CALLAN-LIKE IDENTITIES

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ABSTRACT. In 1998, D. Callan obtained a binomial identity involving the derangement numbers. In this paper, by using the theory of formal series, we extend such an identity to the generalized derangement numbers. Then, by using the same technique, we obtain other identities of the same kind for the generalized arrangement numbers, the generalized Laguerre polynomials, the generalized Hermite polynomials, the generalized exponential polynomials and the generalized Bell numbers, the hyperharmonic numbers, the Lagrange polynomials and the Gegenbauer polynomials.

1. INTRODUCTION

In 1998, David Callan proposed [4] the identity

$$\sum_{k=0}^{n} \binom{n}{k} d_{m+n-k} = n! \sum_{k=0}^{\min(m,n)} \binom{n}{k} \binom{m+n-k}{n} d_{m-k} \tag{1}$$

involving the *derangement numbers* [7, p. 182] [19, A000166]

$$d_n = \sum_{k=0}^{n} \frac{n!}{k!} (-1)^k. \tag{2}$$

This identity has been proved in several ways in [5] and then in [1] by employing MacMahon's operator. In particular, it can be easily proved by using Taylor's formula.

In this paper, by using Taylor's formula in the context of the theory of formal power series (Theorem 1), we extend identity (1) to the generalized derangement numbers. Then, by using the same technique, we obtain similar identities for the generalized arrangement numbers, the generalized Laguerre polynomials, the generalized Hermite polynomials, the generalized exponential polynomials and the generalized Bell numbers, the hyperharmonic numbers, the Lagrange polynomials and the Gegenbauer polynomials.

In the rest of the paper, we will use extensively the properties of formal power series. Here, we limit ourselves to recall some of them and some basic definitions. For a

Date: October 20, 2019.

1991 *Mathematics Subject Classification.* Primary: 05A19; Secondary: 05A15, 33C45.

Key words and phrases. combinatorial sum, generalized derangement number, generalized arrangement number, generalized Laguerre polynomial, generalized Hermite polynomial, generalized exponential polynomial, generalized Bell number, hyperharmonic number, Lagrange polynomial, Gegenbauer polynomial, Lah number, Stirling number.

general introduction to the theory and the combinatorics of formal power series, see, for instance, [7, 9, 17].

The *falling factorials* are the polynomials defined by

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)\,,$$

while the *rising factorials* are defined by the *Pochhammer symbol*

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1)\,.$$

The (*generalized*) *binomial coefficients* and the *multi-insiemistic coefficients* are respectively given by

$$\binom{x}{n} = \frac{x^{\underline{n}}}{n!} \qquad \text{and} \qquad \left(\!\!\binom{x}{n}\!\!\right) = \binom{x+n-1}{n} = \frac{(x)_n}{n!}\,.$$

The product of two exponential series corresponds to the binomial convolution of their coefficients [9, p. 365], that is

$$\sum_{n\geq 0} f_n \frac{t^n}{n!} \cdot \sum_{n\geq 0} g_n \frac{t^n}{n!} = \sum_{n\geq k} \left[\sum_{k=0}^{n} \binom{n}{k} f_k\, g_{n-k}\right] \frac{t^n}{n!} \tag{3}$$

while the product of two ordinary series corresponds to the ordinary convolution of their coefficients, that is

$$\sum_{n\geq 0} f_n t^n \cdot \sum_{n\geq 0} g_n t^n = \sum_{n\geq k} \left[\sum_{k=0}^{n} f_k\, g_{n-k}\right] t^n\,. \tag{4}$$

The generalized geometric series is given by

$$\sum_{n\geq k} \binom{\lambda+n-k}{n-k} t^n = \frac{t^k}{(1-t)^{\lambda+1}}\,. \tag{5}$$

This formula is valid for any $k \in \mathbb{N}$ and any abstract symbol λ. In particular, we have

$$\sum_{n\geq 0} \left(\!\!\binom{\lambda}{k}\!\!\right) t^n = \frac{1}{(1-t)^{\lambda}}\,. \tag{6}$$

For an exponential series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$, we have

$$t^k f(t) = \sum_{n\geq k} \binom{n}{k} k!\, f_{n-k} \frac{t^n}{n!}\,. \tag{7}$$

while, for an ordinary series $f(t) = \sum_{n\geq 0} f_n\, t^n$, we simply have

$$t^k f(t) = \sum_{n\geq k} f_{n-k}\, t^n\,. \tag{8}$$

For an exponential series $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$, we have

$$D^m f(t) = \sum_{n\geq 0} f_{n+m} \frac{t^n}{n!} \tag{9}$$

while, for an ordinary series $f(t) = \sum_{n\geq 0} f_n\, t^n$, we have

$$D^m f(t) = m! \sum_{n\geq 0} \binom{m+n}{n} f_{n+m}\, t^n \,. \tag{10}$$

Finally, we have the following theorem, which will be our main tool.

Theorem 1 (Taylor's Formula)**.** *For any formal power series $f(t)$, the exponential generating series of the successive derivatives $D^m f(t)$, where $D = \frac{d}{dt}$ denotes the formal derivative with respect to t, is*

$$\sum_{m\geq 0} D^m f(t)\, \frac{u^m}{m!} = f(t+u)\,. \tag{11}$$

Notice that this theorem is valid both when $f(t)$ is an exponential series and when $f(t)$ is an ordinary series.

2. Generalized derangement numbers

The *generalized derangement numbers* $d_n^{(\nu)}$ are defined by

$$d_n^{(\nu)} = \sum_{k=0}^{n} \binom{\nu+n-k}{n-k} \frac{n!}{k!}\, (-1)^k \tag{12}$$

and have exponential generating series

$$d^{(\nu)}(t) = \sum_{n\geq 0} d_n^{(\nu)} \frac{t^n}{n!} = \frac{\mathrm{e}^{-t}}{(1-t)^{\nu+1}}\,. \tag{13}$$

For $\nu = 0$, we have the ordinary derangement numbers d_n defined by (2). The *generalized rencontres polynomials* are defined as the *Appell polynomials* [18, p. 86] [16] associated to the numbers $d_n^{(\nu)}$, that is as

$$D_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(\nu)} x^k\,, \tag{14}$$

and have exponential generating series

$$D^{(\nu)}(x;t) = \sum_{n\geq 0} D_n^{(\nu)}(x) \frac{t^n}{n!} = \frac{\mathrm{e}^{(x-1)t}}{(1-t)^{\nu+1}}\,. \tag{15}$$

We start by computing the successive derivatives of series (13).

Lemma 2. *For every $m \in \mathbb{N}$, we have the identity*

$$D^m \frac{\mathrm{e}^{-t}}{(1-t)^{\nu+1}} = \frac{D_m^{(\nu)}(t)\, \mathrm{e}^{-t}}{(1-t)^{\nu+m+1}}\,. \tag{16}$$

Proof. By Taylor's formula (11) applied to series (13) and by series (15), we have

$$\begin{aligned}\sum_{m\geq 0} D^m d^{(\nu)}(t)\frac{u^m}{m!} &= d^{(\nu)}(t+u) = \frac{\mathrm{e}^{-t-u}}{(1-t-u)^{\nu+1}}\\ &= \frac{\mathrm{e}^{-t}}{(1-t)^{\nu+1}}\frac{\mathrm{e}^{(t-1)\frac{u}{1-t}}}{(1-\frac{u}{1-t})^{\nu+1}} = \frac{\mathrm{e}^{-t}}{(1-t)^{\nu+1}}\, D^{(\nu)}\Big(t;\frac{u}{1-t}\Big)\\ &= \frac{\mathrm{e}^{-t}}{(1-t)^{\nu+1}}\sum_{m\geq 0}\frac{D_m^{(\nu)}(t)}{(1-t)^m}\frac{u^m}{m!} = \sum_{m\geq 0}\frac{D_m^{(\nu)}(t)\,\mathrm{e}^{-t}}{(1-t)^{\nu+m+1}}\frac{u^m}{m!}\end{aligned}$$

from which we have identity (16). □

Using the previous lemma, we have the following result, extending the original identity (1).

Theorem 3. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k} d_{m+n-k}^{(\nu)} = n! \sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{\nu+m+n-k}{n-k} d_{m-k}^{(\nu)}\,. \tag{17}$$

Proof. By identities (3), (16), (14) and (5), we have

$$\begin{aligned}\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k} d_{m+n-k}^{(\nu)}\right]\frac{t^n}{n!} &= \sum_{n\geq 0}\frac{t^n}{n!}\cdot\sum_{n\geq 0} d_{n+m}^{(\nu)}\frac{t^n}{n!}\\ &= \mathrm{e}^t\cdot D^m d^{(\nu)}(t) = \mathrm{e}^t\,\frac{D_m^{(\nu)}(t)\,\mathrm{e}^{-t}}{(1-t)^{\nu+m+1}} = \frac{D_m^{(\nu)}(t)}{(1-t)^{\nu+m+1}}\\ &= \sum_{k=0}^{m}\binom{m}{k} d_{m-k}^{(\nu)}\frac{t^k}{(1-t)^{\nu+m+1}} = \sum_{k=0}^{m}\binom{m}{k} d_{m-k}^{(\nu)}\sum_{n\geq 0}\binom{\nu+m+n-k}{n-k} t^n\\ &= \sum_{n\geq 0}\left[n!\sum_{k=0}^{m}\binom{m}{k}\binom{\nu+m+n-k}{n-k} d_{m-k}^{(\nu)}\right]\frac{t^n}{n!}\,.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the first and last series, we have identity (17). □

A second identity, similar to identity (17), can be derived by using the following elementary result.

Lemma 4. *For every $k \in \mathbb{N}$, we have the identity*

$$D^k\frac{1}{(1-t)^{\nu+1}} = \binom{\nu+k}{k}\frac{k!}{(1-t)^{\nu+k+1}}\,. \tag{18}$$

Proof. By Taylor's formula (11) and by series (5), we have

$$\begin{aligned}\sum_{k\geq 0} D^k \frac{1}{(1-t)^{\nu+1}}\, \frac{u^k}{k!} &= \frac{1}{(1-t-u)^{\nu+1}}\\ &= \frac{1}{(1-t)^{\nu+1}} \frac{1}{(1-\frac{u}{1-t})^{\nu+1}} = \sum_{k\geq 0} \binom{\nu+k}{k} \frac{k!}{(1-t)^{\nu+k+1}}\, \frac{u^k}{k!}\end{aligned}$$

from which we have identity (18). □

Now, we can prove the following result.

Theorem 5. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} d^{(\nu)}_{m+n-k} = \sum_{k=0}^{m} \binom{m}{k} (\nu+1)_{n+k} (-1)^{m-k}\,. \tag{19}$$

Proof. Using formulas (3), (5), (18) and Leinbiz's rule, we have

$$\begin{aligned}\sum_{n\geq 0} \left[\sum_{k=0}^{n} \binom{n}{k} d^{(\nu)}_{m+n-k}\right] \frac{t^n}{n!} &= \sum_{n\geq 0} \frac{t^n}{n!} \cdot \sum_{n\geq 0} d^{(\nu)}_{n+m} \frac{t^n}{n!} = \mathrm{e}^t \cdot D^m \frac{\mathrm{e}^{-t}}{(1-t)^{\nu+1}}\\ &= \mathrm{e}^t \sum_{k=0}^{m} \binom{m}{k} D^k \frac{1}{(1-t)^{\nu+1}} \cdot D^{m-k} \mathrm{e}^{-t}\\ &= \mathrm{e}^t \sum_{k=0}^{m} \binom{m}{k} \binom{\nu+k}{k} \frac{k!}{(1-t)^{\nu+k+1}}\, (-1)^{m-k} \mathrm{e}^{-t}\\ &= \sum_{k=0}^{m} \binom{m}{k} \binom{\nu+k}{k} k!\, (-1)^{m-k} \sum_{n\geq 0} \binom{\nu+n+k}{n} t^n\\ &= \sum_{n\geq 0} \left[\sum_{k=0}^{m} \binom{m}{k} (\nu+n+k)^{\underline{n}}\, (\nu+k)^{\underline{k}}\, (-1)^{m-k}\right] \frac{t^n}{n!}\\ &= \sum_{n\geq 0} \left[\sum_{k=0}^{m} \binom{m}{k} (\nu+1)_{n+k} (-1)^{m-k}\right] \frac{t^n}{n!}\,.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the first and last series, we have identity (19). □

Identity (17) can be generalized in the following way.

Theorem 6. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{\lambda+k}{k} k!\, d^{(\nu)}_{m+n-k} = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} k!\, d^{(\nu)}_{m-k}\, d^{(\lambda+\nu+m+1)}_{n-k}\,. \tag{20}$$

In particular, for $\lambda = m = n$, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} k!\, d^{(\nu)}_{2n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 k!\, d^{(\nu)}_{n-k}\, d^{(\nu+2n+1)}_{n-k}\,. \tag{21}$$

Proof. By identities (3), (16), (14), (5) and (7), we have

$$
\begin{aligned}
\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}\binom{\lambda+k}{k}k!\,d_{m+n-k}^{(\nu)}\right]\frac{t^n}{n!} &= \sum_{n\geq 0}\binom{\lambda+n}{n}n!\,\frac{t^n}{n!}\cdot\sum_{n\geq 0}d_{n+m}^{(\nu)}\frac{t^n}{n!} \\
&= \frac{1}{(1-t)^{\lambda+1}}\,D^m d^{(\nu)}(t) = \frac{1}{(1-t)^{\lambda+1}}\,\frac{D_m^{(\nu)}(t)\,\mathrm{e}^{-t}}{(1-t)^{\nu+m+1}} = \frac{D_m^{(\nu)}(t)\,\mathrm{e}^{-t}}{(1-t)^{\lambda+\nu+m+2}} \\
&= D_m^{(\nu)}(t)\;d^{(\lambda+\nu+m+1)}(t) = \sum_{k=0}^{m}\binom{m}{k}d_{m-k}^{(\nu)}t^k\,d^{(\lambda+\nu+m+1)}(t) \\
&= \sum_{k=0}^{m}\binom{m}{k}d_{m-k}^{(\nu)}\sum_{n\geq 0}\binom{n}{k}k!\,d_{n-k}^{(\lambda+\nu+m+1)}\frac{t^n}{n!} \\
&= \sum_{n\geq 0}\left[\sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}k!\,d_{m-k}^{(\nu)}d_{n-k}^{(\lambda+\nu+m+1)}\right]\frac{t^n}{n!}\,.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the first and last series, we have identity (20). □

Remark 7. A slightly different generalization of the derangement numbers $d_n^{(\nu)}$ is given by the numbers $D_n^{(\nu)}$ defined [6, 12] by the exponential generating series

$$\sum_{n\geq 0}D_n^{(\nu)}\frac{t^n}{n!} = \frac{\nu!\,\mathrm{e}^{-t}}{(1-t)^{\nu+1}} \qquad (\nu\in\mathbb{N})\,.$$

Identity (17) holds also for these numbers, while identity (19) changes in

$$\sum_{k=0}^{n}\binom{n}{k}D_{m+n-k}^{(\nu)} = \nu!\sum_{k=0}^{m}\binom{m}{k}(\nu+1)_{n+k}(-1)^{m-k}\,. \tag{22}$$

Similarly, identity (20) changes for a factor $(\lambda+\nu+m+1)!$ on the left-hand side.

Remark 8. Another sequence related to that of the generalized derangement numbers, is given by the *generalized arrangement numbers* $a_n^{(\nu)}$, defined by

$$a_n^{(\nu)} = \sum_{k=0}^{n}\binom{\nu+n-k}{n-k}\frac{n!}{k!}$$

and with exponential generating series

$$a^{(\nu)}(t) = \sum_{n\geq 0}a_n^{(\nu)}\frac{t^n}{n!} = \frac{\mathrm{e}^t}{(1-t)^{\nu+1}}\,.$$

When $\nu = 0$, we have the ordinary arrangement numbers a_n [7, p. 75] [19, A000522]. The Appell polynomials associated to the numbers $a_n^{(\nu)}$ are given by

$$A_n^{(\nu)}(x) = \sum_{k=0}^{n}\binom{n}{k}a_{n-k}^{(\nu)}x^k$$

and have exponential generating series

$$A^{(\nu)}(x;t) = \sum_{n\geq 0} A_n^{(\nu)}(x)\frac{t^n}{n!} = \frac{\mathrm{e}^{(x+1)t}}{(1-t)^{\nu+1}}\,.$$

Using the same approach employed in the proof of Lemma 2, we obtain the identity

$$D^m \frac{\mathrm{e}^t}{(1-t)^{\nu+1}} = \frac{A_m^{(\nu)}(-t)\,\mathrm{e}^t}{(1-t)^{\nu+m+1}} \tag{23}$$

and, consequently, the Callan-like identity

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_{m+n-k}^{(\nu)} = n!\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{\nu+m+n-k}{n-k}(-1)^k a_{m-k}^{(\nu)}\,. \tag{24}$$

Furthermore, as in Theorems 5 and 6, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_{m+n-k}^{(\nu)} = \sum_{k=0}^{m}\binom{m}{k}(\nu+1)_{n+k} \tag{25}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{\lambda+k}{k}k!\,a_{m+n-k}^{(\nu)} = \sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}k!\,(-1)^k a_{m-k}^{(\nu)}\,a_{n-k}^{(\lambda+\nu+m+1)}\,. \tag{26}$$

We also have the following identities relating the generalized derangement numbers and the generalized arrangement numbers.

Theorem 9. *For every $m,n\in\mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n}\binom{n}{k}a_k^{(\lambda)}d_{m+n-k}^{(\nu)} = n!\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{\lambda+\nu+m+n-k+1}{n-k}d_{m-k}^{(\nu)} \tag{27}$$

$$\sum_{k=0}^{n}\binom{n}{k}d_k^{(\lambda)}a_{m+n-k}^{(\nu)} = n!\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{\lambda+\nu+m+n-k+1}{n-k}(-1)^k a_{m-k}^{(\nu)}\,. \tag{28}$$

Proof. By identities (3), (16) and (5), we have

$$\begin{aligned}
\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}a_k^{(\lambda)}d_{m+n-k}^{(\nu)}\right]\frac{t^n}{n!} &= \sum_{n\geq 0}a_n^{(\lambda)}\frac{t^n}{n!}\cdot\sum_{n\geq 0}d_{n+m}^{(\nu)}\frac{t^n}{n!}\\
&= a^{(\lambda)}(t)\cdot D^m d^{(\lambda)}(t) = \frac{\mathrm{e}^t}{(1-t)^{\lambda+1}}\frac{D_m^{(\nu)}(t)\,\mathrm{e}^{-t}}{(1-t)^{\nu+m+1}} = \frac{D_m^{(\nu)}(t)}{(1-t)^{\lambda+\nu+m+2}}\\
&= \sum_{k=0}^{m}\binom{m}{k}d_{m-k}^{(\nu)}\frac{t^k}{(1-t)^{\lambda+\nu+m+2}} = \sum_{k=0}^{m}\binom{m}{k}d_{m-k}^{(\nu)}\sum_{n\geq 0}\binom{\lambda+\nu+m+n-k+1}{n-k}t^n\\
&= \sum_{n\geq 0}\left[n!\sum_{k=0}^{m}\binom{m}{k}\binom{\lambda+\nu+m+n-k+1}{n-k}d_{m-k}^{(\nu)}\right]\frac{t^n}{n!}\,.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the first and last series, we have identity (27). In a completely similar way, we also obtain identity (28). $\square$

3. Generalized Laguerre polynomials

The *generalized Laguerre polynomials* [18, p. 108] are defined by

$$L_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n+\nu}{n-k} \frac{n!}{k!} (-1)^k x^k \tag{29}$$

and have exponential generating series

$$L^{(\nu)}(x;t) = \sum_{n\geq 0} L_n^{(\nu)}(x) \frac{t^n}{n!} = \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\nu+1}} \,. \tag{30}$$

The *Lah polynomials* are defined by

$$L_n(x) = \sum_{k=0}^{n} \left| \begin{matrix} n \\ k \end{matrix} \right| x^k$$

where the coefficients $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right| = \binom{n-1}{k-1}\frac{n!}{k!}$ are the *Lah numbers* [10, 13] [19, A271703]. Since their exponential generating series is

$$\sum_{n\geq 0} L_n(x) \frac{t^n}{n!} = e^{\frac{xt}{1-t}}$$

we have the relation

$$L_n(x) = L_n^{(-1)}(-x) \,.$$

For the successive derivatives of series (30), we have the following result.

Lemma 10. *For every $m \in \mathbb{N}$, we have the identity*

$$D^m L^{(\nu)}(x;t) = L_m^{(\nu)}\Big(\frac{x}{1-t}\Big) \, \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\nu+m+1}} \tag{31}$$

where D denotes the derivative with respect to t.

Proof. By Taylor's formula (11) applied to series (30), we have

$$\begin{aligned}
\sum_{m\geq 0} D^m L^{(\nu)}(x;t) \, \frac{u^m}{m!} &= L^{(\nu)}(x;t+u) = \frac{e^{-x\frac{t+u}{1-t-u}}}{(1-t-u)^{\nu+1}} = \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\nu+1}} \, \frac{e^{-\frac{x}{1-t}\frac{\frac{u}{1-t}}{1-\frac{u}{1-t}}}}{(1-\frac{u}{1-t})^{\nu+1}} \\
&= L^{(\nu)}(x;t) \; L^{(\nu)}\Big(\frac{x}{1-t}; \frac{u}{1-t}\Big) = L^{(\nu)}(x;t) \; \sum_{m\geq 0} L_m^{(\nu)}\Big(\frac{x}{1-t}\Big) \frac{1}{(1-t)^m} \, \frac{u^m}{m!} \\
&= \sum_{m\geq 0} L_m^{(\nu)}\Big(\frac{x}{1-t}\Big) \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\nu+m+1}} \, \frac{u^m}{m!}
\end{aligned}$$

from which we have identity (31). $\square$

Then, we have the following theorem.

Theorem 11. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} L_k^{(\lambda)}(-x)\, L_{m+n-k}^{(\nu)}(x) = \\ = n! \sum_{k=0}^{m} \binom{\lambda+\nu+m+n+k+1}{n} \binom{m+\nu}{m-k} \frac{m!}{k!} (-1)^k x^k . \tag{32}$$

In particular, for $\lambda = -1$, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} L_k(x)\, L_{m+n-k}^{(\nu)}(x) = n! \sum_{k=0}^{m} \binom{\nu+m+n+k}{n} \binom{m+\nu}{m-k} \frac{m!}{k!} (-1)^k x^k . \tag{33}$$

Moreover, for $\lambda = \nu = -1$, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} L_k(-x)\, L_{m+n-k}(x) = n! \sum_{k=0}^{m} \left| \begin{matrix} m \\ k \end{matrix} \right| \binom{m+n+k-1}{n} x^k \tag{34}$$

or, equivalently,

$$\sum_{k=0}^{n} \binom{n}{k} L_k(-x)\, L_{m+n-k}(x) = \sum_{k=0}^{m} \left| \begin{matrix} m \\ k \end{matrix} \right| (m+k)_n\, x^k . \tag{35}$$

Proof. By identities (3), (31), (29) and (5), we have

$$\begin{aligned}
\sum_{n\geq 0} \left[\sum_{k=0}^{n} \binom{n}{k} L_k^{(\lambda)}(-x)\, L_{m+n-k}^{(\nu)}(x) \right] \frac{t^n}{n!} &= \sum_{n\geq 0} L_n^{(\lambda)}(-x) \frac{t^n}{n!} \cdot \sum_{n\geq 0} L_{n+m}^{(\nu)}(x) \frac{t^n}{n!} \\
&= L^{(\lambda)}(-x;t) \cdot D^m L^{(\nu)}(x;t) = \frac{e^{\frac{xt}{1-t}}}{(1-t)^{\lambda+1}}\, L_m^{(\nu)}\!\left(\frac{x}{1-t}\right) \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\nu+m+1}} \\
&= L_m^{(\nu)}\!\left(\frac{x}{1-t}\right) \frac{1}{(1-t)^{\lambda+\nu+m+2}} = \sum_{k=0}^{m} \binom{m+\nu}{m-k} \frac{m!}{k!} (-1)^k x^k \frac{1}{(1-t)^{\lambda+\nu+m+k+2}} \\
&= \sum_{k=0}^{m} \binom{m+\nu}{m-k} \frac{m!}{k!} (-1)^k x^k \sum_{n\geq 0} \binom{\lambda+\nu+m+n+k+1}{n} t^n \\
&= \sum_{n\geq 0} \left[n! \sum_{k=0}^{m} \binom{\lambda+\nu+m+n+k+1}{n} \binom{m+\nu}{m-k} \frac{m!}{k!} (-1)^k x^k \right] \frac{t^n}{n!} .
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the first and last series, we get identity (32). $\square$

We also have the following relation.

Theorem 12. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{\lambda+k}{k} k!\, L_{m+n-k}^{(\nu)}(x) = \sum_{k=0}^{m} \binom{m+\nu}{m-k} \frac{m!}{k!} (-1)^k x^k\, L_n^{(\lambda+\nu+m+k+1)}(x) . \tag{36}$$

Proof. By identities (3), (5), (31) and (29), we have

$$\begin{aligned}
\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}\binom{\lambda+k}{k}k!\,L_{m+n-k}^{(\nu)}(x)\right]\frac{t^n}{n!} &= \sum_{n\geq 0}\binom{\lambda+n}{n}n!\,\frac{t^n}{n!}\cdot\sum_{n\geq 0}L_{n+m}^{(\nu)}(x)\frac{t^n}{n!}\\
&= \frac{1}{(1-t)^{\lambda+1}}\,D^m L^{(\nu)}(x;t) = \frac{1}{(1-t)^{\lambda+1}}\,L_m^{(\nu)}\Big(\frac{x}{1-t}\Big)\,\frac{\mathrm{e}^{-\frac{xt}{1-t}}}{(1-t)^{\nu+m+1}}\\
&= L_m^{(\nu)}\Big(\frac{x}{1-t}\Big)\,\frac{\mathrm{e}^{-\frac{xt}{1-t}}}{(1-t)^{\lambda+\nu+m+2}} = \sum_{k=0}^{m}\binom{m+\nu}{m-k}\frac{m!}{k!}\,(-1)^k x^k\,\frac{\mathrm{e}^{-\frac{xt}{1-t}}}{(1-t)^{\lambda+\nu+m+k+2}}\\
&= \sum_{k=0}^{m}\binom{m+\nu}{m-k}\frac{m!}{k!}\,(-1)^k x^k\,L^{(\lambda+\nu+m+n+k+1)}(x;t)\\
&= \sum_{n\geq 0}\left[\sum_{k=0}^{m}\binom{m+\nu}{m-k}\frac{m!}{k!}\,(-1)^k x^k L_n^{(\lambda+\nu+m+n+k+1)}(x)\right]\frac{t^n}{n!}\,.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the first and last series, we get identity (36). □

4. Generalized Hermite polynomials

The *generalized Hermite polynomials* $H_n^{(\nu)}(x)$, [11, Vol. 2, p. 192], are defined by

$$H_n^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}(-\nu)^k k!(2x)^{n-2k} \tag{37}$$

and have exponential generating series

$$H^{(\nu)}(x;t) = \sum_{n\geq 0}H_n^{(\nu)}(x)\,\frac{t^n}{n!} = \mathrm{e}^{2xt-\nu t^2}. \tag{38}$$

For $\nu = 0$, we have the powers $(2x)^n$. For $\nu = 1$, we have the ordinary *Hermite polynomials* $H_n(x)$.

For the successive derivatives of series (38), we have what follows.

Lemma 13. *For every $m \in \mathbb{N}$, we have the identity*

$$D^m H^{(\nu)}(x;t) = H^{(\nu)}(x;t)\sum_{k=0}^{m}\binom{m}{k}(-2\nu t)^k H_{m-k}^{(\nu)}(x) \tag{39}$$

where D denotes the derivative with respect to t.

Proof. By Taylor's formula (11) applied to series (38), we have

$$\begin{aligned}
\sum_{m\geq 0}D^m H^{(\nu)}(x;t)\,\frac{u^m}{m!} &= \mathrm{e}^{2x(t+u)-\nu(t+u)^2} =\\
&= \mathrm{e}^{2xt-\nu t^2}\mathrm{e}^{2xu-\nu u^2}\mathrm{e}^{-2\nu tu} = H^{(\nu)}(x;t)\,H^{(\nu)}(x;u)\,\mathrm{e}^{-2\nu tu}
\end{aligned}$$

from which we have identity (39). □

Then, we have the following theorem.

Theorem 14. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} H_k^{(\lambda)}(z)\, H_{m+n-k}^{(\nu)}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} (-2\nu)^k k!\, H_{m-k}^{(\nu)}(x)\, H_{n-k}^{(\lambda+\nu)}(x+z)\,. \tag{40}$$

In particular, for $\lambda = 0$, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} (2z)^k H_{m+n-k}^{(\nu)}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} (-2\nu)^k k!\, H_{m-k}^{(\nu)}(x)\, H_{n-k}^{(\nu)}(x+z)\,. \tag{41}$$

Proof. By identities (3), (39), (38) and (7), we have

$$\begin{aligned}
\sum_{n\geq 0} \left[\sum_{k=0}^{n} \binom{n}{k} H_k^{(\lambda)}(z)\, H_{m+n-k}^{(\nu)}(x) \right] \frac{t^n}{n!} &= \sum_{n\geq 0} H_n^{(\lambda)}(z) \frac{t^n}{n!} \cdot \sum_{n\geq 0} H_{n+m}^{(\nu)}(x) \frac{t^n}{n!} \\
&= H^{(\lambda)}(z;t) \cdot D^m H^{(\nu)}(x;t) = H^{(\lambda)}(z;t)\, H^{(\nu)}(x;t) \sum_{k=0}^{m} \binom{m}{k} (-2\nu t)^k H_{m-k}^{(\nu)}(x) \\
&= \sum_{k=0}^{m} \binom{m}{k} (-2\nu)^k H_{m-k}^{(\nu)}(x)\; t^k\, H^{(\lambda+\nu)}(x+z;t) \\
&= \sum_{k=0}^{m} \binom{m}{k} (-2\nu)^k H_{m-k}^{(\nu)}(x) \sum_{n\geq k} \binom{m}{k} k!\, H_{n-k}^{(\lambda+\nu)}(x+z)\, \frac{t^n}{n!} \\
&= \sum_{n\geq 0} \left[\sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} (-2\nu)^k k!\, H_{m-k}^{(\nu)}(x)\, H_{n-k}^{(\lambda+\nu)}(x+z) \right] \frac{t^n}{n!}\,.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the first and last series, we obtain identity (40). □

Remark 15. For $z = 0$, identity (42) becomes identity [11, Vol. 2, p. 195]

$$H_{m+n}^{(\nu)}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} (-2\nu)^k k! H_{m-k}^{(\nu)}(x) H_{n-k}^{(\nu)}(x)\,. \tag{42}$$

Remark 16. Let i_n be the *involution numbers* [19, A000085] and let

$$I_n(x) = \sum_{k=0}^{n} \binom{n}{k} i_{n-k} x^k$$

be the associated Appell polynomials, with exponential generating series

$$\sum_{n\geq 0} I_n(x)\, \frac{t^n}{n!} = \mathrm{e}^{(x+1)t + t^2/2}\,.$$

By series (38), we have the relation

$$I_n(x) = H_n^{(-1/2)}\Big(\frac{x+1}{2}\Big)$$

and consequently, replacing z by $z/2$, identity (42) becomes

$$\sum_{k=0}^{n} \binom{n}{k} z^k I_{m+n-k}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, I_{m-k}(x)\, I_{n-k}(x+z)\,. \tag{43}$$

In particular, for $z = 0$, we have the identity

$$I_{m+n}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, I_{m-k}(x)\, I_{n-k}(x)\,. \tag{44}$$

5. Generalized exponential polynomials

The *generalized Stirling numbers* (of the second kind) $S_{n,k}^{(\nu)}$ are defined by the exponential generating series

$$\sum_{n\geq k} S_{n,k}^{(\nu)} \frac{t^n}{n!} = e^{\nu t} \frac{(e^t - 1)^k}{k!}\,. \tag{45}$$

The *generalized exponential polynomials* $S_n^{(\nu)}(x)$ are defined by

$$S_n^{(\nu)}(x) = \sum_{k=0}^{n} S_{n,k}^{(\nu)}\, x^k \tag{46}$$

and have exponential generating series

$$S^{(\nu)}(x;t) = \sum_{n\geq 0} S_n^{(\nu)}(x)\, \frac{t^n}{n!} = e^{\nu t} e^{x(e^t - 1)}\,. \tag{47}$$

Finally, the *generalized Bell numbers* $b_n^{(\nu)}$ are defined by

$$b_n^{(\nu)} = S_n^{(\nu)}(1) = \sum_{k=0}^{n} S_{n,k}^{(\nu)} \tag{48}$$

and have exponential generating series

$$\sum_{n\geq 0} b_n^{(\nu)} \frac{t^n}{n!} = e^{\nu t} e^{e^t - 1}\,. \tag{49}$$

For $\nu = 0$, we have the ordinary Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ [7, p. 310] [9, p. 244] [19, A008277], the ordinary exponential polynomials $S_n(x)$ [18, p. 63] and the ordinary Bell numbers b_n [7, p. 210] [19, A000110]. More generally, for $\nu = r \in \mathbb{N}$, we have the r-Stirling numbers of the second kind [3], the r-exponential polynomials and the r-Bell numbers [14]. In particular, we have $b_n^{(1)} = b_{n+1}$ and $S_{n,k}^{(1)} = \left\{ {n+1 \atop k+1} \right\}$.

For the successive derivatives of series (47), we have the following result.

Lemma 17. *For every $m \in \mathbb{N}$, we have the identity*

$$D^m S^{(\nu)}(x;t) = S^{(\nu)}_m(x\mathrm{e}^t)\, S^{(\nu)}(x;t) \tag{50}$$

where D denotes the derivative with respect to t.

Proof. By Taylor's formula (11) applied to series (47), we have

$$\begin{aligned}\sum_{m\geq 0} D^m S^{(\nu)}(x;t)\frac{u^m}{m!} &= S^{(\nu)}(x;t+u) = \mathrm{e}^{\nu(t+u)}\mathrm{e}^{x(\mathrm{e}^{t+u}-1)} = \mathrm{e}^{\nu t}\mathrm{e}^{\nu u}\mathrm{e}^{x(\mathrm{e}^t\mathrm{e}^u-1)}\\ &= \mathrm{e}^{\nu t}\mathrm{e}^{\nu u}\mathrm{e}^{x(\mathrm{e}^t(\mathrm{e}^u-1)+\mathrm{e}^t-1)} = \mathrm{e}^{\nu t}\mathrm{e}^{x(\mathrm{e}^t-1)}\,\mathrm{e}^{\nu u}\mathrm{e}^{x\mathrm{e}^t(\mathrm{e}^u-1)} = S^{(\nu)}(x;t)\, S^{(\nu)}(x\mathrm{e}^t;u)\\ &= S^{(\nu)}(x;t)\sum_{m\geq 0} S^{(\nu)}_m(x\mathrm{e}^t)\,\frac{u^m}{m!} = \sum_{m\geq 0} S^{(\nu)}_m(x\mathrm{e}^t)\, S^{(\nu)}(x;t)\,\frac{u^m}{m!}\,.\end{aligned}$$

Confronting the coefficients of $\frac{u^m}{m!}$ in the first and last series, we obtain identity (50). □

Then, we have the following theorem.

Theorem 18. *For every $m,n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k} S^{(\lambda)}_k(z)\, S^{(\nu)}_{m+n-k}(x) = \sum_{k=0}^{m} S^{(\nu)}_{m,k}\, x^k\, S^{(\lambda+\nu+k)}_n(x+z)\,. \tag{51}$$

Proof. By identities (3), (50) and (47), we have

$$\begin{aligned}\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k} S^{(\lambda)}_k(z)\, S^{(\nu)}_{m+n-k}(x)\right]\frac{t^n}{n!} &= \sum_{n\geq 0} S^{(\lambda)}_n(z)\,\frac{t^n}{n!}\cdot\sum_{n\geq 0} S^{(\nu)}_{n+m}(x)\,\frac{t^n}{n!}\\ &= S^{(\lambda)}(z;t)\cdot D^m S^{(\nu)}(x;t) = S^{(\lambda)}(z;t)\, S^{(\nu)}(x;t)\; S^{(\nu)}_m(x\mathrm{e}^t)\\ &= \mathrm{e}^{\lambda t}\mathrm{e}^{z(\mathrm{e}^t-1)}\,\mathrm{e}^{\nu t}\mathrm{e}^{x(\mathrm{e}^t-1)}\sum_{k=0}^{m} S^{(\nu)}_{m,k}\, x^k\, \mathrm{e}^{kt} = \sum_{k=0}^{m} S^{(\nu)}_{m,k}\, x^k\, \mathrm{e}^{(\lambda+\nu+k)t}\mathrm{e}^{(x+z)(\mathrm{e}^t-1)}\\ &= \sum_{k=0}^{m} S^{(\nu)}_{m,k}\, x^k\, S^{(\lambda+\nu+k)}(x+z;t) = \sum_{k=0}^{m} S^{(\nu)}_{m,k}\, x^k \sum_{n\geq 0} S^{(\lambda+\nu+k)}_n(x+z)\frac{t^n}{n!}\\ &= \sum_{n\geq 0}\left[\sum_{k=0}^{m} S^{(\nu)}_{m,k}\, x^k\, S^{(\lambda+\nu+k)}_n(x+z)\right]\frac{t^n}{n!}\,.\end{aligned}$$

Confronting the coefficients of $\frac{t^n}{n!}$ in the first and last series, we obtain identity (51). □

REMARK 19. Identity (51) can be specialized as follows. For $z=0$, we have

$$\sum_{k=0}^{n}\binom{n}{k}\lambda^k\, S^{(\nu)}_{m+n-k}(x) = \sum_{k=0}^{m} S^{(\nu)}_{m,k}\, x^k\, S^{(\lambda+\nu+k)}_n(x)\,. \tag{52}$$

For $z = 0$ and $x = 1$:

$$\sum_{k=0}^{n} \binom{n}{k} \lambda^k b_{m+n-k}^{(\nu)} = \sum_{k=0}^{m} S_{m,k}^{(\nu)} b_n^{(\lambda+\nu+k)}. \tag{53}$$

For $z = 0$, $x = 1$, $\lambda = 1$ and $\nu = 0$:

$$\sum_{k=0}^{n} \binom{n}{k} b_{m+n-k} = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} b_n^{(k+1)}. \tag{54}$$

For $z = 0$, $x = 1$, $\lambda = \nu = 1$:

$$\sum_{k=0}^{n} \binom{n}{k} b_{m+n-k+1} = \sum_{k=0}^{m} \left\{ {m+1 \atop k+1} \right\} b_n^{(k+2)}. \tag{55}$$

For $z = 0$ and $\lambda = 0$:

$$S_{m+n}^{(\nu)}(x) = \sum_{k=0}^{m} S_{m,k}^{(\nu)} x^k S_n^{(\nu+k)}(x). \tag{56}$$

For $z = 0$, $x = 1$ and $\lambda = 0$:

$$b_{m+n}^{(\nu)} = \sum_{k=0}^{m} S_{m,k}^{(\nu)} b_n^{(\nu+k)}. \tag{57}$$

For $z = 0$, $x = 1$ and $\lambda = \nu = 0$:

$$b_{m+n} = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} b_n^{(k)}. \tag{58}$$

For $z = 0$, $x = 1$, $\lambda = 0$ and $\nu = 1$:

$$b_{m+n+1} = \sum_{k=0}^{m} \left\{ {m+1 \atop k+1} \right\} b_n^{(k+1)}. \tag{59}$$

6. Hyperharmonic numbers

The *hyperharmonic numbers* [8, pp. 143, 258–259] [2, 15] are defined by

$$\mathcal{H}_n^{(\nu)} = \sum_{k=1}^{n} \binom{\nu+n-k}{n-k} \frac{1}{k} \tag{60}$$

and have ordinary generating series

$$\mathcal{H}^{(\nu)}(t) = \sum_{n \geq 0} \mathcal{H}_n^{(\nu)} t^n = \frac{1}{(1-t)^{\nu+1}} \ln \frac{1}{1-t}\,. \tag{61}$$

For $\nu = 0$, we have the ordinary *harmonic numbers*

$$\mathcal{H}_n = \sum_{k=1}^{n} \frac{1}{k}$$

and, for $\nu \in \mathbb{N}$, we have the relation

$$\mathcal{H}_n^{(\nu)} = \binom{\nu+n}{n} (\mathcal{H}_{\nu+n} - \mathcal{H}_\nu) .$$

For the successive derivatives of series (61), we have the following result.

Lemma 20. *For every $m \in \mathbb{N}$, we have the identity*

$$D^m \mathcal{H}^{(\nu)}(t) = \binom{\nu+m}{m} \frac{m!\, \mathcal{H}^{(\nu)}(t)}{(1-t)^m} + \frac{m!\, \mathcal{H}_m^{(\nu)}}{(1-t)^{\nu+m+1}} . \tag{62}$$

Proof. By Taylor's formula (11) applied to series (61), we have

$$\begin{aligned}
\sum_{m\geq 0} D^m \mathcal{H}^{(\nu)}(t)\, \frac{u^m}{m!} &= \mathcal{H}^{(\nu)}(t+u) = \frac{1}{(1-t-u)^{\nu+1}} \ln \frac{1}{1-t-u} \\
&= \frac{1}{(1-t)^{\nu+1}} \frac{1}{(1-\frac{u}{1-t})^{\nu+1}} \ln \frac{1}{(1-t)\big(1-\frac{u}{1-t}\big)} \\
&= \frac{1}{(1-t)^{\nu+1}} \frac{1}{\big(1-\frac{u}{1-t}\big)^{\nu+1}} \ln \frac{1}{1-t} + \frac{1}{(1-t)^{\nu+1}} \frac{1}{\big(1-\frac{u}{1-t}\big)^{\nu+1}} \ln \frac{1}{1-\frac{u}{1-t}} \\
&= \frac{\mathcal{H}^{(\nu)}(t)}{\big(1-\frac{u}{1-t}\big)^{\nu+1}} + \frac{1}{(1-t)^{\nu+1}}\, \mathcal{H}^{(\nu)}\Big(\frac{u}{1-t}\Big) \\
&= \mathcal{H}^{(\nu)}(t) \sum_{m\geq 0} \binom{\nu+m}{m} \frac{u^m}{(1-t)^m} + \frac{1}{(1-t)^{\nu+1}} \sum_{m\geq 0} \mathcal{H}_m^{(\nu)} \frac{u^m}{(1-t)^m} \\
&= \sum_{m\geq 0} \left[\binom{\nu+m}{m} \frac{m!\mathcal{H}^{(\nu)}(t)}{(1-t)^m} + \frac{m!\mathcal{H}_m^{(\nu)}}{(1-t)^{\nu+m+1}} \right] \frac{u^m}{m!}
\end{aligned}$$

from which we have identity (62). $\square$

Then, we have the following theorem.

Theorem 21. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{\lambda+k}{k} \binom{m+n-k}{n-k} \mathcal{H}_{m+n-k}^{(\nu)} = \\
&\qquad = \binom{\lambda+\nu+m+n+1}{n} \mathcal{H}_m^{(\nu)} + \binom{\nu+m}{m} \sum_{k=0}^{n} \binom{\lambda+m+k}{k} \mathcal{H}_{n-k}^{(\nu)} .
\end{aligned} \tag{63}$$

Proof. By identities (4), (10), (62) and (5), we have

$$
\begin{aligned}
\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{\lambda+k}{k}\binom{m+n-k}{n-k}\mathcal{H}^{(\nu)}_{m+n-k}\right]t^n &= \sum_{n\geq 0}\binom{\lambda+n}{n}t^n\cdot\sum_{n\geq 0}\binom{m+n}{n}\mathcal{H}^{(\nu)}_{n+m}t^n \\
&= \frac{1}{(1-t)^{\lambda+1}}\frac{D^m\mathcal{H}^{(\nu)}(t)}{m!} = \binom{\nu+m}{m}\frac{\mathcal{H}^{(\nu)}(t)}{(1-t)^{\lambda+m+1}} + \frac{\mathcal{H}^{(\nu)}_m}{(1-t)^{\lambda+\nu+m+2}} \\
&= \binom{\nu+m}{m}\sum_{k\geq 0}\mathcal{H}^{(\nu)}_k\frac{t^k}{(1-t)^{\lambda+m+1}} + \frac{\mathcal{H}^{(\nu)}_m}{(1-t)^{\lambda+\nu+m+2}} \\
&= \binom{\nu+m}{m}\sum_{k\geq 0}\mathcal{H}^{(\nu)}_k\sum_{n\geq 0}\binom{\lambda+m+n-k}{n-k}t^n + \mathcal{H}^{(\nu)}_m\sum_{n\geq 0}\binom{\lambda+\nu+m+n+1}{n}t^n \\
&= \sum_{n\geq 0}\left[\binom{\nu+m}{m}\sum_{k=0}^{n}\binom{\lambda+m+n-k}{n-k}\mathcal{H}^{(\nu)}_k + \binom{\lambda+\nu+m+n+1}{n}\mathcal{H}^{(\nu)}_m\right]t^n\,.
\end{aligned}
$$

Comparing the coefficients of t^n in the first and last series, we have identity (63). □

Remark 22. In particular, for $\nu = 0$ and $\lambda = m = n$, we have the identity

$$\sum_{k=0}^{n}\binom{n+k}{k}\binom{2n-k}{n-k}\mathcal{H}_{2n-k} = \binom{3n+1}{2n+1}\mathcal{H}_n + \sum_{k=0}^{n}\binom{2n+k}{k}\mathcal{H}_{n-k}\,. \tag{64}$$

7. Lagrange polynomials

The *Lagrange polynomials* (in one variable) [11, Vol. II, p. 267] are defined by

$$g^{(\alpha,\beta)}_n(x) = \sum_{k=0}^{n}\binom{\alpha+n-k}{n-k}\binom{\beta+k}{k}x^k \tag{65}$$

and have ordinary generating series

$$g^{(\alpha,\beta)}(x;t) = \sum_{n\geq 0}g^{(\alpha,\beta)}_n(x)t^n = \frac{1}{(1-t)^{\alpha+1}(1-xt)^{\beta+1}}\,. \tag{66}$$

The Lagrange polynomials satisfy the following property

$$g^{(\lambda,\nu)}(x;t)g^{(\alpha,\beta)}(x;t) = g^{(\lambda+\alpha+1,\nu+\beta+1)}(x;t)\,. \tag{67}$$

For the successive derivatives of series (66), we have the following result.

Lemma 23. *For every $m\in\mathbb{N}$, we have the identity*

$$D^m g^{(\alpha,\beta)}(x;t) = \sum_{k=0}^{m}\binom{\alpha+m-k}{m-k}\binom{\beta+k}{k}x^k g^{(\alpha+m-k,\beta+k)}(x;t) \tag{68}$$

where D denotes the derivative with respect to t.

Proof. By Taylor's formula (11) applied to series (66), we have

$$\begin{aligned}\sum_{m\geq 0} D_t^m g^{(\alpha,\beta)}(x;t)\,\frac{u^m}{m!} &= g^{(\alpha,\beta)}(x;t+u)\\ &= \frac{1}{(1-t-u)^{\alpha+1}(1-x(t+u))^{\beta+1}}\\ &= \frac{1}{(1-t)^{\alpha+1}(1-xt)^{\beta+1}}\frac{1}{(1-\frac{u}{1-t})^{\alpha+1}(1-\frac{xu}{1-xt})^{\beta+1}}\\ &= g^{(\alpha,\beta)}(x;t)\,\frac{1}{(1-\frac{u}{1-t})^{\alpha+1}(1-\frac{xu}{1-xt})^{\beta+1}}\\ &= g^{(\alpha,\beta)}(x;t)\sum_{n\geq 0}\left[\sum_{k=0}^{m}\binom{\alpha+m-k}{m-k}\binom{\beta+k}{k}\frac{m!x^k}{(1-t)^{m-k}(1-xt)^k}\right]\frac{u^m}{m!}\end{aligned}$$

from which we have identity (68). □

Then, we have the following theorem.

Theorem 24. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\begin{aligned}\sum_{k=0}^{n}\binom{m+n-k}{n-k}g_k^{(\lambda,\nu)}(x)\,g_{m+n-k}^{(\alpha,\beta)}(x) &=\\ &= \sum_{k=0}^{m}\binom{\alpha+m-k}{m-k}\binom{\beta+k}{k}x^k g_n^{(\lambda+\alpha+m-k+1,\nu+\beta+k+1)}(x)\,.\end{aligned} \tag{69}$$

Proof. By identities (4), (10), (68) and (67), we have

$$\begin{aligned}\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{m+n-k}{n-k}g_k^{(\lambda,\nu)}(x)\,g_{m+n-k}^{(\alpha,\beta)}(x)\right]t^n &=\\ &= \sum_{n\geq 0} g_n^{(\lambda,\nu)}(x)\,t^n\cdot\sum_{n\geq 0}\binom{m+n}{n}g_n^{(\alpha,\beta)}(x)\,t^n\\ &= g^{(\lambda,\nu)}(x;t)\cdot\frac{D^m g_k^{(\alpha,\beta)}(x;t)}{m!}\\ &= g^{(\lambda,\nu)}(x;t)\sum_{k=0}^{m}\binom{\alpha+m-k}{m-k}\binom{\beta+k}{k}x^k g^{(\alpha+m-k,\beta+k)}(x;t)\\ &= \sum_{k=0}^{m}\binom{\alpha+m-k}{m-k}\binom{\beta+k}{k}x^k g^{(\lambda+\alpha+m-k+1,\nu+\beta+k+1)}(x;t)\\ &= \sum_{k=0}^{m}\binom{\alpha+m-k}{m-k}\binom{\beta+k}{k}x^k\sum_{n\geq 0} g_n^{(\lambda+\alpha+m-k+1,\nu+\beta+k+1)}(x)\,t^n\\ &= \sum_{n\geq 0}\left[\sum_{k=0}^{m}\binom{\alpha+m-k}{m-k}\binom{\beta+k}{k}x^k g_n^{(\lambda+\alpha+m-k+1,\nu+\beta+k+1)}(x)\right]t^n\,.\end{aligned}$$

Comparing the coefficients of t^n in the first and last series, we have identity (69). $\square$

8. Gegenbauer polynomials

The *Gegenbauer polynomials* $C_n^{(\nu)}(x)$ [11, Vol. II, p. 175] are defined by

$$C_n^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\!\!\binom{\nu}{n-k}\!\!\right) \binom{n-k}{k} (-1)^k (2x)^{n-2k} \tag{70}$$

and have ordinary generating series

$$C^{(\nu)}(x;t) = \sum_{n \geq 0} C_n^{(\nu)}(x)\, t^n = \frac{1}{(1-2xt+t^2)^\nu}\,. \tag{71}$$

For $\nu = 1$, we have the *Chebyshev polynomials of the second kind* $U_n(x)$, while, for $\nu = 1/2$, we have the *Legendre polynomials* $P_n(x)$.

The Gegenbauer polynomials have the following property

$$C^{(\lambda)}(x;t)\, C^{(\nu)}(x;t) = C^{(\lambda+\nu)}(x;t)\,. \tag{72}$$

For the successive derivatives of series (71), we have what follows.

Lemma 25. *For every $m \in \mathbb{N}$, we have the identity*

$$D^m C^{(\nu)}(x;t) = m! \sum_{k=0}^{\lfloor m/2 \rfloor} \left(\!\!\binom{\nu}{m-k}\!\!\right) \binom{m-k}{k} (-1)^k (2x-2t)^{m-2k} C^{(\nu+m-k)}(x;t) \tag{73}$$

where D denotes the derivative with respect to t.

Proof. By Taylor's formula (11) applied to series (71) and by series (6), we have

$$\begin{aligned}
\sum_{m\geq 0} D_t^m C^{(\nu)}(x;t)\, \frac{u^m}{m!} &= C^{(\nu)}(x;t+u) \\
&= \frac{1}{(1-2x(t+u)+(t+u)^2)^\nu} \\
&= \frac{1}{(1-2xt+t^2-2xu+2tu+u^2)^\nu} \\
&= \frac{1}{(1-2xt+t^2)^\nu}\, \frac{1}{\left(1-\frac{(2x-2t-u)u}{1-2xt+t^2}\right)^\nu} \\
&= \frac{1}{(1-2xt+t^2)^\nu} \sum_{k\geq 0} \left(\!\!\binom{\nu}{k}\!\!\right) \frac{(2x-2t-u)^k u^k}{(1-2xt+t^2)^k} \\
&= \sum_{k\geq 0} \left(\!\!\binom{\nu}{k}\!\!\right) \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{(2x-2t)^{k-i}\, u^{i+k}}{(1-2xt+t^2)^{\nu+k}} \qquad (m = i+k) \\
&= \sum_{m\geq 0} \left[m! \sum_{i=0}^{\lfloor m/2 \rfloor} \left(\!\!\binom{\nu}{m-i}\!\!\right) \binom{m-i}{i} (-1)^i (2x-2t)^{k-i} C^{(\nu+m-i)}(x;t) \right] \frac{u^m}{m!}
\end{aligned}$$

from which we have identity (73). □

In conclusion, we have the following theorem.

Theorem 26. *For every $m, n \in \mathbb{N}$, we have the identity*

$$\begin{aligned}\sum_{k=0}^{n} \binom{m+n-k}{n-k} C_k^{(\lambda)}(x)\, C_{m+n-k}^{(\nu)}(x) &= \\ &= \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} \left(\!\!\binom{\nu}{m-k}\!\!\right) \sum_{i=0}^{m-2k} \binom{m-2k}{i} (-1)^{i+k} x^{m-2k-i} C_{n-i}^{(\lambda+\nu+m-k)}(x)\,.\end{aligned} \tag{74}$$

Proof. By identities (4), (10), (73) and (72), we have

$$\begin{aligned}\sum_{n\geq 0} \left[\sum_{k=0}^{n} \binom{m+n-k}{n-k} C_k^{(\lambda)}(x)\, C_{m+n-k}^{(\nu)}(x)\right] \frac{t^n}{n!} &= \\ &= \sum_{n\geq 0} C_n^{(\lambda)}(x)\, \frac{t^n}{n!} \cdot \sum_{n\geq 0} \binom{m+n}{n} C_{n+m}^{(\nu)}(x)\, \frac{t^n}{n!} = C^{(\lambda)}(x;t) \cdot \frac{D^m C^{(\nu)}(x;t)}{m!} \\ &= \sum_{k=0}^{\lfloor m/2 \rfloor} \left(\!\!\binom{\nu}{m-k}\!\!\right) \binom{m-k}{k} (-1)^k (2x-2t)^{m-2k} C^{(\lambda+\nu+m-k)}(x;t) \\ &= \sum_{k=0}^{\lfloor m/2 \rfloor} \left(\!\!\binom{\nu}{m-k}\!\!\right) \binom{m-k}{k} (-1)^k 2^{m-2k} \times \\ &\qquad \sum_{i=0}^{m-2k} \binom{m-2k}{i} (-1)^i x^{m-2k-i} t^i C^{(\lambda+\nu+m-k)}(x;t)\,.\end{aligned}$$

By applying formula (8), we obtain identity (74). □

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