}^{k_i-1} \sum_{\substack{l_j=2,\\ j\in G}}^{k_j-2} \sum_{\substack{l_s=0,\\ s\in I\setminus(B\cup G)}}^{k_s-1} \prod_{i\in B} l_i^{-1} \left| \sum_{\substack{m_j=1,\\ j\in G}}^{[l_j/2]} \left(\Delta^G_{m_G}(\Delta_1^{I\setminus B} a_k\right) \prod_{j\in G} m_j^{-1}\right|.\end{aligned}$$

If $B = \emptyset$, denote $\delta^k_{\emptyset;G}(a) = \tau^k_G(a)$, i.e.,

$$\begin{aligned}&\tau^k_G(a) = \tau^{(k_1,\dots,k_n)}_G\\ &:= \sum_{\substack{l_j=2,\\ j\in G}}^{k_j-2} \sum_{\substack{l_s=0,\\ s\in I\setminus(B\cup G)}}^{k_s-1} \left| \sum_{\substack{m_j=1,\\ j\in G}}^{[l_j/2]} \left(\Delta^G_{m_G}(\Delta_1^{I\setminus B} a_k\right) \prod_{j\in G} m_j^{-1}\right|,\end{aligned}$$

while for $G = \emptyset$, set

$$\delta^k_{B;\emptyset}(a) = \eta^k_B(a) := \sum_{\substack{l_i=1,\\ i\in B}}^{k_i-1} \sum_{\substack{l_s=0,\\ s\in I\setminus(B\cup G)}}^{k_s-1} \prod_{i\in B} l_i^{-1} \left| \Delta_1^{I\setminus B} a_k \right|.$$

Further set

$$q_{l_j,k_j} = \min([l_j/2], [(k_j - l_j)/2]),$$

and

$$\begin{aligned}&\bar\delta^{k;q}_{B;G} =:\\ &\sum_{\substack{l_i=1,\\ i\in B}}^{k_i-1} \sum_{\substack{l_j=2,\\ j\in G}}^{k_j-2} \sum_{\substack{l_s=0,\\ s\in I\setminus(B\cup G)}}^{k_s-1} \prod_{i\in B} l_i^{-1} \left| \sum_{\substack{m_j=1,\\ j\in G}}^{q(l_j,k_j)} \left(\Delta^G_{m_G}(\Delta_1^{I\setminus B} a_k\right) \prod_{j\in G} m_j^{-1}\right|,\\ &\bar\delta^{k;q}_{\emptyset;G}(a) = \bar\tau^{k;q}_G(a), \quad \bar\tau^{k;q}_\emptyset(a) = \tau^k_\emptyset(a), \quad \bar\delta^{k;q}_{B;\emptyset}(a) = \tau^k_B(a).\end{aligned}$$

We are now in a position to formulate the following generalization of Theorem 0.2.

Theorem 10.2. *If a matrix $\lambda = \{\lambda_{N,l}\} = \{\lambda_l\}$, $l, N \in \mathbf{Z}_+^n$, and $\lambda_l = 0$ for $l \in \mathbf{Z}_+^n \setminus P^n_{N-1}$ satisfies the inequality*

$$\sum_{D,G\subset I} \bar\delta^{k;q}_{D;G}(\lambda(D)) \le C,$$

where $\lambda(D)$ *denotes the sequence* $\{\lambda_{k_D-l_D,l_{I\subset D}}\}$, $D\cap G=\phi$ *and* $D\neq I$, *then for convergence of* $L_N^\lambda(f;x)$ *at each point* x *for every continuous function* f *it is necessary and sufficient that* $\lim\limits_{N\to\infty}\lambda_{N,k}=1$ *for every* $k\in\mathbf{Z}_+^n$ *and*

$$\eta_I^N(\lambda(I))=\sum_{l=1}^{N-1}|\lambda_{N-l}|\prod_{j\in I}l_j^{-1}\leq C.$$

Actually, necessity and sufficiency in this theorem may be represented as two separate results, both of certain interest by themselves.

Theorem 10.3. *We have the lower bound*

$$L_N^\lambda\geq C\sum_{\ell=0}^{N-1}|\lambda_{N,\ell}|\prod_{j\in I}(N_j-\ell_j)^{-1}.$$

Proof. Let

$$\Delta^n=\{z=(z_1,z_2,\ldots,z_n):|z_j|=|x_j+iy_j|<1,\ j=1,2,\ldots,n\}$$

be the unit poly-disk. Denote $z^k=z_1^{k_1}\ldots z_n^{k_n}$ and correspondingly $r^k=r_1^{k_1}\ldots r_n^{k_n}$, and let

$$re^{it}=(r_1e^{it_1},r_2e^{it_2},\ldots,r_ne^{it_n}).$$

Let $0\leq r\leq 1$ mean that $0\leq r_j\leq 1$, $j=1,2,\ldots,n$); and let H_1 be the Hardy space of functions $f(z)$ analytic in Δ^n so, that

$$\sup_{0\leq r<1}\int_{\mathbf{T}^n}|f(re^{it})|dt<\infty.$$

The following generalization of Hardy's inequality is true (see, e.g., [Z1, **?**]): if we have

$$\Phi(z)=\sum_{k\in\mathbf{Z}_+^n}b_kz^k\in H_1,$$

then

$$\sum_{k\in\mathbf{Z}_+^n}|b_k|\prod_{j\in I}(k_j+1)^{-1}\leq 2^{-n}\int_{\mathbf{T}^n}|\Phi(e^{it})|dt<\infty.$$

Since

$$P_{2N}(z)=\sum_{k=0}^{2(N-1)}\lambda_{N,k-N+1}z^k\in H_1,$$

we obtain

$$\begin{aligned}
L_N^\lambda &= (2\pi)^{-n} \int\limits_{\mathbf{T}^n} \left| \sum_{k=-N+1}^{N-1} \lambda_{N,k} e^{ikx} \right| dx \\
&= (2\pi)^{-n} \int\limits_{\mathbf{T}^n} \left| \sum_{k=-N+1}^{N-1} \lambda_{N,k} e^{i(k+N-1)x} \right| dx \\
&= (2\pi)^{-n} \int\limits_{\mathbf{T}^n} |P_{2N}(e^{it})| dt \\
&\geq C \sum_{k=0}^{2(N-1)} |\lambda_{N,k-N+1}| \prod_{j\in I} (k_j+1)^{-1} \\
&> C \sum_{k=0}^{N-1} |\lambda_{N,k}| \prod_{j\in I} (N_j-k_j)^{-1}
\end{aligned}$$

which completes the proof. □

Theorem 10.4. *We have the upper bound*

$$L_N^\lambda \leq C \Bigg(\sum_{\substack{D,G\subset I\\ D\neq I}} \overline{\delta}_{D;G}^{N;q}(\lambda(D)) + \eta_I(\lambda(I)) \Bigg).$$

Proof. We will give only an idea of the proof, since actually it is an n-dimensional repeating of the one-dimensional proof. The point that should be stressed is that integrability results are applied to (cf., e.g., Theorem 9.3 from the previous section) in order to obtain the estimate

$$L_N^\lambda \leq C \sum_{B\setminus G\subset I} \sum_{\substack{D\subset B\setminus G,\\ G\subset I}} \sum_{B\supset B\setminus G} \delta_{D;G}^{\infty}(\lambda_{N,k}^{(B)})$$

for $D\cap G=\emptyset$. To complete the proof, the following estimate is needed

$$\sum_{B\setminus G\subset I} \sum_{\substack{D\subset B\setminus G,\\ G\subset I}} \sum_{B\subset B\setminus G} \delta_{D;G}^{\infty}(\lambda_{N,k}^{(B)}) \leq C \sum_{D,G\subset I} \overline{\delta}_{D;G}^{N;q}(\lambda(D)),$$

which, in turn, is based on the estimate

$$\delta_{D;G}^{\infty}(a) \leq C \sum_{\substack{k_i=1,\\ i\in D}}^{\infty} \sum_{\substack{k_j=1,\\ j\in G}}^{\infty} \sum_{\substack{k_s=0,\\ s\in I\setminus(D\cup G)}}^{\infty} \prod_{i\in D} k_i^{-1} \prod_{j\in G} k_j |\Delta_1^G(\Delta_1^{I\setminus D} a_k)|$$

for any sequence $\{a_k\}$, $k\in \mathbf{Z}_+^n$, and $D\subset I$, $G\subset I$, $D\cap G=\emptyset$.

Both estimates are very technical and no new tools are needed as compared with the one-dimensional prototype [Te]. □

The results given here are, in our opinion, representative for the topic considered. Note that in many cases it is very difficult to compare such types of multi-dimensional results. Sometimes it is merely a problem of complicated notation, but it also occurs that they are essentially incomparable.

11 More results

In this section some results concerning Lebesgue constants are collected which are not, at least explicitly, in the context of Fourier transforms. **11.1.** The following result obtained by

A. Yudin and V. Yudin [YY1] (see Theorem 11.1 below) is closely connected to the result of Podkorytov given above in Theorem 1.2. We mention that Theorem 11.1 was elaborated for estimates from above of the Lebesgue constants of hyperbolic partial sums (see Section 7). It turned out that Theorem 1.2 is also well adjusted to this (Podkorytov's private communication).

Let $U \subset \mathbf{Z}^n$ be a bounded set and $t \in \mathbf{Z}^n$. Set

$$U_t = \{k \in \mathbf{Z}^n : k - t \in U\}$$

and

$$\omega(t, U) = 2|U| - |U \cap U_t| - |U \cap U_{-t}|,$$

where $|U|$ denotes the number of points in U.

Theorem 11.1. *Let numbers $L_1 \le L_2$ be such that*

$$\omega(he_j, U) \le L_1 h \qquad \textit{and} \qquad \omega(he_r, U) \le L_2 h$$

for some natural numbers $r, j \in I$, where $r \ne j$, and every natural number h. Then

$$||S_U|| \le (1/2)(L_1/2)^{1/2} \log_2(L_2/L_1) + (3/(2 - \sqrt{2}))L_1^{1/2}.$$

11.2. The Lebesgue constants of step hyperbolic crosses have been considered in many papers, together with various applications of such estimates. These problems were discussed by Temlyakov [Tm1, Tm2], E. Galeev [Ga1, Ga2], and E. Belinsky (see, e.g., [Be8]). For example, the following was proved by Belinsky in [Be6].

Let H_N be defined as

$$H_N = \bigcup \{m \in \mathbf{Z}^n : 2^{s_j} \le |m_j| < 2^{s_j+1}\}$$

for $s \in \mathbf{Z}^n_+$ such that $0 \le s_1 + ... + s_n \le N$, and $N = 1, 2, ...$

Theorem 11.2. *The following ordinal estimate holds*

$$||S_{H_N}|| \asymp N^{n+(n-1)/2}$$

as $N \to \infty$.

Proof. Observe, that no one exponent is considered if at least one coordinate is zero. As in Section 7, the operator of taking such series of smaller dimension is bounded and thus has no effect on the final estimate.

To obtain the estimate from above, make use of Theorem 10.1. Here all $N_j = 2^N$, $j = 1, 2, ..., n$, and $\lambda_k = 1$ or 0 according to whether k is within or outside the step hyperbolic cross, respectively. This cross consists of at most N^{n-1} rectangular parallelepipeds. It is easy to see that the mixed difference Δ_h does not vanish only in the case when k is in a neighborhood of the vertex of the parallelepiped; namely, this neighborhood is the n-dimensional parallelepiped with length $2h_j$, $j = 1, 2, ..., n$, in the jth direction. Hence the total number of points at which the mixed difference does not vanish, for a given s, does not exceed

$$2^n \prod_{j=1}^{n} 2^N 2^{-s_j} N^{n-1}.$$

Therefore, the right-hand side of (10.1) admits the following estimate, up to some constant,

$$2^{-Nn/2} \sum_{0 \le s_j \le N+1} 2^{\sum s_j/2} 2^{n/2} 2^{Nn/2} 2^{-\sum s_j/2} N^{(n-1)/2} \le CN^{n+(n-1)/2}.$$

The estimate from above is proved.

We now come to the estimate from below. Recall that $||S_{H_N}||$ is an integral (0.4) with corresponding Dirichlet kernel, namely, the kernel

$$\sum_{\substack{0 \le s_1 + \dots \\ + s_n \le N}} \prod_{j=1}^{n} \frac{\sin(2^{s_j} - 1/2)x_j - \sin(2^{s_j+1} - 1/2)x_j}{2\sin(x_j/2)}.$$

Since on $\mathbf{T}$ the ratio $t/(2\sin(t/2))$ is uniformly bounded, we will estimate the following integral with the kernel somewhat different from that given above

$$\int_{\mathbf{T}^n} \left| \sum_{\substack{0 \le s_1 + \dots \\ + s_n \le N}} \prod_{j=1}^{n} \frac{\sin(2^{s_j} - 1/2)x_j - \sin(2^{s_j+1} - 1/2)x_j}{x_j} \right| dx.$$

Denote

$$\varphi_N(x) = \sum_{\substack{0 \le s_1 + \dots \\ + s_n \le N}} \prod_{j=1}^{n} \frac{\sin(2^{s_j} - 1/2)x_j - \sin(2^{s_j+1} - 1/2)x_j}{x_j},$$

$$\Delta^1 \varphi_N(x) = \varphi_N(x) - \varphi_{N+1}(x),$$

and

$$\Delta^m \varphi_N = \Delta(\Delta^{m-1} \varphi_N).$$

We will continue the proof by induction on dimension. We wish to prove a slightly more general result, namely, that for any integer $m \ge 0$ we have

$$\int_{\mathbf{T}^n} |\Delta^m \varphi_N(x)| \, dx \ge C_{m,n} N^{n+(n-1)/2}.$$

In particular, the estimate needed is the one for which $m = 0$. Starting with $n = 1$, we obtain

$$\begin{aligned}
\int_0^\pi |\Delta^m \varphi_N(x_1)| \, dx_1 &= \int_0^\pi \left| \sum_{p=0}^{m} (-1)^p \binom{m}{p} \right. \\
&\qquad \times \left. \sum_{0 \le s_1 \le N+p} \frac{\sin(2^{s_1} - 1/2)x_1 - \sin(2^{s_1+1} - 1/2)x_1}{x_1} \right| dx_1 \\
&= \int_0^\pi \left| \sum_{p=0}^{m} (-1)^p \binom{m}{p} \frac{\sin(2^{N+p+1} - 1/2)x_1 - \sin(x_1/2)}{x_1} \right| dx_1.
\end{aligned}$$

Since

$$\sum_{p=0}^{m}(-1)^p\binom{m}{p}=0,$$

we have

$$\int_0^\pi |\Delta^m\varphi_N(x_1)|\,dx_1=\int_0^\pi\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}x_1^{-1}\sin(2^{N+p+1}-1/2)x_1\right|dx_1$$

$$\geq C\int_0^\pi\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}x_1^{-1}\sin 2^{N+p+1}x_1\right|dx_1-C_m.$$

Substituting the variable $2^N x_1\to x_1$, we obtain

$$\int_0^{2^N\pi}\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\sin 2^{p+1}x_1\right|x_1^{-1}\,dx_1$$

$$\geq\int_\pi^{2^N\pi}\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\sin 2^{p+1}x_1\right|x_1^{-1}\,dx_1$$

$$\geq\sum_{k=1}^{2^N-1}(k+1)^{-1}\int_{k\pi}^{(k+1)\pi}\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\sin 2^{p+1}x_1\right|dx_1$$

$$=\int_0^\pi\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\sin 2^{p+1}x_1\right|dx_1\sum_{k=1}^{2^N-1}(k+1)^{-1}.$$

The right-hand side is equivalent to N, and the one-dimensional case is proved.

Suppose now that our assertion is true for any dimension not exceeding n. Then we have

$$\int_{\mathbf{T}^{n+1}}\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\varphi_{N+p}(x_1,x_2,...,x_{n+1})\right|dx_1\,dx$$

$$=\int_{\mathbf{T}^n}2\int_0^\pi\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\sum_{0\leq s_1\leq N+p}\frac{\sin(2^{s_1}-1/2)x_1-\sin(2^{s_1+1}-1/2)x_1}{x_1}\right.$$

$$\left.\times\,\varphi_{N+p-s_1}(x_2,...,x_{n+1})\right|dx_1\,dx,$$

where obviously $dx=dx_2...dx_{n+1}$. Using the known estimate for the Lebesgue constants, we represent the last integral in the form

$$\int_{\mathbf{T}^n}\int_0^\pi\left|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\sum_{0\leq s_1\leq N+p}\frac{\sin 2^{s_1}x_1-\sin 2^{s_1+1}x_1}{x_1}\right.$$

$$\left.\times\,\varphi_{N+p-s_1}(x_2,...,x_{n+1})\right|dx_1\,dx$$

$$+\,O(N^{n+1+(n-1)/2}).$$

Splitting the inner sum, we obtain the difference of the leading term and the remainder term as follows:

$$\int_{\mathbf{T}^n} \sum_{k=0}^{N-1} \int_{\pi/2^{k+1}}^{\pi/2^k} \left| \sum_{p=0}^{m} (-1)^p \binom{m}{p} \right.$$
$$\left. \times \sum_{k \le s_1 \le N+p} \frac{\sin 2^{s_1} x_1 - \sin 2^{s_1+1} x_1}{x_1} \varphi_{N+p-s_1}(x_2, ..., x_{n+1}) \right| dx_1\, dx$$
$$- \int_{\mathbf{T}^n} \sum_{k=0}^{N-1} \int_{\pi/2^{k+1}}^{\pi/2^k} \left| \sum_{p=0}^{m} (-1)^p \binom{m}{p} \right.$$
$$\left. \times \sum_{0 \le s_1 < k} \frac{\sin 2^{s_1} x_1 - \sin 2^{s_1+1} x_1}{x_1} \varphi_{N+p-s_1}(x_2, ..., x_{n+1}) \right| dx_1\, dx.$$

Using the estimate from above already proved, we obtain the following bound for the remainder term

$$\sum_{p=0}^{m} \binom{m}{p} \sum_{k=0}^{N-1} \pi 2^{-k-1} \sum_{0 \le s_1 < k} 2^{s_1} (N + p - s_1)^{n+(n-1)/2} = O(N^{n+1+(n-1)/2}).$$

Substituting the variable in the leading term $2^k x_1 \to x_1$, we transform it to be of the form

$$\sum_{k=0}^{N-1} \int_{\mathbf{T}^n} \int_{\pi/2}^{\pi} \left| \sum_{p=0}^{m} (-1)^p \binom{m}{p} \sum_{k \le s_1 \le N+p} \frac{\sin 2^{s_1-k} x_1 - \sin 2^{s_1+1-k} x_1}{x_1} \right.$$
$$\left. \times \varphi_{N+p-s_1}(x_2, ..., x_{n+1}) \right| dx_1\, dx.$$

Changing the indices $s_1 - k \to s_1$ and $N - 1 - k \to k$, and estimating x_1 roughly, we arrive at the integral

$$\sum_{k=0}^{N-1} \int_{\mathbf{T}^n} \int_{\pi/2}^{\pi} \left| \sum_{p=0}^{m} (-1)^p \binom{m}{p} \sum_{0 \le s_1 \le p+k+1} (\sin 2^{s_1} x_1 - \sin 2^{s_1+1} x_1) \right.$$
$$\left. \times \varphi_{k+p-s_1+1}(x_2, ..., x_{n+1}) \right| dx_1\, dx.$$

For this value, consider again the difference of the leading term and the remainder term

$$\sum_{k=0}^{N-1}\int_{\mathbf{T}^n}\int_{\pi/2}^{\pi}\Big|\sum_{0\le s_1\le k+1}(\sin 2^{s_1}x_1-\sin 2^{s_1+1}x_1)$$
$$\times\sum_{p=0}^{m}(-1)^p\binom{m}{p}\varphi_{k+p-s_1+1}(x_2,...,x_{n+1})\Big|\,dx_1\,dx$$
$$-\sum_{k=0}^{N-1}\int_{\mathbf{T}^n}\int_{\pi/2}^{\pi}\Big|\sum_{p=0}^{m}(-1)^p\binom{m}{p}\sum_{k+1<s_1\le p+k+1}(\sin 2^{s_1}x_1-\sin 2^{s_1+1}x_1)$$
$$\times\varphi_{k+p-s_1+1}(x_2,...,x_{n+1})\Big|\,dx_1\,dx.$$

In view of the obtained estimate from above, the remainder term is estimated by $O(N^{n+1+(n-1)/2})$. Apply the Abel transform to the inner sum in the leading term. We obtain

$$\sum_{k=0}^{N-1}\int_{\mathbf{T}^n}\int_{\pi/2}^{\pi}\Big|\sum_{0\le s_1\le k+1}(\sin 2^{s_1}x_1-\sin 2^{s_1+1}x_1)$$
$$\times\,\Delta^m\varphi_{k-s_1+1}(x_2,...,x_{n+1})\Big|\,dx_1\,dx$$
$$\ge\sum_{k=0}^{N-1}\int_{\mathbf{T}^n}\int_{\pi/2}^{\pi}\Big|\sum_{0\le s_1\le k}\sin 2^{s_1}x_1\Delta^{m+1}\varphi_{k-s_1+1}(x_2,...,x_{n+1})\Big|\,dx_1\,dx$$
$$+\,O(N^{n+1+(n-1)/2}).$$

Consider now the integral in x_1. We have the lacunary trigonometric polynomial within the signs of absolute value. Hence (see [Zg, Ch.V, §6]) the right-hand side in the last inequality is greater, up to a constant, than

$$\sum_{k=0}^{N-1}\int_{\mathbf{T}^n}\Big(\sum_{0\le s_1\le k}\Big|\Delta^{m+1}\varphi_{k-s_1+1}(x_2,...,x_{n+1})\Big|^2\Big)^{1/2}dx.$$

In view of the generalized Minkowski's inequality, this is greater than

$$\sum_{k=0}^{N-1}\Big(\sum_{0\le s_1\le k}\Big(\int_{\mathbf{T}^n}\Big|\Delta^{m+1}\varphi_{s_1+1}(x_2,...,x_{n+1})\Big|\,dx\Big)^2\Big)^{1/2}.$$

Using the inductive hypothesis, we get the estimate

$$\sum_{k=0}^{N-1}\Big(\sum_{0\le s_1\le k}s_1^{2(n+(n-1)/2)}\Big)^{1/2}$$

which is equivalent to $N^{n+1+n/2}$. This completes the proof. □

11.3. We now have to introduce some new notation to formulate certain results by Dyachenko. Let A_2 be the class of bounded sets $U \subset \mathbf{Z}^n$ such that if $m \in U$, then

$$\mathbf{Z}^n \bigcap \prod_{j=1}^{n} [\min(m_j, 0), \max(m_j, 0)] \subset U,$$

and let us define A_1 by

$$A_1 = \{U \bigcap (0, \infty)^n, \quad \text{where} \quad U \in A_2\}.$$

We also define M_1 as the class of n-dimensional sequences

$$a = \{a_m\} = \{a_{m_1,...,m_n}\},$$

where $1 \le m_1, ..., m_n < \infty$, such that $1 \le k_j \le m_j$ implies that $a_k \ge a_m \ge 0$. Set also

$$\prod(x) = \prod_{j=1}^{n} (|x_j| + 1),$$

and it is then possible to give the following assertions.

Theorem 11.3. *Given $U \in A_1$ or $U \in A_2$ and a number $p \in [1, 2n/(n+1))$, then there holds the estimate*

$$||S_U||_{L^p} \le C_{p,n} \max_{m \in U} \Big(\prod(m) \Big)^{(1-1/n)/2}.$$

We note that Theorem 11.3 yields the upper bound in Theorem 7.1 in the case **1)** for $\alpha = 0$ and $\gamma_1 = ... = \gamma_n = 1$, that is, for the Lebesgue constants of hyperbolically symmetric partial sums.

Corollary 11.1. *The following inequalities are satisfied under the hypotheses of Theorem 11.3:*

$$||S_U||_{L^p} \le C_{p,n} |U|^{(1-1/n)/2}$$

and

$$||S_U||_{L^p} \le C_{p,n} \bigg(\sum_{m \in U} \Big(\prod(m) \Big)^{-2/(n+1)} \bigg)^{(1+1/n)/2}.$$

The first inequality of Corollary 11.1 was proved in [Dy2] for $p = 1$ and $U \in A_1$ with the constant $C_{1,n} = 50n^3$. In [Dy1] a similar estimate was obtained with an additional logarithmic factor. Some other estimates for $p > 1$ as well as some open problems can be found in the survey [Dy3, Sect.3].

We mention also Ustina's results on the Lebesgue constants of the two-dimensional Hausdorff method (see [U]) - this is a partial extension of the one-dimensional result from [LN] to the case of two dimensions (cf. also Proposition 5.1).

One more point of interest closely related to our subject is point-wise behavior of spherical Dirichlet kernels

$$\sum_{|k| \le N} e^{ikx}.$$

Such results depend on very delicate number theory techniques and go back to investigations of Walfisch and Landau. One can find a kind of a brief survey of these questions in [Dy3]. We mention especially the paper by K. I. Babenko [Ba4] and a recent paper by A. Yudin [YA].

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