ON SOME POLYNOMIALS APPLIED TO THE THEORY OF HYPERBOLIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study a class of sequences of polynomials linked to the sequence of Bell polynomials. Some sequences of this class have applications on the theory of hyperbolic differential equations and other sequences generalize Laguerre polynomials and associated Lah polynomials. We discuss, for these polynomials, their explicit expressions, relations to the successive derivatives of a given function, real zeros and recurrence relations. Some known results are significantly simplified.

Keywords: New class of polynomials, recurrence relations, real zeros, Bell polynomials, Laguerre polynomials, associated Lah polynomials.
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1. INTRODUCTION

Many polynomials having applications on the hyperbolic partial differential equations

$$Au_{xx}+2Bu_{xy}+Cu_{yy}+Du_x+Eu_y+F=0 \text{ with } AD>BC,$$

for which the following two sequences of polynomials $(U_n(x))$ and $(V_n(x))$ defined by

$$\begin{aligned}\sum_{n\geq 0}U_n(x)\frac{t^n}{n!} &= (1-t)^{-1/2}\exp\left(x\left((1-t)^{-1/2}-1\right)\right),\\ \sum_{n\geq 0}V_n(x)\frac{t^n}{n!} &= (1-t)^{-3/2}\exp\left(x\left((1-t)^{-1/2}-1\right)\right)\end{aligned}$$

are considered, see [10, pp. 257–258] and [7, pp. 391–398]. They can be written as

$$\begin{aligned}U_n(x) &= xe^{-x}\left(\frac{d}{d(x^2)}\right)^n\left(x^{2n-1}e^x\right),\\ V_n(x) &= \frac{e^{-x}}{x}\left(\frac{d}{d(x^2)}\right)^n\left(x^{2n+1}e^x\right).\end{aligned}$$

Recently, some studies of the sequence of polynomials $(U_n(x))$ are given in [15, 25, 26]. Motivated by these works, to give more properties of these polynomials, we prefer to

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consider their generalized sequence of polynomials $\left(\mathbf{L}_n^{(\alpha,\beta)}(x)\right)$ defined by

$$\sum_{n\geq 0}\mathbf{L}_n^{(\alpha,\beta)}(x)\frac{t^n}{n!}=(1-t)^\alpha\exp\left(x\left((1-t)^\beta-1\right)\right),\ \ \alpha,\beta\in\mathbb{R},\ \beta\neq 0.$$

The first few values of the sequence $\left(\mathbf{L}_n^{(\alpha,\beta)}\left(-\frac{x}{\beta}\right);n\geq 0\right)$ are to be

$$\begin{aligned}
\mathbf{L}_0^{(\alpha,\beta)}\left(-\frac{x}{\beta}\right) &= 1,\\
\mathbf{L}_1^{(\alpha,\beta)}\left(-\frac{x}{\beta}\right) &= x-\alpha,\\
\mathbf{L}_2^{(\alpha,\beta)}\left(-\frac{x}{\beta}\right) &= x^2-(2\alpha+\beta-1)x+(\alpha)_2,\\
\mathbf{L}_3^{(\alpha,\beta)}\left(-\frac{x}{\beta}\right) &= x^3-3(\alpha+\beta-1)x^2+\left(3\alpha^2+(3\alpha+\beta-1)(\beta-2)\right)x-(\alpha)_3,
\end{aligned}$$

where $(\alpha)_n:=\alpha(\alpha-1)\cdots(\alpha-n+1)$ if $n\geq 1$ and $(\alpha)_0:=1$.
We use also the notation $\langle\alpha\rangle_n:=\alpha(\alpha+1)\cdots(\alpha+n-1)$ if $n\geq 1$ and $\langle\alpha\rangle_0:=1$, and we set in the rest of the paper by convention $0^0=1$.
The paper is organized as follows. In the next section we give different expressions for $\mathbf{L}_n^{(\alpha,\beta)}(x)$. In the third section we give special expressions for $\mathbf{L}_n^{(\alpha,\beta)}(x)$ and we show that it has only real zeros under certain conditions on α and β. In the fourth section, we give some recurrence relations, and, in the last section apply the obtained results to some particular polynomials.

2. Explicit expressions for the polynomials $\mathbf{L}_n^{(\alpha,\beta)}$

In this section, we give some explicit expressions for $\mathbf{L}_n^{(\alpha,\beta)}(x)$. Two expressions of $\mathbf{L}_n^{(\alpha,\beta)}(x)$ related to Dobinski's formula and generalized Stirling numbers are given by the following proposition.

Proposition 1. *There hold*

$$\begin{aligned}
\mathbf{L}_n^{(\alpha,\beta)}(x) &= e^{-x}\sum_{k\geq 0}\langle-\alpha-\beta k\rangle_n\frac{x^k}{k!},\\
\mathbf{L}_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n}S_{\alpha,\beta}(n,k)\,x^k,
\end{aligned}$$

where

$$S_{\alpha,\beta}(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\langle-\alpha-\beta j\rangle_n.$$

To prove Proposition 1, the following theorem may be necessary.

Theorem 2. [30, th. 7.50] *Suppose that $c_{m,n} \in \mathbb{C}$ for each $(m,n) \in \mathbb{N} \times \mathbb{N}$ and that ϕ in any one-to-one mapping of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. If any of the three sums*

$$(i) \quad \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\left|c_{m,n}\right|\right), \quad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\left|c_{m,n}\right|\right), \quad \sum_{k=1}^{\infty}\left|c_{\phi(k)}\right|$$

is finite, then all of the series

$$\begin{aligned} &(ii) \quad \sum_{n=1}^{\infty} c_{m,n} \quad (m = 1, 2, \ldots), \\ &(iii) \quad \sum_{m=1}^{\infty} c_{m,n} \quad (n = 1, 2, \ldots), \\ &(iv) \quad \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} c_{m,n}\right), \quad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} c_{m,n}\right), \quad \sum_{k=1}^{\infty} c_{\phi(k)} \end{aligned}$$

are absolutely convergent and the three series in (iv) all have the same sum, where $\mathbb{C}$ and $\mathbb{N}$ are, respectively, the sets of complex and natural numbers.

Proof of Proposition 1. By definition we have

$$\begin{aligned} \sum_{n\geq 0} \mathbf{L}_n^{(\alpha,\beta)}(x)\frac{t^n}{n!} &= e^{-x}(1-t)^{\alpha}\exp\left(x(1-t)^{\beta}\right) \\ &= e^{-x}\sum_{m\geq 0} x^m \frac{(1-t)^{\alpha+\beta m}}{m!} \\ &= e^{-x}\sum_{m\geq 0}\left(\sum_{n\geq 0}\langle -\alpha-\beta m\rangle_n \frac{x^m}{m!}\frac{t^n}{n!}\right), \quad |t|<1, \end{aligned}$$

but for $|t| < 1$ if we set $c_{m,n} = e^{-x}\langle -\alpha-\beta m\rangle_n \frac{x^m}{m!}\frac{t^n}{n!}$ we get

$$|c_{m,n}| = e^{-x}\left|\langle -\alpha-\beta m\rangle_n\right| \frac{|x|^m}{m!}\frac{|t|^n}{n!} \leq e^{-x}\langle |\alpha| + |\beta| m\rangle_n \frac{|x|^m}{m!}\frac{|t|^n}{n!} := C_{m,n}$$

and

$$\begin{aligned} \sum_{m\geq 0}\left(\sum_{n\geq 0} C_{m,n}\right) &= e^{-x}\sum_{m\geq 0}\frac{|x|^m}{m!}\sum_{n\geq 0}\langle |\alpha|+|\beta| m\rangle_n \frac{|t|^n}{n!} \\ &= e^{-x}\sum_{m\geq 0}\frac{|x|^m}{m!}(1-|t|)^{|\alpha|+|\beta| m} \\ &= e^{-x}(1-|t|)^{|\alpha|}\exp\left(|x|(1-|t|)^{|\beta|}\right) \end{aligned}$$

which is finite. Then, by Theorem 2, it follows

$$\begin{aligned}\sum_{n\geq 0}\mathbf{L}_n^{(\alpha,\beta)}(x)\frac{t^n}{n!} &= e^{-x}\sum_{m\geq 0}\left(\sum_{n\geq 0}\langle -\alpha-\beta m\rangle_n\frac{x^m}{m!}\frac{t^n}{n!}\right)\\ &= \sum_{n\geq 0}\left(e^{-x}\sum_{m\geq 0}\langle -\alpha-\beta m\rangle_n\frac{x^m}{m!}\right)\frac{t^n}{n!},\end{aligned}$$

from which the first identity follows.
The second identity follows from the first by expansion e^{-x} in power series. ∎

Proposition 3. *There holds*

$$x^n=\sum_{k=0}^{n}\widetilde{S}_{\alpha,\beta}(n,k)\,\mathbf{L}_k^{(\alpha,\beta)}(x)\quad with\quad \widetilde{S}_{\alpha,\beta}(n,k)=(-1)^{n-k}S_{-\frac{\alpha}{\beta},\frac{1}{\beta}}(n,k).$$

Proof. Upon using the explicit expression of $S_{\alpha,\beta}(n,k)$ given in Proposition 1, we get for $|t|<1$:

$$\begin{aligned}\sum_{n\geq 0}S_{\alpha,\beta}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{n\geq 0}\langle -\alpha-\beta j\rangle_n\frac{t^n}{n!}\\ &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(1-t)^{\alpha+\beta j}\\ &= \frac{1}{k!}(1-t)^{\alpha}\left((1-t)^{\beta}-1\right)^k,\end{aligned}$$

Then, if we set $H_k(t)=(1-t)^{-\alpha/\beta}\left((1-t)^{1/\beta}-1\right)^k$ we get

$$\begin{aligned}\sum_{k=0}^{n}\widetilde{S}_{\alpha,\beta}(n,k)\,\mathbf{L}_k^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n}(-1)^{n-k}S_{-\frac{\alpha}{\beta},\frac{1}{\beta}}(n,k)\,\mathbf{L}_k^{(\alpha,\beta)}(x)\\ &= \sum_{k=0}^{n}(-1)^{n-k}\mathbf{L}_k^{(\alpha,\beta)}(x)\left(\frac{d}{dt}\right)_{t=0}^{n}\left(\frac{H_k(t)}{k!}\right)\\ &= (-1)^n\left(\frac{d}{dt}\right)_{t=0}^{n}\left(\sum_{k=0}^{n}\mathbf{L}_k^{(\alpha,\beta)}(x)\frac{H_k(t)}{k!}\right)\\ &= (-1)^n\left(\frac{d}{dt}\right)_{t=0}^{n}\left(\sum_{k\geq 0}\mathbf{L}_k^{(\alpha,\beta)}(x)\frac{H_k(t)}{k!}\right)\\ &\quad -(-1)^n\left(\frac{d}{dt}\right)_{t=0}^{n}\left(\sum_{k\geq n+1}\mathbf{L}_k^{(\alpha,\beta)}(x)\frac{H_k(t)}{k!}\right).\end{aligned}$$

So, by definition of the sequence $\left(\mathbf{L}_k^{(\alpha,\beta)}(x);k\geq 0\right)$ we have

$$(-1)^n\left(\frac{d}{dt}\right)_{t=0}^n\left(\sum_{k\geq 0}\mathbf{L}_k^{(\alpha,\beta)}(x)\frac{H_k(t)}{k!}\right)=(-1)^n\left(\frac{d}{dt}\right)_{t=0}^n(\exp(-xt))=x^n,$$

and since for $k\in\{0,1,\ldots,n\}$ the coefficient of t^k in the power series $H_k(t)$ is zero, it follows that

$$(-1)^n\left(\frac{d}{dt}\right)_{t=0}^n\left(\sum_{k\geq n+1}\mathbf{L}_k^{(\alpha,\beta)}(x)\frac{H_k(t)}{k!}\right)=0.$$

Hence $\sum\limits_{k=0}^n\widetilde{S}_{\alpha,\beta}(n,k)\,\mathbf{L}_k^{(\alpha,\beta)}(x)=x^n.$ ∎

Corollary 4. *There holds*

$$\langle-\alpha-\beta x\rangle_n=\sum_{j=0}^n S_{\alpha,\beta}(n,j)\,(x)_j\,.$$

Proof. Let $\langle-\alpha-\beta x\rangle_n=\sum\limits_{j=0}^n\delta(n,j)\,(x)_j$. Then, from Proposition 1 we get

$$\mathbf{L}_n^{(\alpha,\beta)}(x)=e^{-x}\sum_{k\geq 0}\langle-\alpha-\beta k\rangle_n\frac{x^k}{k!}=\sum_{j=0}^n\delta(n,j)\left(e^{-x}\sum_{k\geq j}(k)_j\frac{x^k}{k!}\right)=\sum_{j=0}^n\delta(n,j)\,x^j,$$

which gives $\delta(n,j)=S_{\alpha,\beta}(n,j)$. ∎

If $\mathcal{B}_n$ denote the n-th Bell polynomial, then when we replace t by $1-e^t$ in the generating function of the sequence $\left(\mathbf{L}_n^{(\alpha,\beta)}(x)\right)$, then $\mathbf{L}_n^{(\alpha,\beta)}(x)$ can be written in the basis $\{1,\mathcal{B}_1(x),\ldots,\mathcal{B}_n(x)\}$ as follows:

Proposition 5. *There holds*

$$\sum_{k=0}^n(-1)^k S(n,k)\,\mathbf{L}_k^{(\alpha,\beta)}(x)=\sum_{k=0}^n\binom{n}{k}\alpha^{n-k}\beta^k\mathcal{B}_k(x),$$

or equivalently

$$\mathbf{L}_n^{(\alpha,\beta)}(x)=\sum_{j=0}^n\beta^j\left(\sum_{k=j}^n(-1)^k|s(n,k)|\,\alpha^{k-j}\right)\mathcal{B}_j(x),$$

where $s(n,k)$ and $S(n,k)$ are, respectively, the Stirling numbers of the first and second kind, see for instance [6].

Let $B_{n+r,k+r}^{(r)}((a_i,i\geq 1);(b_i,i\geq 1))$ are the partial r-Bell polynomials [5, 21, 28] defined by

$$\sum_{n\geq k}B_{n+r,k+r}^{(r)}(a_l;b_l)\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{j\geq 1}a_j\frac{t^j}{j!}\right)^k\left(\sum_{j\geq 0}b_{j+1}\frac{t^j}{j!}\right)^r$$

and $B_{n,k}\left((a_i, i\geq 1)\right) = B_{n,k}^{(0)}\left((a_i, i\geq 1);(b_i, i\geq 1)\right)$ are the partial Bell polynomials [1, 6, 18, 19]. An expression of $\mathbf{L}_n^{(\alpha,\beta)}(x)$ in terms of the partial r-Bell polynomials is as follows.

Proposition 6. *For any non-negative integer r, there hold*

$$S_{r\alpha,\beta}(n,k) = B_{n+r,k+r}^{(r)}\left(\langle -\beta\rangle_j ; \langle -\alpha\rangle_{j-1}\right),$$

which imply

$$\mathbf{L}_n^{(r\alpha,\beta)}(x) = \sum_{k=0}^{n} B_{n+r,k+r}^{(r)}\left(\langle -\beta\rangle_j ; \langle -\alpha\rangle_{j-1}\right) x^k.$$

Proof. From the proof of Proposition 3 we have

$$\sum_{n\geq 0} S_{\alpha,\beta}(n,k)\frac{t^n}{n!} = \frac{1}{k!}(1-t)^{\alpha}\left((1-t)^{\beta}-1\right)^k.$$

Then, we get

$$\begin{aligned}\sum_{n\geq 0} S_{r\alpha,\beta}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\left((1-t)^{\beta}-1\right)^k (1-t)^{r\alpha}\\ &= \frac{1}{k!}\left(\sum_{n\geq 1}\langle -\beta\rangle_n \frac{t^n}{n!}\right)^k \left(\sum_{n\geq 0}\langle -\alpha\rangle_n \frac{t^n}{n!}\right)^r\\ &= \sum_{n\geq k} B_{n+r,k+r}^{(r)}\left(\langle -\beta\rangle_j ; \langle -\alpha\rangle_{j-1}\right)\frac{t^n}{n!},\end{aligned}$$

and this expansion gives $S_{r\alpha,\beta}(n,k) = B_{n+r,k+r}^{(r)}\left(\langle -\beta\rangle_j ; \langle -\alpha\rangle_{j-1}\right)$. ∎

3. Some properties of the polynomials $\mathbf{L}_n^{(\alpha,\beta)}$

In this section, we show the link of the sequence of polynomials $\left(\mathbf{L}_n^{(\alpha,\beta)}(x)\right)$ to the successive derivatives of a given function and we give sufficient conditions on α and β for which the polynomial $\mathbf{L}_n^{(\alpha,\beta)}$ has only real zeros.

Lemma 7. *There holds*

$$\mathbf{L}_{n+1}^{(\alpha,\beta)}(x) = (n-\alpha-\beta x)\,\mathbf{L}_n^{(\alpha,\beta)}(x) - \beta x\frac{d}{dx}\mathbf{L}_n^{(\alpha,\beta)}(x).$$

Proof. One can verify that the function

$$F_{\alpha,\beta}(t,x) = (1-t)^{\alpha}\exp\left(x\left((1-t)^{\beta}-1\right)\right)$$

is a solution of the partial differential equation

$$(1-t)\frac{d}{dt}Y + \beta x\frac{d}{dx}Y + (\alpha+\beta x)\,Y = 0$$

from which it results the desired identity. ∎

Theorem 8. *For $x>0$ and $n\geq 0$ we have*

$$\begin{aligned}\mathbf{L}_n^{(\alpha,\beta)}\left(x^{\beta}\right) &= (-1)^n x^{n-\alpha}e^{-x^{\beta}}\left(\frac{d}{dx}\right)^n\left(x^{\alpha}e^{x^{\beta}}\right),\\ \mathbf{L}_n^{(\alpha,\beta)}\left(x^{-\beta}\right) &= x^{\alpha+1}e^{-x^{-\beta}}\left(\frac{d}{dx}\right)^n\left(x^{n-1-\alpha}e^{x^{-\beta}}\right).\end{aligned}$$

Proof. For the first identity, Lemma 7 gives

$$\mathbf{L}_n^{(\alpha,\beta)}(x)=(n-1-\alpha-\beta x)\,\mathbf{L}_{n-1}^{(\alpha,\beta)}(x)-\beta x\frac{d}{dx}\mathbf{L}_{n-1}^{(\alpha,\beta)}(x),$$

and if we set $f_n^{(\alpha,\beta)}(x):=(-1)^n x^{\frac{\alpha-n}{\beta}}e^x\mathbf{L}_n^{(\alpha,\beta)}(x)$, the last identity can also be written as

$$f_n^{(\alpha,\beta)}(x)=\beta x^{1-\frac{1}{\beta}}\frac{d}{dx}f_{n-1}^{(\alpha,\beta)}(x)=\frac{d}{d\left(x^{1/\beta}\right)}f_{n-1}^{(\alpha,\beta)}(x)$$

which implies $f_n^{(\alpha,\beta)}(x)=\left(\frac{d}{d\left(x^{1/\beta}\right)}\right)^n f_0^{(\alpha,\beta)}(x)=\left(\frac{d}{d\left(x^{1/\beta}\right)}\right)^n\left(x^{\frac{\alpha}{\beta}}e^x\right)$. So, we get

$$\mathbf{L}_n^{(\alpha,\beta)}(x)=(-1)^n x^{\frac{n-\alpha}{\beta}}e^{-x}\left(\frac{d}{d\left(x^{1/\beta}\right)}\right)^n\left(x^{\frac{\alpha}{\beta}}e^x\right)$$

or equivalently $\mathbf{L}_n^{(\alpha,\beta)}\left(y^{\beta}\right)=(-1)^n y^{n-\alpha}e^{-y^{\beta}}\left(\frac{d}{dy}\right)^n\left(y^{\alpha}e^{y^{\beta}}\right)$.
For the second identity, we proceed as follows

$$\begin{aligned}\left(\frac{d}{d\left(x^{-1/\beta}\right)}\right)^n\left(x^{\frac{\alpha+1-n}{\beta}}e^x\right) &= \left(\frac{d}{dy}\right)^n_{y=x^{-1/\beta}}\left(y^{n-\alpha-1}e^{y^{-\beta}}\right)\\ &= \left(\frac{d}{dy}\right)^n_{y=x^{-1/\beta}}\sum_{k\geq 0}\frac{1}{k!}y^{n-1-\alpha-k\beta}\\ &= \sum_{k\geq 0}(n-1-\alpha-k\beta)_n\left.\frac{y^{-1-\alpha-k\beta}}{k!}\right|_{y=x^{-1/\beta}}\\ &= x^{\frac{\alpha+1}{\beta}}\sum_{k\geq 0}\langle-\alpha-k\beta\rangle_n\frac{x^k}{k!}\\ &= x^{\frac{\alpha+1}{\beta}}e^x\mathbf{L}_n^{(\alpha,\beta)}(x),\end{aligned}$$

i.e. $\mathbf{L}_n^{(\alpha,\beta)}(x)=x^{-\frac{\alpha+1}{\beta}}e^{-x}\left(\frac{d}{d\left(x^{-1/\beta}\right)}\right)^n\left(x^{\frac{\alpha+1-n}{\beta}}e^x\right)$ which is equivalent to the desired identity. ∎

Corollary 9. *There holds*

$$\mathbf{L}_n^{(\alpha,\beta)}(x)=(-1)^n\,\mathbf{L}_n^{(n-1-\alpha,-\beta)}(x)$$

and

$$S_{\alpha,\beta}(n,k) = (-1)^n S_{n-1-\alpha,-\beta}(n,k).$$

Proof. Theorem 8 remains true for any complex number x. Then, use its second identity to obtain

$$\mathbf{L}_n^{(n-1-\alpha,-\beta)}\left(x^\beta\right) = x^{n-\alpha}e^{-x^\beta}\left(\frac{d}{dx}\right)^n\left(x^\alpha e^{x^\beta}\right) = (-1)^n \mathbf{L}_n^{(\alpha,\beta)}\left(x^\beta\right),$$

which this gives the first identity.
The second identity follows from the first one and Proposition 1. ∎

Theorem 8 proves that $\mathcal{B}_n\left(x^\beta\right)$ can also be written in a similar form as follows.

Proposition 10. *For $x > 0$ and $n \geq 0$ we have*

$$\mathcal{B}_n\left(\lambda + x^\beta\right) = x^{-\alpha}e^{-x^\beta}\left(x\frac{d}{dx} - \frac{\alpha}{\beta} + \lambda\right)^n\left(x^\alpha e^{x^\beta}\right), \ \ \beta \neq 0,$$

or equivalently

$$\mathcal{B}_n\left(\lambda + e^{\beta y}\right) = e^{-e^{\beta y}-\alpha y}\left(\frac{d}{dy} - \frac{\alpha}{\beta} + \lambda\right)^n\left(e^{e^{\beta y}+\alpha y}\right), \ \ \beta \neq 0.$$

Proof. From Proposition 5 we get

$$\frac{1}{\alpha^n}\sum_{k=0}^{n}(-1)^k S(n,k)\,\mathbf{L}_k^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\beta}{\alpha}\right)^k \mathcal{B}_k(x),$$

which implies by using Theorem 8

$$\begin{aligned}
\left(\frac{\beta}{\alpha}\right)^n \mathcal{B}_n\left(x^\beta\right) &= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{\alpha^k}\binom{n}{k}\sum_{j=0}^{k}S(k,j)(-1)^j\,\mathbf{L}_j^{(\alpha,\beta)}\left(x^\beta\right) \\
&= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{\alpha^k}\binom{n}{k}\sum_{j=0}^{k}S(k,j)\,x^{j-\alpha}e^{-x^\beta}\left(\frac{d}{dx}\right)^j\left(x^\alpha e^{x^\beta}\right) \\
&= (-1)^n x^{-\alpha}e^{-x^\beta}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{\alpha}\right)^k\sum_{j=0}^{k}S(k,j)\,x^j\left(\frac{d}{dx}\right)^j\left(x^\alpha e^{x^\beta}\right).
\end{aligned}$$

On using the identity $\sum_{j=0}^{k}S(k,j)\,x^j\left(\frac{d}{dx}\right)^j = \left(x\frac{d}{dx}\right)^k$ [11], it follows

$$\begin{aligned}
\mathcal{B}_n\left(x^\beta\right) &= \left(-\frac{\alpha}{\beta}\right)^n x^{-\alpha}e^{-x^\beta}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{x}{\alpha}\frac{d}{dx}\right)^k\left(x^\alpha e^{x^\beta}\right) \\
&= \left(-\frac{\alpha}{\beta}\right)^n x^{-\alpha}e^{-x^\beta}\left(1 - \frac{x}{\alpha}\frac{d}{dx}\right)^n\left(x^\alpha e^{x^\beta}\right) \\
&= x^{-\alpha}e^{-x^\beta}\left(x\frac{d}{dx} - \frac{\alpha}{\beta}\right)^n\left(x^\alpha e^{x^\beta}\right),
\end{aligned}$$

Then, since the sequence of polynomials $(\mathcal{B}_n(x); n \geq 0)$ is of binomial type, we can write

$$\begin{aligned}\mathcal{B}_n\left(\lambda + x^{\beta}\right) &= \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_{n-j}(\lambda)\, \mathcal{B}_j\left(x^{\beta}\right) \\ &= x^{-\alpha} e^{-x^{\beta}} \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \left(x\frac{d}{dx} - \frac{\alpha}{\beta}\right)^{j} \left(x^{\alpha} e^{x^{\beta}}\right) \\ &= x^{-\alpha} e^{-x^{\beta}} \left(x\frac{d}{dx} - \frac{\alpha}{\beta} + \lambda\right)^{n} \left(x^{\alpha} e^{x^{\beta}}\right),\end{aligned}$$

which remains true for $\alpha = 0$. ∎

To study the real zeros of $\mathbf{L}_n^{(\alpha,\beta)}$, we use the following known theorem. Indeed, let P_1 and P_2 be two polynomials having only real zeros and let $x_n \leq \cdots \leq x_1$ and $y_m \leq \cdots \leq y_1$ be the zeros of P_1 and P_2, respectively. Following [31], we say that P_2 interlaces P_1 if $m = n - 1$ and

$$x_n \leq y_{n-1} \leq x_{n-1} \leq \cdots \leq y_1 \leq x_1$$

and that P_2 alternates left of P_1 if $m = n$ and

$$y_n \leq x_n \leq y_{n-1} \leq x_{n-1} \leq \cdots \leq y_1 \leq x_1.$$

Theorem 11. [32, Th. 1] *Let $a_1, a_2,\ b_1,\ b_2$ be real numbers, let P_1, P_2 be two polynomials whose leading coefficients have the same sign and let $P(x) = (a_1 x + b_1) P_1(x) + (a_2 x + b_2) P_2(x)$. Suppose that $P_1,\ P_2$ have only real zeros and P_2 interlaces P_1 or P_2 alternates left of P_1. Then, if $a_1 b_2 \leq b_1 a_2$, $P(x)$ has only real zeros.*

Theorem 12. *Let*

$$\begin{aligned} A &= \left\{(\alpha,\beta) \in \mathbb{R}^2 : (\beta-1)^2 + 4\alpha\beta \geq 0,\ \ \beta < 0,\ \alpha \leq 2\right\}, \\ \widetilde{A} &= \left\{(\alpha,\beta) \in \mathbb{R}^2 : \beta > 0,\ \alpha \geq 1\right\}.\end{aligned}$$

Then, for $(\alpha,\beta) \in A$, the polynomial $\mathbf{L}_n^{(\alpha,\beta)}$ has only real zeros, $n \geq 1$, and, for $(\alpha,\beta) \in \widetilde{A}$, the polynomials $\mathbf{L}_1^{(\alpha,\beta)}, \ldots, \mathbf{L}_{\lceil\alpha\rceil}^{(\alpha,\beta)}$ has only real zeros, where $\lceil\alpha\rceil$ is the smallest integer $\geq \alpha$.

Proof. We proceed by induction on $n \geq 1$. For $n = 1$, the polynomial $\mathbf{L}_1^{(\alpha)}(x) = -\beta x - \alpha$ has a real zero, and for $n = 2$, the polynomial

$$\mathbf{L}_2^{(\alpha)}(x) = \beta^2 x^2 + \beta(2\alpha + \beta - 1)x + \alpha(\alpha - 1)$$

has only real zeros when $(\beta - 1)^2 + 4\alpha\beta \geq 0$ and $\beta < 0$.
Assume that $\mathbf{L}_n^{(\alpha,\beta)}(x)$ has n (≥ 2) real zeros different from zero, since the heading coefficient of $\mathbf{L}_n^{(\alpha,\beta)}(x)$ is $S_{\alpha,\beta}(n,n) = (-\beta)^n$ and the heading coefficient of $\frac{d}{dx}\mathbf{L}_n^{(\alpha,\beta)}(x)$ is $nS_{\alpha,\beta}(n,n) = n(-\beta)^n$, then they are of the same sign. Also, since $\frac{d}{dx}\mathbf{L}_n^{(\alpha,\beta)}(x)$

interlaces $\mathbf{L}_n^{(\alpha,\beta)}(x)$ it follows from Theorem 11 that if $-\beta(n-\alpha)\geq 0$, $\mathbf{L}_{n+1}^{(\alpha,\beta)}$ has only real zeros. The condition $-\beta(n-\alpha)\geq 0$ is satisfied when $(\alpha,\beta)\in A$ because $-\beta(n-\alpha)\geq -\beta(2-\alpha)\geq 0$. It is also satisfied when $n\in[1,\lceil\alpha\rceil-1]$ and $(\alpha,\beta)\in\widetilde{A}$ because $-\beta(n-\alpha)\geq -\beta(\lceil\alpha\rceil-1-\alpha)\geq 0$. ∎

Corollary 13. *For $\alpha\leq 0$ and $\beta<0$ the sequence $\left(S_{\alpha,\beta}(n,k);\ 0\leq k\leq n\right)$ is strictly log-concave, more precisely*

$$\left(S_{\alpha,\beta}(n,k)\right)^2\geq\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right)S_{\alpha,\beta}(n,k+1)\,S_{\alpha,\beta}(n,k-1),\ \ 1\leq k\leq n-1.$$

Proof. For $\alpha\leq 0$ and $\beta<0$ the polynomial $\mathbf{L}_n^{(\alpha,\beta)}$ has only real zeros and its coefficients $S_{\alpha,\beta}(n,k)$ are non-negative, so Newton's inequality [13, pp. 52] completes the proof. ∎

4. Recurrence relations

In [15, Sec. 2] (see also [25]), the authors give a differential equation having as solution the function

$$(1-t)^{-1/2}\exp\left(x\left((1-t)^{-1/2}-1\right)\right)$$

from which they conclude a generalized recurrence relation for the sequence $(U_n(x))$. The results of this section simplify and generalize these results.

Lemma 14. *Let m be a positive integer. The function*

$$F_{\alpha,\beta}(t,x)=(1-t)^{\alpha}\exp\left(x\left((1-t)^{\beta}-1\right)\right)$$

satisfies

$$(1-t)^m\left(\frac{d}{dt}\right)^m F_{\alpha,\beta}(t,x)=F_{\alpha,\beta}(t,x)\,\mathbf{L}_m^{(\alpha,\beta)}\left(x(1-t)^{\beta}\right).$$

Proof. From the definition of $F_{\alpha,\beta}(t,x)$ and Corollary 4 we obtain

$$\begin{aligned}
\left(\frac{d}{dt}\right)^m F_{\alpha,\beta}(t,x) &= \left(\frac{d}{dt}\right)^m \left((1-t)^\alpha \exp\left(x\left((1-t)^\beta-1\right)\right)\right)\\
&= e^{-x}\sum_{k\geq 0}\frac{x^k}{k!}\left(\frac{d}{dt}\right)^m (1-t)^{\alpha+k\beta}\\
&= e^{-x}\sum_{k\geq 0}\frac{x^k}{k!}\left\langle -\alpha-k\beta\right\rangle_m (1-t)^{\alpha+k\beta-m}\\
&= e^{-x}(1-t)^{\alpha-m}\sum_{j=0}^{m} S_{\alpha,\beta}(m,j)\sum_{k\geq 0}(k)_j\frac{x^k(1-t)^{k\beta}}{k!}\\
&= e^{-x}(1-t)^{\alpha-m}\sum_{j=0}^{m} S_{\alpha,\beta}(m,j)\,x^j(1-t)^{j\beta}\sum_{k\geq 0}\frac{x^k(1-t)^{k\beta}}{k!}\\
&= (1-t)^{\alpha-m}\exp\left(x\left((1-t)^\beta-1\right)\right)\sum_{j=0}^{n} S_{\alpha,\beta}(m,j)\,x^j(1-t)^{j\beta}\\
&= F_{\alpha,\beta}(t,x)(1-t)^{-m}\sum_{j=0}^{m} S_{\alpha,\beta}(m,j)\,x^j(1-t)^{j\beta}\\
&= F_{\alpha,\beta}(t,x)(1-t)^{-m}\,\mathbf{L}_m^{(\alpha,\beta)}\left(x(1-t)^\beta\right).
\end{aligned}$$

∎

The next corollary gives an expression of $\mathbf{L}_{n+m}^{(\alpha,\beta)}(x)$ in terms of the family $\left(x^k\mathbf{L}_j^{(\alpha,\beta)}(x)\right)$. The obtained expression is similar to the expression of the Bell number $\mathcal{B}_{n+m} := \mathcal{B}_{n+m}(1)$ given in [29], Bell polynomial $\mathcal{B}_{n+m}(x)$ given in [3, 12] and several generalizations given later, see [14, 16, 17, 20, 33].

Corollary 15. *For $n,m=0,1,2,\ldots,$ we have*

$$\mathbf{L}_{n+m}^{(\alpha,\beta)}(x)=\sum_{j=0}^{n}\sum_{k=0}^{m}\binom{n}{j}\left\langle m-\beta k\right\rangle_{n-j} S_{\alpha,\beta}(m,k)\,x^k\mathbf{L}_j^{(\alpha,\beta)}(x).$$

In particular, for $m=1$, we obtain

$$\mathbf{L}_{n+1}^{(\alpha,\beta)}(x)=-\sum_{j=0}^{n}\binom{n}{j}\left(\alpha(n-j)!+\beta x\left\langle 1-\beta\right\rangle_{n-j}\right)\mathbf{L}_j^{(\alpha,\beta)}(x).$$

Proof. On using Lemma 14, Proposition 1 and the expansion

$$(1-t)^{-x}=\sum_{j\geq 0}\left\langle x\right\rangle_j\frac{t^j}{j!},\quad |t|<1,$$

we obtain

$$\begin{aligned}
\sum_{n\geq 0}\mathbf{L}_{n+m}^{(\alpha,\beta)}(x)\frac{t^n}{n!} &= \left(\frac{d}{dt}\right)^m F_{\alpha,\beta}(t,x)\\
&= F_{\alpha,\beta}(t,x)(1-t)^{-m}\mathbf{L}_m^{(\alpha,\beta)}\left(x(1-t)^{\beta}\right)\\
&= \left(\sum_{i\geq 0}\mathbf{L}_i^{(\alpha,\beta)}(x)\frac{t^i}{i!}\right)\left(\sum_{k=0}^{m}S_{\alpha,\beta}(m,k)x^k(1-t)^{-m+\beta k}\right)\\
&= \sum_{k=0}^{m}S_{\alpha,\beta}(m,k)x^k\left(\sum_{i\geq 0}\mathbf{L}_i^{(\alpha,\beta)}(x)\frac{t^i}{i!}\right)\left(\sum_{j\geq 0}\langle m-\beta k\rangle_j\frac{t^j}{j!}\right)\\
&= \sum_{k=0}^{m}S_{\alpha,\beta}(m,k)x^k\sum_{n\geq 0}\left(\sum_{j=0}^{n}\binom{n}{j}\langle m-\beta k\rangle_{n-j}\mathbf{L}_j^{(\alpha,\beta)}(x)\right)\frac{t^n}{n!}\\
&= \sum_{n\geq 0}\left(\sum_{j=0}^{n}\sum_{k=0}^{m}\binom{n}{j}\langle m-\beta k\rangle_{n-j}S_{\alpha,\beta}(m,k)x^k\mathbf{L}_j^{(\alpha,\beta)}(x)\right)\frac{t^n}{n!}
\end{aligned}$$

which follows gives the desired identity. ∎

Remark 16. *For $n=1$ in Corollary 15 we get*

$$\mathbf{L}_{m+1}^{(\alpha,\beta)}(x)=\sum_{j=0}^{m+1}\left((m-\alpha-\beta j)S_{\alpha,\beta}(m,j)-\beta S_{\alpha,\beta}(m,j-1)\right)x^j.$$

So, since from Proposition 1 we have $\mathbf{L}_{m+1}^{(\alpha,\beta)}(x)=\sum_{j=0}^{m+1}S_{\alpha,\beta}(m+1,j)x^j$, it results

$$S_{\alpha,\beta}(m+1,j)=(m-\alpha-\beta j)S_{\alpha,\beta}(m,j)-\beta S_{\alpha,\beta}(m,j-1),$$

with $S_{\alpha,\beta}(m+1,j)=0$ if $j<0$ or $j>m+1$.

Proposition 17. *There holds*

$$\mathbf{L}_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}(-1)^k S_{\alpha-\frac{\alpha'}{\beta'}\beta,\frac{\beta}{\beta'}}(n,k)\mathbf{L}_k^{(\alpha',\beta')}(x).$$

In particular, for $(\alpha',\beta')=(\alpha/\lambda,\beta/\lambda)$, $(1,1)$ or $(0,1)$, we get

$$\begin{aligned}
\mathbf{L}_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n}(-1)^k B_{n,k}\left(\langle-\lambda\rangle_j\right)\mathbf{L}_k^{(\alpha/\lambda,\beta/\lambda)}(x),\ \ \lambda\neq 0,\\
\mathbf{L}_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n}S_{\alpha-\beta,\beta}(n,k)(x+k)x^{k-1},\ \ n\geq 1,\\
\mathbf{L}_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n}S_{\alpha,\beta}(n,k)x^k.
\end{aligned}$$

Proof. From Proposition 3 we have $x^k = \sum_{j=0}^{k} \widetilde{S}_{\alpha',\beta'}(k,j)\,\mathbf{L}_j^{(\alpha',\beta')}(x)$.

So, use the identity $\mathbf{L}_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} S_{\alpha,\beta}(n,k)\,x^k$ of Proposition 1 to obtain

$$\begin{aligned}\mathbf{L}_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n} S_{\alpha,\beta}(n,k)\left(\sum_{j=0}^{k} \widetilde{S}_{\alpha',\beta'}(k,j)\,\mathbf{L}_j^{(\alpha',\beta')}(x)\right)\\ &= \sum_{j=0}^{n}\left(\sum_{k=j}^{n} S_{\alpha,\beta}(n,k)\,\widetilde{S}_{\alpha',\beta'}(k,j)\right)\mathbf{L}_j^{(\alpha',\beta')}(x).\end{aligned}$$

Now, since from the proof of Proposition 3, we have

$$\sum_{n\geq k} S_{\alpha,\beta}(n,k)\frac{t^n}{n!} = \frac{(-1)^k}{k!}\left(1-(1-t)^{\beta}\right)^k (1-t)^{\alpha},$$

it follows that $M(n,j) := \sum_{k=j}^{n} S_{\alpha,\beta}(n,k)\,\widetilde{S}_{\alpha',\beta'}(k,j) = (-1)^j\, S_{\alpha-\frac{\alpha'}{\beta'}\beta,\frac{\beta}{\beta'}}(n,j)$.

Indeed, since

$$\begin{aligned}S_{\alpha,\beta}(n,k) &= \frac{(-1)^k}{k!}\left(\frac{d}{du}\right)^n_{u=0}\left(\left(1-(1-u)^{\beta}\right)^k(1-u)^{\alpha}\right),\\ \widetilde{S}_{\alpha',\beta'}(k,j) &= \frac{(-1)^k}{j!}\left(\frac{d}{dv}\right)^k_{v=0}\left(\left(1-(1-v)^{\frac{1}{\beta'}}\right)^j(1-v)^{-\frac{\alpha'}{\beta'}}\right),\end{aligned}$$

we get

$$S_{\alpha,\beta}(n,k) = 0 \text{ if } k \geq n+1,\ \ \widetilde{S}_{\alpha',\beta'}(k,j) = 0 \text{ if } j \geq k+1$$

and

$$\begin{aligned}M(n,j) &= \sum_{k\geq 0} S_{\alpha,\beta}(n,k)\,\widetilde{S}_{\alpha',\beta'}(k,j)\\ &= \frac{1}{j!}\left(\frac{d}{du}\right)^n_{u=0}\left(\sum_{k\geq 0}\left(\frac{d}{dv}\right)^k_{v=0} H(v)\,\frac{\left(1-(1-u)^{\beta}\right)^k(1-u)^{\alpha}}{k!}\right),\end{aligned}$$

where $H(v) = \left(1-(1-v)^{\frac{1}{\beta'}}\right)^j(1-v)^{-\frac{\alpha'}{\beta'}}$. But by the Maclaurin formula we have

$$\begin{aligned}\sum_{k\geq 0}\left(\frac{d}{dv}\right)^k_{v=0} H(v)\,\frac{\left(1-(1-u)^{\beta}\right)^k}{k!} &= H\left(1-(1-u)^{\beta}\right)\\ &= \left(1-(1-u)^{\frac{\beta}{\beta'}}\right)^j(1-u)^{-\frac{\alpha'}{\beta'}\beta},\end{aligned}$$

so, we get

$$M(n,j)=\frac{1}{j!}\left(\frac{d}{du}\right)_{u=0}^{n}\left[\left(1-(1-u)^{\frac{\beta}{\beta'}}\right)^{j}(1-u)^{\alpha-\frac{\alpha'}{\beta'}\beta}\right]=(-1)^{j}S_{\alpha-\frac{\alpha'}{\beta'}\beta,\frac{\beta}{\beta'}}(n,j).$$

∎

As a consequence of Proposition 17, by combining it with Propositions 1, it results:

Corollary 18. *For any real numbers $\alpha,\alpha',\beta,\beta'$ such that $\beta'\neq 0$, there hold*

$$\begin{aligned}\langle -\alpha-\beta x\rangle_{n} &= \sum_{j=0}^{n}(-1)^{j}S_{\alpha-\frac{\alpha'}{\beta'}\beta,\frac{\beta}{\beta'}}(n,j)\langle -\alpha'-\beta' x\rangle_{j},\\ S_{\alpha,\beta}(n,k) &= \sum_{j=k}^{n}(-1)^{j}S_{\alpha-\frac{\alpha'}{\beta'}\beta,\frac{\beta}{\beta'}}(n,j)S_{\alpha',\beta'}(j,k).\end{aligned}$$

5. Application to particular polynomials

5.1. **Application to the polynomials U_n and V_n.** For $n\geq 1$, the polynomials $U_n=\mathbf{L}_n^{(-1/2,-1/2)}$ and $V_n=\mathbf{L}_n^{(-3/2,-1/2)}$ defined above, Propositions 1, 5 and 6 give

$$U_n(x)=e^{-x}\sum_{k\geq 0}\left\langle\frac{k+1}{2}\right\rangle_n\frac{x^k}{k!},\qquad V_n(x)=e^{-x}\sum_{k\geq 0}\left\langle\frac{k+3}{2}\right\rangle_n\frac{x^k}{k!},$$

$$U_n(x)=\sum_{k=0}^{n}S_{-1/2,-1/2}(n,k)x^k,\qquad V_n(x)=\sum_{k=0}^{n}S_{-3/2,-1/2}(n,k)x^k,$$

$$U_n(x)=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\frac{|s(n,k)|}{2^k}\right)\mathcal{B}_j(x),\qquad V_n(x)=\sum_{j=0}^{n}\frac{1}{3^j}\left(\sum_{k=j}^{n}|s(n,k)|\left(\frac{3}{2}\right)^k\right)\mathcal{B}_j(x),$$

$$U_n(x)=\sum_{k=0}^{n}B_{n+1,k+1}^{(1)}\left(\left\langle\frac{1}{2}\right\rangle_j;\left\langle\frac{1}{2}\right\rangle_{j-1}\right)x^k,\quad V_n(x)=\sum_{k=0}^{n}B_{n+1,k+1}^{(1)}\left(\left\langle\frac{1}{2}\right\rangle_j;\left\langle\frac{3}{2}\right\rangle_{j-1}\right)x^k.$$

Theorem 12 proves that the polynomials U_n and V_n, $n\geq 1$, have only real zeros and Theorem 8 shows that, for $x>0$, there hold

$$\begin{aligned}U_n\left(\frac{1}{\sqrt{x}}\right) &= (-1)^n x^n\sqrt{x}e^{-\frac{1}{\sqrt{x}}}\left(\frac{d}{dx}\right)^n\left(\frac{1}{\sqrt{x}}e^{\frac{1}{\sqrt{x}}}\right),\\ V_n\left(\frac{1}{\sqrt{x}}\right) &= (-1)^n x^{n+1}\sqrt{x}e^{-\frac{1}{\sqrt{x}}}\left(\frac{d}{dx}\right)^n\left(\frac{1}{x\sqrt{x}}e^{\frac{1}{\sqrt{x}}}\right)\end{aligned}$$

and

$$\begin{aligned}U_n(\sqrt{x}) &= \sqrt{x}e^{-\sqrt{x}}\left(\frac{d}{dx}\right)^n\left(x^{n-1}\sqrt{x}e^{\sqrt{x}}\right),\\ V_n(\sqrt{x}) &= \sqrt{x}e^{-x\sqrt{x}}\left(\frac{d}{dx}\right)^n\left(x^{n-1}\sqrt{x}e^{x\sqrt{x}}\right).\end{aligned}$$

5.2. **Application to the generalized Laguerre polynomials.** We note here that the sequence of generalized Laguerre polynomials $\left(\mathbb{L}_n^{(\lambda)}(x)\right)$ (see for example [4, 9, 27]) defined by

$$\sum_{n\geq 0}\mathbb{L}_n^{(\lambda)}(x)\,t^n = (1-t)^{-\lambda-1}\exp\left(-\frac{xt}{1-t}\right)$$

presents a particular case of the sequence $\left(\mathbf{L}_n^{(\alpha,\beta)}(x)\right)$, i.e. $\mathbb{L}_n^{(\lambda)}(x) = \frac{1}{n!}\mathbf{L}_n^{(-\lambda-1,-1)}(x)$. Propositions 1, 5 and 6 give

$$\begin{aligned}
\mathbb{L}_n^{(\lambda)}(x) &= \frac{e^{-x}}{n!}\sum_{k\geq 0}\langle \lambda+1+k\rangle_n\frac{x^k}{k!},\\
\mathbb{L}_n^{(\lambda)}(x) &= \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\langle \lambda+1+k\rangle_{n-k}\,x^k,\\
\mathbb{L}_n^{(\lambda)}(x) &= \frac{1}{n!}\sum_{j=0}^{n}\left(\sum_{k=j}^{n}|s(n,k)|\,(\lambda+1)^{k-j}\right)\mathcal{B}_j(x),\\
\mathbb{L}_n^{(\lambda)}(x) &= \frac{1}{n!}\sum_{k=0}^{n}B_{n+1,k+1}^{(1)}\left(\langle 1\rangle_j;\langle \lambda+1\rangle_{j-1}\right)x^k.
\end{aligned}$$

Corollary 9 gives

$$\begin{aligned}
\mathbb{L}_n^{(\lambda)}(x) &= \frac{1}{n!}\mathbf{L}_n^{(-\lambda-1,-1)}(x)\\
&= \frac{(-1)^n}{n!}\mathbf{L}_n^{(n+\lambda,1)}(x)\\
&= \frac{(-1)^n}{n!}\left(\frac{d}{dt}\right)^n_{t=0}\left((1-t)^{n+\lambda}e^{-xt}\right)\\
&= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\lambda)_{n-k}\,x^k.
\end{aligned}$$

To write $\mathbf{L}_n^{(\alpha,\beta)}(x)$ in the basis $\left\{1,\mathbb{L}_1^{(\lambda)}(x),\ldots,\mathbb{L}_n^{(\lambda)}(x)\right\}$, set $(\alpha',\beta') = (-\lambda-1,-1)$ in Proposition 17 to obtain

$$\mathbf{L}_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n}(-1)^j j!S_{\alpha-(\lambda+1)\beta,-\beta}(n,j)\,\mathbb{L}_j^{(\lambda)}(x).$$

Theorem 12 proves the known property on the generalized Laguerre polynomials $\mathbb{L}_n^{(\lambda)}$, $n\geq 1$, that have only real zeros (here for $\lambda\geq -2$), for more information about the real zeros of Laguerre polynomials see for example [8]. Theorem 8 shows that, for $x>0$, there hold

$$\begin{aligned}\mathbb{L}_n^{(\lambda)}\left(\frac{1}{x}\right) &= \frac{(-1)^n}{n!}x^{n+1+\lambda}e^{-\frac{1}{x}}\left(\frac{d}{dx}\right)^n\left(x^{-\lambda-1}e^{\frac{1}{x}}\right),\\ \mathbb{L}_n^{(\lambda)}(x) &= \frac{x^{-\lambda}e^{-x}}{n!}\left(\frac{d}{dx}\right)^n\left(x^{n+\lambda}e^{x}\right).\end{aligned}$$

We remark that for $\lambda = 2r-1$ be a positive odd integer, we obtain

$$\mathbb{L}_n^{(2r-1)}\left(\frac{1}{x}\right) = \frac{(-1)^n}{n!}x^{n+2r}e^{-\frac{1}{x}}\left(\frac{d}{dx}\right)^n\left(\frac{1}{x^{2r}}e^{\frac{1}{x}}\right) = \frac{1}{n!}\sum_{k=0}^{n}\frac{L_r(n+r,k+r)}{x^k},$$

where $L_r(n,k)$ is the (n,k)-th r-Lah number, see [4, 22, 24].

5.3. **Application to the associated Lah polynomials.** Let m be a positive integer. The sequence of the associated Lah polynomials $\left(\mathcal{L}_n^{(m)}(x)\right)$ are studied in [2, 23] and are defined by

$$\sum_{n\geq 0}\mathcal{L}_n^{(m)}(x)\frac{t^n}{n!} = \exp\left(x\left((1-t)^{-m}-1\right)\right).$$

This shows that $\mathcal{L}_n^{(m)}(x) = \mathbf{L}_n^{(0,-m)}(x)$. Propositions 1, 5 and 6 give

$$\begin{aligned}\mathcal{L}_n^{(m)}(x) &= e^{-x}\sum_{k\geq 0}\langle mk\rangle_n\frac{x^k}{k!}, & \mathcal{L}_n^{(m)}(x) &= \sum_{k=0}^{n}S_{0,-m}(n,k)\,x^k,\\ \mathcal{L}_n^{(m)}(x) &= \sum_{j=0}^{n}m^j\left|s(n,j)\right|\mathcal{B}_j(x), & \mathcal{L}_n^{(m)}(x) &= \sum_{k=0}^{n}B_{n,k}\left(\langle m\rangle_j\right)x^k.\end{aligned}$$

To write $\mathbf{L}_n^{(\alpha,\beta)}(x)$ in the basis $\left\{1,\mathcal{L}_1^{(m)}(x),\ldots,\mathcal{L}_n^{(m)}(x)\right\}$, set $(\alpha',\beta') = (0,-m)$ in Proposition 17 to obtain

$$\mathbf{L}_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n}(-1)^j S_{\alpha,-\frac{\beta}{m}}(n,j)\,\mathcal{L}_j^{(m)}(x).$$

Theorem 12 proves a known property of the associated Lah polynomials $\mathcal{L}_n^{(m)}$, $n\geq 1$, that have only real zeros and Theorem 8 shows that, for $x>0$, there hold

$$\begin{aligned}\mathcal{L}_n^{(m)}\left(\frac{1}{x^m}\right) &= (-1)^n x^n e^{-1/x^m}\left(\frac{d}{dx}\right)^n\left(e^{1/x^m}\right),\\ \mathcal{L}_n^{(m)}(x^m) &= xe^{-x^m}\left(\frac{d}{dx}\right)^n\left(x^{n-1}e^{x^m}\right).\end{aligned}$$

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