# ASYMPTOTICS OF THE EULERIAN NUMBERS REVISITED: A LARGE DEVIATION ANALYSIS

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ABSTRACT. Using the Saddle point method and multiseries expansions, we obtain from the generating function of the Eulerian numbers $A_{n,k}$ and Cauchy's integral formula, asymptotic results in non-central region. In the region $k = n - n^{\alpha}, \quad 1 > \alpha > 1/2$, we analyze the dependence of $A_{n,k}$ on $\alpha$. This paper fits within the framework of Analytic Combinatorics.

## 1. INTRODUCTION

The Eulerian numbers $A_{n,k}$ have been the object of renewed interest recently (see, for instance, Janson [10]). They are defined by the recurrence

$$A_{n+1,k} = (n-k+2)A_{n,k-1} + kA_{n,k}, \quad 0 \le k \le n.$$

The initial conditions vary in the litterature. We choose here $A_{0,0} = 0, A_{0,1} = 1$ (This is used, for instance, in Bender [2] and Flajolet and Sedgewick [6], ch.$VIII$). The Eulerian numbers correspond, for instance, to runs in permutations. Let us mention OEIS [1], A008292, which stores these numbers and gives many references to the litterature. The bivariate generating function (exponential in $z$, ordinary in $w$) is given by

$$g(z,w) := \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{A_{n,k}}{n!} z^n w^k = \frac{w(1-w)}{e^{(w-1)z} - w}.$$

We have

$$\sum_{k=0}^{\infty} \frac{A_{n,k}}{n!} = 1,$$

hence we can define a random variable (RV) $J_n$ (this corresponds to the number of runs in a random permutation) such that

$$\mathbb{P}[J_n = k] = \frac{A_{n,k}}{n!}.$$

From Flajolet and Sedgewick [6], ch.$IX$, we know that the roots of the denominator are

$$h_j(w) = f(w)^{-1} + \frac{2\mathbf{i}j\pi}{w-1}, j \in \mathbb{Z},$$

with

$$f(w) = \frac{w-1}{\ln(w)}.$$

As $w \to 1$, $f(w)^{-1}$ is close to 1, whereas the other poles $h_j(w)$ with $j \neq 0$ escape to infinity. This fact is consistent with the limit form $g(z,1) = (1-z)^{-1}$ which has only one simple pole at 1. If one restricts $w$ to $|w| \le 2$, there is clearly at most one root of the denominator in $|z| \le 2$, given by $f(w)^{-1}$. Thus we have for $w$ close enough to 1,

$$g(z,w) = \frac{1}{f(w)^{-1} - z} + R(z,w)$$

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with $R(z,w)$ analytic in $|z|\le 2$, and

$$[z^n]g(z,w) = f(w)^{n+1} + \mathcal{O}(2^{-n}).$$

Notice that $f(w)$ does not correspond to a discrete RV, but if we set $w=e^t$, $f(e^t)^{n+1}$ corresponds to a sum of $n+1$ independent RV uniformly distributed on $[0,1]$. (see, for instance, Tanny [14]). In the rest of this paper, we set $m:=n+1, \mu:=\sqrt{m}$. The corresponding mean and variance are given by

$$M = \frac{m}{2},$$
$$\sigma^2 = \frac{m}{12}.$$

The generating function of the mean and second moment are given respectively by

$$\frac{1}{2(1-z)^2},$$
$$\frac{z+2}{6(1-z)^3}.$$

They are obtained either from $M$ and $\sigma^2$ or, more simply, by setting $w=e^t$ in $\frac{1}{f(w)^{-1}-z}$ and differentiating wrt $t$.

Note that the exact mean and second moment generating functions are derived from $g(z,w)$ as

$$\frac{2-2z+z^2}{2(1-z)^2},$$
$$\frac{6-12z+15z^2-7z^3+z^4}{6(1-z)^3}.$$

Of course, asymptotically (by classical singularity analysis), exact and asymptotic moments are the same.

- In the central region $k=M+x\sigma, x=\mathcal{O}(1)$, $J_n$ is asymptotically normal. This has first been proved by David an Barton [4]. Without being exhaustive (a very complete bibliography can be found in Janson [10]), let us also mention Bender [2], Carlitz et al. [3], Tanny [14]. The first two terms of a correction were given in Siraždinov [15] and Nicolas [13]. A complete analysis is given in Gawronski and Neuschel [7]:

  There exists polynomials $q_\nu, \nu\ge 1$, such that, for any $\ell \ge 0$ as $n\to\infty$, uniformly for all $k\in\mathbb{Z}$

$$\frac{A_{n,k}}{n!} = \sqrt{\frac{6}{\pi m}}e^{-x^2/2}\left(1+\sum_{\nu=1}^{\ell}\frac{q_\nu(x)}{m^\nu}\right) + \mathcal{O}\left(m^{-\ell-3/2}\right),$$

(1) $$q_\nu(x) = 12^\nu \sum H_{2\nu+2s}(x)6^s \prod_{j=1}^{\nu}\frac{1}{k_j!}\left(\frac{B_{2j+2}}{(j+1)(2j+2)!}\right)^{k_j},$$

$$x = (k-\frac{m}{2})\sqrt{\frac{12}{m}}$$

  summing over all non-negative integers $(k_1,\dots,k_\nu)$ with $k_1+2k_2+\dots+\nu k_\nu = \nu$ and letting $s = k_1+k_2+\dots+k_\nu$. $B_k$ are the Bernoulli numbers and the Hermite polynomials satisfy

$$H_j(x) = (-1)^j e^{x^2/2}\frac{d^j}{dx^j}e^{-x^2/2}.$$

  A very simple proof is given in Janson [10].
- As far as the large deviation is concerned, let us mention Bender [2], Hwang [9]. Esseen [5] improves Bender's result as follows:

Let $a := k/m$ uniformly in all $0 < k < m$. Let $t(a)$ be the solution of

$$a = \frac{e^{t(a)}}{e^{t(a)}-1} - \frac{1}{t(a)}.$$

Set

$$\delta(a) = \frac{e^{t(a)}-1}{t(a)e^{at(a)}},$$
$$\sigma^2(a) = \frac{1}{t(a)^2} - \frac{e^{t(a)}}{(e^{t(a)}-1)^2}.$$

Then

$$\frac{A_{n,k}}{n!} = \frac{\delta(a)^m}{\sqrt{2\pi m}\sigma(a)}(1+\mathcal{O}(m^{-1})).$$

As noted by Esseen, further terms can be obtained in this asymptotic.

All these papers use the solution $\rho$ of

$$mwf'(w) - kf(w) = 0 \tag{2}$$

which actually corresponds to the Saddle point of the Saddle point method (see Sec.2). In this paper, we are interested in the extreme large deviation case $k = m - m^{\alpha}, 1/2 < \alpha < 1$. (the choice of this range is justified in Sec.3). This range was already the object of our analysis of Stirling numbers of first and second kind (see Louchard [11], [12]).

Let us summarize the motivation of this paper:

- Previous papers simply use $\rho$ as the solution of (2). They don't compute the detailed dependence of $\rho$ on $\alpha$ , neither the precise behaviour of functions of $\rho$ they use.
- We will use multiseries expansions: multiseries are in effect power series (in which the powers may be non-integral but must tend to infinity) and the variables are elements of a scale. The scale is a set of variables of increasing order. The series is computed in terms of the variable of maximum order, the coefficients of which are given in terms of the next-to-maximum order, etc. This is more precise than mixing different terms.

Our work fits within the framework of Analytic Combinatorics.

In Sec.2, we revisit the asymptotic expansion in the central region and in Sec.3, we analyze the non-central region $k = m - m^{\alpha}, \quad \alpha > 1/2, \quad \alpha$ is chosen such that $m^{\alpha}$ is integer. We use Cauchy's integral formula and the Saddle point method. Sec.4 provides a justification of the Saddle point technique we use here.

## 2. Central region

In this section, as a warm-up, we rederive the first terms of the asymptotics (1). We use the Saddle point technique (for a good introduction to this method, see Flajolet and Sedgewick [6], ch.$VIII$). Let $\rho$ be the Saddle point and $\Omega$ the circle $\rho e^{\mathbf{i}\theta}$. By Cauchy's theorem [1],

$$\begin{aligned}\frac{A_{n,k}}{n!} &\sim \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{f(z)^m}{z^{k+1}}dz\\ &= \frac{1}{\rho^k}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{\mathbf{i}\theta})^m e^{-k\mathbf{i}\theta}d\theta\\ &= \frac{1}{\rho^k}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{m\ln(f(\rho e^{\mathbf{i}\theta}))-k\mathbf{i}\theta}d\theta\\ &= \frac{1}{\rho^k}\frac{f(\rho)^m}{2\pi}\int_{-\pi}^{\pi}\exp\left[m\left\{-\frac{1}{2}\kappa_2\theta^2-\frac{\mathbf{i}}{6}\kappa_3\theta^3+\dots\right\}\right]d\theta, \qquad (3)\end{aligned}$$

$$\kappa_i(\rho) = \left(\frac{\partial}{\partial u}\right)^i \ln(f(\rho e^u))\Big|_{u=0}. \qquad (4)$$

See Good [8] for a neat description of this technique.

Let us now turn to the Saddle point computation. We let $\rho$ denote the root (of smallest modulus) of (2) with $k = M + x\sigma$. This amounts to

$$\left[w\left(\frac{1}{\ln(w)}-\frac{w-1}{\ln(w)^2 w}\right)-\frac{w-1}{2\ln(w)}\right]\mu^2-\frac{x3^{1/2}(w-1)}{6\ln(w)}\mu = 0.$$

The solution is given by the asymptotic expansion ($\mu\to\infty$)

$$\rho := 1+\frac{2x3^{1/2}}{\mu}+\frac{6x^2}{\mu^2}+\frac{22x^3 3^{1/2}}{5\mu^3}+\frac{42x^4}{5\mu^4}+\mathcal{O}\left(\frac{1}{\mu^5}\right).$$

In the sequel, we will only give the first terms of our expansions (of course Maple knows much more). This leads to

$$\begin{aligned}T_1 &= -k\ln(\rho)) = -x3^{1/2}\mu - x^2-\frac{x^3 3^{1/2}}{5\mu}-\frac{x^4}{5\mu^2}+\mathcal{O}\left(\frac{1}{\mu^3}\right),\\ T_2 &= \mu^2\ln(f(\rho)) = x3^{1/2}\mu+\frac{x^2}{2}+\frac{x^3 3^{1/2}}{5\mu}+\frac{3x^4}{20\mu^2}+\mathcal{O}\left(\frac{1}{\mu^3}\right),\\ T_1+T_2 &= -\frac{x^2}{2}-\frac{x^4}{20\mu^2}-\frac{11x^6}{1050\mu^4}+\mathcal{O}\left(\frac{1}{\mu^6}\right),\\ T_3 &= \exp\left(T_1+T_2+\frac{x^2}{2}\right) = 1-\frac{x^4}{20\mu^2}+\frac{-11/1050x^6+x^8/800}{\mu^4}+\mathcal{O}\left(\frac{1}{\mu^6}\right).\end{aligned}$$

Now we turn to the integral in (3). The first $\kappa_i$ are given by

$$\begin{aligned}\kappa[2] &= \frac{1}{12}-\frac{x^2}{20\mu^2}+\frac{2x^4}{525\mu^4}+\frac{2x^6}{2625\mu^6}+\mathcal{O}\left(\frac{1}{\mu^8}\right),\\ \kappa[3] &= -\frac{x3^{1/2}}{60\mu}+\frac{79x^3 3^{1/2}}{6300\mu^3}+\mathcal{O}\left(\frac{1}{\mu^5}\right),\\ \kappa[4] &= -\frac{1}{120}+\frac{x^2}{42\mu^2}+\mathcal{O}\left(\frac{1}{\mu^4}\right),\end{aligned}$$

and similar expressions for the next $\kappa_i$ that we don't detail here. We proceed as in Flajolet and Sedgewick [6], ch.$VIII$. Let us choose a splitting value $\theta_0$ such that $m\kappa_2\theta_0^2\to\infty, m\kappa_3\theta_0^3\to 0, n\to\infty$. For instance, we can use $\theta_0 = \mu^{-1/2}$. We must prove that the integral

$$K_{m,k} = \int_{\theta_0}^{2\pi-\theta_0} e^{m\ln(f(\rho e^{\mathbf{i}\theta}))-k\mathbf{i}\theta}d\theta$$

[1] here and in the sequel, $a_n\sim b_n$ means $\frac{a_n}{b_n}\to 1, n\to\infty$

is such that $|K_{m,k}|$ is exponentially small. This is done in the Appendix.

Now we use the classical trick of setting

$$m\left[-\kappa_2\theta^2/2!+\sum_{l=3}^{\infty}\kappa_\ell(\mathbf{i}\theta)^\ell/\ell!\right]=-u^2/2.$$

Computing $\theta$ as a series in $u$, this gives, by Lagrange's inversion,

$$\theta=\sum_1^6 u^i a[i]/\mu=3^{1/2}\left[u\left(2+\frac{3x^2}{5\mu^2}+\mathcal{O}\left(\frac{1}{\mu^4}\right)\right)+u^2\left(\frac{2\mathbf{i}x}{5\mu^2}+\frac{94\mathbf{i}x^3}{525\mu^4}+\mathcal{O}\left(\frac{1}{\mu^6}\right)\right)+\mathcal{O}(u^3)\right]\Big/\mu.$$

This expansion is valid in the dominant integration domain

$$|u|\le\frac{\mu}{a_1}\theta_0=\mu^{1/2}.$$

Setting $d\theta=\frac{d\theta}{du}du$, we integrate on $u=[-\infty..\infty]$: this extension of the range is justified as in Flajolet and Sedgewick [6], ch.$VIII$. This gives

$$T_4:=\frac{3^{1/2}2^{1/2}}{\pi^{1/2}\mu}\left[1+\left(\frac{3x^2}{10}-\frac{3}{20}\right)\Big/\mu^2+\left(\frac{157x^4}{1400}-\frac{27x^2}{280}-\frac{13}{1120}\right)\Big/\mu^4+\mathcal{O}\left(\frac{1}{\mu^6}\right)\right].$$

Now it remains to compute

$$\begin{aligned}T_5:=T_3T_4=\frac{3^{1/2}2^{1/2}}{\pi^{1/2}\mu}&\left[1+\left(-\frac{x^4}{20}+\frac{3x^2}{10}-\frac{3}{20}\right)\Big/\mu^2\right.\\&\left.+\left(\frac{x^8}{800}-\frac{107x^6}{4200}+\frac{67x^4}{560}-\frac{27x^2}{280}-\frac{13}{1120}\right)\Big/\mu^4+\mathcal{O}\left(\frac{1}{\mu^6}\right)\right].\end{aligned}$$

Note that the coefficient of the exponential term is asymptotically equivalent to the dominant term of $\frac{1}{\sqrt{2\pi}\sigma}$, as expected. The first three terms correspond to (1). Note that our derivation is simpler then Nicolas' computation in [13].

## 3. LARGE DEVIATION, $k=m-m^\alpha$, $1>\alpha>1/2$, $m^\alpha$ INTEGER, $m\to\infty$

We have $k=m-m^\alpha$, $m^\alpha$ integer. We set [2],

$$\begin{aligned}&\varepsilon:=m^{\alpha-1},\\&\frac{1}{\varepsilon}=m^{1-\alpha}\ll\mu\ll\exp(1/\varepsilon),\quad \mu=\sqrt{m}.\end{aligned}$$

The multiseries' scale is here $\{m^{1-\alpha},\mu,\exp(1/\varepsilon)\}$. Set $\tau=\exp(-1/\varepsilon)$. Our result can be summarized in the following local limit theorem:

**Theorem 3.1.** *With $1/2<\alpha<1$, the rate of growth of $A_{n,k}/n!$ is the following:*

$$\frac{A_{n,k}}{n!}=e^m\varepsilon^m T_5(1+T_6\tau+T_7\tau^2+\mathcal{O}(\tau^3)),\tag{5}$$

(6) *with*

$$\begin{aligned}T_5&=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{\varepsilon\mu}-\frac{1}{12\varepsilon\mu^3}\right),\\T_6&=-\mu^2-\frac{1}{\varepsilon}+\frac{1}{2\varepsilon^2}+\left(-\frac{1}{8\varepsilon^4}+\frac{5}{6\varepsilon^3}-\frac{1}{\varepsilon^2}\right)\Big/\mu^2+\mathcal{O}\left(\frac{1}{\mu^4}\right),\\T_7&=\frac{\mu^4}{2}+\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-\frac{1}{2}\right)\mu^2+\frac{1}{\varepsilon^4}-\frac{10}{3\varepsilon^3}+\frac{3}{\varepsilon^2}-\frac{1}{\varepsilon}+\mathcal{O}\left(\frac{1}{\mu^2}\right).\end{aligned}$$

[2]here and in the sequel, $a_n\ll b_n$ means $a_n=o(b_n)$

**Proof.** We have

$$k = m - m^{\alpha} = m(1-\varepsilon).$$

The Saddle point equation (2) becomes

$$-w + 1 + \ln(w) + \ln(w)\varepsilon w - \ln(w)\varepsilon = 0. \tag{7}$$

To first order, we have $\ln(w) \sim 1/\varepsilon$. So we set

$$\rho = e^{\xi}, \xi = \frac{1}{\varepsilon}(1+\eta),$$

we now have to the next order

$$\varepsilon(\rho^{-1}-1)(1-\eta) + \rho^{-1} + (1-\rho^{-1})\varepsilon = 0,$$

or

$$-\eta\varepsilon\rho^{-1} + \eta\varepsilon + \rho^{-1} = 0,$$

hence

$$\eta \sim -\frac{\rho^{-1}}{\varepsilon} \ll \varepsilon.$$

This gives

$$\rho^{-1} = e^{-\xi} = e^{-1/\varepsilon - \eta/\varepsilon} \sim \tau(1-\frac{\eta}{\varepsilon}),$$

hence

$$\eta \sim -\frac{\tau(1-\frac{\eta}{\varepsilon})}{\varepsilon} \sim -\frac{\tau}{\varepsilon}(1+\frac{\tau}{\varepsilon^2}) = -\frac{\tau}{\varepsilon} - \frac{\tau^2}{\varepsilon^3}.$$

We derive, by bootstrapping from (7) (again we only provide a few terms here, we use more terms in our expansions)

$$\eta = -\frac{1}{\varepsilon}\tau + \left(-\frac{1}{\varepsilon^3} + \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\tau^2 + \left(-\frac{3}{2\varepsilon^5} + \frac{3}{\varepsilon^4} - \frac{4}{\varepsilon^3} + \frac{2}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\tau^3 + \mathcal{O}(\tau^4),$$

and, successively,

$$\begin{aligned}
\rho &= \frac{\exp(\eta/\varepsilon)}{\tau}, \text{ i.e.}\\
\rho &= \frac{1}{\tau} - \frac{1}{\varepsilon^2} + \left(-\frac{1}{2\varepsilon^4} + \frac{1}{\varepsilon^3} - \frac{1}{\varepsilon^2}\right)\tau + \left(-\frac{2}{2\varepsilon^6} + \frac{2}{\varepsilon^5} - \frac{3}{\varepsilon^4} + \frac{2}{\varepsilon^3} - \frac{1}{\varepsilon^2}\right)\tau^2 + \mathcal{O}(\tau^3),\\
\ln(\rho) &= \frac{1}{\varepsilon}(1+\eta), \text{ i.e.}\\
\ln(\rho) &= \frac{1}{\varepsilon} - \frac{1}{\varepsilon^2}\tau + \left(-\frac{1}{\varepsilon^4} + \frac{1}{\varepsilon^3} - \frac{1}{\varepsilon^2}\right)\tau^2 + \left(-\frac{3}{2\varepsilon^6} + \frac{3}{\varepsilon^5} - \frac{4}{\varepsilon^4} + \frac{2}{\varepsilon^3} - \frac{1}{\varepsilon^2}\right)\tau^3 + \mathcal{O}(\tau^4),\\
f(\rho) &= \frac{1}{\tau} + 1 - \varepsilon - \frac{1}{\varepsilon} + \left(-\frac{1}{2\varepsilon^3} + \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\tau + \mathcal{O}(\tau^2),\\
\ln(f(\rho)) &= \ln(\varepsilon) + \frac{1}{\varepsilon} + \left(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - 1\right)\tau + \mathcal{O}(\tau^2).
\end{aligned}$$

For the first part of the Cauchy's integral, we have

$$T_1 = \ln(f(\rho)) - (1-\varepsilon)\ln(\rho) = \ln(\varepsilon) + 1 - \tau + \left(-\frac{1}{2\varepsilon^2} - \frac{1}{2}\right)\tau^2 + \mathcal{O}(\tau^3),$$

now we extract the dominant part $\ln(\varepsilon) + 1$,

$$T_2 = \exp(\mu^2(T_1 - (\ln(\varepsilon)+1))) = 1 - \mu^2\tau + \left(-\frac{1+\varepsilon^2}{2\varepsilon^2}\mu^2 + \frac{1}{2}\mu^4\right)\tau^2 + \mathcal{O}(\tau^3).$$

Also

$$\kappa[2] = \varepsilon^2 + (-1+2\varepsilon)\tau + \mathcal{O}(\tau^2),$$
$$\kappa[3] = -2\varepsilon^3 + (1-6\varepsilon^2)\tau + \mathcal{O}(\tau^2).$$

Again we must choose a splitting value $\theta_0 = m^\beta, \beta < 0$ such that $m\kappa_2\theta_0^2 \to \infty, m\kappa_3\theta_0^3 \to 0, n \to \infty$. This leads to

$$\beta > \frac{1}{2} - \alpha, \beta < \frac{2}{3} - \alpha,$$

that is why we restrict the range to $1/2 < \alpha < 1$. We can then use $\theta_0 = m^{1/2-\alpha}$. We must prove that the integral

$$K_{m,k} = \int_{\theta_0}^{2\pi-\theta_0} e^{m\ln(f(\rho e^{\mathbf{i}\theta}))-k\mathbf{i}\theta} d\theta$$

is such that $|K_{m,k}|$ is exponentially small. This is done in the Appendix. Proceeding further, we derive

$$\theta = \sum_1^\infty u^i a[i]/\mu = \frac{u}{\varepsilon\mu} + \frac{\mathbf{i}u^2}{3\varepsilon\mu^2} - \frac{u^3}{36\varepsilon\mu^3} + \frac{\mathbf{i}u^4}{270\varepsilon\mu^4} + \mathcal{O}(\tau).$$

This expansion is valid in the dominant integration domain

$$|u| \le \frac{\theta_0\mu}{a_1} = m.$$

Setting $d\theta = \frac{d\theta}{du}du$, we integrate on $u = [-\infty..\infty]$ This gives successively

$$T_3 = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{1}{\varepsilon\mu} - \frac{1}{12\varepsilon\mu^3}\right) + \left(\frac{\frac{1}{2\varepsilon^3} - \frac{1}{\varepsilon^2}}{\mu} + \frac{-\frac{1}{8\varepsilon^5} + \frac{5}{6\varepsilon^4} - \frac{25}{24\varepsilon^3} + \frac{1}{12\varepsilon^2}}{\mu^3}\right)\tau\right] + \mathcal{O}(\tau^2),$$

$$T_4 = T_3T_2 = T_5(1 + T_6\tau + T_7\tau^2 + \mathcal{O}(\tau^3)),$$

with

$$T_5 = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\varepsilon\mu} - \frac{1}{12\varepsilon\mu^3}\right),$$

$$T_6 = -\mu^2 - \frac{1}{\varepsilon} + \frac{1}{2\varepsilon^2} + \left(-\frac{1}{8\varepsilon^4} + \frac{5}{6\varepsilon^3} - \frac{1}{\varepsilon^2}\right)\Big/\mu^2 + \mathcal{O}\left(\frac{1}{\mu^4}\right),$$

$$T_7 = \frac{\mu^4}{2} + \left(-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} - \frac{1}{2}\right)\mu^2 + \frac{1}{\varepsilon^4} - \frac{10}{3\varepsilon^3} + \frac{3}{\varepsilon^2} - \frac{1}{\varepsilon} + \mathcal{O}\left(\frac{1}{\mu^2}\right).$$

This concludes the proof. Given some desired precision, how many terms must we use in our expansions? It depends on $\alpha$. For instance, in $T_7$, we encounter terms like $\frac{1}{\varepsilon^2}\mu^2$ and $\frac{1}{\varepsilon^4}$. So we must compare $(1-\alpha)+1$ with $4(1-\alpha)$. The critical value is $\alpha = 2/3$.

■

To check the quality of our asymptotic, we have chosen $m \in [90, 500]$ and $\alpha = 2/3$. Figure 1 shows $\ln(A_{n,k})$ (circle) and the ln of expression (5) (line). Figure 2 shows the quotient of $\ln(A_{n,k})$ by the ln of expression (5), without the $\tau$ term in (5) (line) and with this term (circle). Of course the good influence of the $\tau$ term is less effective for large $m$.

Another way is to fix $m$, to 1001 for instance ($n = 1000$). The maximum value for $k$ is $\lfloor m - m^{1/2}\rfloor = 969$. We must set $k$ larger than the central domain, for instance larger than $\lfloor \frac{m}{2} + 2\sqrt{m/12}\rfloor = 518$. But notice that the term $T_6$ starts with two negative terms, $-\mu^2 - \frac{1}{\varepsilon}$, so $k$ must be large so that $\tau$ is small enough to compensate these negative terms. It appears that $k = 860$ is large enough in our case. So our $\alpha$ range is $[1/2, 0.71]$. Figure 3 shows $\ln(A_{n,k})$ (circle) and the ln of expression (5) (line).

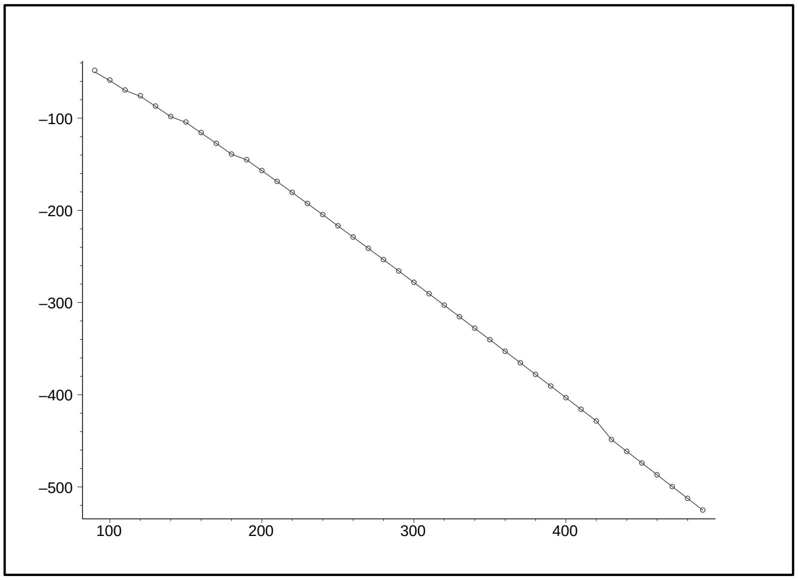


FIGURE 1. $\alpha = 2/3$, $\ln(A_{n,k})$ (circle) and the ln of expression (5) (line) as function of $m$

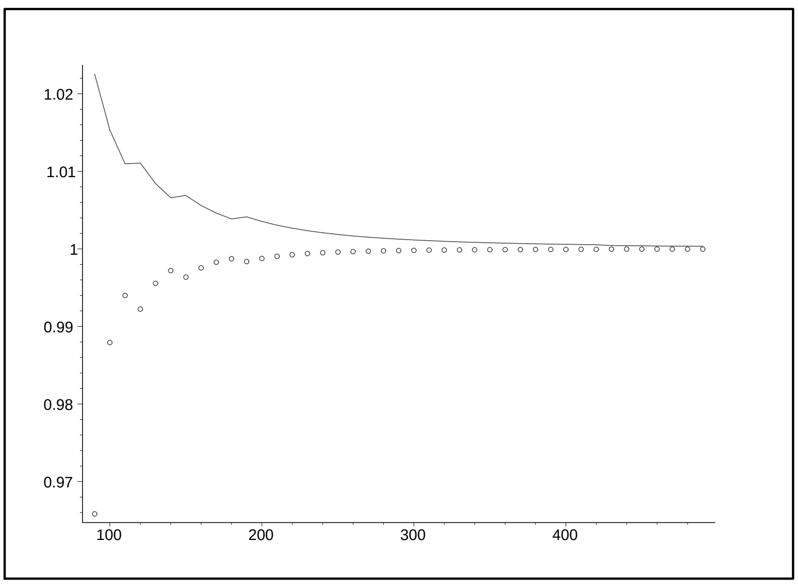


FIGURE 2. $\alpha = 2/3$, quotient of $\ln(A_{n,k})$ by the ln of expression (5), as function of $m$, without the $\tau$ term in (5) (line) and with this term (circle)

Figure 4 shows the quotient of $\ln(A_{n,k})$ by the ln of expression (5), without the $\tau$ term in (5) (line) and with this term (circle).

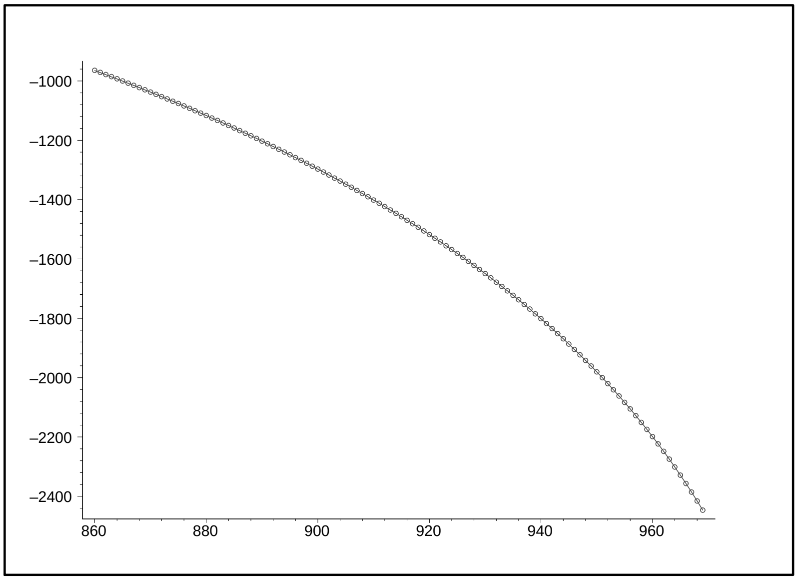


FIGURE 3. $n = 1000$, $\ln(A_{n,k})$ (circle) and the ln of expression (5) (line) as function of $k$

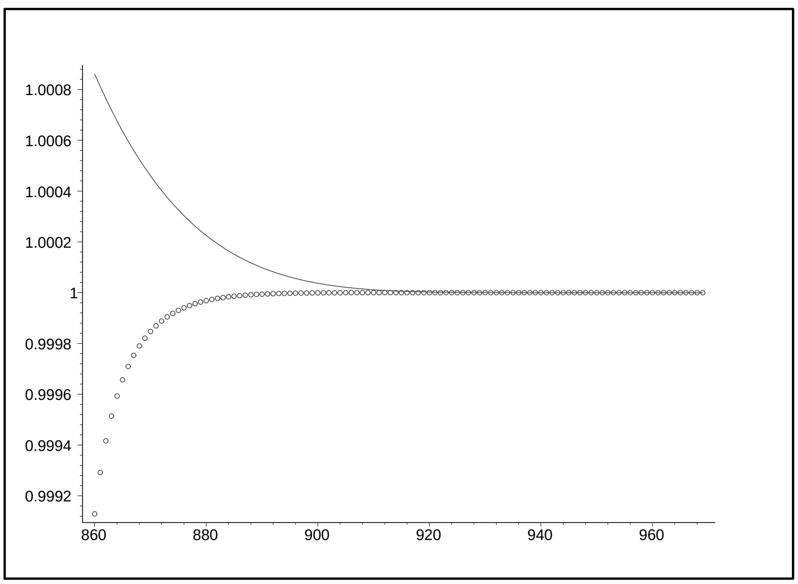


FIGURE 4. $n = 1000$, quotient of $\ln(A_{n,k})$ by the ln of expression (5), as function of $k$, without the $\tau$ term in (5) (line) and with this term (circle)

## 4. APPENDIX. JUSTIFICATION OF THE INTEGRATION PROCEDURE

4.1. **The central region.** We must analyze

$$\Re(m\ln(f(\rho e^{\mathbf{i}\theta})) - k\mathbf{i}\theta).$$

Let us first notice that $k\mathbf{i}\theta$ does not contribute to the analysis. Next, we have

$$\rho = 1 + \frac{2x3^{1/2}}{\mu} + \mathcal{O}\left(\frac{1}{\mu^2}\right).$$

Hence, this leads to analyze

$$\Re\left(\ln\left(\frac{\rho e^{\mathbf{i}\theta}-1}{\ln\left(\rho e^{\mathbf{i}\theta}\right)}\right)\right) = \ln\left(\left|\frac{\rho e^{\mathbf{i}\theta}-1}{\ln\left(\rho e^{\mathbf{i}\theta}\right)}\right|\right)$$
$$= \frac{1}{2}\ln\left(\frac{2(1-\cos(\theta))}{\theta^2}\right) + \frac{x3^{1/2}}{\mu} + \mathcal{O}\left(\frac{1}{\mu^2}\right).$$

The first term has a dominant peak at 0.

4.2. **The non-central region.** Now we must analyze

$$\Re(m\ln(f(\rho e^{\mathbf{i}\theta})) - k\mathbf{i}\theta),$$

Set $\delta = \tau^{-1} = e^{1/\varepsilon}, \quad \ln(\delta) = \frac{1}{\varepsilon}$. We have

$$\rho = \delta - (\ln(\delta))^2 + \mathcal{O}\left(\frac{1}{\delta}\right).$$

Hence

$$\Re\left(\ln\left(\frac{\rho e^{\mathbf{i}\theta}-1}{\ln\left(\rho e^{\mathbf{i}\theta}\right)}\right)\right) = \frac{1}{2}\ln\left(\frac{1-2\rho\cos(\theta)+\rho^2}{\ln(\rho)^2+\theta^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{1}{\ln(\delta)^2+\theta^2}\right) + \ln(\delta) + \mathcal{O}\left(\frac{1}{\delta}\right)$$
$$= \ln(\delta) - \ln(\ln(\delta)) - \frac{\theta^2}{2\ln(\delta)^2} + \mathcal{O}\left(\frac{1}{\ln(\delta)^4}\right)$$
$$= \frac{1}{\varepsilon} + \ln(\varepsilon) - \frac{\varepsilon^2\theta^2}{2} + \mathcal{O}(\varepsilon^4).$$

The $\theta$ contribution has a dominant peak at 0.

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