# ON MUTUALLY ORTHOGONAL CERTAIN GRAPH SQUARES

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Abstract. This paper gives some new results on mutually orthogonal graph squares (MOGS). These generalize mutually orthogonal Latin squares in an interesting way. As such, the topic is quite nice and should have broad appeal. MOGS have strong connections to core fields of finite algebra, cryptography, finite geometry, and design of experiments. We are concerned with the Kronecker product of mutually orthogonal graph squares to get new results of the mutually orthogonal certain graphs squares.

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## 1. Introduction

### 1.1. Nomenclature.

| | |
|---|---|
| $C_k$ | Cycle of length $k$ |
| $P_k$ | Path on $k$-vertices |
| $mG$ | $m$ disjoint copies of $G$ |
| $G \cup H$ | Disjoint union of $G$ and $H$ |
| $K_m$ | Complete graph on $m$ vertices |
| $\mathbb{P}_k(G)$ | $G$-path obtained by replacing each edge in $P_k$ by the graph $G$ |
| $K_{m,n}$ | Complete bipartite graph with independent sets of sizes $m$ and $n$ |
| $E(G)$ | The edge set of the graph $G$ |
| $L(x,y)$ | Entry in row $x$ and column $y$ of the square matrix $L$. |

1.2. **Latin squares.** A Latin square of order $n$ is an $n \times n$ array over a set of $n$ symbols such that every symbol appears exactly once in each row and exactly once in each column. Latin squares encode features of algebraic structures. When an algebraic structure passes certain “Latin square tests”, it is a candidate for use in the construction of cryptographic systems. Latin squares have been studied by mathematicians since ancient times. The origin of the Latin square is not known for certain. The name “Latin square” was inspired by some work conducted by Euler in the late 18th century, who used Latin characters as the symbols [1, 2, 3].

In this paper, the elements of $\mathbb{Z}_p \times \mathbb{Z}_2$ are used for labeling the vertices of $K_{p,p}$. The product $(v, j) \in \mathbb{Z}_p \times \mathbb{Z}_2$ will be written as $v_j$, which refers to the corresponding vertex and the edge $\{c_\gamma, d_\delta\} \in E(K_{p,p})$ if and only if $\gamma \neq \delta$ for all $c, d \in \mathbb{Z}_p$ and $\gamma, \delta \in \mathbb{Z}_2$. Let us introduce the basic definitions for the graph squares (see also [4]).

**Definition 1.** *Suppose $G$ is a subgraph of $K_{p,p}$. A square matrix $L$ of order $p$ is called a $G$-square if every element in $\mathbb{Z}_p$ occurs exactly $p$ times and the graphs $G_i$ where $i \in \mathbb{Z}_p$ with $E(G_i) = \{(x,y) : L(x,y) = i\}$ are isomorphic to $G$. The index set for the rows and columns of $L$ is $\mathbb{Z}_p$.*

Two graph squares of order $p$ are orthogonal if their superimposition yields all $p^2$ possible ordered pairs $(i,j)$, $0 \le i,j \le p-1$. This definition is naturally extended to sets of $k > 2$ graph squares, which are called mutually orthogonal if they are pairwise orthogonal.

Now, we show the relation between the edge decomposition of complete bipartite graphs and the graph squares. Call the collection $\mathcal{G} = \{G_0, G_1, \ldots, G_{p-1}\}$ an edge decomposition of $K_{p,p}$ by $G$ if $G_i \cong G$, where every edge of $K_{p,p}$ is contained in exactly one element (called page) of $\mathcal{G}$. For the collection $\mathcal{G}$, we have $\left|E(G_i) \cap E(G_j)\right| = 0; 0 \le i < j \le p-1$ and $\bigcup_{i=0}^{p-1} E(G_i) = E(K_{p,p})$. It is clear that the $G$-square represents the edge decomposition $\mathcal{G}$.

The two edge decompositions $\mathcal{L} = \{L_0, L_1, \ldots, L_{p-1}\}$ and $\mathcal{D} = \{D_0, D_1, \ldots, D_{p-1}\}$ of $K_{p,p}$ are orthogonal when $|E(L_i)| = \left|E(D_j)\right| = p$ and $\left|E(L_i) \cap E(D_j)\right| = 1$ for $i,j \in \mathbb{Z}_p$. If $\mathcal{D}_i$ and $\mathcal{D}_j; 0 \le i < j \le p-1$ are orthogonal edge decompositions of $K_{p,p}$ by $G$, then we get a set of $l$ mutually orthogonal $G$-squares (MOGS). Let us denote by $N(p, G)$ the maximal number of mutually orthogonal $G$-squares in the largest possible set of mutually orthogonal $G$-squares of order $p$. Although Latin squares have many useful properties, for some statistical applications these structures are too restrictive. The more general concepts of graph squares and MOGS offer more flexibility. For several applications on MOLS, see [5].

**Example 1.** *Three mutually orthogonal $P_4 \cup 2P_2$-squares are defined as follows,*

$$L^0 = \begin{bmatrix} 0 & 0 & 2 & 4 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 1 & 3 & 2 & 2 & 4 \\ 0 & 2 & 4 & 3 & 3 \\ 4 & 1 & 3 & 0 & 4 \end{bmatrix}, \; L^1 = \begin{bmatrix} 0 & 2 & 1 & 0 & 4 \\ 0 & 1 & 3 & 2 & 1 \\ 2 & 1 & 2 & 4 & 3 \\ 4 & 3 & 2 & 3 & 0 \\ 1 & 0 & 4 & 3 & 4 \end{bmatrix}, \; L^2 = \begin{bmatrix} 0 & 1 & 4 & 2 & 0 \\ 1 & 1 & 2 & 0 & 3 \\ 4 & 2 & 2 & 3 & 1 \\ 2 & 0 & 3 & 3 & 4 \\ 0 & 3 & 1 & 4 & 4 \end{bmatrix}.$$

If the $\lambda$-decompositions of $K_{p,p}$ by $G$ are mutually orthogonal, then the union of these decompositions is called $\lambda$ mutually orthogonal covers of $K_{p,p}$ by $G$. If $\lambda = 2$, then we have two orthogonal decompositions of $K_{p,p}$ by $G$ called orthogonal double cover of $K_{p,p}$ by $G$, see [4].

A decomposition of $K_{p,p}$ by $pK_2$ is represented by a Latin square of order $p$. We can say that the two decompositions $\mathcal{L}$ and $\mathcal{D}$ of $K_{p,p}$ by $pK_2$ are orthogonal if and only if the corresponding Latin squares of order $p$ are orthogonal; and thus $N(p, \; pK_2)$ is the maximum number of mutually orthogonal Latin squares (MOLS) of order $p$. One of the most difficult problems in combinatorial designs is the computation of $N(p, \; pK_2)$. Several papers have been devoted to MOLS problem, see [6, 7, 8]. MOGS have strong connections to core fields of finite algebra, cryptography, finite geometry, and design of experiments. Since $N(p,\ G)$ is considered a natural extension of $N(p,\ pK_2)$, the computation of $N(p,\ G)$ for general graphs is interesting. For $p \geq 2$, the relation $N(p,\ G) \leq p$ has been proved in [4] by El-Shanawany. Also, he proved the following, $(i)$ $N(p,\ K_{1,p}) = 2$, $(ii)$ $N(2,\ P_3) = 2$, $N(3,\ P_4) = 3$, $N(5,\ P_6) = 5$ and $N(6,\ P_7) = 6$, $(iii)$ Suppose $p$ is a prime number, then $N(p,\ K_{1,1} \cup \frac{p-1}{2}P_3) = p$, $(iv)$ Suppose $p$ is a prime number, then $N(p,\ (p-2)K_{1,1} \cup P_3) \geq p-1$, $(v)$ $N(9,\ K_{1,3} \cup 3K_{1,2}) \geq 3$, $(vi)$ $N(7,\ 3K_{1,1} \cup 2K_{1,2}) \geq 4$. Also, he conjectured the following. If $p$ is a prime number, then $N(p,\ P_{p+1}) = p$. This conjecture has been proved by two methods, see [9, 10]. The above results on MOGS of graphs having lower degrees motivate us to consider MOGS of graphs having higher degrees.

## 2. Main Results

MacNeish [11] has proved that if $N(m,\ mK_2) = k_1$ and $N(n,\ nK_2) = k_2$ and $\min\{k_1,\ k_2\} = k$, then there are $k$ MOLS of order $mn$. El-Shanawany [12] has proved that if $N(m,\ mK_2) = k$ and $N(n,\ G) = k$, then $N(mn,\ mG) \geq k$. Hereafter, if we have $N(m,\ G) = k_1$ and $N(n,\ H) = k_2$ and $\min\{k_1,\ k_2\} = k$, then we obtain $N(mn,\ \mathbb{T}) = k$ by Proposition 1, where $\mathbb{T} \cong G \times H$. Hence, Proposition 1 is a generalization to the theorems of MacNeish and El-Shanawany.

**Proposition 1.** ([13]) *If there are k MOGS of order m of the graph G and k MOGS of order n of the graph H, then there are k MOGS of order mn of the graph* $\mathbb{T} \cong G \times H$.

All the following results based on $(i)$ The Kronecker product in Proposition 1 and $(ii)$ The existence of MOGS for some classes of graphs that can be used as ingredients for Kronecker product to obtain new MOGS. These are as follows. Consider addition modulo $n$ for the squares of order $n$ and see [4] for the ingredients $(II)$, $(III)$, $(V)$, and $(VI)$.

$(I)$ The $n$ mutually orthogonal $P_{n+1}$-squares are
$M^s = (a^s_{ij}),\ a^s_{ij} = \alpha, i = \alpha + s\beta - \beta^2, j = \alpha + (s+1)\beta - \beta^2, \alpha, \beta, s \in \mathbb{Z}_n$ where $n$ is a prime $> 2$, see [10].

$(II)$ The $n$ mutually orthogonal $(K_{1,1} \cup \frac{n-1}{2}K_{1,2})$-squares are
$M^s = (a^s_{ij}),\ a^s_{ij} = \alpha, i = \beta, j = \alpha + s\beta + \beta^2$, $n$ is a prime $> 2$ and $s, \alpha, \beta \in \mathbb{Z}_n$.

$(III)$ The $(n-1)$ mutually orthogonal $((n-2)K_{1,1} \cup K_{1,2})$-squares are
$M^s = (a^s_{ij}),\ a^s_{ij} = (s+1)i + j - c_i, s \in \mathbb{Z}_{n-1}, c_i = \begin{cases} 1 & \text{if} \quad i = 1, \\ 0 & \text{otherwise.} \end{cases}$

$(IV)$ The 3 mutually orthogonal $C_4$-squares are $M^s = (a^s_{ij}), s \in \mathbb{Z}_3, a^0_{ij} = 0, i, j \in \mathbb{Z}_2, a^0_{ij} = 1, i \in \mathbb{Z}_2, j \in \{2,3\}, a^0_{ij} = 2, i \in \{2,3\}, j \in \mathbb{Z}_2, a^0_{ij} = 3, i, j \in \{2,3\}, a^1_{ij} = 0, i, j \in \{0,2\}, a^1_{ij} = 1, i \in \{0,2\}, j \in \{1,3\}, a^1_{ij} = 2, i \in \{1,3\}, j \in \{0,2\}, a^1_{ij} = 3, i, j \in \{1,3\}, a^2_{ij} = 0, i, j \in \{0,3\}, a^2_{ij} = 1, i \in \{1,2\}, a^2_{ij} = 2, i \in \{0,3\}, j \in \{1,2\}, a^2_{ij} = 3, i \in \{1,2\}, j \in \{0,3\}$, see [12].

$(V)$ The 3 mutually orthogonal $K_{1,3} \cup 3K_{1,2}$-squares are
$M^s = (a^s_{ij}), s \in \mathbb{Z}_3, a^s_{ij} = \beta, i = \alpha, j = \alpha^2 + s\alpha + \beta, \alpha, \beta \in \mathbb{Z}_9$.

$(VI)$ The 4 mutually orthogonal $3K_{1,1} \cup 2K_{1,2}$-squares are
$M^s = (a^s_{ij}), s \in \mathbb{Z}_4, i, j \in \mathbb{Z}_7$, suppose $\beta \in \mathbb{Z}_7$, then, $a^s_{ij} = j, i = 0, j \in \mathbb{Z}_7, a^s_{ij} = \beta, i = 4, j = 1 + \beta + 4s$,
$a^s_{ij} = \beta, i = 1, j = 2 + \beta + s, a^s_{ij} = \beta, i = 2, j = 4 + \beta + 2s, a^s_{ij} = \beta, i = 5, j = 4 + \beta + 5s, a^s_{ij} = \beta, i = 3, j = 6 + \beta + 3s, a^s_{ij} = \beta, i = 6, j = 6 + \beta + 6s$.
Now, we retrieve the proof of the ingredient $(III)$ by another technique in the following theorem.

**Theorem 2.** *Suppose $n$ is a prime $> 2$, then $N(n, ((n-2)K_{1,1} \cup K_{1,2})) \geq n-1$.*

*Proof.* Suppose $L^k(i, j) = (k+1)i + j, k \in \mathbb{Z}_{n-1}$ are $(n-1)$ mutually orthogonal Latin squares of order $n$. Now, we shall construct $(n-1)$ mutually orthogonal $((n-2)K_{1,1} \cup K_{1,2})$-squares of order $n$ as follows.
$M^k(i, j) = L^k(i, j) - c_i$ for $k \in \mathbb{Z}_{n-1}$ and

$$c_i = \begin{cases} 1 & \text{if} \quad i = 1, \\ 0 & \quad \text{otherwise.} \end{cases}$$

It is easy to check that the squares $(M^p, M^q)$ are orthogonal under the condition $(M^p(i, j), M^q(i, j)) = ((p+1)i + j - c_i, (q+1)i + j - c_i), i, j \in \mathbb{Z}_n; 0 \leq p < q \leq n-1$. Hence, the squares $M^k, k \in \mathbb{Z}_{n-1}$ are mutually orthogonal. Now, we prove that the page obtained from the entries in $M^0$ equal to 0 is isomorphic to $(n-2)K_{1,1} \cup K_{1,2}$. For the other pages in the squares $M^k, k \in \mathbb{Z}_{n-1}$, a similar argument can be applied. In $M^0$, every row contains exactly one 0-entry ($n$ vertices $x_0$ have degree one). Furthermore, there is exactly one column with two 0-entry (one vertex $x_1$ has degree two), $(n-2)$ columns contain one 0-entry ($(n-2)$ vertices $x_1$ have degree one), and one column contains no 0-entry (one vertex $x_1$ has degree zero). ∎

In what follows, we present some results as direct applications to Proposition 1. For all the following results if we construct $k$ mutually orthogonal $\mathbb{T}$-squares, then we prove that the page obtained from the entries in $L^0$ equal to 0 is isomorphic to $\mathbb{T}$. Also, we can easily apply a similar argument to the other pages in $L^s, s \in \mathbb{Z}_k$.

**Theorem 3.** *Suppose $m, n$ are primes $> 2$, then $N(mn, \mathbb{T}^{m,n}_1) \geq \min\{m, n\}$.*

*Proof.* We have $m$ mutually orthogonal $P_{m+1}$-squares (ingredient $(I)$) and $n$ mutually orthogonal $(K_{1,1} \cup \frac{n-1}{2}K_{1,2})$-squares (ingredient $(II)$). If $\min\{m, n\} = k$, then we construct $k$ mutually orthogonal $\mathbb{T}^{m,n}_1$-squares $L^s = (c^s_{ij}), s \in \mathbb{Z}_k$, and $i, j \in \mathbb{Z}_{mn}$ (Proposition 1). We prove that the page obtained from the entries in $L^0$ equal to 0 is isomorphic to $\mathbb{T}^{m,n}_1$. Exactly $\frac{m-1}{2}$ rows (columns) contain two 0-entry ($\frac{m-1}{2}$ vertices $x_0$ $(x_1)$ have degree two), one row (column) contains one 0-entry (one vertex $x_0$ $(x_1)$ has degree one), $\frac{(n-1)(m-1)}{4}$ columns contain four 0-entry ($\frac{(n-1)(m-1)}{4}$ vertices $x_1$ have degree four),

$\frac{(n-1)(m-1)}{2}$ rows contain two 0-entry ($\frac{(n-1)(m-1)}{2}$ vertices $x_0$ have degree two), $\frac{n-1}{2}$ columns contain two 0-entry ($\frac{n-1}{2}$ vertices $x_1$ have degree two), $(n-1)$ rows contain one 0-entry ($(n-1)$ vertices $x_0$ have degree one), $\frac{(m-1)n}{2}$ rows contain no 0-entry ($\frac{(m-1)n}{2}$ vertices $x_0$ have degree zero), and $\frac{3mn-(m+n+1)}{4}$ columns contain no 0-entry ($\frac{3mn-(m+n+1)}{4}$ vertices $x_1$ have degree zero). ∎

**Example 2.** *To illustrate Theorem 3, we have* 3 *mutually orthogonal* $P_4$*-squares* $M^s, s \in \mathbb{Z}_3$ *and* 3 *mutually orthogonal* $(K_2 \cup K_{1,2})$*-squares* $N^s, s \in \mathbb{Z}_3$*. Hence, we get* 3 *mutually orthogonal* $\mathbb{T}_1^{3,3}$*-squares* $L^s, s \in \mathbb{Z}_3$ *which are represented by the graphs in Figures 2.1, 2.2, and 2.3 respectively, where* $G_i^s \cong \mathbb{T}_1^{3,3}, i \in \mathbb{Z}_9$ *is the graph corresponding to the entry* $i$ *in the square* $L^s, s \in \mathbb{Z}_3$*.*

$$M^0 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}, \; M^1 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \; M^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix},$$

$$N^0 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \; N^1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \; N^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$L^0 = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 1 & 2 & 0 & 1 & 5 & 3 & 4 \\ 2 & 0 & 1 & 2 & 0 & 1 & 5 & 3 & 4 \\ 6 & 7 & 8 & 3 & 4 & 5 & 3 & 4 & 5 \\ 8 & 6 & 7 & 5 & 3 & 4 & 5 & 3 & 4 \\ 8 & 6 & 7 & 5 & 3 & 4 & 5 & 3 & 4 \\ 6 & 7 & 8 & 0 & 1 & 2 & 6 & 7 & 8 \\ 8 & 6 & 7 & 2 & 0 & 1 & 8 & 6 & 7 \\ 8 & 6 & 7 & 2 & 0 & 1 & 8 & 6 & 7 \end{bmatrix}, L^1 = \begin{bmatrix} 0 & 1 & 2 & 6 & 7 & 8 & 6 & 7 & 8 \\ 1 & 2 & 0 & 7 & 8 & 6 & 7 & 8 & 6 \\ 0 & 1 & 2 & 6 & 7 & 8 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 \\ 1 & 2 & 0 & 4 & 5 & 3 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 \\ 3 & 4 & 5 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 4 & 5 & 3 & 7 & 8 & 6 \\ 3 & 4 & 5 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix},$$

$$L^2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 \\ 1 & 2 & 0 & 4 & 5 & 3 & 1 & 2 & 0 \\ 3 & 4 & 5 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 4 & 5 & 3 & 7 & 8 & 6 \\ 0 & 1 & 2 & 6 & 7 & 8 & 6 & 7 & 8 \\ 0 & 1 & 2 & 6 & 7 & 8 & 6 & 7 & 8 \\ 1 & 2 & 0 & 7 & 8 & 6 & 7 & 8 & 6 \end{bmatrix}.$$

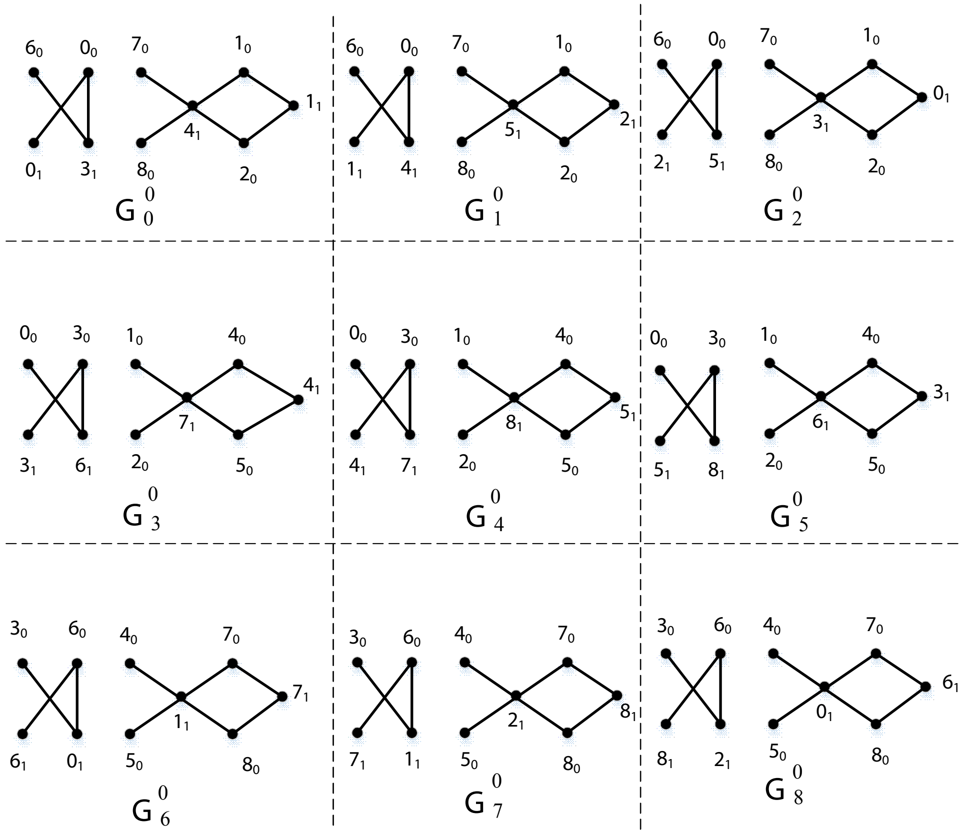


FIGURE 2.1. First edge decomposition of $K_{9,9}$ by $\mathbb{T}_1^{3,3}$.

**Theorem 4.** *Suppose $m, n$ are primes $> 2$, then $N(mn, \mathbb{T}_2^{m,n}) \geq \min\{m,\ n-1\}$.*

*Proof.* We have $m$ mutually orthogonal $P_{m+1}$-squares (ingredient $(I)$) and $(n-1)$ mutually orthogonal $((n-2)K_{1,1} \cup K_{1,2})$-squares (ingredient $(III)$). If $\min\{m,\ n-1\} = k$, then we construct $k$ mutually orthogonal $\mathbb{T}_2^{m,n}$-squares $L^s = (c^s_{ij}), s \in \mathbb{Z}_k$, and $i, j \in \mathbb{Z}_{mn}$ (Proposition 1). We prove that the page obtained from the entries in $L^0$ equal to 0 is isomorphic to $\mathbb{T}_2^{m,n}$. Exactly $\frac{(n-2)(m-1)}{2}$ rows (columns) contain two 0-entry ($\frac{(n-2)(m-1)}{2}$ vertices $x_0$ ($x_1$) have degree two), $(n-2)$ rows (columns) contain one 0-entry ($(n-2)$ vertices $x_0$ ($x_1$) have degree one), $\frac{m-1}{2}$ columns contain four 0-entry ($\frac{m-1}{2}$ vertices $x_1$ have degree four), $(m-1)$ rows contain two 0-entry ($(m-1)$ vertices $x_0$ have degree two), one column contain two 0-entry (one vertex $x_1$ have degree two), two rows contain one 0-entry (two vertices $x_0$ have degree one), $\frac{(m-1)n}{2}$ rows contain no 0-entry ($\frac{(m-1)n}{2}$ vertices $x_0$ have degree zero), and also there are $\frac{n(m-1)+m+1}{2}$ columns contain no 0-entry ($\frac{n(m-1)+m+1}{2}$ vertices $x_1$ have degree zero). ∎

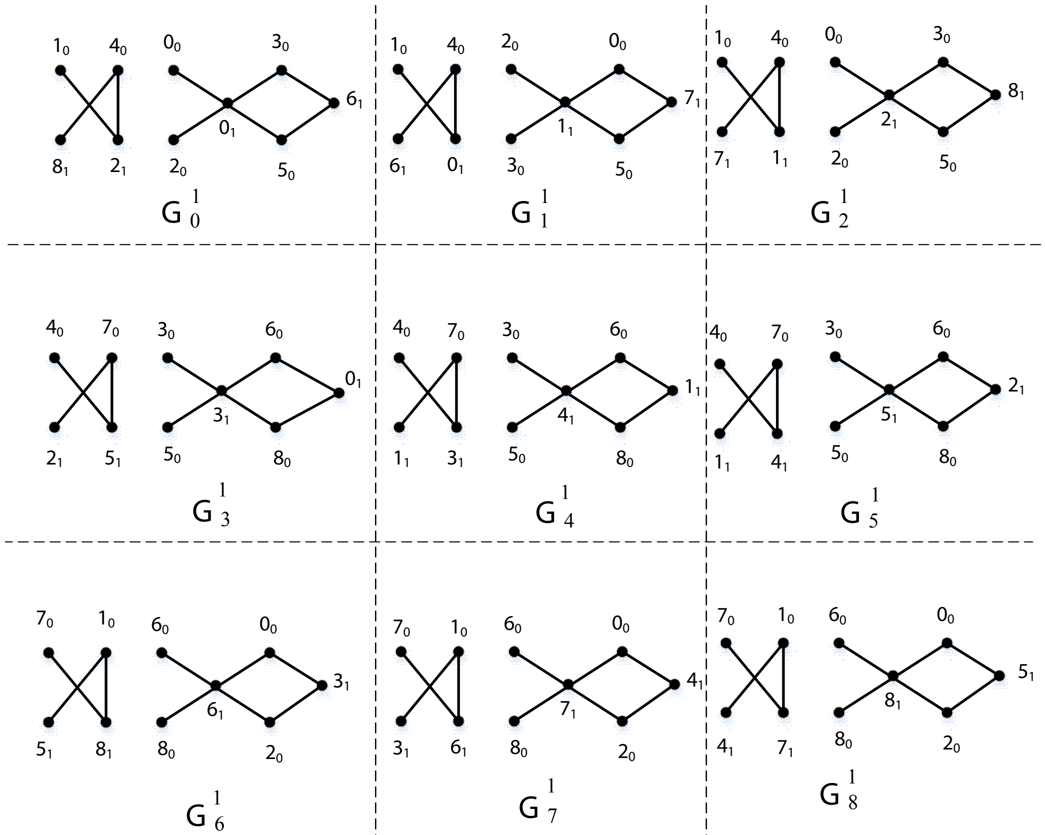


FIGURE 2.2. Second edge decomposition of $K_{9,9}$ by $\mathbb{T}_1^{3,3}$.

**Example 3.** *To illustrate Theorem 4, consider* 3 *mutually orthogonal* $P_4$*-squares* $M^s, s \in \mathbb{Z}_3$ *and* 3 *mutually orthogonal* $(3K_{1,1} \cup K_{1,2})$*-squares* $N^s, s \in \mathbb{Z}_3$*. Hence, we get* 3 *mutually orthogonal* $\mathbb{T}_2^{3,5}$*-squares* $L^s, s \in \mathbb{Z}_3$.

$$M^0 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix},\ M^1 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix},M^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix},$$

$$N^0 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix},\ N^1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{bmatrix},\ N^2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix},$$

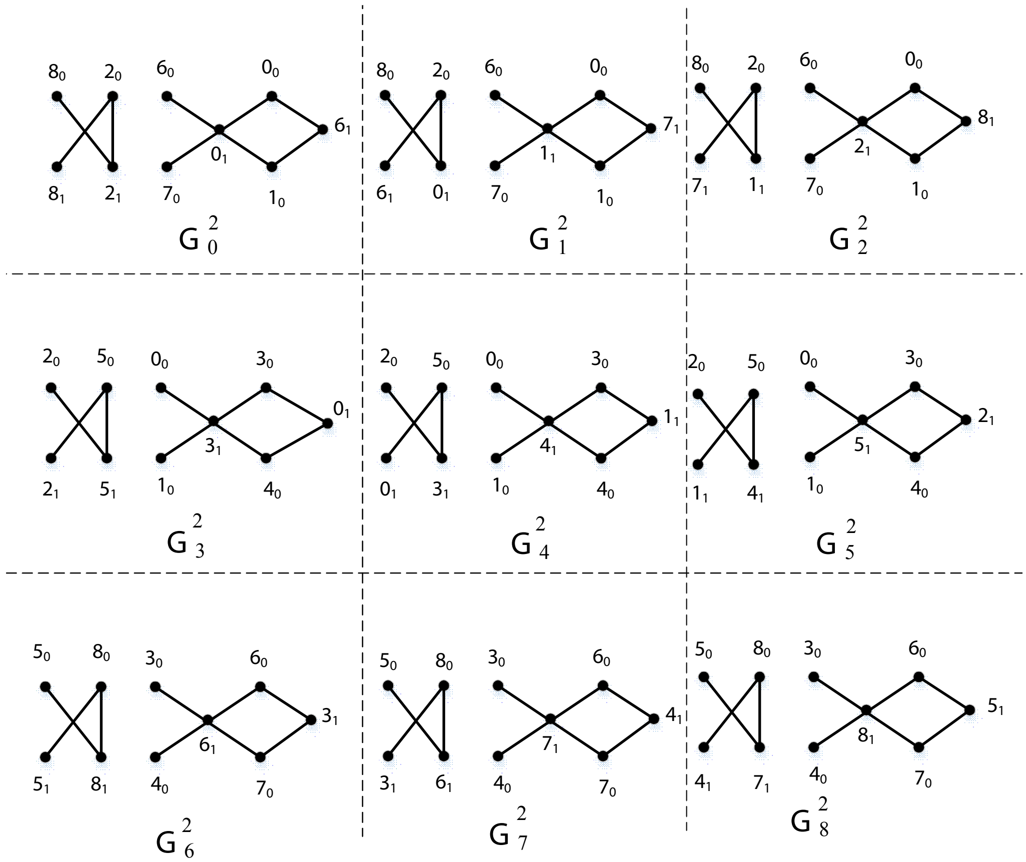


FIGURE 2.3. Third edge decomposition of $K_{9,9}$ by $\mathbb{T}_1^{3,3}$.

$$L^0 = \left[\begin{array}{ccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6 \\
3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 8 & 9 & 5 & 6 & 7 \\
4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 \\
10 & 11 & 12 & 13 & 14 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 & 14 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 \\
12 & 13 & 14 & 10 & 11 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 \\
13 & 14 & 10 & 11 & 12 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 \\
14 & 10 & 11 & 12 & 13 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 \\
10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\
10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\
12 & 13 & 14 & 10 & 11 & 2 & 3 & 4 & 0 & 1 & 12 & 13 & 14 & 10 & 11 \\
13 & 14 & 10 & 11 & 12 & 3 & 4 & 0 & 1 & 2 & 13 & 14 & 10 & 11 & 12 \\
14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 14 & 10 & 11 & 12 & 13
\end{array}\right],$$

$$L^1 = \left[\begin{array}{ccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 & 10 & 11 & 12 & 13 & 14 \\
1 & 2 & 3 & 4 & 0 & 11 & 12 & 13 & 14 & 10 & 11 & 12 & 13 & 14 & 10 \\
4 & 0 & 1 & 2 & 3 & 14 & 10 & 11 & 12 & 13 & 14 & 10 & 11 & 12 & 13 \\
1 & 2 & 3 & 4 & 0 & 11 & 12 & 13 & 14 & 10 & 11 & 12 & 13 & 14 & 10 \\
3 & 4 & 0 & 1 & 2 & 13 & 14 & 10 & 11 & 12 & 13 & 14 & 10 & 11 & 12 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 0 \\
4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 4 & 0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 0 \\
3 & 4 & 0 & 1 & 2 & 8 & 9 & 5 & 6 & 7 & 3 & 4 & 0 & 1 & 2 \\
5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 11 & 12 & 13 & 14 & 10 \\
9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 \\
6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 11 & 12 & 13 & 14 & 10 \\
8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 13 & 14 & 10 & 11 & 12
\end{array}\right],$$

$$L^2 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6 & 2 & 3 & 4 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6 & 2 & 3 & 4 & 0 & 1 \\ 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 12 & 13 & 14 & 10 & 11 \\ 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 11 & 12 & 13 & 14 & 10 \\ 9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 \\ 7 & 8 & 9 & 5 & 6 & 7 & 8 & 9 & 5 & 6 & 12 & 13 & 14 & 10 & 11 \\ 0 & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 & 10 & 11 & 12 & 13 & 14 \\ 2 & 3 & 4 & 0 & 1 & 12 & 13 & 14 & 10 & 11 & 12 & 13 & 14 & 10 & 11 \\ 1 & 2 & 3 & 4 & 0 & 11 & 12 & 13 & 14 & 10 & 11 & 12 & 13 & 14 & 10 \\ 4 & 0 & 1 & 2 & 3 & 14 & 10 & 11 & 12 & 13 & 14 & 10 & 11 & 12 & 13 \\ 2 & 3 & 4 & 0 & 1 & 12 & 13 & 14 & 10 & 11 & 12 & 13 & 14 & 10 & 11 \end{bmatrix}.$$

**Theorem 5.** *Suppose $m$ is a prime $> 2$, then $N(4m, \mathbb{T}_3^m) \geq 3$.*

*Proof.* We have $m$ mutually orthogonal $P_{m+1}$-squares (ingredient $(I)$) and 3 mutually orthogonal $C_4$-squares (ingredient $(IV)$). Then we construct 3 mutually orthogonal $\mathbb{T}_3^m$-squares $L^s = (c_{ij}^s), s \in \mathbb{Z}_3$, and $i, j \in \mathbb{Z}_{4m}$ (Proposition 1). We prove that the page obtained from the entries in $L^0$ equal to 0 is isomorphic to $\mathbb{T}_3^m$. Exactly $(m-1)$ rows (columns) contain four 0-entry ($(m-1)$ vertices $x_0$ ($x_1$) have degree four), two rows (columns) contain two 0-entry (two vertices $x_0$ ($x_1$) have degree two), and $(3m-1)$ rows (columns) contain no 0-entry ($(3m-1)$ vertices $x_0$ ($x_1$) have degree zero). ∎

**Theorem 6.** *Suppose $n$ is a prime $> 3$, then $N(4n, \mathbb{T}_4^n) \geq 3$.*

*Proof.* We have $(n-1)$ mutually orthogonal $((n-2)K_{1,1} \cup K_{1,2})$-squares (ingredient $(III)$) and 3 mutually orthogonal $C_4$-squares (ingredient $(IV)$). Then we construct 3 mutually orthogonal $\mathbb{T}_4^n$-squares $L^s = (c_{ij}^s), s \in \mathbb{Z}_3$, and $i, j \in \mathbb{Z}_{4n}$ (Proposition 1). We prove that the page obtained from the entries in the square $L^0$ equal to 0 is isomorphic to $\mathbb{T}_4^n$. Exactly $(2n-4)$ rows (columns) contain two 0-entry ($(2n-4)$ vertices $x_0$ ($x_1$) have degree two), four rows contain two 0-entry (four vertices $x_0$ have degree two), two columns contain four 0-entry (two vertices $x_1$ have degree four), $2n$ rows contain no 0-entry ($2n$ vertices $x_0$ have degree zero), and $(2n+2)$ columns contain no 0-entry ($(2n+2)$ vertices $x_1$ have degree zero). ∎

**Theorem 7.** *Suppose $n$ is a prime $> 2$, then $N(4n, K_{2,2} \cup \frac{n-1}{2} K_{2,4}) \geq 3$.*

*Proof.* We have $n$ mutually orthogonal $(K_{1,1} \cup \frac{n-1}{2} K_{1,2})$-squares (ingredient $(II)$) and 3 mutually orthogonal $C_4$-squares (ingredient $(IV)$). Then we construct 3 mutually orthogonal $K_{2,2} \cup \frac{n-1}{2} K_{2,4}$-squares $L^s = (c_{ij}^s), s \in \mathbb{Z}_3$, and $i, j \in \mathbb{Z}_{4n}$ (Proposition 1).

We prove that the page obtained from the entries in $L^0$ equal to 0 is isomorphic to $K_{2,2} \cup \frac{n-1}{2} K_{2,4}$. Exactly $2n$ rows contain two 0-entry ($2n$ vertices $x_0$ have degree two), two columns contain two 0-entry (two vertices $x_1$ have degree two), $(n-1)$ columns contain four 0-entry ($(n-1)$ vertices $x_1$ have degree four), $2n$ rows contain no 0-entry ($2n$ vertices $x_0$ have degree zero), and $(3n-1)$ columns contain no 0-entry ($(3n-1)$ vertices $x_1$ have degree zero). ∎

**Theorem 8.** *Suppose $m$ is a prime $> 2$, then $N(9m, \mathbb{T}_5^m) \geq 3$.*

*Proof.* We have $m$ mutually orthogonal $P_{m+1}$-squares (ingredient $(I)$) and 3 mutually orthogonal $(K_{1,3} \cup 3K_{1,2})$-squares (ingredient $(V)$). Then we construct 3 mutually orthogonal $\mathbb{T}_5^m$-squares $L^s = (c_{ij}^s), s \in \mathbb{Z}_3$, and $i, j \in \mathbb{Z}_{9m}$ (Proposition 1). We prove that the page obtained from the entries in $L^0$ equal to 0 is isomorphic to $\mathbb{T}_5^m$. Exactly $\frac{3(m-1)}{2}$ rows contain two 0-entry ($\frac{3(m-1)}{2}$ vertices $x_0$ have degree two), three rows contain one 0-entry (three vertices $x_0$ have degree one), $\frac{m-1}{2}$ columns contain six 0-entry ($\frac{m-1}{2}$ vertices $x_1$ have degree six), one column contains three 0-entry (one vertex $x_1$ has degree three), $3(m-1)$ rows contain two 0-entry ($3(m-1)$ vertices $x_0$ have degree two), six rows contain one 0-entry (six vertices $x_0$ have degree one), $\frac{3(m-1)}{2}$ columns contain four 0-entry ($\frac{3(m-1)}{2}$ vertices $x_1$ have degree four), three columns contain two 0-entry (three vertices $x_1$ have degree two), $\frac{9(m-1)}{2}$ rows contain no 0-entry ($\frac{9(m-1)}{2}$ vertices $x_0$ have degree zero), and $(7m-2)$ columns contain no 0-entry ($(7m-2)$ vertices $x_1$ have degree zero). ∎

**Theorem 9.** *Suppose $m$ is a prime $> 3$, then $N(7m, \mathbb{T}_6^m) \geq 4$.*

*Proof.* We have $m$ mutually orthogonal $P_{m+1}$-squares (ingredient $(I)$) and 4 mutually orthogonal $(3K_{1,1} \cup 2K_{1,2})$-squares (ingredient $(VI)$). Then we construct 4 mutually orthogonal $\mathbb{T}_6^m$-squares $L^s = (c_{ij}^s), s \in \mathbb{Z}_4$, and $i, j \in \mathbb{Z}_{7m}$ (Proposition 1). We prove that the page obtained from the entries in $L^0$ equal to 0 is isomorphic to $\mathbb{T}_6^m$. Exactly $\frac{7(m-1)}{2}$ rows contain two 0-entry ($\frac{7(m-1)}{2}$ vertices $x_0$ have degree two), seven rows contain one 0-entry (seven vertices $x_0$ have degree one), $(m-1)$ columns contain four 0-entry ($(m-1)$ vertices $x_1$ have degree four), $\frac{3m+1}{2}$ columns contain two 0-entry ( $\frac{3m+1}{2}$ vertices $x_1$ have degree two), three columns contain one 0-entry (three vertices $x_1$ have degree one), $\frac{7(m-1)}{2}$ rows contain no 0-entry ($\frac{7(m-1)}{2}$ vertices $x_0$ have degree zero), and $\frac{9m-5}{2}$ columns contain no 0-entry ($\frac{9m-5}{2}$ vertices $x_1$ have degree zero). ∎

**Example 4.** *To illustrate Theorem 5, we have 3 mutually orthogonal $P_4$-squares $M^s, s \in \mathbb{Z}_3$ and 3 mutually orthogonal $C_4$-squares $N^s, s \in \mathbb{Z}_3$. Hence, we get 3 mutually orthogonal $\mathbb{T}_3^3$-squares $L^s, s \in \mathbb{Z}_3$ which are represented by the graphs in Figures 2.4, 2.5, and 2.6, respectively, where $G_i^s \cong \mathbb{T}_3^3, i \in \mathbb{Z}_{12}$ is the graph corresponding to the entry $i$ in the square $L^s, s \in \mathbb{Z}_3$.*

$$M^0 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix},\ M^1 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix},\ M^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix},$$

$$N^0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{bmatrix},\ N^1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix},\ N^2 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 3 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 0 & 2 & 2 & 0 \end{bmatrix},$$

$$L^0 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\ 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \\ 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \\ 8 & 8 & 9 & 9 & 4 & 4 & 5 & 5 & 4 & 4 & 5 & 5 \\ 8 & 8 & 9 & 9 & 4 & 4 & 5 & 5 & 4 & 4 & 5 & 5 \\ 10 & 10 & 11 & 11 & 6 & 6 & 7 & 7 & 6 & 6 & 7 & 7 \\ 10 & 10 & 11 & 11 & 6 & 6 & 7 & 7 & 6 & 6 & 7 & 7 \\ 8 & 8 & 9 & 9 & 0 & 0 & 1 & 1 & 8 & 8 & 9 & 9 \\ 8 & 8 & 9 & 9 & 0 & 0 & 1 & 1 & 8 & 8 & 9 & 9 \\ 10 & 10 & 11 & 11 & 2 & 2 & 3 & 3 & 10 & 10 & 11 & 11 \\ 10 & 10 & 11 & 11 & 2 & 2 & 3 & 3 & 10 & 10 & 11 & 11 \end{bmatrix},$$

$$L^1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 8 & 9 & 8 & 9 & 8 & 9 & 8 & 9 \\ 2 & 3 & 2 & 3 & 10 & 11 & 10 & 11 & 10 & 11 & 10 & 11 \\ 0 & 1 & 0 & 1 & 8 & 9 & 8 & 9 & 8 & 9 & 8 & 9 \\ 2 & 3 & 2 & 3 & 10 & 11 & 10 & 11 & 10 & 11 & 10 & 11 \\ 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 2 & 3 & 2 & 3 \\ 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 2 & 3 & 2 & 3 \\ 4 & 5 & 4 & 5 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 \\ 6 & 7 & 6 & 7 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 \\ 4 & 5 & 4 & 5 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 \\ 6 & 7 & 6 & 7 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 \end{bmatrix},$$

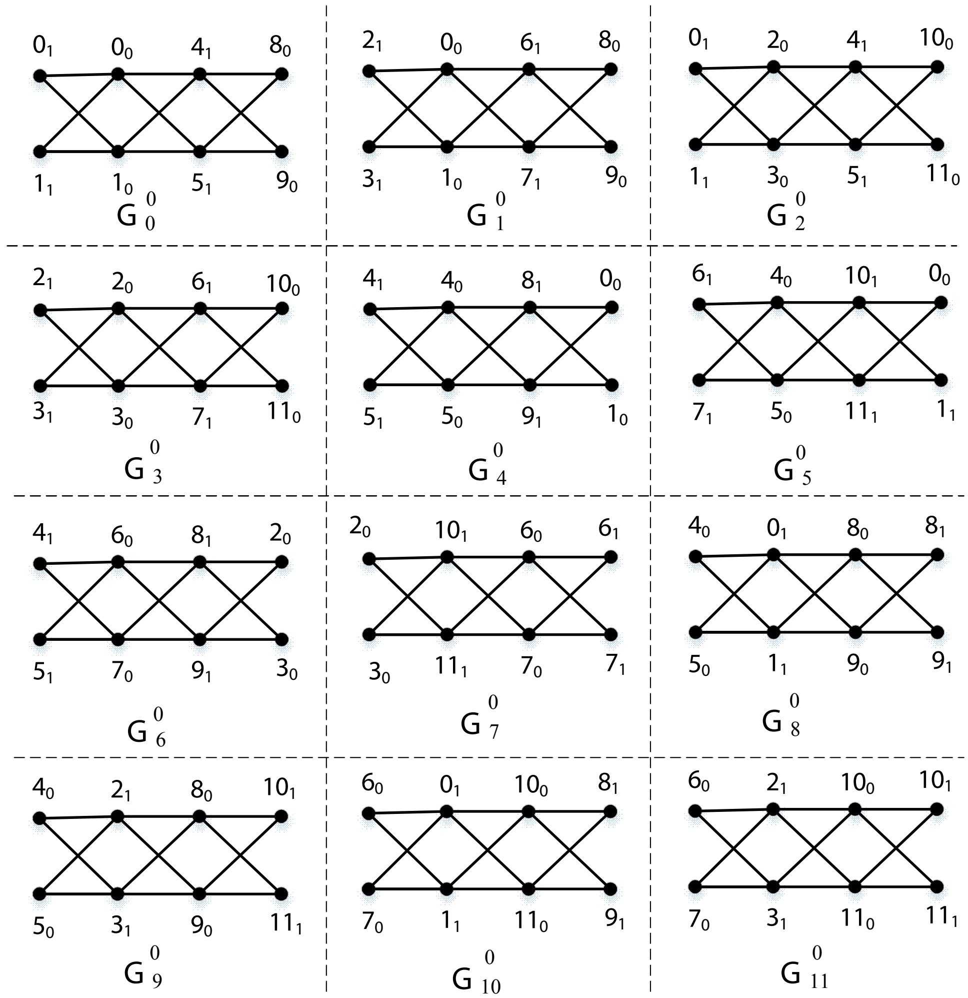


FIGURE 2.4. First edge decomposition of $K_{12,12}$ by $\mathbb{T}_3^3$.

$$L^2 = \begin{bmatrix} 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 0 & 2 & 2 & 0 \\ 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 3 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 3 & 1 & 1 & 3 \\ 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 0 & 2 & 2 & 0 \\ 4 & 6 & 6 & 4 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 \\ 7 & 5 & 5 & 7 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 \\ 7 & 5 & 5 & 7 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 \\ 4 & 6 & 6 & 4 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 \\ 0 & 2 & 2 & 0 & 8 & 10 & 10 & 8 & 8 & 10 & 10 & 8 \\ 3 & 1 & 1 & 3 & 11 & 9 & 9 & 11 & 11 & 9 & 9 & 11 \\ 3 & 1 & 1 & 3 & 11 & 9 & 9 & 11 & 11 & 9 & 9 & 11 \\ 0 & 2 & 2 & 0 & 8 & 10 & 10 & 8 & 8 & 10 & 10 & 8 \end{bmatrix}.$$

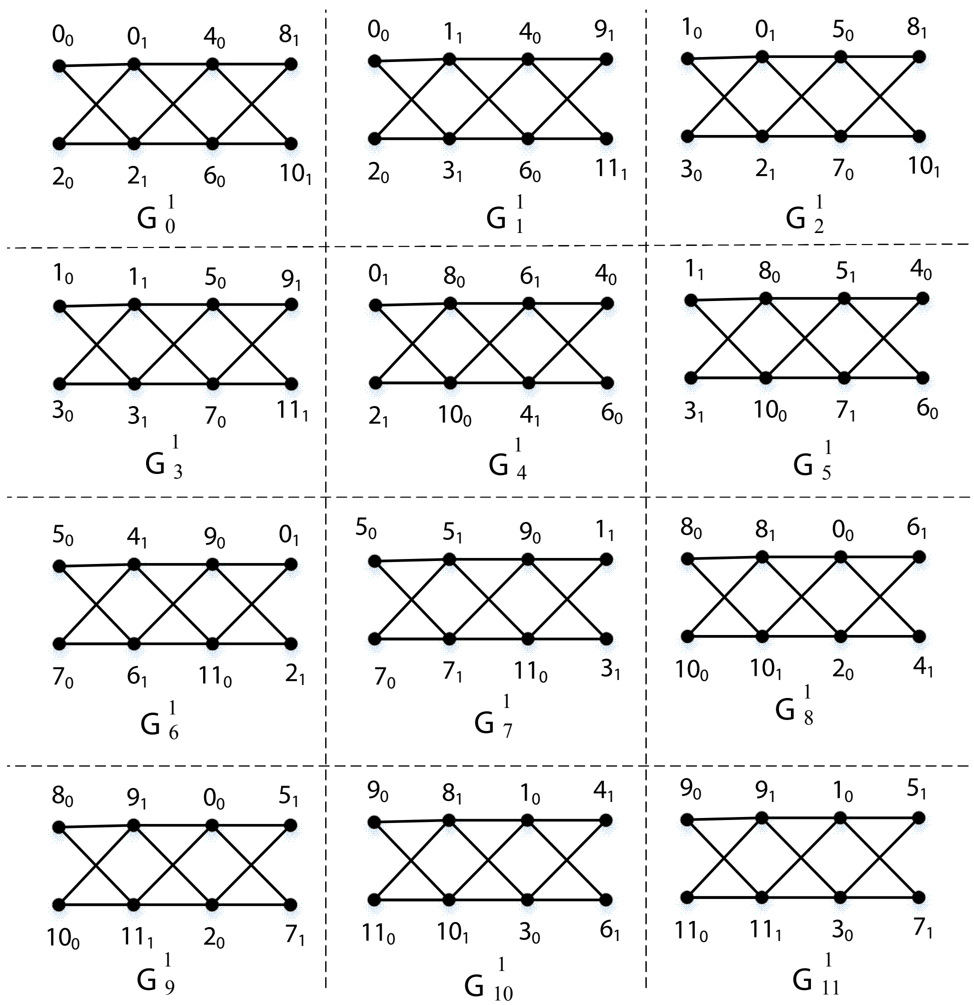


FIGURE 2.5. Second edge decomposition of $K_{12,12}$ by $\mathbb{T}_3^3$.

**Example 5.** *To illustrate Theorem 7, we have* 3 *mutually orthogonal* $K_2 \cup K_{1,2}$*-squares and* 3 *mutually orthogonal* $C_4$*-squares. Hence, we get* 3 *mutually orthogonal* $K_{2,2} \cup K_{2,4}$*-squares* $L^s, s \in \mathbb{Z}_3$ *which are represented by the graphs in Figures 2.7, 2.8, and 2.9, respectively, where* $G_i^s \cong K_{2,2} \cup K_{2,4}, i \in \mathbb{Z}_{12}$ *is the graph corresponding to the entry i in the square* $L^s, s \in \mathbb{Z}_3$*.*

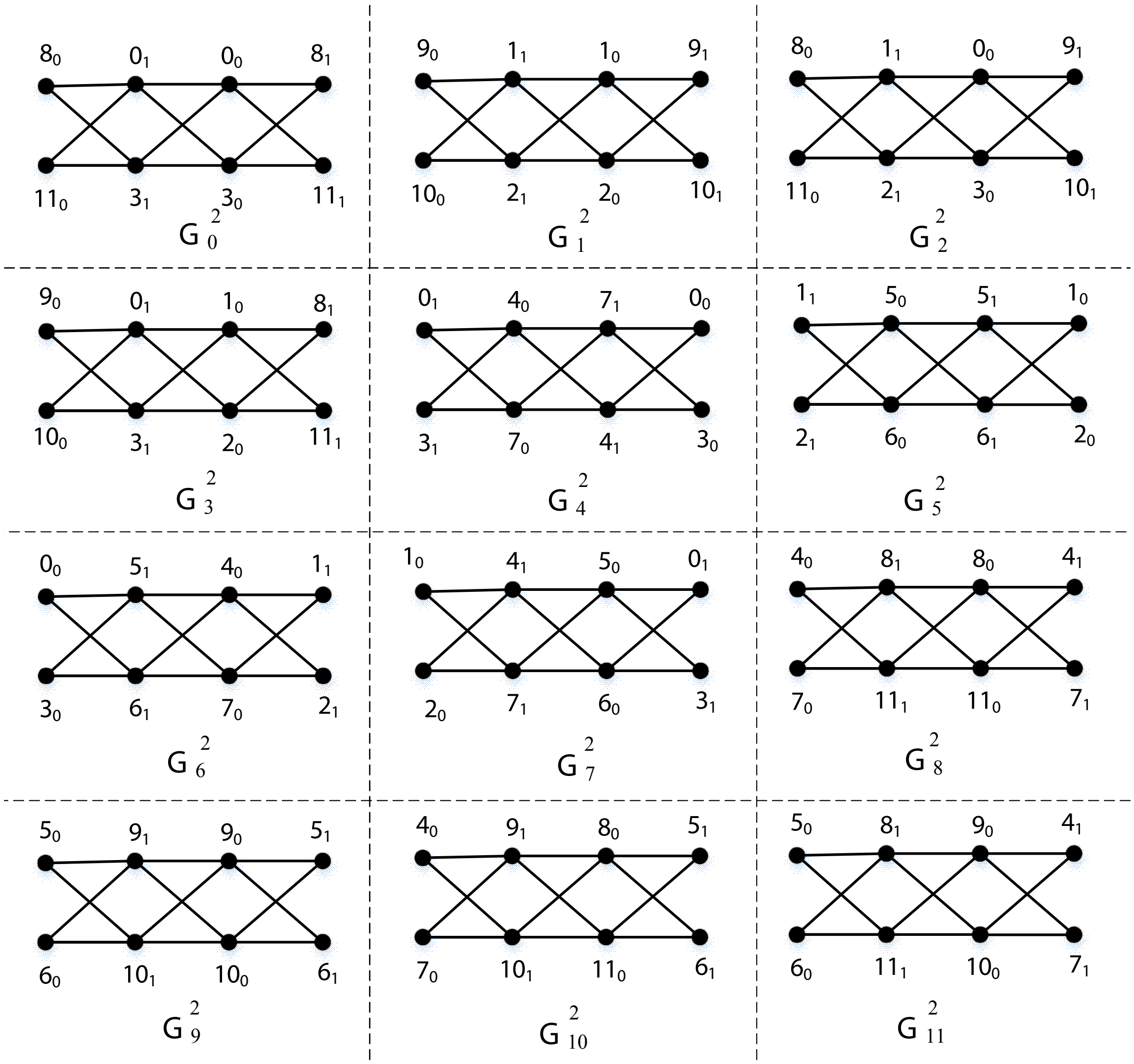


FIGURE 2.6. Third edge decomposition of $K_{12,12}$ by $\mathbb{T}_3^3$.

$$M^0 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix},\ M^1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix},\ M^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$N^0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{bmatrix},\ N^1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix},\ N^2 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 3 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 0 & 2 & 2 & 0 \end{bmatrix},$$

$$L^0 = \begin{bmatrix} 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 & 8 & 8 & 9 & 9 \\ 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 & 8 & 8 & 9 & 9 \\ 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 & 10 & 10 & 11 & 11 \\ 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 & 10 & 10 & 11 & 11 \\ 8 & 8 & 9 & 9 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\ 8 & 8 & 9 & 9 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\ 10 & 10 & 11 & 11 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \\ 10 & 10 & 11 & 11 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \\ 8 & 8 & 9 & 9 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\ 8 & 8 & 9 & 9 & 0 & 0 & 1 & 1 & 4 & 4 & 5 & 5 \\ 10 & 10 & 11 & 11 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \\ 10 & 10 & 11 & 11 & 2 & 2 & 3 & 3 & 6 & 6 & 7 & 7 \end{bmatrix},$$

$$L^1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 \\ 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 \\ 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 \\ 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 \\ 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 & 0 & 1 & 0 & 1 \\ 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 & 2 & 3 & 2 & 3 \\ 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 & 0 & 1 & 0 & 1 \\ 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 & 2 & 3 & 2 & 3 \\ 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 \\ 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 \\ 0 & 1 & 0 & 1 & 4 & 5 & 4 & 5 & 8 & 9 & 8 & 9 \\ 2 & 3 & 2 & 3 & 6 & 7 & 6 & 7 & 10 & 11 & 10 & 11 \end{bmatrix},$$

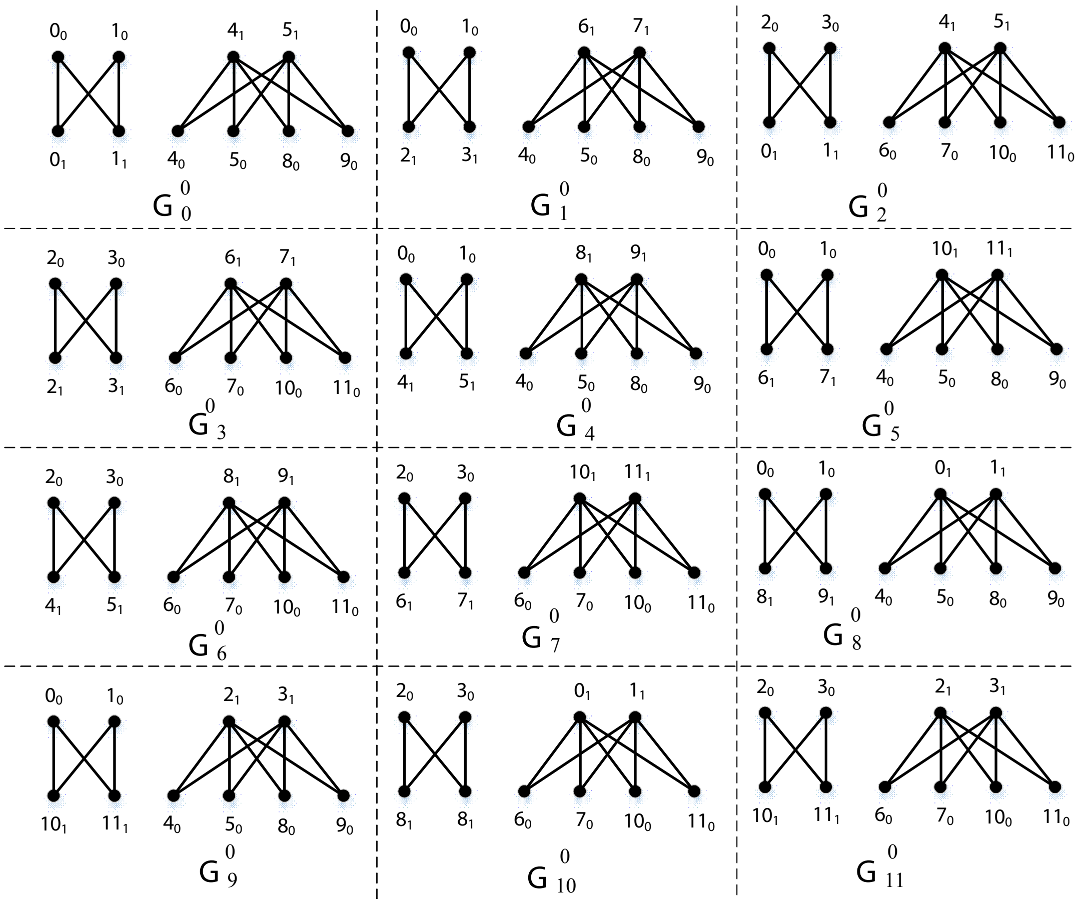


FIGURE 2.7. First edge decomposition of $K_{12,12}$ by $K_{2,2} \cup K_{2,4}$.

$$L^2 = \begin{bmatrix} 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 \\ 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 \\ 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 \\ 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 \\ 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 \\ 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 \\ 3 & 1 & 1 & 3 & 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 \\ 0 & 2 & 2 & 0 & 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 \\ 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 & 0 & 2 & 2 & 0 \\ 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 & 3 & 1 & 1 & 3 \\ 7 & 5 & 5 & 7 & 11 & 9 & 9 & 11 & 3 & 1 & 1 & 3 \\ 4 & 6 & 6 & 4 & 8 & 10 & 10 & 8 & 0 & 2 & 2 & 0 \end{bmatrix}.$$

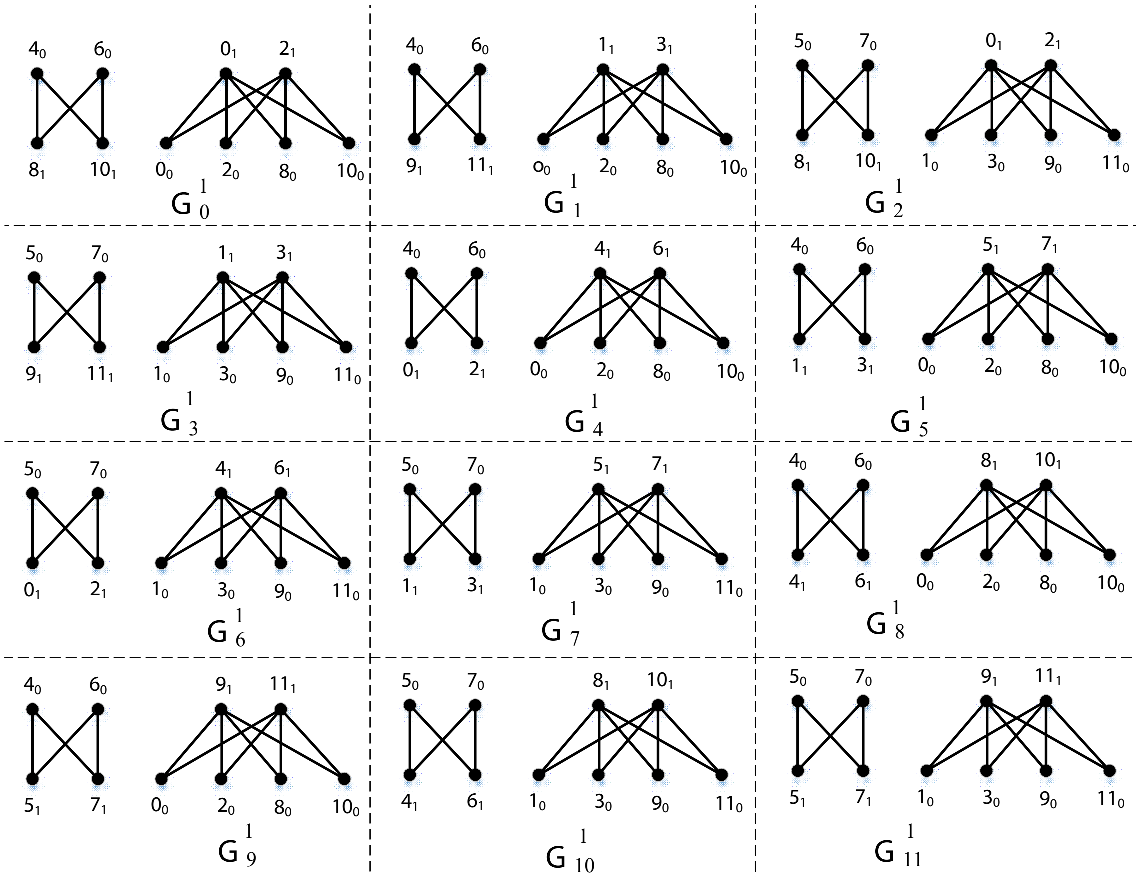


FIGURE 2.8. Second edge decomposition of $K_{12,12}$ by $K_{2,2} \cup K_{2,4}$.

## 3. CONCLUSION

In this paper, we obtained new results for the MOGS which are summarized in Table 1.

| |
|---|
| 1-$N(mn, \mathbb{T}_1^{m,n}) \geq \min\{m, n\}$ |
| 2-$N(mn, \mathbb{T}_2^{m,n}) \geq \min\{m, n-1\}$ |
| 3-$N(4m, \mathbb{T}_3^{m}) \geq 3$ |
| 4-$N(4n, \mathbb{T}_4^{n}) \geq 3$ |
| 5-$N(4n, K_{2,2} \cup \frac{n-1}{2} K_{2,4}) \geq 3$ |
| 6-$N(9m, \mathbb{T}_5^{m}) \geq 3$ |
| 7-$N(7m, \mathbb{T}_6^{m}) \geq 4$. |

Table 1: Summary of the results.

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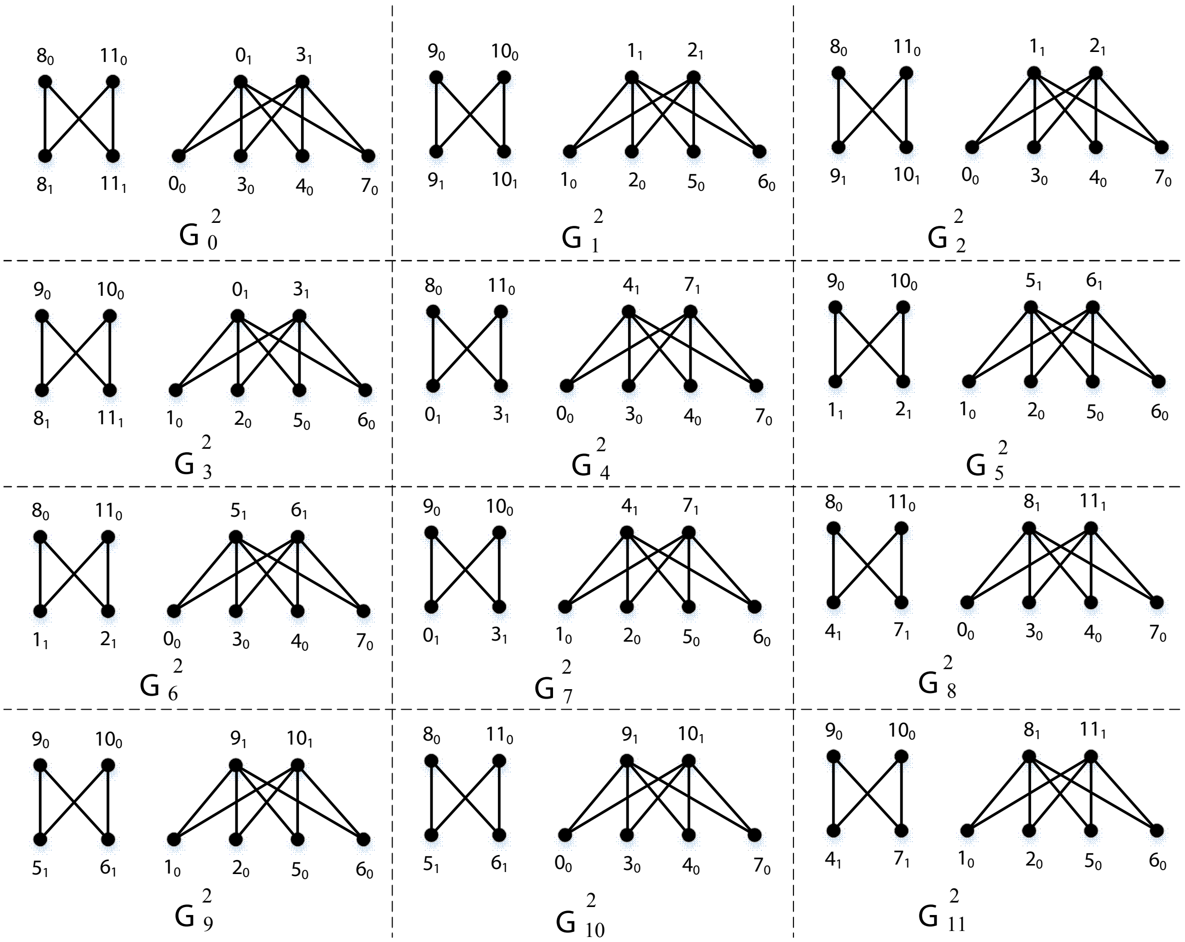


FIGURE 2.9. Third edge decomposition of $K_{12,12}$ by $K_{2,2} \cup K_{2,4}$.

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