# NOTE ON THE COEFFICIENTS OF RATIONAL EHRHART QUASI-POLYNOMIALS OF MINKOWSKI-SUMS

MARTIN HENK AND EVA LINKE

ABSTRACT. By extending former results of Ehrhart, it was shown by Peter McMullen that the number of lattice points in the Minkowski-sum of dilated rational polytopes is a quasi-polynomial function in the dilation factors. Here we take a closer look at the coefficients of these quasi-polynomials and show that they are piecewise polynomials themselves and that they are related to each other by a simple differential equation. As a corollary, we obtain a refinement of former results on lattice points in vector dilated polytopes.

## 1. INTRODUCTION

Let $\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \dots, x_n)^\intercal : x_i \in \mathbb{R}\}$ be the $n$-dimensional Euclidean space and let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the integral lattice consisting of all points with integral coordinates. The origin of an appropriate dimension will be denoted by $\mathbf{0}$. The volume, i.e., the $n$-dimensional Lebesgue measure of a subset $X \subset \mathbb{R}^n$ is denoted by $\mathrm{vol}(X)$, and by $\mathrm{vol}_{\dim(X)}(X)$ we mean the $\dim(X)$-dimensional volume of $X$ measured with respect to its affine hull.

A polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely many points, i.e., $P = \mathrm{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ with $\boldsymbol{v}_i \in \mathbb{R}^n$, $1 \le i \le k$. It is called integral if the points $\boldsymbol{v}_i$ can be chosen to be in $\mathbb{Z}^n$, and it is called rational if $\boldsymbol{v}_i$ can be chosen to be in $\mathbb{Q}^n$. Equivalently, $P$ is a rational polytope if and only if there are $A \in \mathbb{Q}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{Q}^m$ with $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \le \boldsymbol{b}\}$. It is clear that $A$ and $\boldsymbol{b}$ can also be chosen to be integral.

For a rational polytope $P$, the smallest positive integral (resp. rational) number $\rho$ such that $\rho P$ is an integral polytope is called the *(rational) denominator of* $P$ and it is denoted by $\tau_{\mathbb{Z}}(P)$ (resp. $\tau_{\mathbb{Q}}(P)$). A function $p : \mathbb{Z}_{\ge 0} \to \mathbb{R}$ ($p : \mathbb{Q}_{\ge 0} \to \mathbb{R}$) is called a *(rational) quasi-polynomial with period* $\tau \in \mathbb{Z}$ *(*$\tau \in \mathbb{Q}$*)* of degree (at most) $n$ if there exist periodic functions $p_i : \mathbb{Z}_{\ge 0} \to \mathbb{R}$ ($p_i : \mathbb{Q}_{\ge 0} \to \mathbb{R}$), $i = 0, \dots, n$, with period $\tau$ such that $p(r) = \sum_{i=0}^n p_i(r) r^i$.

Now, for given rational polytopes $P_1, \dots, P_k \subset \mathbb{R}^n$, $k \in \mathbb{N}_{\ge 1}$, we are interested in the number of lattice points contained in their (non-negative) rational Minkowski-sums, i.e., we consider the function $\mathrm{Q}(P_1, \dots, P_k, \cdot) : \mathbb{Q}^k_{\ge 0} \to \mathbb{N}$ given by

$$\mathrm{Q}(P_1, \dots, P_k, \boldsymbol{r}) = \#\left(\left(\sum_{i=1}^k r_i P_i\right) \cap \mathbb{Z}^n\right), \quad \text{for } \boldsymbol{r} = (r_1, \dots, r_k) \in \mathbb{Q}^k_{\ge 0}.$$

For one polytope $P$, i.e., $k = 1$, such considerations go back to Ehrhart [8]. One of Ehrhart's fundamental theorems states that $\mathrm{Q}(P_1, l)$, for integers $l \in \mathbb{N}_{\ge 1}$, is a quasi-polynomial with period $\tau_{\mathbb{Z}}(P)$ of degree $\dim(P)$. The leading coefficient of this quasi-polynomial is given by $\mathrm{vol}_{\dim(P)}(P)$ for all integers $l \in \mathbb{N}_{\ge 1}$ such that $\mathrm{aff}(l P)$ contains integral points. Thus, if $P$ is full-dimensional, the leading coefficient is constant and equals $\mathrm{vol}(P)$.

---

*Date*: March 9, 2015.

2010 *Mathematics Subject Classification.* 52B20, 52C07, 11P21, 05A15.

*Key words and phrases.* Minkowski-sum, lattice points, Ehrhart theory, rational polytopes, quasi-polynomials.

Second author is supported by the Deutsche Forschungsgemeinschaft within the project He 2272/5-1.

For more information on all of the fascinating aspects of Ehrhart theory we refer to the book by Beck and Robins [6].

This univariate case was generalized to more than one polytope and to non-negative rational dilates by Peter McMullen [12, Theorem 7]. Actually, [12, Theorem 7] is proven only for integral dilates, but the proof carries easily over to the rational case and we have learnt, that this fact seems to be folklore. For an explicit treatment of rational dilates in the case of one polytope we refer to [10]. For another approach to this rational Ehrhart theory we refer to Baldoni et al. [1, 2, 3]. They study intermediate sums, interpolating between integrals and discrete sums over certain integral points in polytopes which also results in a rational version of Ehrhart's Theorem for such intermediate valuations.

In order to present McMullen's result we need some more notation. For $\boldsymbol{x} \in \mathbb{R}^k$ and a non-negative integral vector $\boldsymbol{a} \in \mathbb{N}^k$ we denote by $\boldsymbol{x}^{\boldsymbol{a}}$ the monomial $\boldsymbol{x}^{\boldsymbol{a}} = \prod_{i=1}^k x_i^{a_i}$, and as an abbreviation we set $I(k,n) = \{\boldsymbol{l} \in \mathbb{N}^k : |\boldsymbol{l}|_1 \leq n\}$, where $|\cdot|_1$ denotes the 1-norm. The *Hadamard product* $\boldsymbol{r} \odot \boldsymbol{s}$ of two rational vectors $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{Q}^k$ is the coordinate-wise product $\boldsymbol{r} \odot \boldsymbol{s} = (r_1 s_1, \ldots, r_k s_k)$.

**Definition 1.1** (Multivariate Rational Quasi-polynomial)**.** Let $k \in \mathbb{N}_{\geq 1}$. A function $p : \mathbb{Q}_{\geq 0}^k \to \mathbb{Q}$ is called a *rational quasi-polynomial of degree (at most) $n$ with (rational) period* $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k) \in \mathbb{Q}^k$ if for all $\boldsymbol{l} \in I(k,n)$ there exist periodic functions $p_{\boldsymbol{l}} : \mathbb{Q}_{\geq 0}^k \to \mathbb{Q}$ with period $\tau_i$ in the $i$th argument, $1 \leq i \leq k$, such that

$$p(\boldsymbol{r}) = \sum_{\boldsymbol{l} \in I(k,n)} p_{\boldsymbol{l}}(\boldsymbol{r})\, \boldsymbol{r}^{\boldsymbol{l}}.$$

We call $p_{\boldsymbol{l}}(\cdot)$ the $\boldsymbol{l}$th coefficient of $p$.

With these notations McMullen's result can be stated as

**Theorem 1.2** ([12, Theorem 7])**.** *Let $P_1, \ldots, P_k \subset \mathbb{R}^n$ be rational polytopes. Then the function* $\mathrm{Q}(P_1, \ldots, P_k, \cdot) : \mathbb{Q}_{\geq 0}^k \to \mathbb{N}$ *is a rational quasi-polynomial of degree* $\dim(P_1 + \ldots + P_k)$ *with period* $\boldsymbol{\tau} = (\tau_{\mathbb{Q}}(P_1), \ldots, \tau_{\mathbb{Q}}(P_k))$.

$\mathrm{Q}(P_1, \ldots, P_k, \cdot)$ is called the *rational Ehrhart quasi-polynomial of* $P_1, \ldots, P_k$, and the $\boldsymbol{l}$th coefficient of $\mathrm{Q}(P_1, \ldots, P_k, \cdot)$ is denoted by $\mathrm{Q}_{\boldsymbol{l}}(P_1, \ldots, P_k, \cdot)$.

As in the univariate case, the leading coefficients are constants and admit a nice geometric interpretation; namely, for all $\boldsymbol{l} \in I(k,n)$ with $|\boldsymbol{l}|_1 = \dim(P_1 + \ldots + P_k) = d$ and for all $\boldsymbol{r} \in \mathbb{Q}_{>0}^k$ such that $\mathrm{aff}\big(\sum_{i=1}^k r_i P_i\big) \cap \mathbb{Z}^n \neq \emptyset$ we have

$$\mathrm{Q}_{\boldsymbol{l}}(P_1, \ldots, P_k, \boldsymbol{r}) = \frac{d!}{l_1! \cdots l_k!} \mathrm{V}_{\boldsymbol{l}}(P_1, \ldots, P_k). \tag{1.1}$$

Here, $\mathrm{V}_{\boldsymbol{l}}(P_1, \ldots, P_k)$ is the $\boldsymbol{l}$th *mixed volume* of the polytopes $P_1, \ldots, P_k$, and it depends only on those polytopes with $l_i > 0$. For a detailed introduction to mixed volumes we refer to Schneider [14, Chapter 5]. Here we just mention that the $d$-dimensional volume of $\sum_{i=1}^k r_i P_i$ is a homogenous polynomial in $\boldsymbol{r}$ of degree $d$, and its coefficients are the so-called *mixed volumes* $\mathrm{V}_{\boldsymbol{l}}(P_1, \ldots, P_k)$ – up to the constant $\frac{d!}{l_1! \cdots l_k!}$. Since the leading term of the Ehrhart quasi-polynomial $\mathrm{Q}(P_1, \ldots, P_k, \boldsymbol{r})$ is the volume of $r_1 P_1 + \ldots + r_k P_k$, the mixed volumes appear as coefficients of $\mathrm{Q}(P_1, \ldots, P_k, \cdot)$. We remark that the leading term of a multivariate polynomial $p(\boldsymbol{r}) = \sum_{\boldsymbol{l} \in I(k,d)} p_{\boldsymbol{l}}(\boldsymbol{r}) \boldsymbol{r}^{\boldsymbol{l}}$ of degree $d$ is the polynomial $\sum_{\boldsymbol{l} \in I(k,d), |\boldsymbol{l}|_1 = d} p_{\boldsymbol{l}}(\boldsymbol{r}) \boldsymbol{r}^{\boldsymbol{l}}$.

Our main result is the following structural statement about the coefficients of the rational Ehrhart quasi-polynomial.

**Theorem 1.3.** *Let* $k \in \mathbb{N}_{\geq 1}$, *let* $P_1, \dots, P_k \subset \mathbb{R}^n$ *be rational polytopes with* $\dim(P_1 + \dots + P_k) = n$ *and let* $\boldsymbol{l} \in I(k,n)$. *Then* $\mathrm{Q}_{\boldsymbol{l}}(P_1, \dots, P_k, \cdot)$ *is a piecewise polynomial function of degree at most* $n - |\boldsymbol{l}|_1$. *Moreover, for all* $\boldsymbol{l} \in I(k, n-1)$ *and for all* $\boldsymbol{r} \in \mathbb{Q}^k_{>0}$ *such that* $\mathrm{Q}(P_1, \dots, P_k, \cdot)$ *is continuous at* $\boldsymbol{r} + \boldsymbol{u} \odot (\tau_{\mathbb{Q}}(P_1), \dots, \tau_{\mathbb{Q}}(P_k))^\intercal$, $\boldsymbol{u} \in \mathbb{N}^k$, *it holds*

$$(l_j + 1)\,\mathrm{Q}_{\boldsymbol{l}+\boldsymbol{e}_j}(P_1, \dots, P_k, \boldsymbol{r}) + \frac{\partial}{\partial r_j}\mathrm{Q}_{\boldsymbol{l}}(P_1, \dots, P_k, \boldsymbol{r}) = 0, \tag{1.2}$$

*where* $\boldsymbol{e}_j \in \mathbb{R}^k$ *denotes the* $j$*th unit vector.*

In words, the coefficients of the rational quasi-polynomial are piecewise polynomials themselves and they are related to each other by a simple differential equation. In particular, the theorem implies that knowing $\mathrm{Q}_{\boldsymbol{0}}(P_1, \dots, P_k, \boldsymbol{r})$ is equivalent to knowing all coefficients of the Ehrhart quasi-polynomial. For $k = 1$ this result was proven by Linke [10]. In Section 2 we give an example (cf. Example 2.1) illustrating the above theorem.

As a corollary of Theorem 1.3 we can slightly extend a statement about lattice points in vector dilations of polytopes. In order to state the result, we introduce some more notation: For a given integral $(m \times n)$-matrix $A$, let $P_A(\boldsymbol{b}) := \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}$, $\boldsymbol{b} \in \mathbb{Q}^m$. We want to count the number of lattice points in $P_A(\boldsymbol{b})$ as a function in $\boldsymbol{b}$ for a fixed matrix $A$.

To this end, we may consider only matrices $A \in \mathbb{Z}^{m \times n}$ such that $P_A(\boldsymbol{b})$ is bounded for all $\boldsymbol{b} \in \mathbb{Q}^m$, that is, we always assume $\mathrm{pos}(A^\intercal) = \mathbb{R}^n$. Here, $\mathrm{pos}(A^\intercal)$ is the cone generated by the rows of $A$, that is, the set of all nonnegative linear combinations of rows of $A$. We denote the number of lattice points in $P_A(\boldsymbol{b})$ by

$$\Phi(A, \boldsymbol{b}) = \#(P_A(\boldsymbol{b}) \cap \mathbb{Z}^n), \quad \boldsymbol{b} \in \mathbb{Q}^m.$$

Since we cannot expect uniform behavior of $\Phi(A, \boldsymbol{b})$ when the combinatorics of the polytope $P_A(\boldsymbol{b})$ changes, we consider subsets of $\mathbb{Q}^m$ on which the polytopes $P_A(\cdot)$ are so-called *locally similar*. To introduce this notation, we have to consider the possible normal fans of $P_A(\boldsymbol{b})$ which we define next.

For a fixed vertex $\boldsymbol{v}$ of a polytope $P_A(\boldsymbol{b})$, the *normal cone* $U_{\boldsymbol{v}}$ of $\boldsymbol{v}$ is the set of all directions $\boldsymbol{u} \in \mathbb{R}^n$, such that the function $\boldsymbol{x} \mapsto \boldsymbol{u}^\intercal \boldsymbol{x}$, $\boldsymbol{x} \in P_A(b)$, is maximized by the vertex $\boldsymbol{v}$. By the definition of vertices as 0-dimensional faces of a polytope, the normal cone $U_{\boldsymbol{v}}$ of a vertex $\boldsymbol{v}$ is full-dimensional.

We call the set of the normal cones of $P_A(\boldsymbol{b})$ the *normal fan*, denoted by $N_A(\boldsymbol{b})$. This notation differs from the usual definition of the normal fan, which is a polyhedral subdivision of $\mathbb{R}^n$ and hence also contains lower dimensional normal cones. In our case it is enough to only consider the maximal cells. Still, the union of all normal cones in $N_A(\boldsymbol{b})$ is $\mathbb{R}^n$, and the interiors of two normal cones in $N_A(\boldsymbol{b})$ have no points in common. We refer to Ziegler [19, Chapter 7] for an introduction to polyhedral fans.

Now, two polytopes are called *locally similar*, if their normal fans coincide. For a given matrix $A$ and varying $\boldsymbol{b}$ there are only finitely many possible normal fans and in the following let $N$ be an arbitrary, but fixed normal fan, and let

$$C_N = \mathrm{cl}\left\{\boldsymbol{b} \in \mathbb{R}^m : N_A(\boldsymbol{b}) = N\right\}$$

be the closure of all right hand side vectors $\boldsymbol{b}$ having the same fixed normal fan $N$. Then $C_N$ is a rational polyhedral cone (cf. Lemma 3.1).

Dahmen and Micchelli, 1988, [7, Theorem 3.1] gave a structural result for $\Phi(A, \boldsymbol{b})$ for a fixed matrix $A$ and suitable integral vectors $\boldsymbol{b} \in C_N$. As a corollary [7, Corollary 3.1], they obtained that $\Phi(A, \cdot)$ is a polynomial in the integral variable $\boldsymbol{b} \in C_N \cap \mathbb{Z}^m$, if $P_A(\boldsymbol{b})$ is integral. Sturmfels, 1995, [15] gave a formula for $\Phi(A, \boldsymbol{b})$, which is also valid if $P_A(\boldsymbol{b})$ is not integral. He uses tools from the theory of polyhedral splines and representation techniques of groups. Altogether, the mentioned references lead to the following well-known theorem:

**Theorem 1.4** ([7, 15]). *Let* $C_N = \mathrm{pos}\{\boldsymbol{h}_1,\dots,\boldsymbol{h}_k\}$ *with* $\boldsymbol{h}_i \in \mathbb{Q}^m$, $1 \le i \le k$. *Then* $\Phi(A,\boldsymbol{b})$ *is a rational quasi-polynomial in* $\boldsymbol{b} \in C_N \cap \mathbb{Q}^m$, *that is,*

$$\Phi(A,\boldsymbol{b}) = \sum_{\boldsymbol{l}\in I(m,n)} \Phi_{\boldsymbol{l}}(A,\boldsymbol{b})\boldsymbol{b}^{\boldsymbol{l}},$$

*where* $\Phi_{\boldsymbol{l}}(A,\boldsymbol{b}) = \Phi_{\boldsymbol{l}}(A,\boldsymbol{b} + [\tau_{\mathbb{Z}}(P_A(\boldsymbol{h}_i))]\ \boldsymbol{h}_i)$ *for* $\boldsymbol{l} \in I(m,n)$ *and* $1 \le i \le k$.

For an approach to this *parametric* problem via generating functions as well as for algorithmic questions related to computing the function $\Phi(A,\boldsymbol{b})$, we refer to the work of Köppe&Verdoolaege [9], Verdoolaege&Woods [17], Verdoolaege et al. [16] and the references within. Mount, 1998, [13] described methods for actually calculating the polynomials $\Phi(A,\boldsymbol{b})$ and normal cones, if $A$ is unimodular and $\boldsymbol{b}$ integral. To this end, Mount gave an alternative argument for the result [7, Corollary 3.1] by Dahmen and Micchelli. His approach makes use of known results on lattice points in Minkowski-sums of polytopes. We will adopt this approach in order to obtain (via Theorem 1.3) the following structural refinement of the theorem above:

**Corollary 1.5.** *In addition to Theorem 1.4 it holds:* $\Phi_{\boldsymbol{l}}(A,\boldsymbol{b})$ *is a piecewise polynomial function of degree* $n - |\boldsymbol{l}|_1$ *in* $\boldsymbol{b}$. *Moreover, for all* $\boldsymbol{l} \in I(m,n-1)$ *we have*

$$(l_j+1)\Phi_{\boldsymbol{l}+\boldsymbol{e}_j}(A,\boldsymbol{b}) + \frac{\partial}{\partial b_j}\Phi_{\boldsymbol{l}}(A,\boldsymbol{b}) = 0.$$

Beck [4, 5] gave a more elementary proof of the quasi-polynomiality of $\Phi(A,\boldsymbol{b})$, if $\boldsymbol{b}$ is integral. He also proved an Ehrhart reciprocity law for vector dilated polytopes, that is, $\Phi(A,-\boldsymbol{b}) = \#(\mathrm{int}(P_A(\boldsymbol{b}))\cap\mathbb{Z}^n)$, for $\boldsymbol{b}\in\mathbb{Z}^n$; here, $\mathrm{int}(\cdot)$ denotes the interior. Since $\Phi(A,\boldsymbol{b}) = \Phi(tA,t\boldsymbol{b})$ for all $t\in\mathbb{Q}_{\ge 0}$, $A\in\mathbb{Z}^{m\times n}$ and $\boldsymbol{b}\in\mathbb{Q}^m$, this statement immediately carries over to rational vectors $\boldsymbol{b}$ and we have

$$\Phi(A,-\boldsymbol{b}) = \#(\mathrm{int}(P_A(\boldsymbol{b}))\cap\mathbb{Z}^n)$$

for all $\boldsymbol{b}\in\mathbb{Q}^m$.

*Remark* 1.6. Theorem 1.2 and Corollary 1.5 can be extended to real $\boldsymbol{r}\in\mathbb{R}^k_{\ge\boldsymbol{0}}$ or $\boldsymbol{b}\in C_N\cap\mathbb{R}^m$ via approximation by rationals (cf. [10, Remark 2.7]), and therefore they may be considered as part of a *real* Ehrhart theory in the spirit of [1, 2, 3]. Observe, however, the main underlying structure, i.e., the normals of the polytopes are still rational vectors and so lattice points may enter or leave only at rational dilates.

The paper is organized as follows. The proof of our main Theorem 1.3 is given in the next section, and in Section 3 we only outline the proof of Corollary 1.5 since it follows pretty much the approach of Mount [13, Theorem 2] in the integral case $\boldsymbol{b}\in\mathbb{Z}^m$. In both sections we give examples, illustrating the associated quasi-polynomial structures.

## 2. PROOF OF THEOREM 1.3

First, in order to make the statement of Theorem 1.3 more transparent and in order to give a flavour of the structure of the coefficients we start with an example. To this end, for $r\in\mathbb{Q}$ we denote by $\{r\}$ its fractional part, that is, $\{r\} = r - \lfloor r\rfloor$ where $\lfloor\cdot\rfloor$ is the floor-function.

**Example 2.1.** We consider the origin-symmetric square $P_1 = \mathrm{conv}\left\{\binom{1}{1},\binom{-1}{1},\binom{-1}{-1},\binom{1}{-1}\right\}$ of edge length 2 and the triangle $P_2 = \{\binom{0}{1},\binom{1}{-1},\binom{-1}{-1}\}$ (see Figure 1). Obviously, the rational

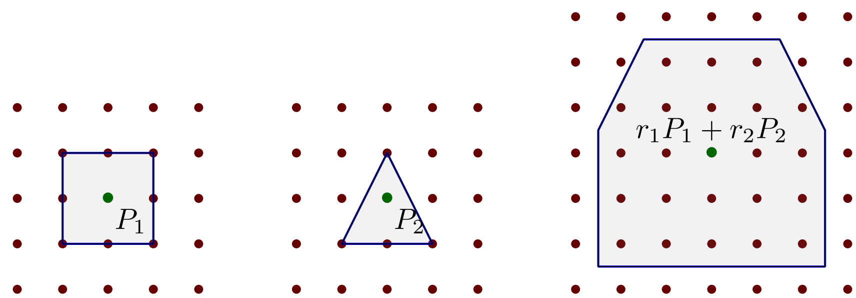


FIGURE 1. $P_1$, $P_2$ and $r_1P_1 + r_2P_2$.

denominators are given by $\tau_{\mathbb{Q}}(P_1) = 1$ and $\tau_{\mathbb{Q}}(P_2) = 1$. For $\boldsymbol{r} = (r_1, r_2)^\intercal \in \mathbb{Q}^2_{>\mathbf{0}}$, the sum $r_1 P_1 + r_2 P_2$ can be written as

$$\begin{aligned} r_1P_1 + r_2P_2 = \operatorname{conv}\Bigg\{ & \begin{pmatrix} r_1 \\ r_1+r_2 \end{pmatrix}, \begin{pmatrix} -r_1 \\ r_1+r_2 \end{pmatrix}, \begin{pmatrix} -(r_1+r_2) \\ r_1-r_2 \end{pmatrix}, \\ & \begin{pmatrix} -(r_1+r_2) \\ -(r_1+r_2) \end{pmatrix}, \begin{pmatrix} (r_1+r_2) \\ -(r_1+r_2) \end{pmatrix}, \begin{pmatrix} r_1+r_2 \\ r_1-r_2 \end{pmatrix} \Bigg\} \\ = \{\boldsymbol{x} \in \mathbb{R}^2 : \quad & -(r_1+r_2) \le x_1 \le r_1+r_2, \\ & -(r_1+r_2) \le x_2 \le r_1+r_2, \\ & \pm 2x_1 + x_2 \le 3r_1 + r_2\}. \end{aligned}$$

After some elementary calculations one obtains:

$$\begin{aligned} &\mathrm{Q}_{(2,0)}(P_1, P_2, \boldsymbol{r}) = 4, \\ &\mathrm{Q}_{(1,1)}(P_1, P_2, \boldsymbol{r}) = 8, \\ &\mathrm{Q}_{(0,2)}(P_1, P_2, \boldsymbol{r}) = 2, \\ &\mathrm{Q}_{(1,0)}(P_1, P_2, \boldsymbol{r}) = -8\{r_1+r_2\} + 4, \\ &\mathrm{Q}_{(0,1)}(P_1, P_2, \boldsymbol{r}) = -2\{3r_1+r_2\} - 2\{r_1+r_2\} + 2, \\ &\mathrm{Q}_{(0,0)}(P_1, P_2, \boldsymbol{r}) = -\frac{1}{2}\left(\{3r_1+r_2\}^2 + \{r_1+r_2\}^2\right) + 3\{3r_1+r_2\}\{r_1+r_2\} - \{r_1+r_2\} \\ &\qquad - \{3r_1+r_2\} + 1 - \begin{cases} \frac{1}{2}, & \lfloor 3r_1+r_2 \rfloor - \lfloor r_1+r_2 \rfloor \text{ odd}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Observe that $\mathrm{Q}_{(2,0)}(P_1, P_2, \boldsymbol{r})$ and $\mathrm{Q}_{(0,2)}(P_1, P_2, \boldsymbol{r})$ are the areas (volumes) of $P_1$ and $P_2$, respectively, i.e., $\mathrm{Q}_{(2,0)}(P_1, P_2, \boldsymbol{r}) = \mathrm{V}_{(2,0)}(P_1, P_2)$, $\mathrm{Q}_{(0,2)}(P_1, P_2, \boldsymbol{r}) = \mathrm{V}_{(0,2)}(P_1, P_2)$, and the coefficient $\mathrm{Q}_{(1,1)}(P_1, P_2, \boldsymbol{r})$ is up to the factor 2 the mixed volume $\mathrm{V}_{(1,1)}(P_1, P_2)$ (cf. (1.1)).

Moreover, all of the coefficients $\mathrm{Q}_{\boldsymbol{l}}(P_1, P_2, \boldsymbol{r})$ are piecewise polynomials of degree $2 - |\boldsymbol{l}|_1$ with period $\boldsymbol{\tau} = (\tau_{\mathbb{Q}}(P_1), \tau_{\mathbb{Q}}(P_2)) = (1, 1)$, and, for instance, regarding the derivatives we have for $r_1, r_2 > 0$ with $3r_1 + r_2 < 1$

$$\begin{aligned} \frac{\partial}{\partial r_1} \mathrm{Q}_{(0,0)}(P_1, P_2, \boldsymbol{r}) &= -(3(3r_1+r_2) + (r_1+r_2)) + 3(3(r_1+r_2) + (3r_1+r_2)) - 4 \\ &= 8(r_1+r_2) - 4 = -\mathrm{Q}_{(1,0)}(P_1, P_2, \boldsymbol{r}) \\ \frac{\partial}{\partial r_2} \mathrm{Q}_{(0,1)}(P_1, P_2, \boldsymbol{r}) &= -4 = (-2)\,\mathrm{Q}_{(0,2)}(P_1, P_2, \boldsymbol{r}). \end{aligned}$$

The main ingredient of the proof of Theorem 1.3 is the following rather technical lemma.

**Lemma 2.2.** *Let $p:\mathbb{Q}^k\to\mathbb{Q}$ be a rational quasi-polynomial of degree $n\geq 1$ with period $\boldsymbol{\tau}\in\mathbb{Q}^k_{>0}$ and constant leading coefficients, that is,*

$$p(\boldsymbol{r})=\sum_{\boldsymbol{l}\in I(k,n)} p_{\boldsymbol{l}}(\boldsymbol{r})\boldsymbol{r}^{\boldsymbol{l}},\boldsymbol{r}\in\mathbb{Q}^k, \tag{2.1}$$

*with $p_{\boldsymbol{l}}(\boldsymbol{r})\in\mathbb{Q}$ for all $\boldsymbol{l}\in I(k,n)$ with $|\boldsymbol{l}|_1=n$, and $p_{\boldsymbol{l}}:\mathbb{Q}^k\to\mathbb{Q}$ are periodic functions with period $\boldsymbol{\tau}$. Suppose there exist a $k$-dimensional open subset $S\subset\mathbb{R}^k$ such that for $(\boldsymbol{r},\boldsymbol{u})\in(S\cap\mathbb{Q}^k)\times\mathbb{N}^k$ the value of $p(\boldsymbol{r}+\boldsymbol{u}\odot\boldsymbol{\tau})$ depends only on $\boldsymbol{u}$, i.e., there exists $c_{\boldsymbol{u}}\in\mathbb{Q}$ for $\boldsymbol{u}\in\mathbb{N}^k$ such that*

$$p(\boldsymbol{r}+\boldsymbol{u}\odot\boldsymbol{\tau})=c_{\boldsymbol{u}},\quad \text{for all } \boldsymbol{r}\in S\cap\mathbb{Q}^k,\ \boldsymbol{u}\in\mathbb{N}^k. \tag{2.2}$$

*Then for $\boldsymbol{l}\in\mathbb{N}^k$ with $|\boldsymbol{l}|_1<n$, the coefficient $p_{\boldsymbol{l}}:S\to\mathbb{Q}$ is a polynomial of degree $n-|\boldsymbol{l}|_1$ satisfying the differential equation*

$$(l_j+1)p_{\boldsymbol{l}+\boldsymbol{e}_j}(\boldsymbol{r})+\frac{\partial}{\partial r_j}p_{\boldsymbol{l}}(\boldsymbol{r})=0. \tag{2.3}$$

First we will show how Theorem 1.3 can be deduced from Lemma 2.2. Afterwards we will proceed with the proof of Lemma 2.2.

*Proof of Theorem 1.3.* By Theorem 1.2 and (1.1) we know that $\mathrm{Q}(P_1,\dots,P_k,\cdot):\mathbb{Q}^k_{\geq\boldsymbol{0}}\to\mathbb{N}$ is a rational quasi-polynomial with period $\boldsymbol{\tau}=(\tau_{\mathbb{Q}}(P_1),\cdots,\tau_{\mathbb{Q}}(P_k))$ and constant leading coefficients, and we may assume that it is of degree $n$.

For a polytope $P\subset\mathbb{R}^n$, let $h(P,\cdot):\mathbb{R}^n\to\mathbb{R}$ be its support function, i.e., $h(P,\boldsymbol{v})=\max\{\boldsymbol{v}^\intercal\boldsymbol{x}:\boldsymbol{x}\in P\}$. We refer to [14, Section 1.7.1] for more information on support functions. Let $\boldsymbol{v}_j\in\mathbb{Z}^n$, $1\leq j\leq q$, be integral outer unit normals of the facets of the rational polytope $P_1+\ldots+P_k$. Observe that for all $\boldsymbol{r}\in\mathbb{R}^k_{>\boldsymbol{0}}$ the facets of the polytope $r_1P_1+\ldots+r_kP_k$ have the same outer normals $\boldsymbol{v}_j$, $1\leq j\leq q$.

Now, for $\boldsymbol{r}\in\mathbb{R}^k_{>\boldsymbol{0}}$ and $\boldsymbol{z}\in\mathbb{Z}^n$ we know $\boldsymbol{z}\in\sum_{i=1}^k r_iP_i$ if and only if $\boldsymbol{v}_j^\intercal\boldsymbol{z}\leq\sum_{i=1}^k r_ih(P_i,\boldsymbol{v}_j)$ for $1\leq j\leq q$. Thus $\mathrm{Q}(P_1,\dots,P_k,\boldsymbol{r})$ is a constant function on the interior of the $k$-dimensional cells induced by the hyperplane arrangement

$$\left\{\left\{\boldsymbol{r}\in\mathbb{R}^k_{\geq 0}:\sum_{i=1}^k r_i\,h(P_i,\boldsymbol{v}_j)=\boldsymbol{v}_j^\intercal\boldsymbol{z}\right\}:\boldsymbol{z}\in\mathbb{Z}^n,j=1,\dots,q\right\}.$$

Let $S$ be the interior of a fixed $k$-dimensional cell given by this section. Then $\mathrm{Q}(P_1,\dots,P_k,\cdot)$ is constant on $S$. Moreover, due to the definition of $\boldsymbol{\tau}$ we have that $S+\boldsymbol{u}\odot\boldsymbol{\tau}$, $\boldsymbol{u}\in\mathbb{N}^k$, lies also inside the interior of a cell of the arrangement. Hence $\mathrm{Q}(P_1,\dots,P_k,\cdot)$ is constant on $S+\boldsymbol{u}\odot\boldsymbol{\tau}$ for a given $\boldsymbol{u}\in\mathbb{N}^k$. Thus, according to Lemma 2.2, we know that for $\boldsymbol{l}\in I(k,n)$, $|\boldsymbol{l}|_1<n$, and for $\boldsymbol{r}$ in the interior of an arbitrary $k$-dimensional cell of the above hyperplane arrangement the coefficient $\mathrm{Q}_{\boldsymbol{l}}(P_1,\dots,P_k,\boldsymbol{r})$ is a polynomial of degree $n-|\boldsymbol{l}|_1$ satisfying the partial differential equation (1.2). Finally, we observe that the assumption in Theorem 1.3 on the continuity of $\mathrm{Q}_{\boldsymbol{l}}(P_1,\dots,P_k,\boldsymbol{r})$ implies that $\boldsymbol{r}$ lies in the interior of a $k$-dimensional cell of the arrangement. □

The proof of Lemma 2.2 is done by induction on the degree, and for readability we outsource a part of the induction step to the next lemma. It just says that if a kind of chain of certain functions is a polynomial, then each of these function has to be a polynomial.

**Lemma 2.3.** *Let $k \geq 1$, $S \subset \mathbb{R}^k$ open, and for $\boldsymbol{l} \in I(k,n)$ let $p_{\boldsymbol{l}} : S \to \mathbb{R}$ functions. For $j \in \{1,\dots,k\}$ and $\boldsymbol{g} \in I(k,n-1)$ let $q_{\boldsymbol{g},j} : S \to \mathbb{R}$ be given by*

$$q_{\boldsymbol{g},j}(\boldsymbol{r}) = \sum_{i=1}^{n-|\boldsymbol{g}|_1} c_{(\boldsymbol{g},i,j)}\, p_{\boldsymbol{g}+i\boldsymbol{e}_j}(\boldsymbol{r}),$$

*where $c_{(\boldsymbol{g},i,j)} \in \mathbb{R}$ are constants. If $q_{\boldsymbol{g},j}$ is a polynomial of degree $n-1-|\boldsymbol{g}|_1$ in $\boldsymbol{r}$, for all $j \in \{1,\dots,k\}$, $\boldsymbol{g} \in I(k,n-1)$, then $p_{\boldsymbol{l}}$ is a polynomial of degree $n-|\boldsymbol{l}|_1$ in $\boldsymbol{r}$ , for all $\boldsymbol{l} \in I(k,n)$, $|\boldsymbol{l}|_1 > 0$.*

*Proof.* We proceed by induction on $|\boldsymbol{l}|_1$ and start with $|\boldsymbol{l}|_1 = n$. Then for $l_m \neq 0$ we have

$$q_{\boldsymbol{l}-\boldsymbol{e}_m,m}(\boldsymbol{r}) = c_{(\boldsymbol{l}-\boldsymbol{e}_m,1,m)}\, p_{\boldsymbol{l}}(\boldsymbol{r}).$$

Hence $p_{\boldsymbol{l}}(\boldsymbol{r})$ is constant. In the same way we find for $|\boldsymbol{l}|_1 \leq n-1$ and $l_m \neq 0$

$$\begin{aligned} q_{\boldsymbol{l}-\boldsymbol{e}_m,m}(\boldsymbol{r}) &= \sum_{i=1}^{n-|\boldsymbol{l}|_1+1} c_{(\boldsymbol{l}-\boldsymbol{e}_m,i,m)}\, p_{\boldsymbol{l}-\boldsymbol{e}_m+i\boldsymbol{e}_m}(\boldsymbol{r}) \\ &= c_{(\boldsymbol{l}-\boldsymbol{e}_m,1,m)}\, p_{\boldsymbol{l}}(\boldsymbol{r}) + \sum_{i=2}^{n-|\boldsymbol{l}|_1+1} c_{(\boldsymbol{l}-\boldsymbol{e}_m,i,m)}\, p_{\boldsymbol{l}-\boldsymbol{e}_m+i\boldsymbol{e}_m}(\boldsymbol{r}). \end{aligned}$$

By our inductive approach the sum on the right hand side is a polynomial of degree $n - |\boldsymbol{l}|_1 - 1$ and by assumption the left side is a polynomial of degree $n-|\boldsymbol{l}|_1$. Hence $p_{\boldsymbol{l}}(\boldsymbol{r})$ is also a polynomial of degree $n-|\boldsymbol{l}|_1$.

□

Now we are ready to prove Lemma 2.2.

*Proof of Lemma 2.2.* First we will prove the polynomiality of the functions $p_{\boldsymbol{l}}$ by induction on $n$. Let $n=1$. By (2.2) and (2.1) we have

$$c_{\boldsymbol{0}} = p(\boldsymbol{r}) = p_{\boldsymbol{0}}(\boldsymbol{r}) + \sum_{\boldsymbol{l}\in I(k,1),|\boldsymbol{l}|_1=1} p_{\boldsymbol{l}}(\boldsymbol{r})\boldsymbol{r}^{\boldsymbol{l}}$$

By assumption, for $|\boldsymbol{l}|_1 = n = 1$ the functions $p_{\boldsymbol{l}}(\boldsymbol{r})$ are constants and so $p_{\boldsymbol{0}}(\boldsymbol{r})$ is a polynomial of degree 1.

Now let $n \geq 2$. To shorten notation, we denote for $\boldsymbol{l} \in I(k,n)$ by $\overline{\boldsymbol{l}} \in I(k-1,n)$ the vector consisting of the first $k-1$ coordinates of $\boldsymbol{l}$, i.e., $\overline{\boldsymbol{l}} = (l_1,\dots,l_{k-1})$. In order to apply induction we consider the function

$$q(\boldsymbol{r}) = p(\boldsymbol{r}+\tau_k\boldsymbol{e}_k) - p(\boldsymbol{r}). \tag{2.4}$$

Due to (2.2) we have

$$q(\boldsymbol{r}+\boldsymbol{u}\odot\boldsymbol{\tau}) = p(\boldsymbol{r}+(\boldsymbol{u}+\boldsymbol{e}_k)\odot\boldsymbol{\tau}) - p(\boldsymbol{r}+\boldsymbol{u}\odot\boldsymbol{\tau}) = c_{\boldsymbol{u}+\boldsymbol{e}_k} - c_{\boldsymbol{u}} \tag{2.5}$$

for all $\boldsymbol{r} \in S \cap \mathbb{Q}^k$, $\boldsymbol{u} \in \mathbb{N}^k$.

Next we observe that $q(\cdot)$ is a polynomial of degree $n-1$:

$$\begin{aligned} q(\boldsymbol{r}) = p(\overline{\boldsymbol{r}}, r_k+\tau_k) - p(\boldsymbol{r}) &= \sum_{\boldsymbol{l}\in I(k,n)} p_{\boldsymbol{l}}(\boldsymbol{r})\overline{\boldsymbol{r}}^{\overline{\boldsymbol{l}}}\left((r_k+\tau_k)^{l_k} - r_k^{l_k}\right) \\ &= \sum_{\boldsymbol{l}\in I(k,n)} p_{\boldsymbol{l}}(\boldsymbol{r})\left(\sum_{i=0}^{l_k-1}\binom{l_k}{i}\tau_k^{l_k-i}\overline{\boldsymbol{r}}^{\overline{\boldsymbol{l}}}r_k^i\right) \\ &= \sum_{(\overline{\boldsymbol{l}},m)\in I(k,n-1)} \overline{\boldsymbol{r}}^{\overline{\boldsymbol{l}}} r_k^m \sum_{i=m+1}^{n-|\overline{\boldsymbol{l}}|} p_{(\overline{\boldsymbol{l}},i)}(\boldsymbol{r})\binom{i}{m}\tau_k^{i-m} \\ &= \sum_{\boldsymbol{l}\in I(k,n-1)} q_{\boldsymbol{l}}(\boldsymbol{r})\boldsymbol{r}^{\boldsymbol{l}}, \end{aligned}$$

with

$$q_{\boldsymbol{l}}(\boldsymbol{r}) = \sum_{i=l_k+1}^{n-|\boldsymbol{l}|_1+l_k} p_{(\overline{\boldsymbol{l}},i)}(\boldsymbol{r})\binom{i}{l_k}\tau_k^{i-l_k} = \sum_{i=1}^{n-|\boldsymbol{l}|_1} p_{\boldsymbol{l}+i\boldsymbol{e}_k}(\boldsymbol{r})\binom{i+l_k}{l_k}\tau_k^i.$$

In particular, for $|\boldsymbol{l}|_1 = n-1$ we get $q_{\boldsymbol{l}}(\boldsymbol{r}) = p_{(\overline{\boldsymbol{l}},l_k+1)}(\boldsymbol{r})(l_k+1)\tau_k$, which by assumption is a constant. Thus we know that $q:\mathbb{Q}^k\to\mathbb{Q}$ is a rational quasi-polynomial of degree $n-1$ with period $\boldsymbol{\tau}$ and constant leading coefficients. In view of (2.5) we get by our induction that $q_{\boldsymbol{l}}(\boldsymbol{r})$ is a polynomial of degree $n-1-|\boldsymbol{l}|_1$ in $\boldsymbol{r}$.

Obviously, this is also true if we replace the index $k$ in the definition of $q$ (cf. (2.4)) by any other index $j\in\{1,\dots,k\}$. Hence, for $j\in\{1,\dots,k\}$ and $\boldsymbol{l}\in I(k,n-1)$ the functions

$$q_{\boldsymbol{l},j}(\boldsymbol{r}) = \sum_{i=1}^{n-|\boldsymbol{l}|_1} p_{\boldsymbol{l}+i\boldsymbol{e}_j}(\boldsymbol{r})\binom{i+l_j}{l_j}\tau_j^i$$

are polynomials of degree $n-1-|\boldsymbol{l}|_1$ in $\boldsymbol{r}$. According to Lemma 2.3 this implies that $p_{\boldsymbol{l}}(\boldsymbol{r})$ is a polynomial of degree $n-|\boldsymbol{l}|_1$ in $\boldsymbol{r}$ for all $\boldsymbol{l}\in I(k,n)$, $|\boldsymbol{l}|_1>0$. The missing case $p_{\boldsymbol{0}}(\boldsymbol{r})$ follows immediately from the identity (2.2)

$$c_{\boldsymbol{0}} = p_{\boldsymbol{0}}(\boldsymbol{r}) + \sum_{\boldsymbol{l}\in I(k,n)\setminus\{\boldsymbol{0}\}} p_{\boldsymbol{l}}(\boldsymbol{r})\boldsymbol{r}^{\boldsymbol{l}}.$$

It remains to show (2.3), i.e.,

$$(l_j+1)p_{\boldsymbol{l}+\boldsymbol{e}_j}(\boldsymbol{r}) + \frac{\partial}{\partial r_j}p_{\boldsymbol{l}}(\boldsymbol{r}) = 0.$$

for $\boldsymbol{l}\in I(k,n)$ with $|\boldsymbol{l}|_1<n$, and for all $\boldsymbol{r}\in S$. To this end we are looking for an explicit formula for $p_{\boldsymbol{l}}(\boldsymbol{r})$. Since we already know that $p_{\boldsymbol{l}}(\boldsymbol{r})$ is a polynomial of degree $n-|\boldsymbol{l}|_1$ we may write it as

$$p_{\boldsymbol{l}}(\boldsymbol{r}) = \sum_{\boldsymbol{g}\in I(k,n-|\boldsymbol{l}|_1)} p_{\boldsymbol{l},\boldsymbol{g}}\,\boldsymbol{r}^{\boldsymbol{g}},$$

for some coefficients $p_{\boldsymbol{l},\boldsymbol{g}}\in\mathbb{Q}$.

By the periodicity of $p_{\boldsymbol{l}}(\boldsymbol{r})$ and (2.2) we have for $\boldsymbol{u}\in\mathbb{N}^k$

$$\begin{aligned} c_{\boldsymbol{u}} = p(\boldsymbol{r}+\boldsymbol{u}\odot\boldsymbol{\tau}) &= \sum_{\boldsymbol{l}\in I(k,n)} p_{\boldsymbol{l}}(\boldsymbol{r})(\boldsymbol{r}+\boldsymbol{u}\odot\boldsymbol{\tau})^{\boldsymbol{l}} \\ &= \sum_{\boldsymbol{l}\in I(k,n)}\left(\sum_{\boldsymbol{g}\in I(k,n-|\boldsymbol{l}|_1)} p_{\boldsymbol{l},\boldsymbol{g}}\,\boldsymbol{r}^{\boldsymbol{g}}\right)(\boldsymbol{r}+\boldsymbol{u}\odot\boldsymbol{\tau})^{\boldsymbol{l}}. \end{aligned} \tag{2.6}$$

Since the right hand side regarded as a polynomial in $\boldsymbol{r} \in S \cap \mathbb{Q}^k$ is constant, all powers of the variables $r_j$ have to vanish and thus

$$c_{\boldsymbol{u}} = p(\boldsymbol{r} + \boldsymbol{u} \odot \boldsymbol{\tau}) = \sum_{\boldsymbol{l} \in I(k,n)} p_{\boldsymbol{l},\boldsymbol{0}} (\boldsymbol{u} \odot \boldsymbol{\tau})^{\boldsymbol{l}}.$$

Setting $\tilde{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{u} \odot \boldsymbol{\tau}$ we may write

$$\begin{aligned}
c_{\boldsymbol{u}} = p(\boldsymbol{r} + \boldsymbol{u} \odot \boldsymbol{\tau}) &= \sum_{\boldsymbol{l} \in I(k,n)} p_{\boldsymbol{l},\boldsymbol{0}} (\tilde{\boldsymbol{r}} - \boldsymbol{r})^{\boldsymbol{l}} \\
&= \sum_{\boldsymbol{l} \in I(k,n)} p_{\boldsymbol{l},\boldsymbol{0}} \sum_{\substack{\boldsymbol{g} \in I(k,n) \\ \boldsymbol{g} \le \boldsymbol{l}}} \left( \prod_{i=1}^{k} \binom{l_i}{g_i} (-1)^{l_i - g_i} \right) \tilde{\boldsymbol{r}}^{\boldsymbol{g}} \boldsymbol{r}^{\boldsymbol{l}-\boldsymbol{g}} \\
&= \sum_{\boldsymbol{g} \in I(k,n)} \left( \sum_{\substack{\boldsymbol{l} \in I(k,n) \\ \boldsymbol{l} \ge \boldsymbol{g}}} \left( \prod_{i=1}^{k} \binom{l_i}{g_i} (-1)^{l_i - g_i} \right) p_{\boldsymbol{l},\boldsymbol{0}} r^{\boldsymbol{l}-\boldsymbol{g}} \right) \tilde{r}^{\boldsymbol{g}} \\
&= \sum_{\boldsymbol{g} \in I(k,n)} \left( \sum_{\substack{\boldsymbol{l} \in I(k,n) \\ |\boldsymbol{l}|_1 \le n - |\boldsymbol{g}|_1}} \left( \prod_{i=1}^{k} \binom{l_i + g_i}{g_i} (-1)^{l_i} \right) p_{\boldsymbol{l}+\boldsymbol{g},\boldsymbol{0}} \boldsymbol{r}^{\boldsymbol{l}} \right) \tilde{\boldsymbol{r}}^{\boldsymbol{g}}.
\end{aligned}$$

Compared with the first equation in (2.6) we conclude

$$p_{\boldsymbol{g}}(\boldsymbol{r}) = \sum_{\substack{\boldsymbol{l} \in I(k,n) \\ |\boldsymbol{l}|_1 \le n - |\boldsymbol{g}|_1}} \left( \prod_{i=1}^{k} \binom{l_i + g_i}{g_i} (-1)^{l_i} \right) p_{\boldsymbol{l}+\boldsymbol{g},\boldsymbol{0}} \boldsymbol{r}^{\boldsymbol{l}},$$

or by interchanging the role of $\boldsymbol{g}$ and $\boldsymbol{l}$

$$p_{\boldsymbol{l}}(\boldsymbol{r}) = \sum_{\substack{\boldsymbol{g} \in I(k,n) \\ |\boldsymbol{g}|_1 \le n - |\boldsymbol{l}|_1}} \left( \prod_{i=1}^{k} \binom{g_i + l_i}{l_i} (-1)^{g_i} \right) p_{\boldsymbol{l}+\boldsymbol{g},\boldsymbol{0}} \boldsymbol{r}^{\boldsymbol{g}}.$$

It remains to calculate the partial derivative. To this end let $|\boldsymbol{l}|_1 < n$ and for short we set $\alpha(i,j) := \binom{i+j}{j}(-1)^i$. Then

$$\begin{aligned}
\frac{\partial}{\partial r_j} p_{\boldsymbol{l}}(\boldsymbol{r}) &= \sum_{\substack{\boldsymbol{g} \in I(k,n) \\ |\boldsymbol{g}|_1 \le n - |\boldsymbol{l}|_1, g_j \ge 1}} \left( \prod_{i=1}^{k} \alpha(g_i, l_i) \right) p_{\boldsymbol{l}+\boldsymbol{g},\boldsymbol{0}}\, g_j\, \boldsymbol{r}^{\boldsymbol{g}-\boldsymbol{e}_j} \\
&= \sum_{\substack{\boldsymbol{g} \in I(k,n) \\ |\boldsymbol{g}|_1 \le n - |\boldsymbol{l}|_1 - 1}} \left( \prod_{\substack{i=1 \\ i \ne j}}^{k} \alpha(g_i, l_i) \right) (-1)^{g_j+1} \binom{g_j + l_j + 1}{l_j} p_{\boldsymbol{l}+\boldsymbol{g}+\boldsymbol{e}_j,\boldsymbol{0}} (g_j + 1) \boldsymbol{r}^{\boldsymbol{g}} \\
&= -(l_j + 1) \sum_{\substack{\boldsymbol{g} \in I(k,n) \\ |\boldsymbol{g}|_1 \le n - |\boldsymbol{l}|_1 - 1}} \left( \prod_{\substack{i=1 \\ i \ne j}}^{k} \alpha(g_i, l_i) \right) (-1)^{g_j} \binom{g_j + l_j + 1}{l_j + 1} p_{\boldsymbol{l}+\boldsymbol{g}+\boldsymbol{e}_j,\boldsymbol{0}} \boldsymbol{r}^{\boldsymbol{g}} \\
&= -(l_j + 1) p_{\boldsymbol{l}+\boldsymbol{e}_j}(\boldsymbol{r}),
\end{aligned}$$

which finishes the proof. □

## 3. Sketch of the proof of Corollary 1.5

The proof of Corollary 1.5 follows the general approach of Mount [13, Theorem 2] in the integral case $b \in \mathbb{Z}^m$. Therefore, we will only describe the main steps here in order to prove Corollary 1.5.

First we recall some notation. For $A \in \mathbb{Z}^{m\times n}$, $\boldsymbol{b} \in \mathbb{Q}^m$ let $P_A(\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \le \boldsymbol{b}\}$, where we assume that the polytope $P_A(\boldsymbol{b})$ is bounded for all right hand sides. For given $\boldsymbol{b} \in \mathbb{Q}^m$, the number of lattice points $P_A(\boldsymbol{b})$ is denoted by $\Phi(A,\boldsymbol{b})$, i.e., $\Phi(A,\boldsymbol{b}) = \#(P_A(\boldsymbol{b}) \cap \mathbb{Z}^n)$.

Furthermore, $N$ was a fixed, but arbitrary, normal fan of $P_A(\boldsymbol{b})$, and $C_N$ is the rational cone consisting of all vectors $\boldsymbol{b} \in \mathbb{R}^m$ for which all the polytopes $P_A(\boldsymbol{b})$, $b \in C_N$, have the normal fan $N$, i.e., for which the polytopes are locally similar. Moreover, it is known that

**Lemma 3.1** (McMullen [11, Section 2, Section 6])**.** *$C_N$ is a polyhedral cone, and for $\boldsymbol{b},\boldsymbol{c} \in C_N$ we have that $P_A(\boldsymbol{b}) + P_A(\boldsymbol{c}) = P_A(\boldsymbol{b}+\boldsymbol{c})$.*

*Sketch of the proof of Corollary 1.5.* Due to Lemma 3.1 let $C_N = \mathrm{pos}\{\boldsymbol{h}_1,\dots,\boldsymbol{h}_k\}$ with $\boldsymbol{h}_i \in \mathbb{Q}^m$, $1 \le i \le k$. In order to get results on lattice points in vector dilated polytopes via Minkowski-sums of polytopes we just follow the approach of Mount [13, Theorem 2] in the integral case, i.e., we perform the following steps

- Fix $\boldsymbol{b} \in C_N \cap \mathbb{Q}^m$ and write $\sum_{i=1}^k \lambda_i \boldsymbol{h}_i = \boldsymbol{b}$ with $\lambda_i \in \mathbb{Q}_{\ge 0}$, $1 \le i \le k$. By Lemma 3.1 we have
$$\Phi(A,\boldsymbol{b}) = \#(P_A(\boldsymbol{b}) \cap \mathbb{Z}^n) = \#\left(\sum_{i=1}^m \lambda_i P_A(\boldsymbol{h}_i) \cap \mathbb{Z}^n\right).$$
- Apply statements on lattice points in Minkowski-sums to the latter sum.
- Make the relation between $\boldsymbol{b}$ and scalars $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_m)$ one-to-one by fixing a certain canonical choice for $\boldsymbol{\lambda}$. With this bijection, the statements about lattice points in Minkowski-sums can be transformed into statements with right-hand-side variable $\boldsymbol{b}$.

Using this strategy, Theorem 1.2 yields Theorem 1.4. Analogously, combining Theorem 1.2 with Theorem 1.3 gives the refinement stated in Corollary 1.5.

Since the leading term of $\mathrm{Q}(P_A(\boldsymbol{h}_1),\dots,P_A(\boldsymbol{h}_k),\boldsymbol{\lambda})$ is the volume of the Minkowski-sum $\sum_{i=1}^k \lambda_i P_A(v h_i)$ we also get that the leading term of $\Phi_A(\boldsymbol{b})$ is the volume of $P_A(\boldsymbol{b})$. This implies, in particular, that $\mathrm{vol}(P_A(\boldsymbol{b}))$ is a homogeneous polynomial of degree $n$ in $\boldsymbol{b}$ (see also the next example). We refer to [18] for a closed formula of this polynomial. $\square$

**Example 3.2.** As an example, we consider the following polytope $P_A(\boldsymbol{b})$ in dimension $n = 2$ with $m = 4$ inequalities (see Figure 2):

$$P_A(\boldsymbol{b}) = \{\boldsymbol{x}^\intercal \in \mathbb{R}^2 : \quad \begin{array}{rcl} 2x_1 + x_2 & \le & b_1, \\ -2x_1 + x_2 & \le & b_2, \\ x_2 & \le & b_3, \\ -x_2 & \le & b_4 \end{array} \}.$$

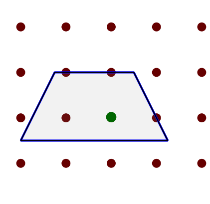

FIGURE 2. $P_A((2, \frac{7}{2}, 1, \frac{1}{2}))$

The intersection point of both non-horizontal inequalities is $\boldsymbol{v} = \left(\frac{b_1-b_2}{4}, \frac{b_1+b_2}{2}\right)$. This polytope is nonempty, whenever $-b_4 \le b_3$ and $v_2 \ge -b_4$, that is $b_1 + b_2 + 2d_4 \ge 0$. If $v_2 \le b_3$, that is $b_1 + b_2 - 2b_3 \le 0$, then $P(\boldsymbol{b})$ is a triangle. Otherwise, that is, if $b_1 + b_2 - 2b_3 > 0$, $P$ is a proper quadrangle. Hence, there are the following possible normal fans:

$$C_{\text{point}} = \{\boldsymbol{b} \in \mathbb{R}^4 : b_3 + b_4 \ge 0, b_1 + b_2 + 2b_4 = 0\}$$

for a single point,

$$C_{\text{line}} = \{\boldsymbol{b} \in \mathbb{R}^4 : b_3 + b_4 = 0, b_1 + b_2 + 2b_4 \ge 0\}$$

for a line-segment,

$$C_{3\text{-gon}} = \{\boldsymbol{b} \in \mathbb{R}^4 : b_3 + b_4 \ge 0, b_1 + b_2 + 2b_4 \ge 0, b_1 + b_2 - 2b_3 \le 0\}$$

for the triangle and

$$C_{4\text{-gon}} = \{\boldsymbol{b} \in \mathbb{R}^4 : b_3 + b_4 \ge 0, b_1 + b_2 + 2b_4 \ge 0, b_1 + b_2 - 2b_3 \ge 0\}$$

for the quadrangle.

Since $C_{\text{point}} \subset C_{3\text{-gon}}$ and $C_{\text{line}} \subset C_{4\text{-gon}}$ it is sufficient to investigate $C_{3\text{-gon}}$ and $C_{4\text{-gon}}$. Here we get for $\boldsymbol{b} \in C_{4\text{-gon}}$

$$\begin{aligned}
\Phi(A, \boldsymbol{b}) &= \frac{1}{2}\left(b_4^2 - b_3^2 + b_2 b_3 + b_1 b_3 + b_2 b_4 + b_1 b_4\right) + \frac{b_1}{2}\left(1 - \{b_4\} - \{b_3\}\right) \\
&+ \frac{b_2}{2}\left(1 - \{b_4\} - \{b_3\}\right) + \frac{b_3}{2}\left(-\{b_2\} + 2\{b_3\} - \{b_1\}\right) + \frac{b_4}{2}\left(2 - 2\{b_4\} - \{b_2\} - \{b_1\}\right) \\
&+ \Bigg(\left\{\frac{b_3 - b_1}{2}\right\}^2 + \left\{\frac{b_3 - b_2}{2}\right\}^2 - \left\{\frac{b_1 + b_4}{2}\right\}^2 - \left\{\frac{b_2 + b_4}{2}\right\}^2 \\
&+ \left\{\frac{b_1 + b_4}{2}\right\}\{b_1\} + \left\{\frac{b_2 + b_4}{2}\right\}\{b_2\} + \left\{\frac{b_3 - b_1}{2}\right\}\{b_1\} + \left\{\frac{b_3 - b_2}{2}\right\}\{b_2\} \\
&- \left\{\frac{b_3 - b_1}{2}\right\}\{b_3\} - \left\{\frac{b_3 - b_2}{2}\right\}\{b_3\} + \left\{\frac{b_2 + b_4}{2}\right\}\{b_4\} + \left\{\frac{b_1 + b_4}{2}\right\}\{b_4\} \\
&- \left\{\frac{b_3 - b_2}{2}\right\} - \left\{\frac{b_3 - b_1}{2}\right\} - \{b_1\} - \{b_2\} + \{b_3\} - \{b_4\} + 1\Bigg),
\end{aligned}$$

and for $\boldsymbol{b} \in C_{3\text{-gon}}$

$$\begin{aligned}
\Phi(A, \boldsymbol{b}) &= \frac{1}{8}\left(b_1^2 + b_2^2 + 4b_4^2 + 2b_1 b_2 + 4b_1 b_4 + 4b_2 b_4\right) \\
&+ \frac{b_1}{4}\left(2 - \{b_1\} - \{b_2\} - 2\{b_4\}\right) + \frac{b_2}{4}\left(2 - \{b_1\} - \{b_2\} - 2\{b_4\}\right) \\
&+ \frac{b_4}{2}\left(2 - \{b_1\} - \{b_2\} - 2\{b_4\}\right) \\
&+ \Bigg(2\left\{\frac{b_1 - b_2}{4}\right\}^2 - \left\{\frac{b_2 + b_4}{2}\right\}^2 - \left\{\frac{b_1 + b_4}{2}\right\}^2 + \{b_1\}\left\{\frac{b_1 + b_4}{2}\right\} + \{b_2\}\left\{\frac{b_2 + b_4}{2}\right\} \\
&- \{b_1\}\left\{\frac{b_1 - b_2}{4}\right\} - \{b_2\}\left\{\frac{b_2 - b_1}{4}\right\} + \{b_4\}\left\{\frac{b_2 + b_4}{2}\right\} + \{b_4\}\left\{\frac{b_1 + b_4}{2}\right\} \\
&- \{b_4\} - 2\left\{\frac{b_1 - b_2}{4}\right\} + 1\Bigg).
\end{aligned}$$

*Acknowledgement.* We are very grateful to the referee for his/her appropriate and constructive suggestions and corrections.

## References


[1] Velleda Baldoni, Nicole Berline, Matthias Köppe, and Michèle Vergne. Intermediate Sums on Polyhedra: Computation and Real Ehrhart Theory. *Mathematika*, 59(1):1–22, 2013.

[2] Velleda Baldoni, Nicole Berline, Jesùs A. De Loera, Matthias Köppe and Michèle Vergne. Intermediate Sums on Polyhedra II: Bidegree and Poisson Formula. arXiv:1404.0065, 2014.

[3] ______. Three Ehrhart Quasi-polynomials. arXiv:1410.8632, 2014.

[4] Matthias Beck. A closer look at lattice points in rational simplices. *Electronic Journal of Combinatorics*, 6, Research Paper 37, 1999; http://www.combinatorics.org/ojs/index.php/eljc/article/view/v6i1r37/pdf

[5] ______. Multidimensional Ehrhart reciprocity. *Journal of Combinatorial Theory, Series A*, 97(1):187–194, 2002.

[6] Matthias Beck and Sinai Robins. *Computing the Continuous Discretely*. Springer, 2007.

[7] Wolfgang Dahmen and Charles A. Micchelli. The number of solutions to linear Diophantine equations and multivariate splines. *Transactions of the American Mathematical Society*, 308(2):509–532, 1988.

[8] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *Comptes Rendus des Séances de l'Académie des Sciences Série A*, 254:616–618, 1962.

[9] Matthias Köppe and Sven Verdoolaege. Computing Parametric Rational Generating Functions with a Primal Barvinok Algorithm. *The Electronic journal of Combinatorics*, 15, Research Paper 16, 2008; http://www.combinatorics.org.mathscinet.han.med.uni-magdeburg.de/ojs/index.php/eljc/article/view/v15i1r16/pdf

[10] Eva Linke. Rational Ehrhart quasi-polynomials. *Journal of Combinatorial Theory, Series A*, 118(7):1966–1978, 2011.

[11] Peter McMullen. Representations of polytopes and polyhedral sets. *Geometriae Dedicata*, 2:83–99, 1973.

[12] ______, Lattice invariant valuations on rational polytopes. *Archiv der Mathematik*, 31(1):509-516, 1978.

[13] John Mount. Fast unimodular counting. *Combinatorics, Probability and Computing*, 9:277–285, 2000.

[14] Rolf Schneider. *Convex bodies: The Brunn-Minkowski Theory*. 2nd expanded edition, Cambridge University Press, 2013.

[15] Bernd Sturmfels. On vector partition functions. *Journal of Combinatorial Theory, Series A*, 72:302–309, 1995.

[16] Sven Verdoolaege, Rachid Seghir, Kristof Beyls, Vincent Loechner and Maurice Bruynooghe. Counting integer points in parametric polytopes using Barvinok's rational functions. Algorithmica, 48(1):37–66, 2007.

[17] Sven Verdoolaege and Kevin Woods. Counting with rational generating functions. *J. Symbolic Comput.*, 43(2):75–91, 2008.

[18] Zhiqiang Xu. Multivariate splines and polytopes. *Journal of Approximation Theory*, 163(3), 377–387, 2011.

[19] Günter M. Ziegler. *Lectures on polytopes*. Springer, 1995.



TECHNISCHE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK, SEKR. MA 4-1, STRASSE DES 17 JUNI 136, D-10623 BERLIN, GERMANY

*E-mail address*: henk@math.tu-berlin.de, eva.linke@gmail.com