THE COSET AND STABILITY RINGS

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ABSTRACT. We show that if G is a discrete Abelian group and $A \subset G$ has $\|1_A\|_{B(G)} \leqslant M$ then A is $O(\exp(\pi M))$-stable in the sense of Terry and Wolf.

In [TW19] Terry and Wolf, inspired by ideas in model theory, introduce the notion of stability for sets in Abelian groups, and very quickly after there followed a number of papers building on their work *e.g.* [CPT17, AFZ19, Sis18] and [TW18]. In this note we develop a relationship between stability and the Fourier algebra.

Suppose that G is a (possibly infinite) Abelian group. Following [TW18, Definition 1], for $k \in \mathbb{N}$ we say $A \subset G$ has the **k-order property** if there are vectors $a, b \in G^k$ such that $a_i + b_j \in A$ if and only if $i \leqslant j$. If A does not have the k-order property it is said to be **k-stable**. Note that the order property is monotonic so if A has the k-order property then it has the $(k-1)$-order property (for $k \geqslant 2$), and mutatis mutandis for stability.

Write $\mathcal{S}_l(G)$ for the set of subsets of G that are l-stable and $\mathcal{S}(G)$ for their union over all $l \in \mathbb{N}$. We begin with some examples from [TW19]:

Lemma 1.1 (The empty set and cosets)**.** *$\mathcal{S}_1(G) = \{\emptyset\}$ and $\mathcal{S}_2(G) = \mathcal{S}_1(G) \cup \bigcup_{H \leqslant G} G/H$.*

Proof. The first equality is immediate. For the second, from [TW19, Example 1 & Lemma 2] (or Lemma 1.5 later) we have $\bigcup_{H \leqslant G} G/H \subset \mathcal{S}_2(G)$. And conversely if A is 2-stable and $x, y, z \in A$ then putting $a_1 = x$, $a_2 = y$, $b_1 = z - x$ and $b_2 = 0_G$ we see that $a_1 + b_1, a_1 + b_2, a_2 + b_2 \in A$ by design. Since A does *not* have the 2-order property it follows that $y + z - x = a_2 + b_1 \in A$, and so $A + A - A \subset A$ and if A is non-empty it follows (by *e.g.* [Rud90, §3.7.1]) that A is a coset of a subgroup. The result is proved. $\square$

More interesting than the examples, Terry and Wolf show that $\mathcal{S}(G)$ has a ring structure. Recall that $\mathcal{R}$ is a **ring of subsets of** G if $\mathcal{R} \subset \mathcal{P}(G)$ is closed under complements and finite intersections (and hence finite unions). The prototypical example is $\mathcal{P}(G)$ itself; [TW19, Lemmas 1 & 2] give the following.

Theorem 1.2 (Terry–Wolf Stability Ring)**.** *Suppose that G is an Abelian group. Then $\mathcal{S}(G)$ is a translation-invariant ring of subsets of G.*

It may help to compare this with *e.g.* [TZ12, Exercise 8.2.9], the folklore fact that the set of stable formulas is closed under boolean combinations.

We write $\mathcal{W}(G)$ for the **coset ring** of G, that is the minimal translation-invariant ring of sets containing all cosets of subgroups of G. The coset ring has received attention in harmonic analysis (see [Rud90, Chapters 3 and 4]), and in view of Lemma 1.1 and Theorem 1.2 we have $\mathcal{W}(G) \subset \mathcal{S}(G)$; it is natural to ask whether we have equality.

For any $A \subset G$ we have $A = \bigcup_{x\in A}(x + \{0_G\})$ and so if G is finite any set is a finite union of cosets of subgroups of G and in particular $\mathcal{W}(G) = \mathcal{S}(G)$, but if G is not finite then things may be different. To see this we need a new example of sets of low stability.

Following[1] [Cil12, Definition 1] we say a set $A \subset G$ is a **Sidon set** (also known as a B_2-set) if whenever $x - y = z - w$ for some $(x, y, z, w) \in A^4$ we have $x = y$ or $x = z$.

Lemma 1.3 (Sidon sets are 3-stable). *Suppose that G is an Abelian group and $A \subset G$ is a Sidon set. Then A is 3-stable.*

Proof. Suppose that $a, b \in G^3$ witness the 3-order property in A. Then $a_i + b_j \in A$ whenever $i \leqslant j$, and so $(a_1 + b_2, a_1 + b_3, a_2 + b_2, a_2 + b_3) \in A^4$. But then

$$(a_1 + b_2) - (a_1 + b_3) = b_2 - b_3 = (a_2 + b_2) - (a_2 + b_3),$$

so by Sidonicity either $a_1 + b_2 = a_1 + b_3$ and we have $b_2 = b_3$; or $a_1 + b_2 = a_2 + b_2$ and we have $a_1 = a_2$. In the former case we have $a_3 + b_2 = a_3 + b_3 \in A$ – a contradiction. In the latter we have that $a_2 + b_1 = a_1 + b_1 \in A$ – a contradiction. It follows that A is 3-stable. □

On the other hand there are (at least if $|G| > 3$) sets in $\mathcal{S}_3(G)\backslash\mathcal{S}_2(G)$ that are not Sidon, for example a subgroup of size at least 4 with the identity removed.[2]

The set $\mathcal{T} := \{1, 3, 9, 27, \dots\}$ is an example of an infinite Sidon set in the integers. While stability need not be preserved by passing to subsets, Sidonicity is and so every subset of $\mathcal{T}$ is also Sidon and *a fortiori* 3-stable, so $\mathcal{S}(\mathbb{Z})$ is uncountable. On the other hand there are countably many cosets of subgroups of $\mathbb{Z}$ and so $\mathcal{W}(\mathbb{Z})$ is countable, and we conclude that $\mathcal{S}(\mathbb{Z}) \neq \mathcal{W}(\mathbb{Z})$.

In view of the above discussion it is tempting to ask for families of sets in $\mathcal{S}_{k+1}(G)$ that are not in the ring generated by $\mathcal{S}_k(G)$ for $k > 2$ – the 'irreducible' elements of $\mathcal{S}_{k+1}(G)$.

Cohen's idempotent theorem [Rud90, §3.1.3] tells us that $\mathcal{W}(G)$ is equal to the Fourier algebra $\mathcal{A}(G)$. To define the latter we take G to be discrete and write $\widehat{G}$ for the compact Abelian dual group of homomorphisms $G \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and if $\mu \in M(\widehat{G})$ put

$$\widehat{\mu}(x) := \int \lambda(x)d\mu(\lambda) \text{ for all } x \in G.$$

For $f : G \to \mathbb{C}$, if there is some $\mu \in M(\widehat{G})$ such that $f = \widehat{\mu}$ then μ is unique [Rud90, §1.3.6] and we put

$$\|f\|_{B(G)} := \|\mu\| := \int d|\mu|.$$

[1] Deviating from other definitions *e.g.* [TV06, Definition 4.26] if G has 2-torsion.

[2] Such a set is the intersection of a subgroup with the complement of a subgroup. The former is 2-stable (as recorded in Lemma 1.1); the latter is 3-stable since subgroups are 2-stable and the complements of k-stable sets are $(k+1)$-stable by [TW19, Lemma 1]. It follows by [TW19, Lemma 3] that the resulting intersection is 3-stable. On the other hand it is not a coset of a subgroup, and so not 2-stable, and not Sidon since if A is a Sidon subset of a group H then $|A|^2 - |A| + 1 \leqslant |H|$, but $(|H| - 1)^2 - (|H| - 1) + 1 > |H|$.

With this we write

$$\mathcal{A}(G) := \left\{A \subset G : \|1_A\|_{B(G)} < \infty\right\},$$

which it turns out is a translation-invariant [Rud90, §1.3.3 (c)] ring of sets [Rud90, §3.1.2].

Since $\mathcal{A}(G) = \mathcal{W}(G) \subset \mathcal{S}(G)$, we see that if $\|1_A\|_{B(G)} < \infty$ then $A \in \mathcal{S}(G)$, and it is natural to wonder if there is a universal monotonic $F : \mathbb{R}_{\geqslant 1} \to \mathbb{N}$ such that

$$\|1_A\|_{B(G)} \leqslant M \Rightarrow A \in \mathcal{S}_{F(M)}(G). \tag{1.1}$$

There is fairly direct approach via a quantitative version of Cohen's theorem. To describe this we make a definition: given $H \leqslant G$ and $\mathcal{S} \subset G/H$ we write[3] $\mathcal{S}^* := \mathcal{S} \cup \{\neg \bigcup \mathcal{S}\}$, that is the partition of G into cells from $\mathcal{S}$ and an additional cell that is everything else. We say that A has a (k,s)**-representation** if there are subgroups $H_1, \dots, H_k \leqslant G$, and sets $\mathcal{S}_1 \subset G/H_1, \dots, \mathcal{S}_k \subset G/H_k$ of size at most s such that A is the (disjoint) union of some cells in the partition[4] $\mathcal{S}_1^* \wedge \cdots \wedge \mathcal{S}_k^*$.

Theorem 1.4 (Quantitative idempotent theorem, [GS08, Theorem 1.2])**.** *Suppose that $\|1_A\|_{B(G)} \leqslant M$. Then A has a (k,s)-representation where*

$$k \leqslant M + O(1) \text{ and } s \leqslant \exp(\exp(O(M^4))).$$

The arguments of Terry and Wolf are also quantitative, and we record some of them in a slightly stronger form than they state. We begin with a slight extension of [TW19, Example 1].

Lemma 1.5 (Unions of cosets)**.** *Suppose that $H \leqslant G$ and $\mathcal{S} \subset G/H$ has size s. Then $\bigcup \mathcal{S}$ is $(s+1)$-stable.*

Proof. Suppose that $a, b \in G^{s+1}$ witness the $(s+1)$-order property in $\bigcup \mathcal{S}$. By the pigeonhole principle there is some $1 \leqslant i < j \leqslant s+1$ such that $a_1 + b_i$ and $a_1 + b_j$ are in the same coset of H, whence $b_i + H = b_j + H$. Since $a_j + b_j \in \bigcup \mathcal{S}$ we see that $a_j + b_i \in \bigcup \mathcal{S} + H = \bigcup \mathcal{S}$, a contradiction since $j > i$. □

In general the above lemma is best-possible as the next lemma shows when $G = \mathbb{Z}$, $H = \{0\}$, and $\mathcal{S}$ is the set of size-one subsets of an arithmetic progression.

Lemma 1.6 (Arithmetic progressions)**.** *Suppose that A is an arithmetic progression of integers of size r. Then A has the r-order property.*

Proof. Write $A = \{x, x+d, \dots, x+(r-1)d\}$, and let $s_i := x - id$ and $t_i = id$ for $1 \leqslant i \leqslant r$. Then $s_i + t_j = x + (j-i)d \in A$ if and only if $i \leqslant j$, and so the vectors $s, t \in \mathbb{Z}^r$ so defined witness the r-order property in A. (*c.f.* [Sis18, Lemma 6.3].) □

Quantitatively [TW19, Lemma 1] is about as good as one could hope – it says if A is s-stable then $\neg A$ is $(s+1)$-stable – however we shall combine it with a multi-set version of [TW19, Lemma 2].

[3] For $S \subset G$ we write $\neg S := G \setminus S$.

[4] Recall that if $\mathcal{P}$ and $\mathcal{Q}$ are partitions of the same set then $\mathcal{P} \wedge \mathcal{Q} := \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$.

Write $r(k_1,\dots,k_m)$ for the smallest natural number such that in any m colouring of the complete graph on $r(k_1,\dots,k_m)$ vertices there is some $1 \leqslant q \leqslant m$ such that the qth colour class contains a complete graph on k_q vertices.

Lemma 1.7. *Suppose that $\bigcup_{q=1}^m A_q$ has the $r(k_1+1,\dots,k_m+1)+1$-order property. Then there is some $1 \leqslant q \leqslant m$ such that A_q has the k_q-order property.*

Proof. Write $N := r(k_1+1,\dots,k_m+1)$. Since $\bigcup_{q=1}^m A_q$ has the $(N+1)$-order property there are vectors $a,b \in G^{N+1}$ so that we can colour the vertices of the complete graph on $\{1,\dots,N\}$ by giving the edge ij (for $1 \leqslant i < j \leqslant N$) the colour of the smallest q such that $a_{i+1}+b_j \in A_q$ – this is an m-colouring of the complete graph on N vertices.

By definition there is some $1 \leqslant q \leqslant m$ and a sequence $1 \leqslant s_1 < \cdots < s_{k_q+1} \leqslant N$ with $a_{s_i+1}+b_{s_j} \in A_q$ for all $1 \leqslant i < j \leqslant k_q+1$. On the other hand whenever $N \geqslant i \geqslant j \geqslant 1$ we have $a_{i+1}+b_j \notin \bigcup_n A_n$ by the N-order property of $\bigcup_n A_n$, and hence $a_{s_i+1}+b_{s_j} \notin A_q$ whenever $k_q+1 \geqslant i \geqslant j \geqslant 1$. Finally let $a_i' := a_{s_i+1}$ and $b_i' := b_{s_{i+1}}$ for all $1 \leqslant i \leqslant k_q$, and note that $a_i'+b_j' \in A_q$ if and only if $i \leqslant j$ as required. □

We now use this lemma to compute an upper bound on the stability of a set in the coset ring based on the complexity of its representation. To do this we need a bound on the multicolour Ramsey numbers. The usual Erdős-Szekeres argument gives (see *e.g.* [GG55, Corollary 3]) that

$$r(k_1+1,\dots,k_m+1) \leqslant \frac{(k_1+\cdots+k_m)!}{k_1!\cdots k_m!}. \tag{1.2}$$

Suppose now that A has a (k,s)-representation as described before Theorem 1.4. Then each of the sets $\neg\bigcup \mathcal{S}_i$ is $s+2$ stable by Lemma 1.5 and [TW19, Lemma 1], so each cell in the partition $\mathcal{S}_1^* \wedge \cdots \wedge \mathcal{S}_k^*$ is an intersection of k sets of stability at most $s+2$. It follows from Lemma 1.7 and (1.2) that A is t-stable where

$$\begin{aligned} t &\leqslant r(\overbrace{r(\underbrace{s+3,\dots,s+3}_{k\text{ times}})+2,\dots,r(s+3,\dots,s+3)+2}^{(s+1)^k\text{ times}})+2\\ &\leqslant r(\overbrace{k!^{s+2}+2,\dots,k!^{s+2}+2}^{(s+1)^k\text{ times}})+2 \leqslant 2^{k^{7ks}}. \end{aligned}$$

Plugging Theorem 1.4 into this shows that one may take F in (1.1) with

$$F(M) = \exp(\exp(\exp(\exp(O(M^4))))). \tag{1.3}$$

On the other hand, in some situations we can do far better: if A is finite and G is torsion-free then McGehee, Pigno and Smith's solution to Littlewood's conjecture[5] [MPS81, Theorem 2] applies to show that if $\|1_A\|_{B(G)} \leqslant M$ then $|A| = \exp(O(M))$. It follows from Lemma 1.5 that A is $\exp(O(M))$-stable. This is far better than the bound

[5]This theorem extends to $\widehat{G}$ connected as noted in [MPS81, §3, Remark (i)], and $\widehat{G}$ is connected since G is torsion-free by [Rud90, Theorem 2.5.6(c) & Theorem 1.7.2]).

in (1.3) and it is the main purpose of this note to prove a bound of this strength directly and in full generality:

Theorem 1.8. *Suppose that G is a (discrete) Abelian group and $A \subset G$ has $\|1_A\|_{B(G)} \leqslant M$. Then A is $(c_0 \exp(\pi M) + 1)$-stable where $c_0 := 2^{-4}\exp(-\gamma)\pi = 0.110\ldots$ and γ is the Euler–Mascheroni constant.*

Proof. Suppose that $a, b \in G^k$ witness the k-order property and consider

$$P : \ell_2^k \to \ell_2^k; v \mapsto \left(\sum_{m=1}^{k} 1_A(a_l + b_m)v_m\right)_{l=1}^{k},$$

where ℓ_2^k denotes k-dimensional complex Hilbert space.

We compute the trace norm of P in two ways: one showing it is large by direct calculation as it is just the trace norm[6] of (a variant of) the adjacency matrix of the half-graph; on the other hand it is small as a result of the hereditary smallness of the algebra norm.

Since a and b witness the order property, writing Q for the matrix of P with respect to the standard basis we have $Q_{ij} = 1$ if $i \leqslant j$ and 0 otherwise. It happens to be easier to deal with Q^{-1}; for reference (which can be easily checked)

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \text{ and } Q^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

It follows that

$$Q^{-1}(Q^{-1})^t = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Let[7] $\omega := \exp\left(\frac{2\pi i j}{2k+1}\right)$ and $v := (\omega, \ldots, \omega^k)$ so

$$Q^{-1}(Q^{-1})^t(v + \overline{v})^t = (2 - \omega - \omega^{-1})(v + \overline{v})^t.$$

Of course $2 - \omega - \omega^{-1} = 4\cos^2\left(\frac{\pi j}{2k+1}\right)$ which takes k distinct values as $1 \leqslant j \leqslant k$. It follows that the eigenvalues of $P^{-1}(P^{-1})^*$ are exactly the numbers $4\cos^2\left(\frac{\pi j}{2k+1}\right)$ for

[6]The trace norm of the adjacency matrix of a graph is sometimes called the **graph energy** [Gut78].

[7]Similar spectral computations to those here may be found in *e.g.* [BH12, §1.4.4], though we followed [Elk11].

$1 \leqslant j \leqslant k$, and hence the eigenvalues of $(P^{-1}(P^{-1})^*)^{-1} = P^*P$ are the reciprocals of these. These reciprocals, $1/4\cos^2\left(\frac{\pi j}{2k+1}\right)$ for $1 \leqslant j \leqslant k$, are themselves distinct and so have corresponding unit eigenvectors $v^{(1)}, \dots, v^{(k)}$ (of P^*P) which are mutually perpendicular, as are the unit vectors $w^{(1)}, \dots, w^{(k)}$ defined through

$$Pv^{(j)} = \frac{1}{2\cos\left(\frac{\pi j}{2k+1}\right)} w^{(j)} \text{ for } 1 \leqslant j \leqslant k.$$

Since $\|1_A\|_{B(G)} \leqslant M$ there is some $\mu \in M(\widehat{G})$ with $\|\mu\| \leqslant M$ and $1_A = \widehat{\mu}$ so, in particular,

$$1_A(a_l + b_m) = \int \lambda(a_l)\lambda(b_m) d\mu(\lambda) \text{ for all } 1 \leqslant l, m \leqslant k. \tag{1.4}$$

Let $\lambda^a, \lambda^b \in \ell_2^k$ be defined by $\lambda^a_l := \lambda(a_l)$ and $\lambda^b_m := \lambda(-b_m)$ for $1 \leqslant l, m \leqslant k$ so that $\|\lambda^a\|_{\ell_2^k} = \|\lambda^b\|_{\ell_2^k} = \sqrt{k}$. Then by (1.4) and linearity we have

$$\begin{aligned}
\langle w^{(j)}, Pv^{(j)} \rangle_{\ell_2^k} &= \sum_{l=1} w_l^{(j)} \overline{\sum_{m=1}^{k} 1_A(a_l + b_m) v_m^{(j)}} \\
&= \int \left(\sum_{l=1} w_l^{(j)} \lambda(-a_l) \right) \overline{\left(\sum_{m=1}^{k} \lambda(b_m) v_m^{(j)} \right) d\mu(\lambda)} \\
&= \int \langle w^{(j)}, \lambda^a \rangle_{\ell_2^k} \overline{\langle \lambda^b, v^{(j)} \rangle_{\ell_2^k} d\mu(\lambda)} = \int \langle w^{(j)}, \lambda^a \rangle_{\ell_2^k} \langle v^{(j)}, \lambda^b \rangle_{\ell_2^k} \overline{d\mu(\lambda)},
\end{aligned}$$

and hence (noting the left hand side is real so that the inequality makes sense)

$$\begin{aligned}
\sum_{j=1}^{k} \langle w^{(j)}, Pv^{(j)} \rangle_{\ell_2^k} &\leqslant \int \sum_{j=1}^{k} \left| \langle w^{(j)}, \lambda^a \rangle_{\ell_2^k} \langle v^{(j)}, \lambda^b \rangle_{\ell_2^k} \right| d|\mu|(\lambda) \\
&\leqslant \int \left(\sum_{j=1}^{k} |\langle w^{(j)}, \lambda^a \rangle_{\ell_2^k}|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{k} |\langle v^{(j)}, \lambda^b \rangle_{\ell_2^k}|^2 \right)^{\frac{1}{2}} d|\mu|(\lambda) \\
&= \int \|\lambda^a\|_{\ell_2^k} \|\lambda^b\|_{\ell_2^k} d|\mu|(\lambda) \leqslant kM.
\end{aligned} \tag{1.5}$$

This last equality is Parseval's identity (or the generalised Pythagorean theorem) applied with the two orthonormal bases $(w^{(j)})_{j=1}^k$ and $(v^{(j)})_{j=1}^k$ and the vectors λ^a and λ^b respectively.

In the other direction we have

$$\sum_{j=1}^{k}\langle w^{(j)},Pv^{(j)}\rangle_{\ell_2^k}=\sum_{j=1}^{k}\frac{1}{2\left|\cos\frac{j\pi}{2k+1}\right|}=\frac{1}{2}\sum_{l=0}^{k-1}\frac{1}{\sin\frac{(2l+1)\pi}{2(2k+1)}}$$
$$=\frac{2k+1}{\pi}\sum_{l=0}^{k-1}\frac{1}{2l+1}$$
$$+\frac{1}{2}\sum_{l=0}^{k-1}\left(\csc\left(\frac{\pi}{2}\cdot\frac{2l+1}{2k+1}\right)-\frac{2}{\pi\cdot\frac{2l+1}{2k+1}}\right).$$

The function $x\mapsto\csc\left(\frac{\pi}{2}x\right)-\frac{2}{\pi x}$ is an increasing function on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and so we can apply a standard integral estimate (the details of which we omit) to see that

$$\frac{1}{2}\sum_{l=0}^{k-1}\left(\csc\left(\frac{\pi}{2}\cdot\frac{2l+1}{2k+1}\right)-\frac{2}{\pi\cdot\frac{2l+1}{2k+1}}\right)\geqslant\frac{2k+1}{4}\left(\frac{2}{\pi}\log\frac{4}{\pi}-\frac{3}{\pi(2k+1)}\right).$$

Again, omitting details, one can use inequalities of Tóth [PMW$^+$91, Problem E3432(i)] to estimate the harmonic numbers (the nth of which we denote by H_n) and get

$$\frac{2k+1}{\pi}\sum_{l=0}^{k-1}\frac{1}{2l+1}=\frac{2k+1}{\pi}\left(H_{2k}-\frac{1}{2}H_k\right)\geqslant\frac{2k+1}{2\pi}\left(\log k+\log 4+\gamma\right).$$

In view of these two calculations we conclude that

$$\sum_{j=1}^{k}\langle w^{(j)},Pv^{(j)}\rangle_{\ell_2^k}\geqslant\frac{k}{\pi}\left(\log kc_0^{-1}-\frac{1}{k}\right).$$

Finally, $\exp(-x)\geqslant 1-x$ and so the above along with (1.5) rearranges to give the result. □

In the other direction we have the examples afforded by intervals.

Example 1.9. Suppose that $k\in\mathbb{N}$ and $G:=\mathbb{Z}$. Then there is a set $A\subset G$ such that A is at best $(k+1)$-stable (meaning A has the k-order property) and writing $M:=\|1_A\|_{B(G)}$ has

$$k+1\geqslant c_1\exp\left(\frac{\pi^2}{4}M\right)\text{ where}^8\ c_1:=2^{-2}\exp(-\gamma)\prod_{m=1}^{\infty}m^{-\frac{2}{4m^2-1}}=0.087\ldots.$$

Proof. Put $A:=\{1,\ldots,k\}$. A short calculation shows that

$$M=\int_0^1\left|\sum_{n=1}^{k}\exp(2\pi in\theta)\right|d\theta=\int_0^1\frac{|\sin(\pi k\theta)|}{\sin\pi\theta}d\theta.$$

[8] So $\frac{c_1}{c_0}=\frac{4}{\pi}\prod_{m=1}^{\infty}m^{-\frac{2}{4m^2-1}}=2\prod_{m=1}^{\infty}\frac{4m^2-1}{4m^{2\cdot\frac{4m^2}{4m^2-1}}}=0.789\ldots$, where c_0 is the constant in Theorem 1.8.

Szegő in [Sze21, (R)] gives a beautiful evaluation of this quantity (in fact the cited formula is for k odd, but the same argument works for any k as noted in [Sze21, Remark 2, §3]):

$$\begin{aligned}\int_0^1 \frac{|\sin(\pi k\theta)|}{\sin \pi\theta}d\theta &= \frac{16}{\pi^2}\sum_{m=1}^{\infty}\frac{H_{2mk}-\frac{1}{2}H_{mk}}{4m^2-1}\\ &< \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\log mk+\gamma+\log 4+\frac{1}{16(mk)^2}}{4m^2-1}\\ &\leqslant \frac{4}{\pi^2}(\log k+\gamma+\log 4)+\frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\log m}{4m^2-1}+\frac{1}{4\pi^2k^2}\end{aligned}$$

where the first inequality follows from the inequalities of Tóth [PMW+91, Problem E3432(i)]. By Lemma 1.6, A has the k-order property and the result is proved. □

Thus certainly the exponent π in Theorem 1.8 cannot be improved past $\frac{\pi^2}{4}$. That being said, the fact that these two numbers are close leads one to wonders if the proof of Theorem 1.8 above is amenable to improvement by direct analysis in the case that we are close to equality in the inequalities used.

As far as we know a better result than Theorem 1.8 may be true in the model setting $G = \mathbb{F}_2^n$ where there are no large arithmetic progressions – the set used in Example 1.9.

Since $\mathcal{S}(\mathbb{Z}) \neq \mathcal{A}(\mathbb{Z})$ we know that there is no converse to Theorem 1.8. The following example (*c.f.* [Fab93]) shows that this is so in essentially the worst possible way.

Example 1.10. Suppose q is a prime power. Then for $G := \mathbb{Z}/(q^2+q+1)\mathbb{Z}$ there is a Sidon (and *a fortiori* 3-stable) set $A \subset G$ of size $q+1$ such that $\|1_A\|_{B(G)} \geqslant \sqrt{|A|-1+o_{|A|\rightarrow\infty}(1)}$.

Proof. The perfect difference set construction of Singer [Sin38, p381] gives a set A of size $q+1$ in $G := \mathbb{Z}/(q^2+q+1)\mathbb{Z}$ that is a Sidon set and a direct calculation shows that $1_A = \widehat{\mu}$ for $\mu \in M(\widehat{G})$ with

$$\int 1_\Gamma d|\mu| = \frac{\sqrt{q}}{q^2+q+1}|\Gamma\backslash\{0_{\widehat{G}}\}| + \frac{q+1}{q^2+q+1}|\Gamma \cap \{0_{\widehat{G}}\}|.$$

It follows that

$$\|1_A\|_{B(G)} = \frac{q+1}{q^2+q+1} + \frac{q^2+q}{q^2+q+1}\sqrt{q} = \sqrt{q+o_{q\rightarrow\infty}(1)},$$

as claimed. □

Note if A is 2-stable then it is a coset of a subgroup (or empty) and so $\|1_A\|_{B(G)} \leqslant 1$. On the other hand a careful accounting of the constants in [Bou93, (3.3)] shows that any finite $A \subset G$ has $\|1_A\|_{B(G)} \leqslant \sqrt{|A|-1+o_{|A|\rightarrow\infty}(1)}$ which matches our bound above up to the little-o term. In other words 3-stable sets can have algebra norm essentially as large as possible for their size.

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