LOG-CONCAVITY OF TWO SEQUENCES RELATED TO CAUCHY NUMBERS OF TWO KINDS

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Abstract. For Cauchy numbers of the first kind $\{a_n\}_{n\geq 0}$ and Cauchy numbers of the second kind $\{b_n\}_{n\geq 0}$, we prove that two sequences $\{\sqrt[n]{|a_n|}\}_{n\geq 2}$ and $\{\sqrt[n]{b_n}\}_{n\geq 1}$ are log-concave. In addition, we show that two sequences $\left\{\frac{1}{\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ and $\left\{\frac{1}{\sqrt[n]{b_n}}\right\}_{n\geq 1}$ are log-balanced.

Keywords: Cauchy numbers, log-convexity, log-concavity, log-balancedness.
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1. Introduction

Let a_n and b_n denote the n-th Cauchy numbers of the first and the second kind, respectively. The values of a_n and b_n are defined by the following integrals (see Comtet [3])

$$a_n = \int_0^1 (x)_n dx, \quad b_n = \int_0^1 \langle x\rangle_n dx,$$

where

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x-1)\cdots(x-n+1), & n \geq 1, \end{cases}$$

$$\langle x\rangle_n = \begin{cases} 1, & n = 0, \\ x(x+1)\cdots(x+n-1), & n \geq 1. \end{cases}$$

The first few Cauchy numbers of two kinds are as follows:

n	0	1	2	3	4	5	6	7	8	9
a_n	1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{4}$	$-\frac{19}{30}$	$\frac{9}{4}$	$-\frac{863}{84}$	$\frac{1375}{24}$	$-\frac{33953}{90}$	$\frac{57281}{20}$
b_n	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{9}{4}$	$\frac{251}{30}$	$\frac{475}{12}$	$\frac{19087}{84}$	$\frac{36799}{24}$	$\frac{1070017}{90}$	$\frac{2082753}{20}$

Cauchy numbers of two kinds are related to Bernoulli numbers, harmonic numbers, and Stirling numbers of two kinds. For properties and applications of Cauchy numbers of two kinds, see for instance [3, 7, 10, 11, 12, 13, 14, 15, 1, 5, 6, 8]. The object of this paper is to study log-concavity of two sequences involving $\{a_n\}_{n\geq 0}$ (or $\{b_n\}_{n\geq 0}$).

Let us recall some definitions in combinatorics. For a positive sequence $\{z_n\}_{n\geq 0}$, it is said to be *log-convex* (or *log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ (or $z_n^2 \geq z_{n-1}z_{n+1}$) for all $n \geq 1$. It is well known that a sequence $\{z_n\}_{n\geq 0}$ is log-concave if and only if $\{\frac{1}{z_n}\}_{n\geq 0}$ is log-convex. A log-convex sequence $\{z_n\}_{n\geq 0}$ is said to be *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave (Došlić [4] gave this definition). It is clear that log-balancedness is a special case of log-convexity and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $(n+1)z_n^2 - nz_{n-1}z_{n+1} \geq 0$ for each $n \geq 1$. Log-convex (or log-concave) sequences appear in many subjects such as combinatorics, algebra, and geometry. See for instance Brenti [2] or Stanley [9]. Log-behavior of a sequence is often instrumental in obtaining its growth rate and asymptotic behavior. Moreover, log-behavior is an important source of inequalities in combinatorics. Since log-balancedness of a sequence is related to log-convexity and log-concavity, it is useful for us to find more inequalities. Many famous sequences in combinatorics, including Motzkin numbers, Fine numbers, and central Delannoy numbers, are log-balanced. For more log-balanced sequences, see Došli'c [4]. Hence the log-behavior of a sequence deserves to be studied.

Recently, the log-behavior of some sequences involving Cauchy numbers has been studied. For instance, Zhao [15] showed that two sequences $\{|a_n|\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$ are log-convex. Zhang and Zhao [11] proved that $\{\sqrt{|a_n|}\}_{n\geq 2}$ and $\{\sqrt{b_n}\}_{n\geq 1}$ are log-balanced. In this paper, we mainly discuss the log-behavior of $\{\sqrt[n]{|a_n|}\}_{n\geq 1}$ and $\{\sqrt[n]{b_n}\}_{n\geq 1}$. In the next section, we prove that $\{\sqrt[n]{|a_n|}\}_{n\geq 2}$ and $\{\sqrt[n]{b_n}\}_{n\geq 1}$ are log-concave. In addition, we investigate the log-balancedness of the sequences $\left\{\frac{1}{\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ and $\left\{\frac{1}{\sqrt[n]{b_n}}\right\}_{n\geq 1}$.

2. The log-concavity of two sequences $\{\sqrt[n]{|a_n|}\}_{n\geq 2}$ and $\{\sqrt[n]{b_n}\}_{n\geq 1}$

The following lemmas will be used later on.

Lemma 2.1. ([11]) For $n \geq 0$, let $x_n = \frac{|a_{n+1}|}{|a_n|}$ and $y_n = \frac{b_{n+1}}{b_n}$. Then

$$n - \frac{1}{2} < x_n < n, \quad (n \geq 2), \tag{2.1}$$

$$n + \frac{1}{2} < y_n < n+1, \quad (n \geq 1). \tag{2.2}$$

Lemma 2.2. *For Cauchy numbers of two kinds $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, we have*

$$|a_n| \;<\; \frac{(n-1)!}{6}, \quad (n \geq 3), \tag{2.3}$$

$$b_n \;<\; \frac{n!}{2}, \quad (n \geq 2). \tag{2.4}$$

Proof. For $0 \leq x \leq 1$, it is evident that

$$\begin{aligned} x(1-x)\cdots(n-1-x) &\leq (n-1)!x(1-x), \quad (n \geq 3), \\ x(x+1)\cdots(x+n-1) &\leq n!x, \quad (n \geq 2). \end{aligned}$$

Then we have

$$\begin{aligned} \int_0^1 x(1-x)\cdots(n-1-x)dx &< (n-1)!\int_0^1 x(1-x)dx, \\ &= \frac{(n-1)!}{6}, \quad (n \geq 3), \\ \int_0^1 x(x+1)\cdots(x+n-1)dx &< n!\int_0^1 xdx \\ &= \frac{n!}{2}, \quad (n \geq 2). \end{aligned}$$

On the other hand, we observe that

$$|a_n| = \int_0^1 x(1-x)\cdots(n-1-x)dx.$$

Hence the inequalities (2.3) and (2.4) hold. □

Lemma 2.3. *For $n \geq 1$,*

$$n! < \left(\frac{n}{e}\right)^n \sqrt{2e\pi n}. \tag{2.5}$$

Proof. We prove by induction that (2.5) holds. We observe that

$$k! < \left(\frac{k}{e}\right)^k \sqrt{2e\pi k} \quad \text{for} \quad k = 1, 2, 3.$$

For $k \geq 3$, assume that $k! < \left(\frac{k}{e}\right)^k \sqrt{2e\pi k}$. Next we prove that

$$(k+1)! < \left(\frac{k+1}{e}\right)^{k+1} \sqrt{2e\pi(k+1)}$$

for $k \geq 3$. It is clear that

$$(k+1)! < \left(\frac{k}{e}\right)^k \sqrt{2e\pi k}(k+1).$$

Now we show that

$$\left(\frac{k}{e}\right)^k \sqrt{2e\pi k}(k+1) < \left(\frac{k+1}{e}\right)^{k+1} \sqrt{2e\pi(k+1)}.$$

We note that

$$\left(\frac{k}{e}\right)^k \sqrt{2e\pi k}(k+1) < \left(\frac{k+1}{e}\right)^{k+1} \sqrt{2e\pi(k+1)} \quad (k \geq 3)$$

if and only if

$$k \ln k - k \ln(k+1) + \frac{\ln k}{2} - \frac{\ln(k+1)}{2} + 1 < 0 \quad (k \geq 3). \tag{2.6}$$

Now we prove that (2.6) holds. For $x > 0$, define a function

$$f(x) = x \ln x - x \ln(x+1) + \frac{\ln x}{2} - \frac{\ln(x+1)}{2} + 1.$$

By computation, we have

$$\begin{aligned} f'(x) &= \ln x + 1 - \ln(x+1) - \frac{x}{x+1} + \frac{1}{2x} - \frac{1}{2(x+1)}, \\ f''(x) &= -\frac{1}{2x^2(x+1)^2}. \end{aligned}$$

Since $f''(x) < 0$, $f'(x)$ is monotonic decreasing on $(0, +\infty)$. We find that $\lim_{x \to +\infty} f'(x) = 0$. This implies that $f'(x) > 0$ for $x > 0$. Then $f(x)$ is monotonic increasing on $(0, +\infty)$. On the other hand, we observe that $\lim_{x \to +\infty} f(x) = 0$. This means that $f(x) < 0$ for $x > 0$. Naturally, (2.6) holds. Then we have

$$(k+1)! < \left(\frac{k+1}{e}\right)^{k+1} \sqrt{2e\pi(k+1)}.$$

This completes the proof of (2.5). □

Now we give the main results of this paper.

Theorem 2.4. *The sequences* $\{\sqrt[n]{|a_n|}\}_{n \geq 2}$ *and* $\{\sqrt[n]{b_n}\}_{n \geq 1}$ *are log-concave.*

Proof. For $n \geq 0$, let $x_n = \frac{|a_{n+1}|}{|a_n|}$ and $y_n = \frac{b_{n+1}}{b_n}$. For $n \geq 2$, we have

$$\begin{aligned}
&\frac{2\ln|a_n|}{n} - \frac{\ln|a_{n-1}|}{n-1} - \frac{\ln|a_{n+1}|}{n+1} \\
&= \frac{2(n^2-1)\ln|a_n| - (n^2+n)\ln|a_{n-1}| - (n^2-n)\ln|a_{n+1}|}{n(n-1)(n+1)} \\
&= \frac{-2\ln|a_n| + (n^2+n)\ln x_{n-1} - (n^2-n)\ln x_n}{n(n-1)(n+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2\ln b_n}{n} - \frac{\ln b_{n-1}}{n-1} - \frac{\ln b_{n+1}}{n+1} \\
&= \frac{-2\ln b_n + (n^2+n)\ln y_{n-1} - (n^2-n)\ln y_n}{n(n-1)(n+1)}.
\end{aligned}$$

For $n \geq 2$, put

$$\begin{aligned}
S_n &= -2\ln|a_n| + (n^2+n)\ln x_{n-1} - (n^2-n)\ln x_n, \\
T_n &= -2\ln b_n + (n^2+n)\ln y_{n-1} - (n^2-n)\ln y_n.
\end{aligned}$$

Then we get

$$\begin{aligned}
\frac{2\ln|a_n|}{n} - \frac{\ln|a_{n-1}|}{n-1} - \frac{\ln|a_{n+1}|}{n+1} &= \frac{S_n}{n(n-1)(n+1)}, \\
\frac{2\ln b_n}{n} - \frac{\ln b_{n-1}}{n-1} - \frac{\ln b_{n+1}}{n+1} &= \frac{T_n}{n(n-1)(n+1)}.
\end{aligned}$$

In order to prove that $\{\sqrt[n]{|a_n|}\}_{n\geq 2}$ and $\{\sqrt[n]{b_n}\}_{n\geq 1}$ are log-concave, we need to show that $S_n \geq 0$ $(n \geq 3)$ and $T_n \geq 0$ $(n \geq 2)$. It follows from (2.1)–(2.2) and (2.3)–(2.5) that

$$\begin{aligned}
S_n &> 2\ln 6 - 2\ln(n-1)! + (n^2+n)\ln\left(n - \frac{3}{2}\right) - (n^2-n)\ln n \\
&= 2\ln 6 - 2\ln(n-1)! + 2n\ln n - (n^2+n)\ln\left(1 + \frac{3}{2n-3}\right) \\
&> \ln 18 - 1 - \ln\pi + \ln(n-1) + 2n\ln\left(1 + \frac{1}{n-1}\right) + 2(n-1) \\
&\quad -(n^2+n)\ln\left(1 + \frac{3}{2n-3}\right)
\end{aligned}$$

and

$$\begin{aligned} T_n &> -2\ln b_n + (n^2+n)\ln\left(n-\frac{1}{2}\right) - (n^2-n)\ln(n+1) \\ &> \ln 2 - 1 - \ln\pi + 2n - \ln n - (n^2+n)\ln\left(1+\frac{1}{2n-1}\right) \\ &\quad -(n^2-n)\ln\left(1+\frac{1}{n}\right). \end{aligned}$$

Since

$$\frac{x}{1+x} < \ln(1+x) < x, \quad (x>0), \tag{2.7}$$

we obtain

$$\begin{aligned} S_n &> \ln 18 - \ln\pi + \ln(n-1) + 2n - 1 - \frac{3(n^2+n)}{2n-3} \\ &> 1.7 + \ln(n-1) + \frac{n^2-11n+3}{2n-3} \\ &> 0 \quad (n\geq 6), \end{aligned}$$

$$\begin{aligned} T_n &> \ln 2 - \ln\pi + n - \ln n - \frac{n^2+n}{2n-1} \\ &= \ln 2 - \ln\pi + \frac{n}{2} - \ln n - \frac{3n}{2(2n-1)}. \end{aligned}$$

For $x>0$, define a function

$$g(x) = \ln 2 - \ln\pi + \frac{x}{2} - \ln x - \frac{3x}{2(2x-1)}.$$

Since

$$\begin{aligned} g'(x) &= \frac{x-2}{2x} + \frac{3}{2(2x-1)^2} \\ &> 0 \quad (x>2), \end{aligned}$$

$g(x)$ is monotonic increasing on $[2,+\infty)$. Due to $g(7)>0$, $g(x)>0$ for $x\geq 7$. This implies that $T_n = g(n) > 0$ for $n\geq 7$. On the other hand, we note that $S_n>0$ for $3\leq n\leq 5$ and $T_n>0$ for $2\leq n\leq 6$. Hence we have $S_n>0$ $(n\geq 3)$ and $T_n>0$ $(n\geq 2)$. □

In the rest of this section, we discuss the log-balancedness of $\left\{\frac{1}{\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ and $\left\{\frac{1}{\sqrt[n]{b_n}}\right\}_{n\geq 1}$.

Theorem 2.5. *The sequences* $\left\{\frac{1}{\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ *and* $\left\{\frac{1}{\sqrt[n]{b_n}}\right\}_{n\geq 1}$ *are log-balanced.*

Proof. It follows from Theorem 2.4 that the sequences $\left\{\frac{1}{\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ and $\left\{\frac{1}{\sqrt[n]{b_n}}\right\}_{n\geq 1}$ are log-convex. Now we prove that $\left\{\frac{1}{n!\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ and $\left\{\frac{1}{n!\sqrt[n]{b_n}}\right\}_{n\geq 1}$ are log-concave. In order to prove the log-concavity of $\left\{\frac{1}{n!\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ and $\left\{\frac{1}{n!\sqrt[n]{b_n}}\right\}_{n\geq 1}$, we need to show that

$$\begin{aligned} (n+1)|a_{n-1}|^{\frac{1}{n-1}}|a_{n+1}|^{\frac{1}{n+1}} - n|a_n|^{\frac{2}{n}} &\geq 0, \\ (n+1)b_{n-1}^{\frac{1}{n-1}}b_{n+1}^{\frac{1}{n+1}} - nb_n^{\frac{2}{n}} &\geq 0. \end{aligned}$$

We note that $(n+1)|a_{n-1}|^{\frac{1}{n-1}}|a_{n+1}|^{\frac{1}{n+1}} - n|a_n|^{\frac{2}{n}} \geq 0$ if and only if

$$\frac{n(n^2-1)\ln\left(1+\frac{1}{n}\right) + (n^2+n)\ln|a_{n-1}| + (n^2-n)\ln|a_{n+1}| - 2(n^2-1)\ln|a_n|}{n(n^2-1)} \geq 0$$

and $(n+1)b_{n-1}^{\frac{1}{n-1}}b_{n}^{\frac{1}{n+1}} - nb_n^{\frac{2}{n}} \geq 0$ if and only if

$$\frac{n(n^2-1)\ln\left(1+\frac{1}{n}\right) + (n^2+n)\ln b_{n-1} + (n^2-n)\ln b_{n+1} - 2(n^2-1)\ln b_n}{n(n^2-1)} \geq 0$$

For $n \geq 0$, let $x_n = \frac{|a_{n+1}|}{|a_n|}$ and $y_n = \frac{b_{n+1}}{b_n}$. For $n \geq 2$, we have

$$\begin{aligned} & n(n^2-1)\ln\left(1+\frac{1}{n}\right) + (n^2+n)\ln|a_{n-1}| + (n^2-n)\ln|a_{n+1}| - 2(n^2-1)\ln|a_n| \\ = {} & n(n^2-1)\ln\left(1+\frac{1}{n}\right) + 2\ln|a_n| + (n^2-n)\ln x_n - (n^2+n)\ln x_{n-1} \end{aligned}$$

and

$$\begin{aligned} & n(n^2-1)\ln\left(1+\frac{1}{n}\right) + (n^2+n)\ln b_{n-1} + (n^2-n)\ln b_{n+1} - 2(n^2-1)\ln b_n \\ = {} & n(n^2-1)\ln\left(1+\frac{1}{n}\right) + 2\ln b_n + (n^2-n)\ln y_n - (n^2+n)\ln y_{n-1}. \end{aligned}$$

For $n \geq 2$, put

$$\begin{aligned} U_n &= n(n^2-1)\ln\left(1+\frac{1}{n}\right) + 2\ln|a_n| + (n^2-n)\ln x_n - (n^2+n)\ln x_{n-1}, \\ V_n &= n(n^2-1)\ln\left(1+\frac{1}{n}\right) + 2\ln b_n + (n^2-n)\ln y_n - (n^2+n)\ln y_{n-1}. \end{aligned}$$

It follows from (2.1)–(2.2) that

$$\begin{aligned} U_n &> n(n^2-1)\ln\left(1+\frac{1}{n}\right)+2\ln|a_n|+(n^2-n)\ln\left(n-\frac{1}{2}\right)-(n^2+n)\ln(n-1) \\ &= n(n^2-1)\ln\left(1+\frac{1}{n}\right)+2\ln|a_n|-2n\ln n+(n^2-n)\ln\left(1-\frac{1}{2n}\right) \\ &\quad -(n^2+n)\ln\left(1-\frac{1}{n}\right) \\ &= n(n^2-1)\ln\left(1+\frac{1}{n}\right)+2\ln|a_n|-2n\ln n-(n^2-n)\ln\left(1+\frac{1}{2n-1}\right) \\ &\quad +(n^2+n)\ln\left(1+\frac{1}{n-1}\right) \end{aligned}$$

and

$$\begin{aligned} V_n &> n(n^2-1)\ln\left(1+\frac{1}{n}\right)+2\ln b_n+(n^2-n)\ln\left(n+\frac{1}{2}\right)-(n^2+n)\ln n \\ &= n(n^2-1)\ln\left(1+\frac{1}{n}\right)+2\ln b_n-2n\ln n+(n^2-n)\ln\left(1+\frac{1}{2n}\right). \end{aligned}$$

Owing to (2.7), $\ln|a_n|>0$ $(n\ge 5)$, and $\ln b_n>0$ $(n\ge 3)$, we derive

$$\begin{aligned} U_n &> n(n-1)+2\ln|a_n|-2n\ln n-\frac{n^2-n}{2n-1}+n+1 \\ &= n^2+1+2\ln|a_n|-2n\ln n-\frac{n}{2}+\frac{n}{2(2n-1)} \\ &> n^2+1-2n\ln n-\frac{n}{2} \quad (n\ge 5) \end{aligned}$$

and

$$\begin{aligned} V_n &> n(n-1)-2n\ln n+\frac{n^2-n}{2n+1} \quad (n\ge 3) \\ &= \frac{2n[n^2-1-(2n+1)\ln n]}{2n+1}. \end{aligned}$$

For $x>0$, define two functions

$$\begin{aligned} \varphi(x) &= x^2+1-2x\ln x-\frac{x}{2}, \\ \psi(x) &= x^2-1-(2x+1)\ln x. \end{aligned}$$

It is evident that

$$U_n>\varphi(n) \quad (n\ge 5) \quad \text{and} \quad V_n>\frac{2n\psi(n)}{2n+1} \quad (n\ge 3).$$

We have

$$\varphi'(x) = 2x - \frac{5}{2} - 2\ln x, \quad \varphi''(x) = 2\left(1 - \frac{1}{x}\right),$$
$$\psi'(x) = 2x - 2 - 2\ln x - \frac{1}{x}, \quad \psi''(x) = 2\left(1 - \frac{1}{x}\right) + \frac{1}{x^2}.$$

Since $\varphi''(x) > 0$ $(x > 1)$ and $\psi''(x) > 0$ $(x > 0)$, $\varphi'(x)$ is monotonic increasing on $[1, +\infty)$ and so is $\psi'(x)$ on $(0, +\infty)$. We observe that $\varphi'(2) > 0$ and $\psi'(2) > 0$. Then $\varphi(x)$ and $\psi(x)$ are monotonic increasing on $[2, +\infty)$. We find that $\varphi(2) > 0$ and $\psi(3) > 0$. It is obvious that $\varphi(x) > 0$ $(x \geq 2)$ and $\psi(x) > 0$ $(x \geq 3)$. This implies that $U_n > 0$ $(n \geq 5)$ and $V_n > 0$ $(n \geq 3)$. On the other hand, we observe that $U_j > 0$ $(j = 3, 4)$, and $V_2 > 0$. Since $U_n > 0$ $(n \geq 3)$ and $V_n > 0$ $(n \geq 2)$, we get

$$\begin{aligned}(n+1)|a_{n-1}|^{\frac{1}{n-1}}|a_{n+1}|^{\frac{1}{n+1}} - n|a_n|^{\frac{2}{n}} &\geq 0, \quad n \geq 3,\\ (n+1)b_{n-1}^{\frac{1}{n-1}}b_{n+1}^{\frac{1}{n+1}} - nb_n^{\frac{2}{n}} &\geq 0, \quad n \geq 2.\end{aligned}$$

Hence the sequences $\left\{\frac{1}{n!\sqrt[n]{|a_n|}}\right\}_{n\geq 2}$ and $\left\{\frac{1}{n!\sqrt[n]{b_n}}\right\}_{n\geq 1}$ are log-concave. □

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