A NOTE ON A SUMSET IN $\mathbb{Z}_{2k}$

OCTAVIO A. AGUSTÍN-AQUINO

Abstract. Let A and B be additive sets of $\mathbb{Z}_{2k}$, where A has cardinality k and $B = v.\complement A$ with $v \in \mathbb{Z}_{2k}^{\times}$. In this note some bounds for the cardinality of $A+B$ are obtained, using four different approaches. We also prove that in a special case the bound is not sharp and we can recover the whole group as a sumset.

1. Introduction

In music, a *canon* is typically understood as a musical composition where a melody is imitated by various voices, with a duration offset between them (well known examples are "Row, Row, Row Your Boat" or "Frère Jacques"). A canon like those can be more aptly described as a "pitch canon", in contraposition to the "rhythmic canons" introduced by Oliver Messiaen, where the rhythm is imitated instead of the melody. In this direction, a remarkable and pioneering use of sumsets in music was done by Dan Tudor Vuza, introducing what he called "Regular Complementary Canons of Maximal Category", which are aperiodic sumsets $S, T \subseteq \mathbb{Z}_n$ such that $\mathbb{Z}_n = S + T$, where S (or T) represents the set of duration offsets between rhythmic imitations. For a nice introduction to the fascinating interplay of music and mathematics in this regard, see [3].

In this note, we make further connections between sumsets and the musical realm of *counterpoint*, where canon is but one of its techniques. Thus let U and V be additive subsets of $\mathbb{Z}_{2k}$ with cardinality k, and

$$U + V = \{u + v : u \in U, v \in V\},$$
$$x.U = \{xa : a \in U\}.$$

I stumbled upon the problem of proving that, if k is large enough and under certain hypothesis regarding the structure of U, we have

$$U + V = \mathbb{Z}_{2k}$$

where U is a set closely related to V. Hence U and V are akin to Vuza canons, except for aperiodicity is not required beforehand but some other conditions are to be fulfilled. To be more specific, a very interesting case from the mathematical counterpoint theory perspective is when

$$V = v.\complement U, \quad v \in \mathbb{Z}_{2k}^{\times} \setminus \{-1\}$$

Date: 15 December 2017.

1991 *Mathematics Subject Classification.* 11B13, 11L07.

and, additionally, $\complement U = U + k$ (here $\complement$ stands for the set complement with respect to $\mathbb{Z}_{2k}$). In order to explain why, let $\overrightarrow{GL}(\mathbb{Z}_{2k})$ be the set of bijective functions

$$\begin{aligned} e^u.v : \mathbb{Z}_{2k} &\to \mathbb{Z}_{2k}, \\ x &\mapsto vx + u, \end{aligned}$$

where $v \in \mathbb{Z}_{2k}^{\times}$ and $u \in \mathbb{Z}_{2k}$. If $A \subseteq \mathbb{Z}_{2k}$ is such that $g(A) \neq A$ for every $g \in \overrightarrow{GL}(\mathbb{Z}_{2k})$ except the identity, and $A \cup p(A) = \mathbb{Z}_{2k}$ for a unique $p \in \overrightarrow{GL}(\mathbb{Z}_{2k})$, then it is called a *counterpoint dichotomy* and p is its *polarity*.

Example 1.1. *One interesting example is $A = \{0, 2, 3\} \subseteq \mathbb{Z}_6$, whose polarity is $e^1.-1$, since it is essentially the only one available. Another important specimen is*

$$K = \{0, 3, 4, 7, 8, 9\} \subseteq \mathbb{Z}_{12},$$

with polarity $e^2.5$, for K is the set of consonances in Renaissance counterpoint modulo octave, when the intervals in 12-tone equally tempered scale are interpreted as $\mathbb{Z}_{12}$. The interested reader may consult [9, Part VII] *or* [1] *and references therein for further details.*

Throughout this paper, we will attack (with varying degrees of generality) the following question.

Question 1.2. *Given a subset $A \subseteq \mathbb{Z}_{2k}$ of cardinality k, is it true that*

$$A + v.\complement A = \begin{cases} \mathbb{Z}_{2k}, & v \in \mathbb{Z}_{2k}^{\times} \setminus \{-1\}, \\ \mathbb{Z}_{2k} \setminus \{0\}, & v = -1? \end{cases} \tag{1}$$

When this question can be answered in the affirmative then, for any $e^u.(-v)$ except the identity, there exists $x \in A$ and $y \in \complement A$ such that

$$x + (-v)y = u \quad \text{or} \quad vy + u = x \quad \text{or} \quad e^u.(-v)(y) \in A$$

which means that no element of $\overrightarrow{GL}(\mathbb{Z}_{2k})$ but the identity leaves the set A invariant. If there exists also a $p \in \overrightarrow{GL}(\mathbb{Z}_{2k})$ such that $p(A) = \complement A$, then A is a counterpoint dichotomy.

A set that I have been trying to prove is a counterpoint dichotomy for a long time (some reasons for this are stated in [2]) via answering Question 1.2 is

$$A = \{0, 1\} \cup \{3, 4, \ldots, k-1\} \cup \{k+2\}. \tag{2}$$

It is not difficult to verify that $e^k.1(A) = \complement A$ and to see that

$$A + A = \mathbb{Z}_{2k} \quad \text{and} \quad A - A \supseteq \mathbb{Z}_{2k} \setminus \{k\},$$

since $2 = 1 + 1$, $2k - 1 = (k+2) + (k-3)$ and $k + 1 = (k-2) + 3$ for the first equality. The other one is consequence of $3 - 1 = 2$ and $1 - 3 = -2$.

Although the following three sections do not prove A satisfies the rest of (1), they provide some evidence and results that may be interesting on their own. Moreover, an elementary proof of this fact found by Merlijn Staps is presented in the fifth section. In the last section, some final remarks are made.

2. Using the Ruzsa distance

Let U and V be subsets of an additive group G. A couple of weak bounds for $|U+V|$ can be obtained using Ruzsa's useful notion of "distance" in additive combinatorics

$$d(U,V) = \log \frac{|U-V|}{\sqrt{|U||V|}},$$

which is a seminorm. In particular, it satisfies a triangle inequality

$$d(U,V) \leq d(U,W) + d(W,V).$$

Note now that, regarding the set (2), we have

$$d(A,-A) = \log \frac{|A+A|}{|A|} = \log \frac{2k}{k} = \log 2;$$

the number $\delta(U) = \exp(d(U,-U))$ is the *doubling constant* of the set U, and thus $\delta(A) = 2$.

From the Ruzsa triangle inequality we can deduce [12, p. 61]

$$|U||V-V| \leq |U+V|^2$$

which, for the case of $V = A$ and $U = B$, specializes to

$$|A+B| \geq \sqrt{|B||A-A|} \geq \sqrt{k(2k-1)} = \sqrt{2-\frac{1}{k}}k.$$

On the other hand, again by the triangle inequality

$$\log 2 = d(A,-A) \leq d(A,B) + d(B,-A)$$

and a pigeon-hole argument, either

$$d(A,B) \geq \frac{1}{2}\log 2$$

or

$$d(-A,B) = d(A,-B) \geq \frac{1}{2}\log 2.$$

Equivalently, either

$$|A-B| \geq \sqrt{2}k$$

or

$$|A+B| \geq \sqrt{2}k.$$

We conclude that, for any subsets A and B of the cardinality k such that $\delta(A) = 2$, we have

$$\max\{|A+B|, |A-B|\} \geq \sqrt{2}k.$$

We do not know if there exist pairs of subsets of $\mathbb{Z}_{2k}$ such that A has doubling constant 2 and $|A+B|$ or $|A-B|$ get arbitrarily close to this bound.

3. Using additive energy and a theorem by Olson

Let

$$[P] = \begin{cases} 1, & P \text{ is true}, \\ 0, & \text{otherwise}, \end{cases}$$

be the Iverson bracket [**?**, p. 24], and define the *additive energy* of the subsets U and V of the additive group G by

$$E(U,V) = \sum_{u_1,u_2\in U, v_3,v_4\in V} [u_1 + u_2 = v_3 + v_4].$$

Another well-known inequality [12, p. 63] for the cardinality of $U+V$ is

$$|U \pm V| \geq \frac{(|U||V|)^2}{E(U,V)}.$$

From this we infer another strategy to improve the previous estimates for $|A+B|$, namely finding upper bounds for $E(A,B)$. A good start might be the Cauchy-Schwarz inequality

$$E(A,B) \leq \sqrt{E(A,A)E(B,B)}.$$

This seems promising when $B = v.\complement A$ and $\complement A = A + \{k\}$, since the invertibility of v implies

$$\begin{aligned} E(v.\complement A, v.\complement A) &= \sum_{a_1,a_2,a_3,a_4\in A+\{k\}} [va_1 + va_2 = va_3 + va_4] \\ &= \sum_{a_1,a_2,a_3,a_4\in A} [v(a_1 + a_2) = v(a_3 + a_4)] \\ &= \sum_{a_1,a_2,a_3,a_4\in A} [a_1 + a_2 = v^{-1}v(a_3 + a_4)] \\ &= \sum_{a_1,a_2,a_3,a_4\in A} [a_1 + a_2 = a_3 + a_4] = E(A,A). \end{aligned}$$

Thus $E(A, v.\complement A) \leq E(A,A)$. Nevertheless, this straightforward approach loses some of its charm as soon as we calculate a few values of the energy and the corresponding bounds.

As it is readily seen in Table 1, the quality of the bound is expected to decrease as k increases, although it would remain as a mild improvement with respect the one obtained in the previous section. In fact, assuming $E(A,A)$ is a polynomial in k, from a simple interpolation from the data in Table 1 we find that

$$E(A,A) = \tfrac{2}{3}k^3 - \tfrac{47}{3}k + 80.$$

This means that, for $k \geq 6$, we have $E(A,A) \leq \frac{2}{3}k^3$, and then

$$|A \pm v.\complement A| \geq \tfrac{3}{2}k.$$

k	$E(A,A)$	$\frac{(\lvert A\rvert\lvert v.\complement A\rvert)^2}{E(A,A)} = \frac{k^4}{E(A,A)}$	$\frac{k^3}{2E(A,A)}$
8	296	13.84	0.86
9	425	15.44	0.86
10	590	16.95	0.85
11	795	18.42	0.84
12	1044	19.86	0.83
100	665180	150.34	0.751
1000	666651080	1500.04	0.750

TABLE 1. Additive energy $E(A,A)$ for small $k = |A|$, where A is defined by (2), the corresponding bound for $|A+B|$ and the fraction of $\mathbb{Z}_{2k}$ that is thus guaranteed to be covered by $A+B$.

This bound can be obtained from a theorem due to Olson, and actually it holds for any set B of cardinality k, not only those of the form $v.\complement A$. Before stating Olson's theorem, observe that an additive subset U of G is contained in a coset of a unique smallest subgroup H of G. Denote with $[U]$ such a coset.

Theorem 3.1 (Olson, 1984, [10], [8],[4]). *Let U and V be additive subsets of G. If $U+V \neq G$ and $[U] = G$, then $|U+V| \geq \frac{1}{2}|U| + |V|$.*

Suppose $G = \mathbb{Z}_{2k}$ and $U = A$. Any coset containing A has cardinality at least k. But it cannot have exactly k elements, for the cosets would be forced to be either the set of even elements of $\mathbb{Z}_{2k}$ or its complement, but clearly A is contained in neither. Thus $[A] = \mathbb{Z}_{2k}$, so if $A+B$ is not the whole group, it must consist of at least $\frac{1}{2}k + k = \frac{3}{2}k$ elements.

4. Using trigonometric sums

Let $r_{U+V}(t)$ the number of representations of t as a sum $t = u+v$ for $u \in U$ and $v \in V$, where U and V are additive subsets of a group G. The following is a standard technique using the so-called trigonometric sums in number theory (a readable and short introduction can be found in [5]). Note first that

$$\frac{1}{m}\sum_{\xi=0}^{m-1} e^{2\pi i \xi x/m} = [x \equiv 0 \pmod{m}],$$

so we can write

$$\frac{1}{2k}\sum_{\xi=0}^{2k-1} e^{2\pi i \xi (u+v-\lambda)/(2k)} = [u+v \equiv \lambda \pmod{2k}].$$

If we sum over U and V and exchange the order of summation,

$$\begin{aligned} r_{U+V}(\lambda) &= \sum_{u\in U}\sum_{v\in V}[u+v\equiv\lambda \pmod{2k}]. \\ &= \frac{1}{2k}\sum_{u\in U}\sum_{v\in V}\sum_{\xi=0}^{2k-1} e^{2\pi i\xi(u+v-\lambda)/(2k)} \\ &= \frac{1}{2k}\sum_{\xi=0}^{2k-1}\left(\sum_{u\in U} e^{2\pi i\xi u/(2k)}\sum_{v\in V} e^{2\pi i\xi v/(2k)}\right) e^{-2\pi i\xi\lambda/(2k)}, \end{aligned}$$

and then we extract the $\xi = 0$ term, we conclude

$$r_{U+V}(\lambda) = \frac{k}{2} + E$$

where, by the triangle inequality,

$$|E| \leq \frac{1}{2k}\sum_{\xi=1}^{2k-1}\left|\sum_{u\in U} e^{\pi i\xi u/k}\right|\left|\sum_{v\in V} e^{\pi i\xi v/k}\right| = 2k\sum_{\xi=1}^{2k-1}|\widehat{1}_U(\xi)||\widehat{1}_V(\xi)| \tag{3}$$

and

$$\widehat{f}(\xi) := \frac{1}{|G|}\sum_{x\in G} f(x)\overline{e^{2\pi i\xi x/|G|}}$$

is the Fourier transform. Observe now that $|\widehat{1}_{v.\complement A}(\xi)| = |\widehat{1}_{\complement A}(\xi)| \leq |\widehat{1}_A(\xi)|$, so for $U = A$ and $V = v.\complement A$, we have

$$|E| \leq 2k\sum_{\xi=1}^{2k-1}|\widehat{1}_A(\xi)|^2 \leq k,$$

which is not useful. On the other hand, since (see [13, Lemma 6, Chapter 1])

$$|\widehat{1_A}(\xi)| \leq \frac{1}{2k\sin(\pi\xi/(2k))} + \frac{1}{k}$$

then

$$\begin{aligned} \sum_{\xi=1}^{2k-1}|\widehat{1}_A(\xi)|^2 &\leq \sum_{\xi=1}^{2k-1}\left(\frac{1}{2k\sin(\pi\xi/(2k))} + \frac{1}{k}\right)^2 \\ &= 2\sum_{\xi=1}^{k}\left(\frac{1}{2k\sin(\pi\xi/(2k))} + \frac{1}{k}\right)^2 - \frac{9}{4k^2}. \end{aligned}$$

Now the sequence

$$a_{k,\xi} = \begin{cases} \left(\frac{1}{2k\sin(\pi\xi/(2k))} + \frac{1}{k}\right)^2, & 1 \leq \xi \leq k, \\ 0, & \text{otherwise}, \end{cases}$$

is such that $a_{k,\xi} \geq a_{k+1,\xi}$ and $\sum_{\xi=1}^{\infty} a_{1,\xi} = \frac{9}{4}$. By the monotone convergence theorem, we obtain

$$\lim_{k\to\infty} \sum_{\xi=1}^{2k-1} |\widehat{1}_A(\xi)|^2 = 2 \lim_{k\to\infty} \sum_{\xi=1}^{k-1} \frac{1}{\pi^2\xi^2} = \frac{1}{3},$$

which amounts to estimate $|E| \leq \frac{2}{3}k$ for large k, but that is not enough to ensure that $r_{A+v.\complement A}(\lambda) \geq 0$ for any λ and $v \neq -1$. Furthermore, it suggests that the most we can get this way is $|A + v.\complement A| \geq \frac{5}{6}k$ (see [12, p. 210]).

5. Using a result by Mann

For a penultimate attempt we use the following generalization of the celebrated Cauchy-Davenport theorem.

Theorem 5.1 (Mann, 1965, see [11]). *Let S be a subset of an arbitrary abelian group G. Then one of the following holds:*

(1) *For every subset T such that $S + T \neq G$, we have $|S + T| \geq |S| + |T| - 1$.*
(2) *There exists a proper subgroup H of G such that $|S + H| < |S| + |H| - 1$.*

Thus one of these two alternatives holds:

(1) It is true that $|A + v.\complement A| \geq |A| + |v.\complement A| - 1 = 2k - 1$.
(2) There is proper subgroup H, such that
$$|A + H| < k + |H| - 1.$$

We claim that, for the set A, we have

$$|A + v.\complement A| \geq 2k - 1$$

by discarding the second alternative. In order to do so, suppose $H = \langle d \rangle$ where $0 \leq d \leq k$ and

$$|H + A| < k + |H| - 1.$$

Being that H is proper, we have $|H| \leq k$. Let us suppose that $d \geq 1$ (since the trivial case is evidently false), which implies that $|H| = \frac{2k}{d}$. Thus $A + H$ is the placement of copies of A with spaces of d elements, so it covers all the elements of $\mathbb{Z}_{2k}$ with at most $\frac{2k}{d}$ exceptions, thus

$$k + \frac{2k}{d} - 1 > |A + H| \geq 2k - \frac{2k}{d}.$$

This is possible if, and only if,

$$\frac{2k}{d} + k - 1 > 2k - \frac{2k}{d}$$

or, equivalently,

$$4 > \frac{4k}{k+1} > d,$$

thereof $d = 2$ or $d = 3$. If $d = 2$, we are done, for A has $\{0, 1\}$ as a subset, thus $A + H = \mathbb{Z}_{2k}$, a contradiction.

In the later case (which arises only when 3 divides k), it would be possible that each "slot" of 3 elements $\{3j, 3j+1, 3j+2\}$ determined by H and to be covered by A to have $3j+2$ uncovered. Nevertheless, the "antipodal" slot $\{3j+k, 3j+k+1, 3j+k+2\}$ would not allow this to happen, since the potentially uncovered element must be covered with the translate $(3j+k+2)+k$ of $3j+k+2 \in A+3j$. Moreover: we are certain that a copy of A is placed in k because 3 is one of its factors. So, $A+H$ would leave no element uncovered, for there are an even number of slots, each one paired with its antipode. Hence $H = \langle 3 \rangle$ is also an impossibility.

From the above proof we also obtain that A is aperiodic, i. e., $A+H \neq A$ except for $H = \{0\}$. Invoking Kemperman structure theorem (as stated, for example, in [7, p. 71-72])[1], we conclude that

$$A - \complement A = \mathbb{Z}_{2k} \setminus \{0\}$$

and, furthermore, if $A + v.\complement A \neq \mathbb{Z}_{2k}$, then there exists u such that

$$v.\complement A = u - \complement A.$$

This equivalent to the following: except for for $v = -1$, and $u = 0$ it is true that

$$-v.A + u \neq A,$$

which means exactly that A is a counterpoint dichotomy. Thus, Kemperman's theorem cannot lead us further in relation to the cardinality of $A + v.\complement A$.

6. A proof for a special case

Question 1.2 can be answered affirmatively for the set (2), when $k \geq 10$, as we now show. Let us first identify v with an element in

$$\{-k+1, -k+2, \ldots, -1, 0, 1, \ldots, k-1\}.$$

Observe that for $v = -1$ we have $A + (-1).B = \mathbb{Z}_{2k} \setminus \{k\}$, and for $v = 1$ we have $A + 1.B = A + B = A + k + A = \mathbb{Z}_{2k} + k = \mathbb{Z}_{2k}$. Now suppose $k - 3 \geq |v| > 1$, that is, $k - 3 \geq |v| \geq 3$. Choose

$$X = \{3, 4, \ldots, k-1\}$$

and $Y = X - 3$. We claim that $Y + v.Y = \mathbb{Z}_{2k}$, for this would imply that

$$\begin{aligned} A + v.A &\supset X + v.X \\ &= Y + 3 + v.(Y+3) \\ &= Y + v.Y + 3 + 3v \\ &= \mathbb{Z}_{2k} + 3 + 3v = \mathbb{Z}_{2k}, \end{aligned}$$

and hence $A + v.B = A + v.(A+k) = A + v.A + vk = \mathbb{Z}_{2k}$, as we want.

To prove the claim, we note that the set $Y + v.Y$ contains the multiples $0, v, 2v, \ldots, (k-4)v$. We have

$$|(k-4)v| > |3(k-4)| \geq 2k$$

[1]More specifically, the pair $(A, -v.\complement A)$ is of type IV in the classification stated in [7, p. 71].

and, since $|v| \leq k-3$, the elements between multiples of v are also in $Y + v.Y$, as Y contains $\{0, 1, \ldots, v-1\}$. This means that $Y + v.Y = \mathbb{Z}_{2k}$. The remaining cases we need to deal with are

$$v \in \{\pm(k-1), \pm(k-2)\};$$

note that $\pm(k-2)$ only occurs when k is odd.

For $v = k-1$, we note that $A + (k-1).A$ contains A and $A + (k-1) = A + k - 1 = B - 1$, so $A + (k-1).A$ contains all the elements of $\mathbb{Z}_{2k}$ with the possible exceptions of those in $B \setminus (B-1)$. However

$$\begin{aligned} -1 &= (k+2) + (k-1)3, \\ k+1 &= 4 + (k-1)3, \\ 2 &= 6 + (k-1)4, \end{aligned}$$

proving that all of them belong to $A + (k-1).A$. We must have $4, 6 \in A$ since $k \geq 10$.

For $v = -(k-1) = k+1$ we have $B \setminus (B+1) = \{2, k, k+2\}$, and the analogous certificates are

$$\begin{aligned} 2 &= (k-1) + (k+1)3, \\ k &= (k-4) + (k+1)4, \\ k+2 &= (k-3) + (k+1)4; \end{aligned}$$

for $v = k-2$ we have $B \setminus (B-2) = \{-1, -2, k, 2\}$ and

$$\begin{aligned} -1 &= 1 + (k-2)1, \\ -2 &= 0 + (k-2)1, \\ k &= 6 + (k-2)3, \\ 2 &= 10 + (k-2)4; \end{aligned}$$

finally, for $v = k+2$ we have $B \setminus (B+2) = \{2, k, k+1, k+4\}$ and

$$\begin{aligned} 2 &= k + (k+2)1, \\ k &= k + (k+2)0, \\ k+1 &= (k-7) + (k+2)4, \\ k+4 &= (k-4) + (k+2)4. \end{aligned}$$

7. Some final remarks

The results distilled from Mann's and Kemperman's theorems take us rather close to the goal of proving that (1) holds for the set A defined by (2), but ultimately fail. We can manage to provide an elementary proof of the fact, but we do not know how much this approach can be generalized, or what this means for the classificatory nature of Kemperman's theorem.

Nevertheless, these facts make evident that there is a significant gap between $E(A, v.\complement A)$ and $E(A, A)$. They also point out that, in order to succeed with the use of exponential sums, a very sharp estimate of (3) is required.

8. Acknowledgments

I am indebted to Merlijn Staps at Universiteit Utrecht for his kind help reading an earlier version of this paper and finding a proof of the sumset property of (2).

I sincerely thank my friends José Hernández Santiago at Centro de Ciencias Matemáticas (UNAM) and Marcelino Ramírez Ibáñez at Universidad del Papaloapan for their invaluable feedback regarding early drafts of this paper. I also thank an anonymous reviewer who brought an embarrassing blunder to my attention, as well as another who helped me to polish my prose and to clarify several obscure points.

References

[1] Octavio A. Agustín-Aquino and Julien Junod and Guerino Mazzola, *Computational Counterpoint Worlds*, Springer, (2015).

[2] Octavio A. Agustín-Aquino, *Extended Counterpoint Symmetries and Continuous Counterpoint*, in *The Musical-Mathematical Mind*, pp. 1–7, Pareyon, G. and Pina-Romero, S. and Agustín-Aquino, O.A. and Lluis-Puebla, E., Springer, (2017).

[3] Moreno Andreatta, *Constructing and Formalizing Tiling Rhythmic Canons: A Historical Survey of a "Mathemusical" Problem*, Perspectives of New Music, vol. 49, no. 2, pp. 33–64, (2011).

[4] Thomas Boothby, Matt DeVos, and Amanda Montejano, *A New Proof of Kemperman's Theorem*, arXiv:1301.0095 (2013).

[5] Moubariz Z. Garaev, *Sumas trigonométricas y congruencias aditivas*, Gaceta de la Real Sociedad Matemática Española, vol. 12, no. 1, pp. 129–143, (2009).

[6] Ronald Graham, Donald Knuth, and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, 2nd Ed., (1994).

[7] David J. Grynkiewicz, *A Step Beyond Kemperman's Structure Theorem*, Mathematika, vol. 55, no. 1-2, pp. 67–114, Cambridge University Press, (2009).

[8] Vsevolod F. Lev, *Critical pairs in abelian groups and Kemperman's structure theorem*, International Journal of Number Theory, vol. 2, no. 03, pp. 379–396, World Scientific, (2006).

[9] Guerino Mazzola, *The Topos of Music: Geometric Logic of Concepts, Theory, and Performance*, Birkhüser Verlag, (2002).

[10] John E. Olson, *On the sum of two sets in a group*, Journal of Number Theory, vol. 18, no. 1, pp. 110-120, Elsevier, (1984).

[11] Oriol Serra and Gilles Zémor, *On a generalization of a theorem by Vosper*, INTEGERS, vol. 0, pp. A10, (2000).

[12] Terrence Tao and Van Vu, *Additive Combinatorics*, Cambridge University Press, Cambridge studies in advanced mathematics, no. 105, (2006).

[13] Ivan M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Interscience Publishers, (1954).

Instituto de Física y Matemáticas, Universidad Tecnológica de la Mixteca, Carretera a Acatlima Km. 2.5, Huajuapan León, Oaxaca, México, C.P. 69000

E-mail address: octavioalberto@mixteco.utm.mx