ON CYCLIC ORTHOGONAL DOUBLE COVER OF CIRCULANT GRAPHS BY THE DISJOINT UNION OF CORONAS

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Abstract. In this paper, we present a method to construct a cyclic orthogonal double cover (CODC) of circulant graphs by certain kinds of coronas that model by linear functions.

1. Introduction

Let H be an arbitrary graph with n vertices and let $\mathcal{G} = \{G_0, G_1, \ldots, G_{n-1}\}$ be a collection of n pages (spanning subgraphs) of H. $\mathcal{G}$ is an orthogonal double cover (ODC) of H if there exists a bijective mapping $\phi : V(H) \rightarrow \mathcal{G}$ satisfying the following two conditions:

(1) Every edge of H exists in exactly two of pages $\mathcal{G}$ and
(2) If for any two distinct pages G_i and $G_j \in \mathcal{G}$, $|E(G_i) \cap E(G_j)| = |E(\phi(i)) \cap E(\phi(j))| = 1$, if and only if i and j are adjacent in H.

If all of the graphs $\{G_0, G_1, \ldots, G_{n-1}\}$ are isomorphic to a graph G, $\mathcal{G}$ is called an ODC of H by G.

An elegant method to construct ODCs in [4],[5] and [7] was based on the idea of translating a given subgraph G by a group acting on $V(H)$. If the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$ (the set of all automorphism of $\mathcal{G}$), then an ODC $\mathcal{G}$ of H is a cyclic orthogonal double cover (CODC). This concept is a generalization of the definitions of an ODC of complete graphs, complete bipartite graphs which studied in [1].

All graphs considered in this paper are finite and simple (without multiple edges or loops), and also we use the standard notation: In particular, K_n for the complete graph on n vertices, C_n for the cycle on n vertices, P_n for the path on n vertices. $D \cup F$ for the disjoint union of D and F, rF for r disjoint copies of F. Let F be a graph with n_1 vertices and m_1 edges and let G be a graph with n_2 vertices and m_2 edges, then the corona $F \odot G$ is the graph consists of one copy of F and n_1 copies of G. In $F \odot G$, there is an edge between the ith vertex of F and every vertex in the ith copy of G. Other notations and terminology can be found in [2].

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For a sequence $\{d_1, d_2, \cdots, d_k\}$ of positive integers with $1 \leq d_1 \leq d_2 \leq \cdots \leq d_k \leq \lfloor n/2 \rfloor$, the circulant graph $Circ(n; \{d_1, d_2, \cdots, d_k\})$ with the vertex set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$; two vertices v_1 and v_2 are adjacent, if and only if $v_1 - v_2 = \pm d_i \,(\bmod\, n)$, for some $i \in \{1, 2, \cdots, k\}$. The length of an edge $\{v_1, v_2\}$ in$Circ(n; \{d_1, d_2, \cdots, d_k\})$ is $\min\{|v_1 - v_2|$,
$n - |v_1 - v_2|\}$. Given two edges $e_1 = \{u_1, u_2\}$ and $e_2 = \{v_1, v_2\}$ of the same length l in $Circ(n; \{d_1, d_2, \cdots, d_k\})$, the rotation distance $r(l)$ between e_1 and e_2 is

$$r(l) = \min\{r_1, r_2 : (u_1 + r_1)(u_2 + r_1) = e_2, (v_1 + r_2)(v_2 + r_2) = e_1\},$$

where addition and difference are calculated inside $\mathbb{Z}_n$. Note that if $r(l) = l$, then the edges $e_1$and e_2 are adjacent; if $r(l) \neq l$, then the edges $e_1$and e_2 are non-adjacent.

Gronau et al. [3] introduce the notion of an orthogonal labeling of a graph $G = (V, E)$ with $n-1$ edges. That is if there is a $1-1$ mapping $\Psi : V(G) \to \mathbb{Z}_n$, the following conditions are satisfied:

I. For every $l \in \{1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$, G contains exactly two edges of length l, and exactly one edge of length $(n/2)$ if n is even, and

II. For every $l \in \{1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$, $r(l) = \{1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$.

Theorem 1. *([3]) A CODC of K_n by a graph G exists if and only if there exists an orthogonal labeling of G.*

In [4], Sampathkumar et al. generalized the orthogonal $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$-labeling to an orthogonal $\{d_1, d_2, \cdots, d_k\}$-labeling for a sequence of positive integers $\{d_1, d_2, \cdots, d_k\}$ with $1 \leq d_1 \leq d_2 \leq \cdots \leq d_k \leq \lfloor n/2 \rfloor$ as in the following theorem:

Theorem 2. *([4])A CODC of $Circ(n; \{d_1, d_2, \cdots, d_k\})$ by a graph G exists, if and only if there exists an orthogonal $\{d_1, d_2, \cdots, d_k\}$-labeling of G.*

In [5], Scapellato et al. deals with Cayley graphs of degree 2 and 3 and offers some insights on the circulant graphs. In [6], Sampathkumar et al. introduced a kind of orthogonal labeling called orthogonal σ-labeling, and they found it for some caterpillars of diameters 4. In [4], Sampathkumar et al. completely settled the existence problem of CODCs of 4-regular circulant graphs. In [7], El -Shanawany et al. studied CODCs of circulant graphs with higher degrees, constructed CODC by certain classes of graphs, and introduced an approach to obtain CODCs from a given CODCs. Also, the study of CODC of circulant graphs by linear orthogonal labeling concentrated in El -Shanawany et al. [9]. The above results motivate us to present CODC of circulant graphs using graphs model by linear and nonlinear functions as in the next Section.

2. CODCs of Circulant Graphs by Coronas

The purpose of this section is to study a new class CODCs and we will use graphs modeled by linear and nonlinear function (as in Theorem 3 and Theorem 5, respectively) to construct of CODC of circulant graphs. Theorem 4 is the main result for this Section.

Let α be a positive integer and $f : \mathbb{Z}_n \setminus \{0\} \to \mathbb{Z}_n$, be the linear function defined by $f(x) = \alpha x$. We denote by G_f the spanning subgraph of $Circ(n; \{1, 2, \ldots, \lfloor n/2 \rfloor\})$ modeled by linear function f such that edges set of G_f is defined by $E(G_f) = \{\{x, f(x)\} : x \in \mathbb{Z}_n \setminus \{0\}\}$. It is easy to see that G_f has $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$-orthogonal labeling, since it satisfies (a) For every $l \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, G_f contains exactly two edges of length l as $\pm\{x, \alpha x\} \bmod n$, where x is the solution of equation $(\alpha - 1)x = l \bmod n$. In case of n is even, there exists exactly one edge in G_f of length $(n/2)$ as $\{0, n/2\}$, and (b) $\{r(l) = -(\alpha - 1)^{(-1)}\ l\ (1 + \alpha) \bmod n\ : l \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}\} = \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Hence, we give the following result as an immediate consequence of the Theorem 2.

Theorem 3. *Let α be a positive integer and $\gcd(\alpha^2 - 1, n) = 1$, then, there is a CODC of $Circ(n; \{1, 2, \ldots, \lfloor n/2 \rfloor\})$ by G_f.*

In [8], El-Shanawany et al. approved the existence of ODC of Cayley graph by complete tripartite graph $K_{1,r,s}$ which is isomorphic to the corona $K_{r,s} \odot K_1$. The following Theorem is the new result to construct CODC of circulant graphs by the disjoint union of coronas.

Theorem 4. *Let n, m, α be positive integers and $p \geq 5$ be a prime number such that $n = p\alpha$, $\alpha = 2^m \in \mathbb{Z}_p$, $\gcd(\alpha^2 - 1, n) = 1$ and the multiplication order $o(\alpha) = e$ divisor $p - 1$ with respect to the multiplication group $\mathbb{Z}_p \setminus \{0\}$ (i.e. $p - 1 = qe$), then, there exist CODC of $Circ(n; \{1, 2, \ldots, \frac{n}{2}\})$ by $G_f = q\,(C_e \odot (2^m - 1)\, K_1) \cup (K_1 \odot (2^m - 1)\, K_1)$.*

Proof. Since $\gcd(\alpha^2 - 1, n) = 1$, then applying Theorem 3, $G_f = q\,(C_e \odot (2^m - 1)\, K_1) \cup (K_1 \odot (2^m - 1)\, K_1)$ has an orthogonal $\{1, 2, \ldots, \frac{n}{2}\}$-labeling and G_f is described as follows: The elements for each i^{th} cycle is denoted by $\beta_{ij} = \beta_{i1}(\alpha^{(j-1)})$ where, $1 \leq i \leq q$; $1 \leq j \leq e$. And β_{i1} for each cycle is defined as; for the first cycle, let $\beta_{11} = 2^m$, therefore, $\beta_{1j} = 2^m(\alpha^{(j-1)})$, $1 \leq j \leq e$. For r^{th} cycle, $\beta_{r1} = 2^m \mu$ for $2 \leq r \leq q$ where, $\mu \in \{\mathbb{Z}_p \setminus \{0\}\} \setminus \left\{\frac{1}{2^m}\, \beta_{st} :\ 1 \leq s \leq r - 1,\ 1 \leq t \leq e\right\}$. The vertex with labeling equal α in these set of vertices will be connected to every vertex of the following set of vertices $U = \left\{1 + \frac{n}{\alpha} i :\ 0 \leq i \leq d - 1 \text{ and } \gcd(n, \alpha) = d\right\}$ difference from $\{2^m\ (\mathbb{Z}_p \setminus \{0\})\}$. Since we have q Coronas, denoted by H_i for all $1 \leq i \leq q$, we denote the corona that contains the vertex with labeling equal α as the major corona and denoted by H_1. Then the other vertices in H_1 can be obtained by multiplying the vertices of U by $(p + \alpha)$ and the other Coronas can be obtained by multiplying each vertex of the major corona by β_{ij}. ∎

For more illustration to Theorem 4, for $n = (5)2$ it follows that $\alpha = 2, e = 4$ and $q = 1$. Then, there exist CODC of $Circ(10; \{1, 2, 3, 4, 5\})$ by $G_f = (C_4 \odot K_1) \cup (K_1 \odot K_1)$ as in Figure 2. Also, for $n = (13)\ 2^3$ it follows that $\alpha = 8, e = 4$ and $q = 3$. Then, there exists a CODC of $Circ(104; \{1, 2, \ldots, 52\})$ by $G_f = 3\,(C_4 \odot 7K_1) \cup (K_1 \odot 7K_1)$ as in Figure 2.

The following Theorem is interested in obtaining an orthogonal labelling for CODC of circulant graphs by graph modeled by a nonlinear function.

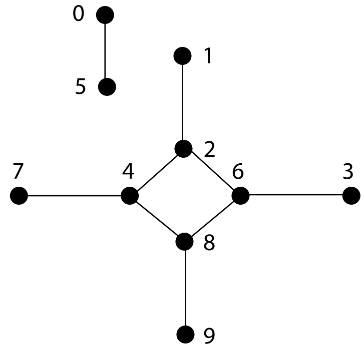

FIGURE 1. An orthogonal $\{1,2,3,4,5\}-$labeling of $G_f = (C_4 \odot K_1) \cup (K_1 \odot K_1)$ w.r.t. $\mathbb{Z}_{10}$.

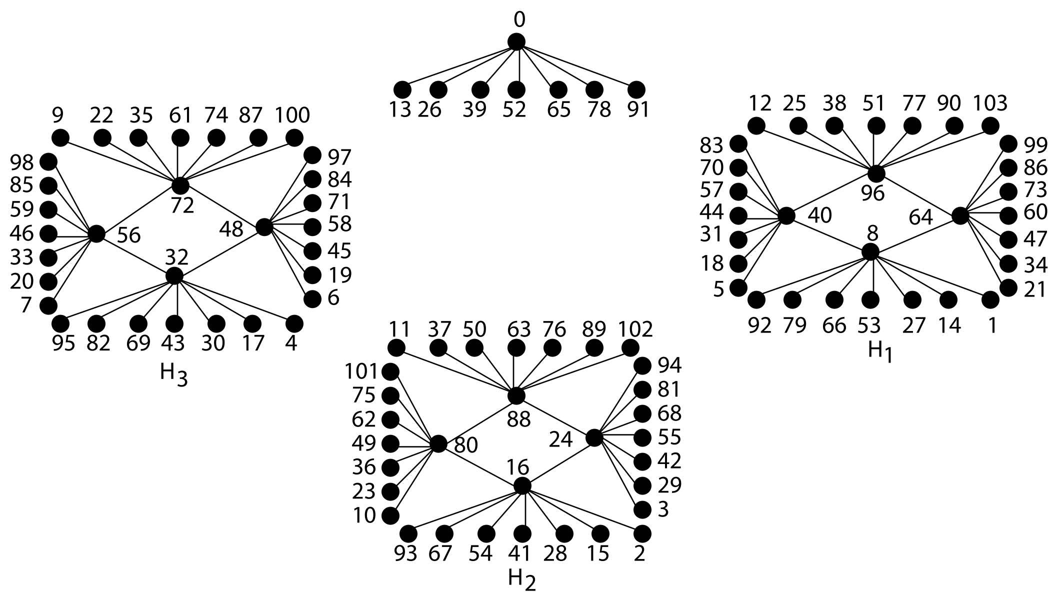

FIGURE 2. An orthogonal $\{1,2,\ldots,52\}$-labeling of $G_f = 3\,(C_4 \odot 7K_1) \cup (K_1 \odot 7K_1)$ w.r.t. $\mathbb{Z}_{104}$.

Theorem 5. *Let n be a positive integer and G_f be the spanning subgraph modeled by nonlinear function $f(x) = -x^2$ such that edges set of G_f is defined by $E(G_f) = \{\{f(x), f(x)+x\} : x \in \mathbb{Z}_n \setminus \{0\}\}$. Then, there exist a CODC of $Circ(n; \{1,2,\ldots,\lfloor n/2 \rfloor\})$ by G_f.*

Proof. (a) For every $x \in \{1,2,\ldots,\lfloor n/2 \rfloor\}$, G_f contains exactly two edges of length x as $\{f(x), f(x) \pm x\}$, in case of n is even, there exists exactly one edge in G_f of length $n/2$ as $\{0, n/2\}$ or $\{n/2, 0\}$. (b) Since every two edges of the same length are adjacent, then $r(x) = x, \forall x \in \{1,2,\ldots,\lfloor n/2 \rfloor\}$. Hence, from conditions (a) and (b), G_f has an orthogonal $\{1,2,\ldots,\lfloor n/2 \rfloor\}$-labeling. ∎

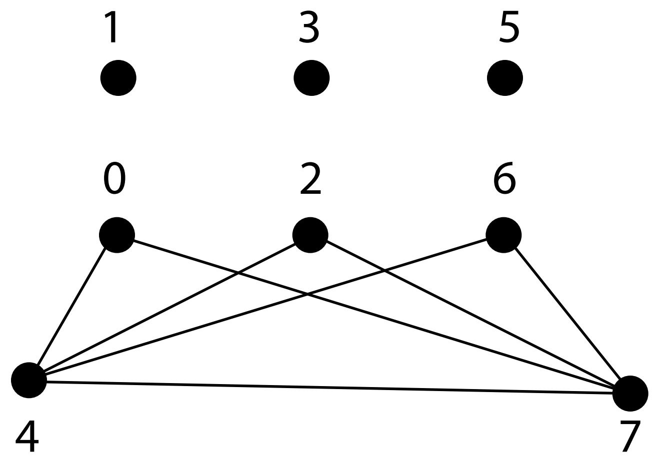

FIGURE 3. An orthogonal $\{1, 2, 3, 4\}$-labeling of $G_f = 3K_1 \cup (P_2 \odot 3K_1)$ w.r.t. $\mathbb{Z}_8$

As a special result of Theorem 5, see Figure 2. Note that which value of n to construct a CODC of circulant graphs by certain kinds of coronas that model by nonlinear functions is still, remaining open.

3. Conclusion

This paper concerns with a construction of CODC of circulant graphs by the disjoint union of specials coronas using graphs model by linear functions. In future, we will construct CODC of circulant graphs by new classes of coronas.

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