THE BINOMIAL COEFFICIENT $\binom{n}{x}$ FOR ARBITRARY x

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Abstract. Binomial coefficients of the form $\binom{\alpha}{\beta}$ for complex numbers α and β can be defined in terms of the gamma function, or equivalently the generalized factorial function. Less well-known is the fact that if n is a natural number, the binomial coefficient $\binom{n}{\beta}$ can be defined in terms of elementary functions. This enables us to investigate the function $\binom{n}{x}$ of the real variable x. The results are completely in line with what one would expect after glancing at the graph of $\binom{3}{x}$, for example, but the techniques involved in the investigation are not the standard methods of calculus. The analysis is complicated by the existence of removable singularities at all of the integer points in the interval $[0, n]$, and requires multiplying, rearranging, and differentiating infinite series.

1. Introduction

The number of different ways of choosing a subset of size k from a set of n objects is denoted by the symbol $\binom{n}{k}$ ("n choose k"). A simple combinatorial argument shows that

(1) $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where n and k are nonnegative integers and $0 \leq k \leq n$. (Here $0! = 1$ by definition. See [6] for a thorough treatment of the combinatorial properties of $\binom{n}{k}$.)

If we replace n by α and rewrite (1) in the form

(2) $$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

at least when $k > 0$, we obtain an expression which makes sense for arbitrary real (or for that matter, complex) values of α. This definition is well-known and very useful, for instance in the study of generating functions. One is therefore encouraged to attempt to define $\binom{\alpha}{\beta}$ for arbitrary complex α and β.

This has in fact been done, the only problematic case occurring when α is a negative integer. (Then $\binom{\alpha}{\beta}$ is undefined unless β is also an integer.) The key is to extend the factorial function to the complex plane minus the negative integers. More precisely, one can show that there is a function $\frac{1}{z!}$ which is defined and analytic on the entire complex plane, and which equals zero exactly when z is a negative integer. For α and

β complex, then, we simply generalize (1) and define

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}. \tag{3}$$

This is understood to be zero if β is a negative integer, or if $\alpha - \beta$ is a negative integer. For a brief description of this construction and of how to deal with the case where α is a negative integer, see [6, pp. 210–211]. A much more detailed treatment of the complex factorial function and the gamma function can be found in [3, Chapter 3]. We note for future reference, however, that the general definition implies that

$$\binom{\alpha}{0} = 1 \tag{4}$$

for all complex α.

The above general definition of $\binom{\alpha}{\beta}$ has the disadvantage of depending on the rather complicated definition of $z!$ for non-integer z. However, D. Fowler notes in [5] that if $\alpha = n$ where n is a nonnegative integer, then (3) reduces to the formula

$$\binom{n}{\beta} = \frac{n!\sin\pi\beta}{\pi\beta(1-\beta)(2-\beta)\cdots(n-\beta)} \tag{5}$$

where β is any complex number such that $\beta \notin \{0, 1, 2, \ldots, n\}$. Actually Fowler takes β to be real, but the identity which allows us to obtain (5) from (3) is

$$\sin\pi\beta = \frac{\pi}{(\beta-1)!(-\beta)!}$$

which holds for all complex β. (See [3].)

In this paper we take formula (5) as our starting point, much as (2) and (4) are routinely considered as defining $\binom{\alpha}{k}$ for noninteger α. For any given nonnegative integer n, our aim is to investigate the function $\binom{n}{x}$, where x is a real variable. This is an elementary function with removable singularities at all the integer values of x from 0 to n, singularities which can be filled via the equation (1). However, standard calculus techniques do not take us very far in our investigation; even determining the sign of the first derivative for a given x is not trivial. We will need to multiply, rearrange, and differentiate infinite series of functions, and to make use of the following formula:

$$\pi\cot\pi z = \lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{z+k} = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{z+k}+\frac{1}{z-k}\right) = \frac{1}{z} + \sum_{k=1}^{\infty}\frac{2z}{z^2-k^2} \tag{6}$$

for noninteger complex z. The usual proof of this formula involves contour integration in the complex plane, but a more elementary proof can be found in [6, Exercise 73, p. 317].

This paper is intended to be accessible to readers (including students) who are comfortable working with infinite series of functions. Familiarity with the gamma function and with analytic functions of a complex variable is not assumed.

In Section 2 we exhibit binomial coefficient identities for coefficients of the form $\binom{n}{\beta}$ which are analogous to the usual identities for coefficients of the form $\binom{\alpha}{k}$ as defined by (2). We calculate the first and second derivatives of $\binom{n}{x}$ in Section 3; since the identical computations work for complex variables, we state and prove all the results in Section 3 for functions $\binom{n}{z}$ of the complex variable z. Our one genuine use of analytic function theory is Proposition 3.1, which enables us to conclude that all derivatives exist and are continuous. The computation of the first two derivatives is routine for noninteger values of z, and for integer values l'Hôpital's rule suffices for the (fairly complicated) computation of these two derivatives.

In Section 4 we investigate the function $\binom{n}{x}$ of the real variable x, looking as usual at regions of increase and decrease, local extrema, points of inflection, and so on. As an example we consider the graph of $y = \binom{3}{x}$ in Figure 1.

We have a bell-shaped curve in the region $0 \leq x \leq n$, which is not surprising when one considers the numbers appearing in a typical row of Pascal's triangle. Note also that $\binom{n}{k} = 0$ for all *integers* k outside this region. This is not to say that $\binom{n}{x} = 0$ for all $x < 0$ and for all $x > n$. Figure 2 shows part of the left tail of the previous graph in more detail.

Figure 1 might lead us to conclude that the investigation of the function $\binom{n}{x}$ should not be overly difficult, at least for x in the interval $(-1, n+1)$, which is after all the interval that we're actually interested in. In fact it turns out to be easier to determine the behavior of the function in the tail regions $x < -1$ and $x > n+1$ than it is for $-1 < x < n+1$. Determination of the sign of the second derivative of $\binom{n}{x}$ for $-1 < x < n+1$ is decidedly nontrivial; it requires multiplying and rearranging infinite series as well as some complicated computations.

Section 5 includes numerical information about the location of the local extrema and the points of inflection of the function $y = \binom{n}{x}$ for some particular values of n. In particular it is shown that we can approximate the two points of inflection of the curve $y = \binom{n}{x}$ on the interval $(-1, n+1)$ by $\frac{n \pm \sqrt{n+1}}{2}$.

Finally, Section 6 contains some concluding remarks.

Some of the infinite series whose rearrangement enables us to compute the sign of the second derivative of $\binom{n}{x}$ for $-1 < x < n+1$ are conditionally convergent. It is well-known that such rearrangement can change the sum of the series, or even cause the resulting series to diverge. We will be careful to ensure that our rearrangements of conditionally convergent series satisfy the conditions of the following result from [4]:

Lemma 1.1. *Let $\sum_{n=0}^{\infty} a_n$ be a convergent series of real numbers. Suppose that, in a rearrangement of the series, there is a fixed positive integer p such that each term of the series that is shifted forward is shifted at most p places. Then the rearranged series converges to the same sum as the original one.*

(For the record, we note that the result in [4] is a weaker version of a theorem proved by A. Borel in [2] and given as an exercise in [7, p. 77].)

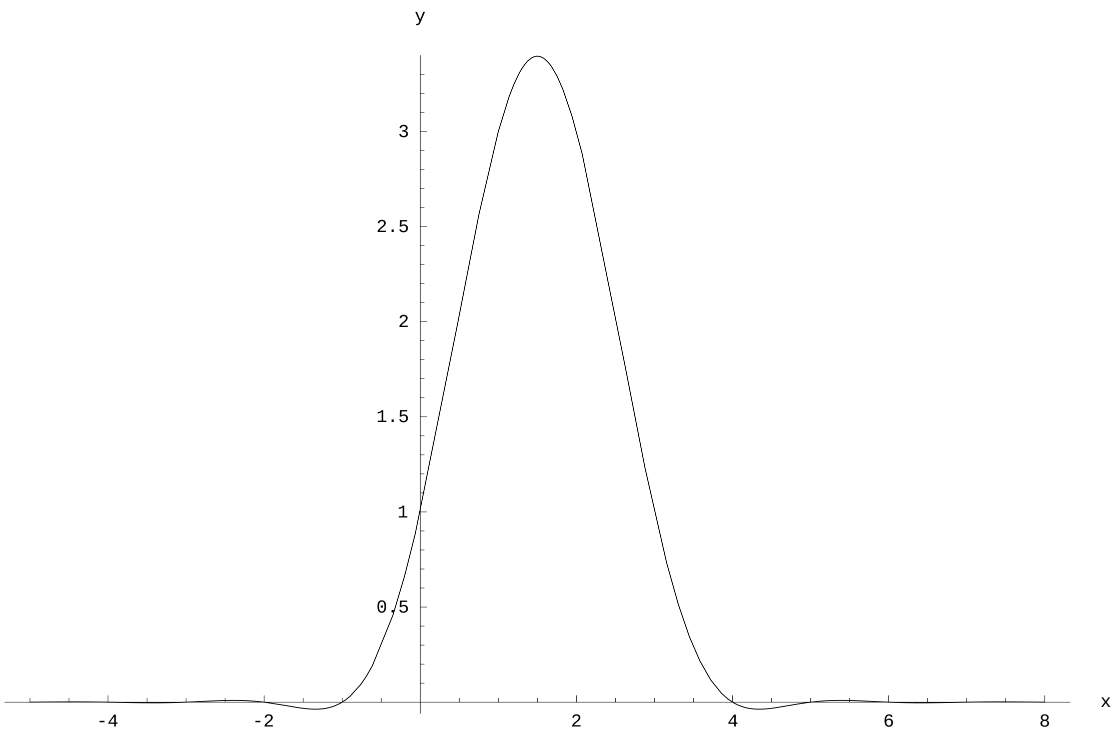

FIGURE 1. The graph of $y = \binom{3}{x}$

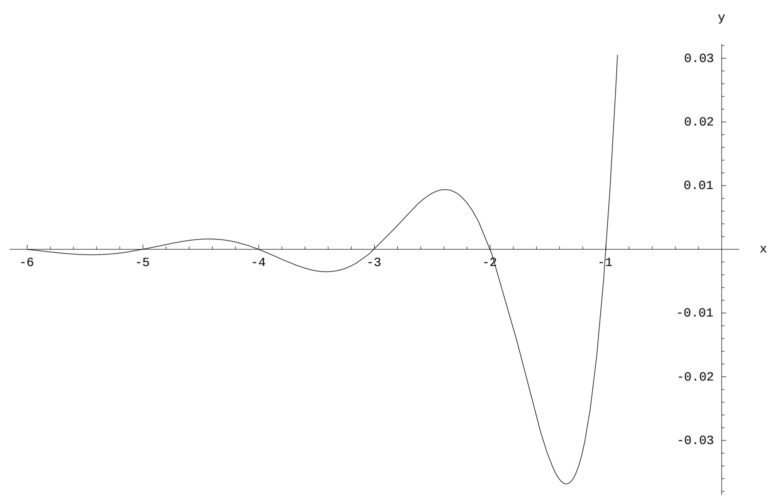

FIGURE 2. The left tail of the graph of $y = \binom{3}{x}$

We will use the following standard symbols in this paper:

Notation 1.2. $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers (including 0), integers, real numbers, and complex numbers, respectively.

2. Combinatorial identities involving $\binom{n}{\beta}$

In Chapter 5 of [6] the authors develop a number of identities for binomial coefficients of the form $\binom{\alpha}{k}$ as defined by (2). In this section we exhibit the analogous identities for coefficients of the form $\binom{n}{\beta}$. The terms for these identities correspond to those given in [6, p. 174].

Theorem 2.1. *Let $n \in \mathbb{N}$ and $\beta \in \mathbb{C}$. Then* *(i) Symmetry:*

$$\binom{n}{\beta} = \binom{n}{n-\beta} \tag{7}$$

(ii) Absorption/extraction:

$$\binom{n+1}{\beta} = \frac{n+1}{\beta}\binom{n}{\beta-1} \tag{8}$$

where $\beta \neq 0$,

(iii) Addition/induction:

$$\binom{n+1}{\beta} = \binom{n}{\beta} + \binom{n}{\beta-1} \tag{9}$$

(iv) Trinomial revision:

$$\binom{n}{\beta}\binom{\beta}{k} = \binom{n}{k}\binom{n-k}{\beta-k} \tag{10}$$

where $k \in \mathbb{Z}$ satisfies the condition $k \leq n$.

Proof. When $\beta \in \mathbb{Z}$ these identities are well-known (and are proved in [6], for example), so suppose $\beta \notin \mathbb{Z}$. We leave the proofs of (7), (8), and (9) to the reader and prove (10). We use the fact that

$$\sin \pi\,(\beta - k) = (-1)^k \sin \pi\beta \tag{11}$$

when $k \in \mathbb{Z}$.

So let $k \leq n$ be an integer. If $k < 0$ then $\binom{\beta}{k} = \binom{n}{k} = 0$ and so (10) holds. Suppose then that $0 \leq k \leq n$. We expand the left-hand side of (10) via (2) and (5):

$$\binom{n}{\beta}\binom{\beta}{k} = \frac{n!\sin \pi\beta}{\pi\beta(1-\beta)(2-\beta)\cdots(n-\beta)} \cdot \frac{\beta(\beta-1)(\beta-2)\cdots(\beta-k+1)}{k!}.$$

Multiplying by $\frac{(n-k)!}{(n-k)!}$, canceling β and regrouping, we have

$$\begin{aligned}
\binom{n}{\beta}\binom{\beta}{k} &= \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!\,(\beta-1)(\beta-2)\cdots(\beta-k+1)\sin \pi\beta}{\pi(1-\beta)(2-\beta)\cdots(n-\beta)} \\
&= \binom{n}{k} \cdot \frac{(n-k)!\,(-1)^{k-1}(1-\beta)(2-\beta)\cdots(k-1-\beta)\sin \pi\beta}{\pi(1-\beta)(2-\beta)\cdots(n-\beta)} \\
&= \binom{n}{k} \cdot \frac{(n-k)!\,(-1)^{k-1}\sin \pi\beta}{\pi(k-\beta)(k+1-\beta)\cdots(n-\beta)}. \\
&= \binom{n}{k} \cdot \frac{(n-k)!\,(-1)^{k}\sin \pi\beta}{\pi(\beta-k)\,[1-(\beta-k)]\,[2-(\beta-k)]\cdots[(n-k)-(\beta-k)]}.
\end{aligned}$$

By (11), the last expression above really is $\binom{n}{k}\binom{n-k}{\beta-k}$. □

The next set of basic binomial coefficient identities considered in [6] involve summation. Two of the corresponding identities for coefficients of the form $\binom{n}{\beta}$ appear in the following result.

Theorem 2.2. *Let $m, n \in \mathbb{N}$ and $\beta \in \mathbb{C}$. Then*

(i) Upper summation:

$$\sum_{0\le k\le n} \binom{k}{\beta} = \binom{n+1}{\beta+1} - \binom{0}{\beta+1} \tag{12}$$

(ii) Parallel summation:

$$\sum_{0\le k\le n} \binom{m+k}{\beta+k} = \binom{m+n+1}{\beta+n} - \binom{m}{\beta-1} \tag{13}$$

Proof. Both identities can be proved by combining induction on n with repeated use of (9). We also note that if $\beta \in \mathbb{Z} \setminus \{-1\}$ then $\binom{0}{\beta+1} = 0$, so (12) takes on the form given in [6]. (If $\beta = -1$ then (12) becomes $0 = 1 - 1$.) Similarly, taking $\beta = 0$ in (13) yields the version of parallel summation which appears in [6]. □

Perhaps the most important of the summation identities is *Vandermonde's convolution*:

$$\sum_{k=0}^{\infty} \binom{\alpha}{k}\binom{\gamma}{\delta-k} = \binom{\alpha+\gamma}{\delta}. \tag{14}$$

This is presented in [6] at first for $\alpha, \gamma \in \mathbb{R}$ and $\delta \in \mathbb{Z}$, but it is noted later that Gauss proved that the identity holds for arbitrary $\alpha, \gamma, \delta \in \mathbb{C}$ satisfying

$$\operatorname{Re}\alpha + \operatorname{Re}\gamma > -1,$$

where γ is not a negative integer unless δ is also an integer. (If $\delta \in \mathbb{Z}$ then there are no restrictions on α and γ.)

For example, when $\alpha = \gamma = \delta = \frac{1}{2}$, we have:

$$\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}\binom{\frac{1}{2}}{\frac{1}{2}-k} = \binom{1}{\frac{1}{2}}. \tag{15}$$

Now our formulas (2) and (5) do not enable us to calculate $\binom{\frac{1}{2}}{\frac{1}{2}-k}$, but we can appeal to the symmetry property implicit in the definition (3) to conclude that

$$\binom{\frac{1}{2}}{\frac{1}{2}-k} = \binom{\frac{1}{2}}{k}. \tag{16}$$

We should at this point warn the reader that symmetry doesn't always hold for general binomial coefficients, because of the complications involved in defining $\binom{\alpha}{\beta}$ when α is a negative integer. (For example, as noted in [6, p. 156], for any $k \in \mathbb{N}$ we have $\binom{-1}{k} = (-1)^k$ by (2), whereas $\binom{-1}{-1-k} = 0$ since $-1-k$ is a negative integer.) But (16) is perfectly valid.

Now (5) implies that

$$\binom{1}{\frac{1}{2}} = \frac{\sin\frac{\pi}{2}}{\pi\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \frac{4}{\pi},$$

so we can write (15) in the form

$$\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}^2 = \frac{4}{\pi}. \tag{17}$$

We leave it to the reader to show using (2) that

$$\binom{\frac{1}{2}}{k} = \frac{2(-1)^{k+1}}{k4^k}\binom{2k-2}{k-1}$$

for all $k \geq 1$. (The computational trick required to do this can be found in [6, p. 186].) Thus (17) is equivalent to the somewhat surprising formula

$$\sum_{k=0}^{\infty} \frac{4}{(k+1)^2\, 16^{k+1}}\binom{2k}{k}^2 = \frac{4}{\pi} - 1.$$

We can do much more by combining (5) with more general forms of Vandermonde's convolution. This idea is developed at length in [8].

3. Derivatives of $\binom{n}{z}$

In this section we use (5) to define, for any given $n \in \mathbb{N}$, a function $b_n(z)$ of the complex variable z:

$$b_n(z) = \binom{n}{z} = \begin{cases} \frac{n!\sin\pi z}{\pi z(1-z)(2-z)\cdots(n-z)}, & z \in \mathbb{C} \setminus \{0,1,2,\ldots,n\}, \\ \frac{n!}{z!(n-z)!}, & z \in \{0,1,2,\ldots,n\}. \end{cases} \tag{18}$$

In particular, when $n = 0$ we have

$$b_0(z) = \binom{0}{z} = \begin{cases} \frac{\sin\pi z}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{19}$$

This is the well-known function sinc z, or sinc πz; both definitions appear in the literature. (We don't need to commit ourselves here as to which definition we prefer, since we use the symbol $b_0(z)$ for this function.)

The Taylor series expansion for $b_0(z)$ around $z = 0$ is given by

$$b_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\pi z)^{2k}}{(2k+1)!},$$

which converges everywhere in the complex plane. Therefore $b_0(z)$ is an entire function. By (9) we can prove by induction on n that:

Proposition 3.1. *For every $n \in \mathbb{N}$, $b_n(z) = \binom{n}{z}$ is an entire function. Thus all of the derivatives of $b_n(z)$ exist and are continuous everywhere in $\mathbb{C}$.*

From this point on we won't need to bring in any more results about analytic functions. The reader who is unfamiliar with functions of a complex variable can take z to be a real variable for the remainder of this section. One can also bypass Proposition 3.1 by proving directly that the function $b_0(x)$ is continuous for all real x, and that its first and second derivatives are also continuous for all real x (in particular for $x = 0$). We won't need any derivatives beyond the second in this paper.

As for the actual computation of $b_n'(z)$, one option is to differentiate the expression $\frac{n!}{z!(n-z)!}$. This in turn entails differentiating the factorial function $z!$ (or equivalently, the gamma function $\Gamma(z)$). Unfortunately there is no simple formula for $\Gamma'(z)$, just as there is none for $\Gamma(z)$ itself; see [3] for more details. On the other hand, (18) is formulated in terms of elementary functions, so we can differentiate it directly, at least for $z \notin \mathbb{Z}$.

Theorem 3.2. *Let $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$. Then*

$$b_n'(z) = \left(\pi \cot \pi z - \frac{1}{z} - \frac{1}{z-1} - \cdots - \frac{1}{z-n} \right) b_n(z). \tag{20}$$

The proof is an exercise in calculus and is left to the reader. In the following theorem we deal with integer values of z.

Theorem 3.3. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.*

(i) If $k \in \mathbb{Z} \setminus \{0, 1, 2, \ldots, n\}$, then

$$b_n'(k) = \frac{(-1)^{n+k}}{(k-n)\binom{k}{n}}. \tag{21}$$

(ii) If $k \in \{0,1,2,\ldots,n\}$ and $k < \frac{n}{2}$, then

$$b_n'(k) = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{n-k}\right)\binom{n}{k}. \tag{22}$$

(iii) If $k \in \{0,1,2,\ldots,n\}$ and $k > \frac{n}{2}$, then

$$b_n'(k) = -\left(\frac{1}{n-k+1} + \frac{1}{n-k+2} + \cdots + \frac{1}{k}\right)\binom{n}{k}. \tag{23}$$

(iv) If n is even and $k = \frac{n}{2}$, then

$$b_n'(k) = 0. \tag{24}$$

Proof. Proposition 3.1 implies that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$b_n'(k) = \lim_{z\to k} b_n'(z)$$

Substituting (20) into this expression yields

$$b_n'(k) = \lim_{z\to k}\left(\pi\cot\pi z - \frac{1}{z} - \frac{1}{z-1} - \cdots - \frac{1}{z-n}\right)\binom{n}{z}. \tag{25}$$

(i) Suppose $k \in \mathbb{Z}\setminus\{0,1,2,\ldots,n\}$. Then (25) implies that

$$b_n'(k) = \lim_{z\to k}(\pi\cot\pi z)\binom{n}{z} - \lim_{z\to k}\left(\frac{1}{z} + \frac{1}{z-1} + \cdots + \frac{1}{z-n}\right)\lim_{z\to k}\binom{n}{z}.$$

Now $\lim_{z\to k}\left(\frac{1}{z} + \frac{1}{z-1} + \cdots + \frac{1}{z-n}\right)$ is finite and $\lim_{z\to k}\binom{n}{z} = \binom{n}{k} = 0$ since $k \in \mathbb{Z}\setminus\{0,1,2,\ldots,n\}$, so the above becomes

$$b_n'(k) = \lim_{z\to k}(\pi\cot\pi z)\binom{n}{z} = \lim_{z\to k}\frac{(\pi\cot\pi z)n!\sin\pi z}{\pi z(1-z)(2-z)\cdots(n-z)}$$
$$= \lim_{z\to k}\frac{n!(-1)^n\cos\pi z}{z(z-1)(z-2)\cdots(z-n)}\,.$$

This limit can be evaluated by simply substituting k for z, so

$$b_n'(k) = \frac{n!(-1)^n(-1)^k}{k(k-1)(k-2)\cdots(k-n)} = \frac{(-1)^{n+k}}{(k-n)\left[\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\right]}\,,$$

or

$$b_n'(k) = \lim_{z\to k}(\pi \cot \pi z)\binom{n}{z} = \frac{(-1)^{n+k}}{(k-n)\binom{k}{n}} \tag{26}$$

as required.

(ii) Suppose now that $k \in \{0, 1, 2, \ldots, n\}$ and $k < \frac{n}{2}$. This time (25) implies that

$$\begin{aligned} b_n'(k) = \lim_{z\to k}\Bigg(\left[\pi \cot \pi z - \frac{1}{z-k}\right] - \Bigg[\frac{1}{z-n} + \frac{1}{z-n+1} + \cdots \\ + \frac{1}{z-k-1} + \frac{1}{z-k+1} + \cdots + \frac{1}{z-1} + \frac{1}{z}\Bigg]\Bigg) \lim_{z\to k}\binom{n}{z}. \end{aligned} \tag{27}$$

We can write this as

$$\begin{aligned} b_n'(k) = \lim_{z\to k}\left[\pi \cot \pi z - \frac{1}{z-k}\right]\binom{n}{k} + \Bigg[\frac{1}{n-k} + \frac{1}{n-k-1} + \cdots \\ + \frac{1}{2} + 1 - 1 - \frac{1}{2} - \cdots - \frac{1}{k-1} - \frac{1}{k}\Bigg]\binom{n}{k}. \end{aligned} \tag{28}$$

Now since $\cot \pi z$ is periodic with period 1, we can compute the remaining limit in the above expression as follows:

$$\begin{aligned} \lim_{z\to k}\left[\pi \cot \pi z - \frac{1}{z-k}\right] = \lim_{z-k\to 0}\left[\pi \cot\left(\pi\left(z-k\right)\right) - \frac{1}{z-k}\right] = \\ = \lim_{z\to 0}\left[\pi \cot \pi z - \frac{1}{z}\right] = \lim_{z\to 0}\frac{\pi z \cos \pi z - \sin \pi z}{z \sin \pi z} = 0 \end{aligned} \tag{29}$$

by two applications of l'Hôpital's rule. Since $k < \frac{n}{2}$, (28) simplifies to (22).

(iii) The proof of (23) is identical, the only difference being in the form which equation (28) takes after simplifying.

(iv) The preceding remark applies here as well. Note that we could also use (7) to prove that (24) holds for arbitrary $n \in \mathbb{N}$. (For that matter, n need not be a natural number. We can take n to be any real number which is not a negative integer.) □

Similarly, we can compute the second derivative of $b_n(z)$. We again begin with the case where z is not an integer.

Theorem 3.4. *Let $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathbb{Z}$. Then*

$$b_n''(z) = \left[-\pi^2 - 2\pi \cot \pi z \left(\frac{1}{z} + \frac{1}{z-1} + \cdots + \frac{1}{z-n} \right) + \left(\frac{1}{z} + \frac{1}{z-1} + \cdots \right.\right.$$

$$\left.\left. + \frac{1}{z-n} \right)^2 + \left(\frac{1}{z^2} + \frac{1}{(z-1)^2} + \cdots + \frac{1}{(z-n)^2} \right) \right] b_n(z). \tag{30}$$

This follows directly from (20) and the identity $\cot^2 \pi z + 1 = \csc^2 \pi z$. The corresponding result for $z = k \in \mathbb{Z}$ is:

Theorem 3.5. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.*

(i) If $k \in \mathbb{Z} \setminus \{0, 1, 2, \ldots, n\}$, then

$$b_n''(k) = \frac{2 \cdot (-1)^{n+k+1}}{(k-n)\binom{k}{n}} \left(\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{k-n} \right). \tag{31}$$

(ii) If $k \in \{0, 1, 2, \ldots, n\}$ and $k \leq \frac{n}{2}$, then

$$b_n''(k) = 2 \left[\frac{-\pi^2}{6} + \sum_{i=1}^{n-k} \frac{1}{i^2} + \sum_{k<i<j\leq n-k} \frac{1}{ij} \right] \binom{n}{k}. \tag{32}$$

(iii) If $k \in \{0, 1, 2, \ldots, n\}$ and $k \geq \frac{n}{2}$, then

$$b_n''(k) = 2 \left[\frac{-\pi^2}{6} + \sum_{i=1}^{k} \frac{1}{i^2} + \sum_{n-k<i<j\leq k} \frac{1}{ij} \right] \binom{n}{k}. \tag{33}$$

(The first sum in (32) is 0 when $k = n$, and the first sum in (33) is 0 when $k = 0$. The second sums in (32) and in (33) are 0 when $|n - 2k| \leq 1$.)

Proof. As in the proof of Theorem 3.3, we again appeal to Proposition 3.1 to conclude that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$b_n''(k) = \lim_{z \to k} b_n''(z).$$

This time we substitute (30) into this expression to get

$$b_n''(k) = \lim_{z \to k} \left[-\pi^2 + \left(\frac{1}{z} + \cdots + \frac{1}{z-n} \right)^2 + \frac{1}{z^2} + \frac{1}{(z-1)^2} + \cdots + \frac{1}{(z-n+1)^2} \right.$$

$$+\frac{1}{(z-n)^2}\Bigg]\binom{n}{z}+\lim_{z\to k}\left(-2\pi\cot\pi z\left(\frac{1}{z}+\cdots+\frac{1}{z-n}\right)\right)\binom{n}{z} \tag{34}$$

(*i*) Suppose $k\in\mathbb{Z}\setminus\{0,1,2,\ldots,n\}$, so $\binom{n}{k}=0$. Then the expression in square brackets in the above equation tends to a finite limit as $z\to k$, and $\lim_{z\to k}\binom{n}{z}=\binom{n}{k}=0$, so the first summand in (34) is 0. Thus $b''_n(k)$ equals the second summand, i.e.

$$b''_n(k)=-2\left(\frac{1}{k}+\frac{1}{k-1}+\cdots+\frac{1}{k-n}\right)\lim_{z\to k}(\pi\cot\pi z)\binom{n}{z}.$$

The result now follows from (26).

(*ii*) Suppose now that $k\in\{0,1,2,\ldots,n\}$ and $k\le\frac{n}{2}$. In this case we can rewrite (34) by treating the expression $\left(\frac{1}{z}+\frac{1}{z-1}+\cdots+\frac{1}{z-n}\right)$ as a binomial in which the first summand is $\frac{1}{z-k}$ and the second summand is the sum of the remaining terms.

$$\begin{aligned}b''_n(k)=\lim_{z\to k}\Bigg[&-\pi^2+\frac{1}{(z-k)^2}+\frac{2}{z-k}\left(\frac{1}{z}+\frac{1}{z-1}+\cdots+\frac{1}{z-k+1}+\right.\\&\left.+\frac{1}{z-k-1}+\cdots+\frac{1}{z-n}\right)+\left(\frac{1}{z}+\cdots+\frac{1}{z-k+1}+\frac{1}{z-k-1}+\cdots\right.\\&\left.+\frac{1}{z-n}\right)^2+\frac{1}{(z-k)^2}+\frac{1}{z^2}+\cdots+\frac{1}{(z-k+1)^2}+\frac{1}{(z-k-1)^2}+\cdots\\&+\frac{1}{(z-n)^2}+\frac{(-2\pi\cot\pi z)}{z-k}-2\pi\cot\pi z\left(\frac{1}{z}+\frac{1}{z-1}+\cdots+\frac{1}{z-k+1}+\right.\\&\left.+\frac{1}{z-k-1}+\frac{1}{z-k-2}+\cdots+\frac{1}{z-n+1}+\frac{1}{z-n}\right)\Bigg]\binom{n}{z}.\end{aligned}$$

Regrouping terms and breaking up the sum into separate limits, we have

$$\begin{aligned}b''_n(k)=\lim_{z\to k}\Bigg[&\left(\frac{2}{z-k}-2\pi\cot\pi z\right)\left(\frac{1}{z}+\cdots+\frac{1}{z-k+1}+\frac{1}{z-k-1}+\cdots\right.\\&\left.+\frac{1}{z-n}\right)\Bigg]\binom{n}{z}+\lim_{z\to k}\left[\frac{2}{(z-k)^2}-\frac{2\pi\cot\pi z}{z-k}\right]\binom{n}{z}+\lim_{z\to k}\Bigg[-\pi^2+\left(\frac{1}{z}+\cdots\right.\\&\left.+\frac{1}{z-k+1}+\frac{1}{z-k-1}+\cdots+\frac{1}{z-n}\right)^2+\frac{1}{z^2}+\cdots+\frac{1}{(z-k+1)^2}+\cdots\\&+\frac{1}{(z-k-1)^2}+\cdots+\frac{1}{(z-n+1)^2}+\frac{1}{(z-n)^2}\Bigg]\binom{n}{z}.\end{aligned} \tag{35}$$

The first limit in (35) is 0, by (29) and the fact that $\lim_{z\to k}\left(\frac{1}{z}+\cdots\right.$ $\left.+\frac{1}{z-k+1}+\frac{1}{z-k-1}+\cdots+\frac{1}{z-n}\right)$ and $\binom{n}{k}$ are finite.

Because $\cot \pi z$ is periodic with period 1, the second limit in (35) is equal to

$$2\lim_{z\to k}\left[\frac{1}{(z-k)^2}-\frac{\pi\cot\pi(z-k)}{z-k}\right]\lim_{z\to k}\binom{n}{z}=2\lim_{z\to 0}\left[\frac{1}{z^2}-\frac{\pi\cot\pi z}{z}\right]\binom{n}{k}=$$

$$=2\lim_{z\to 0}\left[\frac{\sin\pi z-\pi z\cos\pi z}{z^2\sin\pi z}\right]\binom{n}{k}=\frac{2\pi^2}{3}\binom{n}{k}$$

by two applications of l'Hôpital's rule.

The third limit in (35) can be computed by simply substituting k for z. We thus have

$$b_n''(k)=\left[\frac{2\pi^2}{3}-\pi^2+\left(\frac{1}{k}+\frac{1}{k-1}+\cdots+1-1-\frac{1}{2}-\cdots-\frac{1}{n-k}\right)^2+\frac{1}{k^2}+\right.$$

$$\left.+\frac{1}{(k-1)^2}+\cdots+\frac{1}{1^2}+\frac{1}{(-1)^2}+\frac{1}{(-2)^2}+\cdots+\frac{1}{(k-n)^2}\right]\binom{n}{k}. \tag{36}$$

Now we finally use the assumption that $k\le\frac{n}{2}$ to conclude that (32) holds.

(*iii*) Suppose that $k\in\{0,1,2,\ldots,n\}$ and $k\ge\frac{n}{2}$. The computation from the previous case is still valid, right up through (36). This time the inequality $k\ge\frac{n}{2}$ implies that (36) reduces to (33).

□

In Section 5 we will show that (32) enables us to approximate the two points of inflection of the curve $y=b_n(x)$ on the interval $(-1,n+1)$ by $x=\frac{n-\sqrt{n+1}}{2}$ and (by symmetry) $x=\frac{n+\sqrt{n+1}}{2}$.

4. The function $b_n(x)=\binom{n}{x}$ for x real

Let $n\in\mathbb{N}$. We wish to investigate the behavior of the restriction of the function $b_n(z)$ (defined in (20)) to the real line. We will abuse notation and refer to the restriction also as b_n; more precisely, we will write $b_n(x)$, where x is understood to be a *real* variable.

Our method will be to approach the investigation of the function $y=b_n(x)$ as a curve-sketching problem in the spirit of elementary calculus, making heavy use of the results of the previous section. (Figures 1 and 2 in the introduction to this paper give us some advance information regarding the shape of the curve.)

So for given $n\in\mathbb{N}$, we analyze the function

$$(37) \qquad y = b_n(x) = \binom{n}{x} = \begin{cases} \frac{n!\sin \pi x}{\pi x(1-x)(2-x)\cdots(n-x)}, & x \in \mathbb{R} \setminus \{0,1,2,\ldots,n\}, \\ \\ \frac{n!}{x!(n-x)!}, & x \in \{0,1,2,\ldots,n\}. \end{cases}$$

(a)Domain of definition, continuity and differentiability: It follows from Proposition 3.1 that $b_n(x)$ and all of its derivatives are defined and continuous everywhere in $\mathbb{R}$.

(b)Intercepts: Clearly $b_n(0) = 1$. As for the x-intercepts, i.e. the zeros of $b_n(x)$, it is immediate from (37) that $b_n(x) = 0 \;\Leftrightarrow\; x \in \mathbb{Z} \setminus \{0,1,2,\ldots,n\}$.

(c)Symmetry: From (7) we conclude that $x = \frac{n}{2}$ is the axis of symmetry for the curve $y = b_n(x)$.

(d)Asymptotes: Obviously there are no vertical asymptotes. It is also clear from (37) that $\lim_{x\to\infty} b_n(x) = \lim_{x\to-\infty} b_n(x) = 0$, so the x-axis is the horizontal asymptote in both the positive and negative directions.

(e)Sign of $b_n(x)$: This can also be determined completely from (37):

Proposition 4.1. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then*

- *(i)* $b_n(x) > 0$ *in exactly the following cases:*
  - *(a)* $x \in (-1, n+1)$,
  - *(b)* $x \in (-2m-1, -2m)$ *for some positive integer* m,
  - *(c)* $x \in (n+2m, n+2m+1)$ *for some positive integer* m.
- *(ii)* $b_n(x) < 0$ *in exactly the following cases:*
  - *(a)* $x \in (-2m-2, -2m-1)$ *for some* $m \in \mathbb{N}$,
  - *(b)* $x \in (n+2m+1, n+2m+2)$ *for some* $m \in \mathbb{N}$.

(f)Local extrema, regions of increase and decrease: It is clear from Figure 2 that the results here are going to be complicated. We record them as:

Theorem 4.2. *Let $n \in \mathbb{N}$. There exists an infinite sequence $\{\delta_k\}_{k=1}^{\infty}$ of real numbers δ_k (which should really carry a second index to indicate dependence on n, but we will suppress this) such that $0 < \delta_k < 1$ for each k, and such that for all $x \in \mathbb{R}$,*

- *(i)* $b_n'(x) = 0 \Leftrightarrow x = \frac{n}{2}$ *or* $x = -k - \delta_k$ *for some* $k \geq 1$ *or* $x = n + k + \delta_k$ *for some* $k \geq 1$.
- *(ii)* $b_n'(x) > 0 \Leftrightarrow x \in \left(-1-\delta_1, \frac{n}{2}\right)$ *or* $x \in \left(-(2m+1) - \delta_{2m+1}, -2m - \delta_{2m}\right)$ *for some* $m \geq 1$ *or* $x \in \left(n+2m-1+\delta_{2m-1}, n+2m+\delta_{2m}\right)$ *for some* $m \geq 1$. *These are the intervals on which $b_n(x)$ increases.*

(iii) $b_n'(x) < 0 \Leftrightarrow x \in \left(\frac{n}{2}, n+1+\delta_1\right)$ *or* $x \in (-2m-\delta_{2m}, -(2m-1)-\delta_{2m-1})$ *for some* $m \geq 1$ *or* $x \in (n+2m+\delta_{2m}, n+2m+1+\delta_{2m+1})$ *for some* $m \geq 1$. *These are the intervals on which* $b_n(x)$ *decreases.*

(iv) The function $b_n(x)$ *has local maxima at the points* $x = \frac{n}{2}$, $x = -2m - \delta_{2m}$ *for* $m \geq 1$, *and* $x = n + 2m + \delta_{2m}$ *for* $m \geq 1$.

(v) The function $b_n(x)$ *has local minima at the points* $x = -(2m-1) - \delta_{2m-1}$ *for* $m \geq 1$, *and* $x = n + 2m - 1 + \delta_{2m-1}$ *for* $m \geq 1$.

(vi) The point $x = \frac{n}{2}$ *is the global maximum for* $b_n(x)$ *, and the points* $x = -1 - \delta_1$ *and* $x = n + 1 + \delta_1$ *are the global minima.*

Proof. We prove (*i*), (*ii*), and (*iii*) together, using Theorems 3.2 and 3.3. Theorem 3.3 shows that the values of $b_n'(x)$ for $x \in \mathbb{Z}$ are consistent with the claims in Theorem 4.2, so we turn our attention to non-integer values of x. Proposition 4.1 provides us with the sign of $b_n(x)$, so our task is to establish the sign of the expression

$$f_n(x) = \pi \cot \pi x - \frac{1}{x} - \frac{1}{x-1} - \cdots - \frac{1}{x-n} \tag{38}$$

and then use (20). Since we have seen that $x = \frac{n}{2}$ is the axis of symmetry for the curve $y = b_n(x)$, we can confine our attention to non-integer x such that $x \leq \frac{n}{2}$.

Let us first consider the case where $x \in (-k-1, -k)$, k an arbitrary positive integer. Differentiating (38) yields

$$f_n'(x) = -\pi^2 \csc^2 \pi x + \frac{1}{x^2} + \frac{1}{(x-1)^2} + \cdots + \frac{1}{(x-n)^2} \tag{39}$$

which holds for all $x \in (-k-1, -k)$. On this interval we also have $\csc^2 \pi x \geq 1$ and

$$\frac{1}{x^2} + \frac{1}{(x-1)^2} + \cdots + \frac{1}{(x-n)^2} \leq \frac{1}{k^2} + \frac{1}{(k+1)^2} + \cdots + \frac{1}{(k+n)^2} \leq \frac{\pi^2}{6}. \tag{40}$$

The last inequality in (40) is a result of Euler's well-known formula

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}. \tag{41}$$

(See [6, p. 286], for example.) Substituting into (39), we conclude that

$$f_n'(x) \leq -\pi^2 + \frac{\pi^2}{6} = -\frac{5\pi^2}{6} < 0$$

for all $x \in (-k-1, -k)$, which means that $f_n(x)$ is monotonically decreasing on that interval.

Now

$$\lim_{x\to(-k-1)^+} f_n(x) = \lim_{x\to(-k-1)^+} \left(\pi\cot\pi x - \frac{1}{x} - \frac{1}{x-1} - \cdots - \frac{1}{x-n}\right) = +\infty \tag{42}$$

and

$$\lim_{x\to(-k)^-} f_n(x) = \lim_{x\to(-k)^-} \left(\pi\cot\pi x - \frac{1}{x} - \frac{1}{x-1} - \cdots - \frac{1}{x-n}\right) = -\infty. \tag{43}$$

We conclude from the Intermediate Value Theorem that there is a zero for f_n in the interval $(-k-1,-k)$; that is, there is a $\delta_k \in \mathbb{R}$ such that $0 < \delta_k < 1$ and $f_n(-k-\delta_k) = 0$. Since f_n decreases monotonically on $(-k-1,-k)$, we have that

$$f_n(x) > 0 \text{ for } x \in (-k-1, -k-\delta_k) \tag{44}$$

and

$$f_n(x) < 0 \text{ for } x \in (-k-\delta_k, -k)\,. \tag{45}$$

Now

$$b_n'(x) = f_n(x) b_n(x) \text{ for } x \in \mathbb{R}\setminus\mathbb{Z}, \tag{46}$$

so by formulas (44) and (45) and Proposition 4.1, we conclude that if k is even, then

$$b_n'(x) > 0 \text{ for } x \in (-k-1, -k-\delta_k)\,, \text{ and}$$

$$b_n'(x) < 0 \text{ for } x \in (-k-\delta_k, -k)\,.$$

If k is odd, then

$$b_n'(x) < 0 \text{ for } x \in (-k-1, -k-\delta_k)\,, \text{ and}$$

$$b_n'(x) > 0 \text{ for } x \in (-k-\delta_k, -k)\,.$$

We have proved the desired result for all real x except for $x \in (-1, n+1)$. By (7) it suffices to consider $x \in \left(-1, \frac{n}{2}\right)$. Again we wish to establish the sign of $f_n(x)$ in (38), but this time the denominators are no longer all conveniently nonzero.

We circumvent this problem by appealing to (6) in order to rewrite (38) in the form

$$f_n(x) = \frac{1}{x+1} + \frac{1}{x+2} + \cdots + \frac{1}{x+n} + \sum_{j=n+1}^{\infty}\left(\frac{1}{x+j} + \frac{1}{x-j}\right). \tag{47}$$

The series

$$\sum_{j=n+1}^{\infty}\left(\frac{1}{x+j}+\frac{1}{x-j}\right)=\sum_{j=n+1}^{\infty}\frac{2x}{x^2-j^2}$$

is absolutely convergent, and since the number of summands in each term $\left(\frac{1}{x+j}+\frac{1}{x-j}\right)$ is fixed and $\lim_{j\to\infty}\frac{1}{x+j}=\lim_{j\to\infty}\frac{1}{x-j}=0$ for any $x\in\mathbb{R}$, we can dispense with the parentheses in (47) and write

$$f_n(x)=\frac{1}{x+1}+\frac{1}{x+2}+\cdots+\frac{1}{x+n}+\frac{1}{x+n+1}+\frac{1}{x-n-1}+$$

$$+\frac{1}{x+n+2}+\frac{1}{x-n-2}+\frac{1}{x+n+3}+\frac{1}{x-n-3}+\cdots. \tag{48}$$

We would like to rearrange the terms on the right-hand side of (48), but since that series is now only conditionally convergent we have to be careful. Fortunately, in our intended rearrangement, each term which is shifted forward is shifted at most n places, so Lemma 1.1 applies and we can conclude that

$$f_n(x)=\left(\frac{1}{x+1}+\frac{1}{x-(n+1)}\right)+\left(\frac{1}{x+2}+\frac{1}{x-(n+2)}\right)$$

$$+\left(\frac{1}{x+3}+\frac{1}{x-(n+3)}\right)+\left(\frac{1}{x+4}+\frac{1}{x-(n+4)}\right)+\cdots \tag{49}$$

For $x\in\left(-1,\frac{n}{2}\right)$, each summand $\frac{1}{x+j}+\frac{1}{x-(n+j)}=\frac{n-2x}{(x+j)(n+j-x)}$ in (49) is positive, so $f_n(x)$ is positive on this entire interval. By Proposition 4.1, $b_n(x)>0$ on this interval as well, so $b_n'(x)=f_n(x)b_n(x)>0$.

This completes the proof of (*i*), (*ii*), and (*iii*), from which (*iv*) and (*v*) follow immediately. As for (*vi*), still taking $x\le\frac{n}{2}$, it follows from (*ii*) and (*iii*) that $b_n(x)\ge b_n(-1-\delta_1)$ for all $x\in\left(-2-\delta_2,\frac{n}{2}\right)$. Next we note that for $x\in(-\infty,-2]$ we have

$$|b_n(x)|\le\left|\frac{n!}{\pi x(1-x)\cdots(n-x)}\right|\le\frac{n!}{\pi\cdot 2\cdot 3\cdots(n+2)}=\frac{n!}{\pi(n+2)!}. \tag{50}$$

In order to prove that

$$b_n(x)\ge b_n(-1-\delta_1)\text{ for all }x\in(-\infty,-2],$$

then, it suffices to show that

$$b_n(-1-\delta_1)\le\frac{-n!}{\pi(n+2)!}. \tag{51}$$

And since $x = -1 - \delta_1$ is a local minimum for $b_n(x)$ on the interval $\left(-2 - \delta_2, \frac{n}{2}\right)$, we can prove (51) by showing that the equation

$$b_n(x) = \frac{n! \sin \pi x}{\pi x(1-x)(2-x)\cdots(n-x)} = \frac{-n!}{\pi(n+2)!}$$

has a solution in the interval $(-2, -1)$. In fact it has two solutions in that interval, since $b_n(-1) = b_n(-2) = 0$ and

$$b_n(-3/2) = \frac{-n! \sin\left(\frac{-3\pi}{2}\right)}{\pi(\frac{3}{2})(1+\frac{3}{2})(2+\frac{3}{2})\cdots(n+\frac{3}{2})} < \frac{-n!}{\pi \cdot 2 \cdot 3 \cdots (n+2)}\,.$$

We have thus shown that $x = -1 - \delta_1$ is the global minimum for $b_n(x)$ for x restricted to the interval $(-\infty, n/2)$. By (7) we conclude that $x = -1 - \delta_1$ and $x = n + 1 + \delta_1$ are the global minima for $b_n(x)$.

In order to show that $x = \frac{n}{2}$ is the global maximum for $b_n(x)$, it suffices by the above considerations regarding symmetry, the sign of $b_n(x)$, and regions of increase and decrease, to show that $b_n(\frac{n}{2}) > b_n(x)$ for all $x \in (-\infty, -2]$. But for such x we have by (50) that

$$|b_n(x)| \leq \frac{n!}{\pi(n+2)!} < 1 \leq b_n\left(\frac{n}{2}\right)\,.$$

□

The remaining stage in our investigation of $b_n(x)$ involves determining the sign of the second derivative.

(g)Concavity, convexity, and points of inflection: The analogous result to Theorem 4.2 is the following:

Theorem 4.3. *Let $n \in \mathbb{N}$. There exists an $a_0 \in \mathbb{R}$ and an infinite sequence $\{\epsilon_k\}_{k=1}^{\infty}$ of real numbers ϵ_k (again we refrain from adding a second index indicating dependence on n) such that $-1 < a_0 < \frac{n}{2}$, $0 < \epsilon_k < 1$ for each k, and such that for all $x \in \mathbb{R}$,*

(i) $b_n''(x) = 0 \Leftrightarrow x = a_0$ *or* $x = n - a_0$ *or* $x = -k - \epsilon_k$ *for some* $k \geq 1$ *or* $x = n + k + \epsilon_k$ *for some* $k \geq 1$. *These are the points of inflection of the graph of* $y = b_n(x)$.

(ii) $b_n''(x) > 0 \Leftrightarrow x \in (-1 - \epsilon_1, a_0)$ *or* $x \in (a_0, n + 1 + \epsilon_1)$ *or* $x \in (-(2m+1) - \epsilon_{2m+1}, -2m - \epsilon_{2m})$ *for some* $m \geq 1$ *or* $x \in (n + 2m + \epsilon_{2m}, n + 2m + 1 + \epsilon_{2m+1})$ *for some* $m \geq 1$. *These are the intervals on which the graph of* $y = b_n(x)$ *is concave up.*

(iii) $b_n''(x) < 0 \Leftrightarrow x \in (a_0, n - a_0)$ *or* $x \in (-2m - \epsilon_{2m}, -(2m-1) - \epsilon_{2m-1})$ *for some* $m \geq 1$ *or* $x \in (n + 2m - 1 + \epsilon_{2m-1}, n + 2m + \epsilon_{2m})$ *for some* $m \geq 1$. *These are the intervals on which the graph of* $y = b_n(x)$ *is concave down.*

Proof. We deal with all three clauses simultaneously. As in the previous theorem, it suffices to consider non-integer values of x such that $x \leq \frac{n}{2}$. Given $n \in \mathbb{N}$, we define the function

$$g_n(x) = -\pi^2 - 2\pi \cot \pi x \left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-n}\right) + \left(\frac{1}{x} + \frac{1}{x-1} + \cdots \right.$$

$$\left. + \frac{1}{x-n}\right)^2 + \frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x-2)^2} + \cdots + \frac{1}{(x-n)^2}. \tag{52}$$

Thus, for $x \in \mathbb{R} \setminus \mathbb{Z}$, we have

$$b_n''(x) = g_n(x) b_n(x) \tag{53}$$

by (30). We prove:

Claim 4.4. *Let $n \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, the function $g_n(x)$ is strictly decreasing on the interval $(-k-1, -k)$. Furthermore, $\lim_{x \to (-k-1)^+} g_n(x) = \infty$ for all $k \in \mathbb{N}$ and $\lim_{x \to -k^-} g_n(x) = -\infty$ for all $k \geq 1$. On the other hand, $\lim_{x \to 0^-} g_n(x)$ is finite.*

Proof. The last statement is the easiest to prove: For any $n \in \mathbb{N}$, $\lim_{x \to 0^-} g_n(x)$ is finite by (32), (53), and the fact that $b_n(0) = 1$. We prove the remaining statements by induction on n.

Taking $n = 0$ in (52), we have

$$g_0(x) = -\pi^2 - \frac{1}{x} 2\pi \cot \pi x + \frac{1}{x^2} + \frac{1}{x^2}.$$

So

$$g_0(x) = -\pi^2 + \frac{1}{x^2} + \frac{1}{x}\left(\frac{1}{x} - 2\pi \cot \pi x\right).$$

From (6) we conclude that

$$g_0(x) = -\pi^2 + \frac{1}{x^2} + \frac{1}{x}\left(\frac{1}{x} - \left[\frac{2}{x} + \sum_{j=1}^{\infty} \frac{4x}{x^2 - j^2}\right]\right).$$

This simplifies to

$$g_0(x) = -\pi^2 - \sum_{j=1}^{\infty} \frac{4}{x^2 - j^2}. \tag{54}$$

Each summand $\frac{4}{x^2 - j^2}$ has a strictly decreasing denominator on the interval $(-k-1, -k)$, so each summand is strictly increasing on that interval (both for $j \leq k$ and for $j > k$). Thus $\sum_{j=1}^{\infty} \frac{4}{x^2-j^2}$ is strictly increasing on $(-k-1,-k)$, and therefore $g_0(x)$ is strictly decreasing on that interval.

It also follows from (54) that $\lim_{x\to(-k-1)^+} g_0(x) = \infty$ for all $k \in \mathbb{N}$ and $\lim_{x\to -k^-} g_0(x) = -\infty$ for all $k \geq 1$. (Here we need to use the fact that the sums $\sum_{j\geq 1, j\neq k+1} \frac{4}{(k+1)^2-j^2}$ and $\sum_{j\geq 1, j\neq k} \frac{4}{k^2-j^2}$ are finite.)

Assuming now that the claim holds for n, we prove it for $n+1$. From (52) we have

$$g_{n+1}(x) = g_n(x) + \frac{1}{(x-n-1)^2} + \left(\frac{2}{x} + \frac{2}{x-1} + \cdots + \frac{2}{x-n} - 2\pi\cot\pi x\right)\frac{1}{x-n-1} + \frac{1}{(x-n-1)^2},$$

and so

$$g_{n+1}(x) = g_n(x) + \frac{2}{x-n-1}\left(\frac{1}{x} + \frac{1}{x-1} + \cdots + \frac{1}{x-n-1} - \pi\cot\pi x\right).$$

Using (38), we can write this as

$$g_{n+1}(x) = g_n(x) - \frac{2}{x-n-1} f_{n+1}(x). \tag{55}$$

Let $k \in \mathbb{N}$. Then $g_n(x)$ is strictly decreasing on the interval $(-k-1,-k)$ by induction hypothesis. As for the second term on the right-hand side of (55), Lemma 1.1 enables us to rearrange terms and write the formula

$$f_{n+1}(x) = \left(\frac{1}{x+1} + \frac{1}{x-(n+2)}\right) + \left(\frac{1}{x+2} + \frac{1}{x-(n+3)}\right) + \left(\frac{1}{x+3} + \frac{1}{x-(n+4)}\right) + \left(\frac{1}{x+4} + \frac{1}{x-(n+5)}\right) + \cdots,$$

obtained from (49), in the equivalent form

$$f_{n+1}(x) = \left[\frac{1}{x-n-2} + \frac{1}{x-n-3} + \cdots + \frac{1}{x-2n-2}\right] + \left(\frac{1}{x+1} + \frac{1}{x-2n-3}\right) + \left(\frac{1}{x+2} + \frac{1}{x-2n-4}\right) + \left(\frac{1}{x+3} + \frac{1}{x-2n-5}\right) + \cdots$$

which can in turn be rewritten as

$$f_{n+1}(x) = \left[\frac{1}{x-n-2} + \frac{1}{x-n-3} + \cdots + \frac{1}{x-2n-2}\right] + \frac{2x-2n-2}{(x+1)(x-2n-3)} + \frac{2x-2n-2}{(x+2)(x-2n-4)} + \frac{2x-2n-2}{(x+3)(x-2n-5)} + \cdots$$

and therefore

$$\frac{1}{x-n-1}f_{n+1}(x) = \frac{1}{(x-n-1)(x-n-2)} + \frac{1}{(x-n-1)(x-n-3)} + \cdots$$

$$+\frac{1}{(x-n-1)(x-2n-2)} + \sum_{i=1}^{\infty}\frac{2}{(x+i)(x-2n-2-i)}. \tag{56}$$

It is a simple calculus exercise to show that if r and s are real numbers with $r < s$, then the function $\frac{1}{(x-r)(x-s)}$ increases on the intervals $(-\infty, r)$ and $\left(r, \frac{r+s}{2}\right)$, and it decreases elsewhere. One way to see this is to write

$$\frac{1}{(x-r)(x-s)} = \frac{1}{s-r}\left(\frac{1}{x-s} - \frac{1}{x-r}\right). \tag{57}$$

Differentiating now yields the required result.

Thus, in particular, each summand $\frac{1}{(x-n-1)(x-n-i)}$ and $\frac{2}{(x+i)(x-2n-2-i)}$ on the right hand side of (56) increases on the interval $(-k-1, -k)$, so $\frac{1}{x-n-1}f_{n+1}(x)$ is an increasing function on $(-k-1, -k)$. Therefore $\frac{-2}{x-n-1}f_{n+1}(x)$ is strictly decreasing on $(-k-1, -k)$, and by induction hypothesis the same holds for $g_n(x)$. From (55) it follows that $g_{n+1}(x)$ is an increasing function on $(-k-1, -k)$.

Finally, by induction hypothesis we also have $\lim_{x\to(-k-1)^+} g_n(x) = \infty$ for all $k \in \mathbb{N}$ and $\lim_{x\to -k^-} g_n(x) = -\infty$ for all $k \geq 1$. In addition, (42) and (43) hold with $n+1$ in place of n. Since $\frac{-2}{x-n-1}$ is a bounded positive-valued function on $(-k-1, -k)$, we conclude that

$$\lim_{x\to(-k-1)^+} \frac{-2}{x-n-1}f_n(x) = \infty$$

and

$$\lim_{x\to -k^-} \frac{-2}{x-n-1}f_n(x) = -\infty.$$

Thus $\lim_{x\to(-k-1)^+} g_{n+1}(x) = \infty$ for all $k \in \mathbb{N}$ and $\lim_{x\to -k^-} g_{n+1}(x) = -\infty$ for all $k \geq 1$, by (55). This completes the proof of Claim 4.4. $\square$

We now proceed as in the proof of Theorem 4.2, using Claim 4.4 in place of (42) and (43), and replacing δ_k by ϵ_k. This proves Theorem 4.3 for $x < -1$ (and symmetrically for $x > n+1$).

Next we turn our attention to $x \in \left(-1, \frac{n}{2}\right]$. The definition of $g_n(x)$ given in (52) is unwieldy in this interval because of all the removable singularities at integer points. To find a more tractable expression for $g_n(x)$, recall that we defined $f_n(x)$ so that (46) holds. Differentiating both sides of (46), we have

$$b_n''(x) = f_n(x)b_n'(x) + f_n'(x)b_n(x) = f_n(x)f_n(x)b_n(x) + f_n'(x)b_n(x),$$

where the second equation follows from (46). Thus we have

$$b_n''(x) = \left[f_n^2(x) + f_n'(x)\right] b_n(x).$$

Comparing this with (53), we conclude that

$$g_n(x) = f_n^2(x) + f_n'(x). \tag{58}$$

Instead of using the definition of $f_n(x)$ given in (38), we will work with the series expansion in (49). If we rewrite this as

$$f_n(x) = \sum_{i=1}^{\infty} \frac{2x-n}{(x+i)(x-n-i)}, \tag{59}$$

then we have a series expression which is defined for all $x \in (-1, n+1)$ (as well as for all $x \in \mathbb{R} \setminus \mathbb{Z}$); moreover the series converges absolutely for any such x. This means that we can multiply this series by itself to obtain a series for $f_n^2(x)$, which also converges absolutely. (See [1, Theorem 8.44], for example.) Thus we have

$$f_n^2(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{2x-n}{(x+i)(x-n-i)} \cdot \frac{2x-n}{(x+j)(x-n-j)}. \tag{60}$$

Next we find a series for the second summand in (58).

Claim 4.5. *Let $n \in \mathbb{N}$ and $x \in (-1, n+1) \cup (\mathbb{R} \setminus \mathbb{Z})$. Then*

$$f_n'(x) = \sum_{i=1}^{\infty} \left[\frac{-1}{(x+i)^2} + \frac{-1}{(x-n-i)^2}\right]. \tag{61}$$

Proof. This formula results from differentiating (49) term by term. We are primarily interested in the case where $x \in (-1, n+1)$, so we fix such an x. Let a and b be such that $-1 < a < x < b < n+1$. In order to prove that the sum in (61) equals $f_n'(x)$, it suffices to show that this series converges uniformly on (a, b). (See [1, Theorem 9.14].) We do so by means of the Weierstrass M-test ([1, Theorem 9.6].)

Let $t \in (a, b)$ and $i > 0$. Then

$$\left|\frac{-1}{(t+i)^2} + \frac{-1}{(t-n-i)^2}\right| = \frac{1}{(t+i)^2} + \frac{1}{(t-n-i)^2}.$$

For $i = 1$ we have $|t+1| > |a+1|$ and $|t-n-1| > |n+1-b|$, so

$$\frac{1}{(t+1)^2}+\frac{1}{(t-n-1)^2}<\frac{1}{(a+1)^2}+\frac{1}{(n+1-b)^2}\,.$$

For $i>1$, the fact that $t\in(-1,n+1)$ implies that $|t+i|>i-1$ and $|t-n-i|>i-1$, so

$$\frac{1}{(t+i)^2}+\frac{1}{(t-n-i)^2}<\frac{2}{(i-1)^2}\,.$$

Since $\frac{1}{(a+1)^2}+\frac{1}{(n+1-b)^2}+\sum_{i=2}^{\infty}\frac{2}{(i-1)^2}$ converges, the series (61) converges uniformly on (a,b). Thus we can differentiate (49) term by term on all of (a,b), and in particular at x.

The proof of the claim for $x\in\mathbb{R}\setminus\mathbb{Z}$ such that $x<-1$ or $x>n+1$ is very similar and is left to the reader. □

We can rewrite (60) as

$$f_n^2(x)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{x+i}+\frac{1}{x-n-i}\right)\left(\frac{1}{x+j}+\frac{1}{x-n-j}\right). \tag{62}$$

Substituting (62) and (61) into (58), we have

$$\begin{aligned} g_n(x)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Bigg(&\frac{1}{(x+i)(x+j)}+\frac{1}{(x-n-i)(x-n-j)}+\frac{1}{(x+i)(x-n-j)}+\\ &+\frac{1}{(x-n-i)(x+j)}\Bigg)+\sum_{i=1}^{\infty}\left(\frac{-1}{(x+i)^2}+\frac{-1}{(x-n-i)^2}\right). \end{aligned} \tag{63}$$

Since these series converge absolutely, we can freely rearrange terms provided we don't break up the sums enclosed in large brackets. In particular, in the first double sum we can separate the cases $j=i$ and $j\neq i$:

$$\begin{aligned} g_n(x)=&\sum_{i=1}^{\infty}\left(\frac{1}{(x+i)^2}+\frac{1}{(x-n-i)^2}+\frac{2}{(x+i)(x-n-i)}\right)+\\ &+\sum_{i=1}^{\infty}\sum_{j=1,j\neq i}^{\infty}\Bigg(\frac{1}{(x+i)(x+j)}+\frac{1}{(x-n-i)(x-n-j)}+\frac{1}{(x+i)(x-n-j)}+\\ &+\frac{1}{(x-n-i)(x+j)}\Bigg)+\sum_{i=1}^{\infty}\left(\frac{-1}{(x+i)^2}+\frac{-1}{(x-n-i)^2}\right). \end{aligned}$$

We can combine the first and last sigmas and utilize the symmetry between i and j to rewrite this with $i<j$:

$$g_n(x) = \sum_{i=1}^{\infty} \frac{2}{(x+i)(x-n-i)} + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\left(\frac{1}{(x+i)(x+j)}+\right.$$

$$\left.+\frac{1}{(x-n-i)(x-n-j)} + \frac{1}{(x+i)(x-n-j)} + \frac{1}{(x-n-i)(x+j)}\right). \tag{64}$$

We need to make one more rearrangement, and this time we need to break up the expressions in the large brackets belonging to the double sigma. We first fix i and consider the sum over j in the double sigma:

$$\sum_{j=i+1}^{\infty}\left(\frac{1}{(x+i)(x+j)} + \frac{1}{(x-n-i)(x-n-j)} + \frac{1}{(x+i)(x-n-j)}+\right.$$

$$\left.+\frac{1}{(x-n-i)(x+j)}\right). \tag{65}$$

An analogous argument to the one which allowed us to make the transition from (47) to (48) enables us to eliminate the parentheses in (65), so the sum equals

$$\frac{1}{(x+i)(x+i+1)} + \frac{1}{(x-n-i)(x-n-i-1)} + \frac{1}{(x+i)(x-n-i-1)}+$$

$$+\frac{1}{(x-n-i)(x+i+1)} + \frac{1}{(x+i)(x+i+2)} + \frac{1}{(x-n-i)(x-n-i-2)} + \cdots$$

which now converges conditionally rather than absolutely. The next step is to insert parentheses around the first two summands and then around every four summands after that. We conclude that the sum in (65) is equal to

$$\left(\frac{1}{(x+i)(x+i+1)} + \frac{1}{(x-n-i)(x-n-i-1)}\right) + \sum_{j=i+1}^{\infty}\left(\frac{1}{(x+i)(x-n-j)}+\right.$$

$$\left.+\frac{1}{(x-n-i)(x+j)} + \frac{1}{(x+i)(x+j+1)} + \frac{1}{(x-n-i)(x-n-j-1)}\right).$$

Now we can substitute this expression for the sum over j within the double sigma in (64) and move the first two terms (after multiplication by 2) to the first sigma in (64). The resulting equation is

$$g_n(x) = 2\sum_{i=1}^{\infty}\left(\frac{1}{(x+i)(x-n-i)} + \frac{1}{(x+i)(x+i+1)} +\right.$$

$$\left.\frac{1}{(x-n-i)(x-n-i-1)}\right) + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\left(\frac{1}{(x+i)(x+j+1)} +\right.$$

$$+\frac{1}{(x+i)(x-n-j)}+\frac{1}{(x-n-i)(x+j)}+\frac{1}{(x-n-i)(x-n-j-1)}\Bigg). \tag{66}$$

In order to show that $g_n(x)$ is strictly decreasing on $\left(-1,\frac{n}{2}\right]$, it suffices to prove the following claim.

Claim 4.6. *Each summand of each sigma in* (66) *is a strictly decreasing function of x on the interval* $\left(-1,\frac{n}{2}\right]$.

Proof. Consider first an arbitrary summand from the first sigma. We exploit the symmetry of the summand about the axis $x=\frac{n}{2}$ by making the substitution $x=u+\frac{n}{2}$, and obtain the expression

$$\frac{1}{\left(u+\frac{n}{2}+i\right)\left(u-\frac{n}{2}-i\right)}+\frac{1}{\left(u+\frac{n}{2}+i\right)\left(u+\frac{n}{2}+i+1\right)}+$$
$$+\frac{1}{\left(u-\frac{n}{2}-i\right)\left(u-\frac{n}{2}-i-1\right)},$$

which equals

$$\frac{4(-4+4i^2+4in+n^2+12u^2)}{\left[(2i+n)^2-4u^2\right]\left[(2+2i+n)^2-4u^2\right]}. \tag{67}$$

We want to show that this function of u decreases for $u\in\left(-\frac{n}{2}-1,0\right]$, or equivalently that it *increases* for $u\in\left[0,\frac{n}{2}+1\right)$. Now the numerator is positive and increasing for all $u\geq 0$, and the denominator is positive and *decreasing* on $\left[0,\frac{n}{2}+i\right)$, and so in particular on $\left[0,\frac{n}{2}+1\right)$. Thus (67) increases on $\left[0,\frac{n}{2}+1\right)$.

Now we look at an arbitrary summand from the second sigma in (66). This case turns out to be much more complicated that the previous one, partly because of all the factors in the denominator. In particular, the analogous formula to (67) does not give us what we want.

Our first step is to use (57) to rewrite the summand in question as

$$\frac{1}{j-i+1}\left(\frac{1}{x+i}-\frac{1}{x+j+1}\right)+\frac{1}{n+i+j}\left(\frac{1}{x-n-j}-\frac{1}{x+i}\right)+$$
$$+\frac{1}{n+i+j}\left(\frac{1}{x-n-i}-\frac{1}{x+j}\right)+\frac{1}{j-i+1}\left(\frac{1}{x-n-j-1}-\frac{1}{x-n-i}\right).$$

Now we add and subtract the expression $\frac{1}{j-i+1}\left(\frac{1}{x-n-j}-\frac{1}{x+j}\right)$ and rearrange terms to get

$$\begin{aligned}&\left[\frac{1}{j-i+1}\left(\frac{1}{x+i}-\frac{1}{x+j}+\frac{1}{x-n-j}-\frac{1}{x-n-i}\right)+\right.\\&\left.+\frac{1}{n+i+j}\left(\frac{1}{x-n-j}-\frac{1}{x+i}+\frac{1}{x-n-i}-\frac{1}{x+j}\right)\right]+\\&+\left[\frac{1}{j-i+1}\left(\frac{1}{x+j}-\frac{1}{x+j+1}+\frac{1}{x-n-j-1}-\frac{1}{x-n-j}\right)\right].\end{aligned}\tag{68}$$

It is easy to deal with the term in the second set of square brackets: We substitute $x=u+\frac{n}{2}$ and combine, obtaining

$$\frac{8(4j+4j^2+2n+4jn+n^2+4u^2)}{(j-i+1)\left[(2j+n)^2-4u^2\right]\left[(2+2j+n)^2-4u^2\right]}\tag{69}$$

and proceed as with (67) to show that this expression increases on $\left[0,\frac{n}{2}+1\right)$, and so decreases on $\left(-\frac{n}{2}-1,0\right]$.

As for the term in the first set of square brackets in (68), when we substitute $x=u+\frac{n}{2}$ and add, we get

$$\frac{8[-4ij-2in-2jn-n^2+(8j-8i+4)u^2]}{(j-i+1)\left[(2i+n)^2-4u^2\right]\left[(2j+n)^2-4u^2\right]}.\tag{70}$$

The argument we used for (67) and (69) fails here, because the numerator is not always positive (in fact it is negative at $u=0$). Proving that (70) increases on $\left[0,\frac{n}{2}+1\right)$ by differentiating it is an unappealing prospect, to say the least. What we can do is to add a constant to (70) to obtain an expression which is always positive on $\left[0,\frac{n}{2}+1\right)$, since after all we are only interested in showing that (70) increases on this interval. The obvious constant to add is the absolute value of the expression obtained by substituting $u=0$ in (70), namely

$$\frac{8(4ij+2in+2jn+n^2)}{(j-i+1)(2i+n)^2(2j+n)^2}.$$

The result (thank heavens for computer algebra systems, in this case *Mathematica*) is

$$\frac{32u^2(C+4u^2)}{(j-i+1)(2i+n)(2j+n)\left[(2i+n)^2-4u^2\right]\left[(2i+n)^2-4u^2\right]},\tag{71}$$

where

$$C=-4i^2+4ij-8i^2j-4j^2+8ij^2-2in-4i^2n-2jn+4j^2n-n^2-2in^2+2jn^2.\tag{72}$$

Clearly the denominator of (71) is positive and decreasing for $u\in\left[0,\frac{n}{2}+1\right)$. Assuming that $C>0$, we also have that the numerator is positive and *increasing* for all $u\in$

$\left(0, \frac{n}{2}+1\right)$. Thus (71) increases on $\left[0, \frac{n}{2}+1\right)$. This proves the claim, modulo the fact that $C > 0$. We leave the proof of that fact to the next claim. □

Claim 4.7. *Let $n, i, j \in \mathbb{N}$ be such that $1 \leq i < j$. Then*

$$-4i^2 + 4ij - 8i^2j - 4j^2 + 8ij^2 - 2in - 4i^2n - 2jn + 4j^2n - n^2 - 2in^2 + 2jn^2 + 4u^2 > 0.$$

Proof. Write $j = i + l$ where $l \geq 1$. Then (72) can be rewritten as a polynomial in n:

$$C = (-4i^2 - 4il + 8i^2l - 4l^2 + 8il^2) + (-4i - 2l + 8il + 4l^2)n + (-1 + 2l)n^2. \tag{73}$$

We will show that each of the three parenthesized expressions in (73) is positive.

The first of these can be written as

$$4\left[i^2(2l-1) + (2il^2 - il - l^2)\right].$$

Since $i \geq 1$ and $l \geq 1$, we have that $2l - 1 > 0$ and

$$2il^2 - il - l^2 \geq 2il^2 - il^2 - il^2 = 0.$$

So the first parenthesized expression in (73) is positive.

The coefficient of n in (73) is equal to $(4i + 2l)(2l - 1)$ and the coefficient of n^2 is $2l - 1$, both of which are positive. Since $n \geq 0$ as well, the claim is proven. □

Now that we have established that $g_n(x)$ is a strictly decreasing function on $\left(-1, \frac{n}{2}\right]$ and that $\lim_{x \to -1^+} g_n(x) = \infty$, the proof of Theorem 4.3 will be complete once we have shown that $g_n(x_0) < 0$ for some $x_0 \in \left(-1, \frac{n}{2}\right]$. If n is even we take $x_0 = \frac{n}{2} \in \mathbb{N}$. By (53) and (32) we have

$$g_n\left(\frac{n}{2}\right) = 2\left[-\frac{\pi^2}{6} + \sum_{i=1}^{\frac{n}{2}} \frac{1}{i^2}\right] < 0,$$

since $\sum_{i=1}^{\frac{n}{2}} \frac{1}{i^2} < \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$ by (41).

If n is odd, we choose $x_0 = \frac{n-1}{2} \in \mathbb{N}$ and use the same formulas as in the even case to conclude that

$$g_n\left(\frac{n-1}{2}\right) = 2\left[-\frac{\pi^2}{6} + \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i^2}\right] < 0.$$

This (finally!) completes the proof of Theorem 4.3. □

TABLE 1. Values of δ_k and ϵ_k

	n											
	0		1		2		3		20		100	
k	δ_k	ϵ_k	δ_k	ϵ_k	δ_k	ϵ_k	δ_k	ϵ_k	δ_k	ϵ_k	δ_k	ϵ_k
1	.430	.891	.389	.814	.362	.758	.342	.715	.241	.494	.180	.364
2	.459	.930	.430	.877	.408	.834	.390	.799	.284	.576	.209	.420
3	.471	.948	.448	.907	.430	.872	.415	.842	.310	.628	.227	.456
4	.477	.959	.459	.924	.443	.895	.430	.869	.329	.665	.241	.484
5	.482	.966	.466	.937	.452	.911	.441	.888	.344	.694	.252	.506
6	.484	.971	.471	.945	.459	.922	.448	.902	.356	.718	.262	.525
20	.495	.994	.490	.985	.486	.972	.482	.965	.428	.858	.334	.669
50	.498	.996	.496	.992	.494	.988	.492	.984	.465	.930	.392	.785
100	.499	.998	.498	.996	.497	.994	.496	.992	.481	.962	.430	.861

Corollary 4.8. *Let $n \in \mathbb{N}$. Then for each $k \geq 1$, we have*

$$0 < \delta_k < \epsilon_k < 1,$$

where δ_k is as defined in Theorem 4.2 and ϵ_k is as defined in Theorem 4.3.

Proof. For any $k \geq 1$ we have by definition of δ_k that $b_n'(-k-\delta_k) = 0$. Since we have $b_n(-k-\delta_k) \neq 0$, it follows from (46) that $f_n(-k-\delta_k) = 0$. We conclude from (58) and (61) that

$$g_n(-k-\delta_k) = f_n'(-k-\delta_k) < 0.$$

The zero $-k-\epsilon_k$ of g_n on the interval $(-k-1, -k)$ is therefore between $-k-1$ and $-k-\delta_k$, which means that $0 < \delta_k < \epsilon_k < 1$. $\square$

5. Some numbers and estimates

We can compute δ_k and ϵ_k for various values of n via Newton's method applied to (38) and (52), respectively. Better yet, we arrange for *Mathematica* to do so. The results appear in Table 1.

It is clear from the table that for any fixed $n \in \mathbb{N}$, we have $\lim_{k\to\infty} \delta_k = \frac{1}{2}$ and $\lim_{k\to\infty} \epsilon_k = 1$. So if we look at the tail of the graph of $y = b_n(x)$, as in Figure 2 for example, the local extrema will be closer and closer to half-integer values of x and the inflection points will be closer and closer to integer values of x, as $x \to \infty$ or $x \to -\infty$.

Given $n \in \mathbb{N}$, it would be nice to find an estimate for a_0, the inflection point for $y = b_n(x)$ in the interval $\left(-1, \frac{n}{2}\right]$. Setting the expression in (30) equal to zero is not an enticing prospect, and finding the zeros of $g_n(x)$ from (66) is even worse. What we can do is to try to locate the integer part of a_0 by using (32).

It is convenient to use the following standard symbols.

Definition 5.1. Let $x \in \mathbb{R}$.

(*i*) The *floor* of x (or *integer part* of x), denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. That is,

$$\lfloor x \rfloor = \max \{k \in \mathbb{Z} | \, k \leq x\} .$$

(*ii*) The *ceiling* of x, denoted by $\lceil x \rceil$, is the least integer greater than or equal to x:

$$\lceil x \rceil = \min \{k \in \mathbb{Z} | \, k \geq x\} .$$

(Thus, e.g., $\lfloor 4.7 \rfloor = 4$, $\lceil 4.7 \rceil = 5$, $\lfloor -6.8 \rfloor = -7$, $\lceil -6.8 \rceil = -6$, and $\lfloor 9 \rfloor = \lceil 9 \rceil = 9$.)

To compute $b_n''(x)$ at integer values of x, we can use Theorem 3.5. This enables us to prove:

Theorem 5.2. *Let $n \in \mathbb{N}$. The point of inflection a_0 for $y = b_n(x)$ which lies in the interval $\left(-1, \frac{n}{2}\right]$ satisfies*

$$\left\lfloor \frac{(n-1) - \sqrt{n+1}}{2} \right\rfloor \leq a_0 \leq \left\lceil \frac{(n+1) - \sqrt{n+1}}{2} \right\rceil . \tag{74}$$

Proof. Note that the interval in (74) is of length 2 unless $\sqrt{n+1}$ is an integer, in which case it is of length 1. We can do better for small values of n; for example, for $n = 0$ we have by (31) that

$$b_0''(-1) = \frac{2 \cdot (-1)^0}{(-1)\binom{-1}{0}} \cdot \frac{1}{-1} = 2,$$

and

$$b_0''(0) = -\frac{\pi^2}{3}\binom{0}{0} = -\frac{\pi^2}{3}$$

by (32). Thus $\lfloor a_0 \rfloor = -1$. Similarly, $\lfloor a_0 \rfloor = -1$ for $n = 1$ as well.

Now suppose $n \geq 2$. To see where $b_n''(x)$ changes sign on $\left(-1, \frac{n}{2}\right]$, let us first find a condition on $k \in \{0, 1, 2, \ldots, n\}$, where $k \leq \frac{n}{2}$, which guarantees that $b_n''(k) \geq 0$. From (32) we see that k must satisfy the inequality

$$\sum_{i=1}^{n-k} \frac{1}{i^2} + \sum_{k<i<j\leq n-k} \frac{1}{ij} \;\geq\; \frac{\pi^2}{6} .$$

Using (41), we can rewrite this as

$$\sum_{k<i<j\leq n-k} \frac{1}{ij} \;\geq\; \sum_{m=n-k+1}^{\infty} \frac{1}{m^2} \;. \tag{75}$$

Now

$$\sum_{m=n-k+1}^{\infty} \frac{1}{m^2} \leq \sum_{m=n-k+1}^{\infty} \frac{1}{m(m-1)} = \sum_{m=n-k+1}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m}\right).$$

This last sum telescopes and equals $\frac{1}{n-k}$. So if we want k to satisfy (75), it suffices to choose k such that

$$\sum_{k<i<j\leq n-k} \frac{1}{ij} \;\geq \frac{1}{n-k}.$$

Each summand on the left of the above inequality is greater than $\frac{1}{(n-k)^2}$, so it is sufficient for k to satisfy

$$\sum_{k<i<j\leq n-k} \frac{1}{(n-k)^2} \;\geq\; \frac{1}{n-k}\,.$$

There are $\binom{n-2k}{2}$ equal summands on the left of this inequality, that being the number of ways of choosing two numbers i and j (with $i<j$) from the set $\{k+1, k+2, \cdots, n-k\}$. This yields

$$\binom{n-2k}{2} \frac{1}{(n-k)^2} \;\geq\; \frac{1}{n-k}\,,$$

or

$$\frac{(n-2k)(n-2k-1)}{2} \geq n-k.$$

That is,

$$4k^2 + (4-4n)k + \left(n^2 - 3n\right) \geq 0. \tag{76}$$

The roots of the quadratic polynomial (in k) on the left-hand side of (76) are $\frac{n-1 \pm \sqrt{n+1}}{2}$, so we want k to satisfy

$$k \leq \frac{n-1 \;-\; \sqrt{n+1}}{2} \tag{77}$$

or

$$k \geq \frac{n-1 \;+\; \sqrt{n+1}}{2}.$$

Since we specified that $k \le \frac{n}{2}$ earlier, we choose k to satisfy (77). Also k is an integer, so

$$k \le \left\lfloor \frac{n-1-\sqrt{n+1}}{2} \right\rfloor . \tag{78}$$

Next we find a condition on $k \in \{0,1,2,\ldots,n\}$, where $k \le \frac{n}{2}$, which guarantees that $b_n''(k) \le 0$. This corresponds to (75) with the inequality reversed; that is,

$$\sum_{k<i<j\le n-k} \frac{1}{ij} \le \sum_{m=n-k+1}^{\infty} \frac{1}{m^2} . \tag{79}$$

This time we note that

$$\sum_{m=n-k+1}^{\infty} \frac{1}{m^2} \ge \sum_{m=n-k+1}^{\infty} \frac{1}{m(m+1)} = \sum_{m=n-k+1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1}\right) = \frac{1}{n-k+1},$$

so it is sufficient to choose k satisfying

$$\sum_{k<i<j\le n-k} \frac{1}{ij} \le \frac{1}{n-k+1}.$$

Each summand on the left is less than or equal to $\frac{1}{(k+1)(k+2)}$, and there are $\binom{n-2k}{2}$ summands as we noted previously. Thus it suffices that k satisfy

$$\binom{n-2k}{2} \frac{1}{(k+1)(k+2)} \le \frac{1}{n-k+1}.$$

That is,

$$\frac{(n-2k)(n-2k-1)}{2(k+1)(k+2)} \le \frac{1}{n-k+1},$$

and since both denominators are positive, we can multiply through by them without changing the direction of the inequality. The above inequality is thus equivalent to

$$4k^3 - 8nk^2 + \left(5n^2 + n + 4\right)k + \left(4 + n - n^3\right) \ge 0. \tag{80}$$

Rather than try to solve a cubic inequality, we see what happens if we add 1 to the root $\frac{n-1-\sqrt{n+1}}{2}$ of (76) which we found earlier and then substitute into the polynomial on the left-hand side of (80). The result of substituting $k = \frac{n+1-\sqrt{n+1}}{2}$ into this polynomial is (courtesy of *Mathematica*)

$$8 + 4n - 4\sqrt{n+1}. \tag{81}$$

To show that this expression is positive for all $n \in \mathbb{N}$ is equivalent to showing that

$$2 + n \geq \sqrt{n+1},$$

which is obviously true for all $n \in \mathbb{N}$. We conclude that (80), and therefore (79), hold for $k = \frac{n+1-\sqrt{n+1}}{2}$.

We again need to replace this expression for k by an integer, in this case by $k = \left\lceil \frac{n+1-\sqrt{n+1}}{2} \right\rceil$. The inequality (80) remains true for this value of k because $4k^3 - 8nk^2 + \left(5n^2 + n + 4\right)k + \left(4 + n - n^3\right)$ is an increasing function of k in this region. (The local extrema for this polynomial for $n \geq 6$ are at

$$k = \frac{2n}{3} \pm \frac{\sqrt{n^2 - 3n - 12}}{6},$$

both of which are greater than $\frac{n}{2}$. Thus the polynomial increases with k for $0 \leq k \leq \frac{n}{2}$. For $0 \leq n \leq 5$, the polynomial increases for all k.) This completes the proof of Theorem 5.2. □

It turns out that for $n \in \mathbb{N}$, the average of the upper and lower bounds given in (74), without the integer rounding, is a good approximation to a_0. That is, for each $n \in \mathbb{N}$ we have that

$$a_0 \approx \frac{n - \sqrt{n+1}}{2}. \tag{82}$$

Table 5 indicates that this approximate value is slightly larger than the correct value, at least for $n \leq 5000$. Similarly, $\frac{n-\sqrt{n+2}}{2} < a_0$ for these values of n. (Here a_0 is computed by *Mathematica* by setting the expression in (52) equal to zero, using Newton's method.) It would be nice to show that these inequalities hold for all values of $n \in \mathbb{N}$, but the complexity of the expression for $g_n(x)$ makes this a daunting prospect.

6. Final remarks

In this paper we have investigated the function $b_n(x) = \binom{n}{x}$ for $n \in \mathbb{N}$. Many of the results hold for arbitrary nonnegative $n \in \mathbb{R}$ as well. In particular, it appears fairly certain that this is true of Proposition 4.1 and Theorems 4.2 and 4.3, although this author cannot claim to have checked that this is indeed the case.

The formula (82) also seems to be a good approximation to the inflection point a_0 when n is a nonnegative real number. For example, taking $n = 98.6$, *Mathematica* can set the second derivative of its built-in binomial function equal to zero and use Newton's method to obtain the value $a_0 = 44.301486$. The approximation in (82) is 44.310010.

TABLE 2. Values of a_0 and its approximation by $\frac{n-\sqrt{n+1}}{2}$ and $\frac{n-\sqrt{n+2}}{2}$

n	a_0	$\frac{n-\sqrt{n+1}}{2}$	$\frac{n-\sqrt{n+2}}{2}$
0	-0.662586	-0.500000	-0.707107
1	-0.310244	-0.207107	-0.366025
2	0.057774	0.133975	0.000000
3	0.438804	0.500000	0.381966
4	0.830231	0.881966	0.775255
5	1.230013	1.275255	1.177124
10	3.311903	3.341688	3.267949
20	7.688749	7.708712	7.654792
50	21.417146	21.429285	21.394448
100	44.966602	44.975062	44.950248
500	238.804727	238.808485	238.797322
1000	484.177877	484.180708	484.172808
5000	2464.639947	2464.641126	2464.637591

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