SOME DOMINATION PROPERTIES OF THE TOTAL GRAPH OF A MODULE WITH RESPECT TO SINGULAR SUBMODULE

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Abstract.
Let R be a commutative ring with unity and M be an R-module. The total graph of M with respect to the singular submodule $Z(M)$ of M is an undirected graph $T(\Gamma(M))$ with vertex set as M and any two distinct vertices x and y are adjacent if and only if $x + y \in Z(M)$. In this paper the author attempts to study the domination in the graph $T(\Gamma(M))$ and investigate the domination number and the bondage number of $T(\Gamma(M))$ and its induced subgraphs. Some domination parameters of $T(\Gamma(M))$ are also studied. It has been showed that $T(\Gamma(M))$ is excellent, domatically full and well covered under certain conditions.

Keywords: Total graph; bondage number; domination number; module; singular submodule.

Mathematics Subject Classification(2010): 05C25.

1. INTRODUCTION

In 1988, Beck [6] opened up the fascinating insight which relates a graph with the algebraic structure of a ring by introducing the zero-divisor graph of a commutative ring. He was mostly interested in colorings of this graph. This introduction was slightly modified later on by Anderson and Naseer in [2]. Further modification to the concept of the zero-divisor graph was made by Anderson and Livingston in [4]. Many authors studied the zero-divisor graph as Anderson-Livingston did in [4]. Since then, the concept of zero-divisor graphs of rings have played a vital role. Motivating from this well expanded idea of Beck, lots of correspondences of graph with algebraic structures have been introduced with variety of applications. Some of them are total graph of a commutative ring by Anderson and Badawi [3], total graph and regular graph of a commutative ring by Akbari [1], total graph of a finite commutative ring by Shekarriz [15] etc.

In 2008, Anderson and Badawi [3] defined the total graph of a commutative ring R as an undirected graph with vertex set R with any two vertices are adjacent if and only if their ring sum is a zero divisor of R. In that paper, they discussed the characteristics of total graphs and two of their induced subgraphs by considering two cases, namely, the set of zero divisors $Z(R)$ of R is an ideal of R and $Z(R)$ is not an ideal of R. Thereafter, the idea of the total graph has been generalised to many algebraic structures. The author in [9] introduced the notion of singularity of a module over a ring and defined the total graph of a module M with respect to singular submodule $Z(M)$. The line graph of the total graph of a module has also been studied in [10]. Before going to our discussion we recall the following.

Let R be a commutative ring. An element x of R is called a zero-divisor of R if there exists a non-zero element y of R with $xy = 0$. The collection of all zero-divisors of R is denoted by $Z(R)$, and henceforth, we use it. An ideal I of R is an essential ideal if its intersection with any non-zero ideal of R is non-zero. For an R-module M, let $Z(M)$ be the set of those $x \in M$ for which the ideal $\{r \in R | xr = 0\}$ is essential in R, i.e., $Z(M) = \{x \in M | xI = 0, \text{ for some essential ideal } I \text{ of } R\}$. Then $Z(M)$ is a submodule of M, called the singular submodule of M. A module M is said to be singular if $Z(M) = M$. On the other hand M is non-singular if $Z(M) = 0$. For any undefined terminology in rings and modules we refer to [5], [8] and [14].

By a graph G, we mean a simple undirected graph without loops. For a graph G, we denote by $V(G)$ and $E(G)$ the set of all vertices and edges respectively. We recall that a graph is finite if both $V(G)$ and $E(G)$ are finite sets, and we use the symbol $|G|$ to denote the number of vertices in the graph G. A graph G is complete if any two distinct vertices are adjacent. We denote the complete graph on n vertices by K_n. If the vertex set $V(G)$ of the graph G are partitioned into two non-empty disjoint sets X and Y of cardinality $|X| = m$ and $|Y| = n$, and two vertices are not adjacent if they are in the same partite set, then G is called a bipartite graph. A graph G is called a complete bipartite graph if every vertex in X is connected to every vertex in Y. We denote the complete bipartite graph on m and n vertices by $K_{m,n}$.
For a subset $S \subseteq V(G)$, $< S >$ denotes the subgraph of G induced by S. For a vertex $v \in V(G)$, $deg(v)$ is the degree of the vertex v, $N(v) = \{u \in V(G) \mid u \text{ is adjacent to } v\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of $V(G)$ is called a dominating set if every vertex in $V(G) - S$ is adjacent to atleast one vertex in S. A dominating set S is called a strong (or weak) dominating set if for every vertex $u \in V(G) - S$ there is a vertex $v \in S$ with $deg(v) \geq deg(u)$ (or $deg(v) \leq deg(u)$) and u is adjacent to v. The domination number $\gamma(G)$ of G is defined to be minimum cardinality of a dominating set in G and such a dominating set is called γ-set of G. In a similar way, we define the strong domination number γ_s and the weak domination number γ_w. A graph G is called excellent if for every vertex $v \in V(G)$, there exists a γ-set S containing v. A domatic partition of G is a partition of $V(G)$ into dominating sets in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. A graph G is called domatically full if $d(G) = \delta(G) + 1$, which is the maximum possible order of a domatic partition of $V(G)$ where $\delta(G)$ is the minimum degree of a vertex of G. The disjoint domination number $\gamma\gamma(G)$ defined by $\gamma\gamma(G) = \min\{|S_1| + |S_2| : S_1, S_2$

are disjoint dominating sets of $G\}$. Similarly, we can define $ii(G)$ and $\gamma i(G)$. The double domination parameters are referred to [11]. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number. A set of vertices $S \subseteq V(G)$ is said to be independent if no two vertices in S are adjacent in G. The independence number $\beta_0(G)$, is the maximum cardinality of an independent set in G. A graph G is called well-covered if $\beta_0(G) = i(G)$. For basic definitions and results in domination we refer to [12] and for any undefined graph-theoretic terminology we refer to [7].

The concepts of dominating sets and domination numbers are very important terminology of graph theory. Dominating sets are the focus of many books of graph theory, for example see[12] and [13]. But not much research has been done about the domination parameters of graphs associated to algebraic structures such as groups, rings, or modules in terms of algebraic properties. Recently, Chelvam and Asir [17] studied the domination in the total graph of a commutative ring. The domination number of total graph of module has been studied in [16].

Throughout this paper R is a commutative ring with unity and M is an R-module. The author [9] has introduced the total graph of M with respect to $Z(M)$, denoted by $T(\Gamma(M))$, to be an undirected graph with all elements of M as vertices, and for distinct $x, y \in M$, the vertices x and y are adjacent, written $x\ adj\ y$ if and only if $x + y \in Z(M)$. Let $\overline{Z}(M) = M - Z(M)$. Let $Z(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$, with vertices $Z(M)$, and let $\overline{Z}(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $\overline{Z}(M)$.

In this paper the author attempts to study the domination in the graph $T(\Gamma(M))$ and investigate the domination number and the bondage number of $T(\Gamma(M))$ and its induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$. Some domination parameters of $T(\Gamma(M))$ are also studied. It has been showed that $T(\Gamma(M))$ is excellent, domatically full and well covered under certain conditions.

2. Examples

There are certain classes of graphs whose dominating sets and domination numbers are clear. For instance, we state some of them in the following example, where their proofs are straightforward.

Example 1:

(i) If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$; while $\gamma(G) = n$ if and only if $G \cong \overline{K_n}$.

(ii) $\gamma(K_n) = 1$ for a complete graph K_n, but the converse is not true, in general and $\gamma(\overline{K_n}) = n$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph ($r \geq 2$) with partite sets $V_1, V_2, ..., V_r$. If $|V_i| \geq 2$ for $1 \leq i \leq r$, then $\gamma(G) = 2$; because one vertex of V_1 and one vertex of V_2 dominate G. If $|V_i| = 1$ for some i, then $\gamma(G) = 1$.

(iv) $\gamma(K_{1,n}) = 1$ for a star graph $K_{1,n}$.

(v) If G is a partition of disjoint subgraphs $G_1, G_2, ..., G_k$, then $\gamma(G) = \gamma(G_1) + \gamma(G_2) + ... + \gamma(G_k)$.

(vi) Domination number of a bistar graph is 2; because the set consisting of two centres of the graph is a minimal dominating set.

(vii) Let C_n and P_n be a n-cycle and a path with n vertices, respectively. Then $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil = \gamma(P_n)$.

3. Domination number of the graph $T(\Gamma(M))$ and its induced subgraphs

As mentioned in the introduction the author [9] has introduced the total graph of M with respect to singular submodule $Z(M)$, denoted by $T(\Gamma(M))$, to be an undirected graph with all elements of M as vertices, and for distinct $x, y \in M$, the vertices x and y are adjacent, written x *adj* y if and only if $x + y \in Z(M)$. Let $Z(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$, with vertices $Z(M)$, and let $\overline{Z}(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $\overline{Z}(M)$.
In this section, an attempt has been made to study the domination properties of the graph $T(\Gamma(M))$. In particular, the domination number of $T(\Gamma(M))$ and its induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$ have been determined. An equivalent condition describing relationship between the domination number of $T(\Gamma(M))$ and the same of $\overline{Z}(\Gamma(M))$ has also been established.

We begin with the following examples.

Example 3.1:
Let $R = Z_8$ be the ring of integers modulo 8 and $M = Z_4$ be the module of integers modulo 4. Then the essential ideals of R are $I = \{0, 2, 4, 6\}$ and R itself. We have $Z(M) = \{0, 2\}$ and therefore $\overline{Z}(M) = \{1, 3\}$.
Now we can easily observe that the induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$ are K_2 each. Thus, we have $\gamma(Z(\Gamma(M))) = \gamma(\overline{Z}(\Gamma(M))) = 1$.
Also, we can see that the total graph $T(\Gamma(M))$ is the union of two disjoint K_2's. Clearly, $\{0, 1\}$ is a γ-set of $T(\Gamma(M))$. Hence, we have $\gamma(T(\Gamma(M))) = 2$. Here, $\{2, 3\}$ is another γ-set of $T(\Gamma(M))$.

Example 3.2:
Let $R = Z_4$ be the ring of integers modulo 4 and $M = Z_8$ be the module of integers modulo 8. Then M is an R-module with the usual operations, and $Z(M) = \{0, 2, 4, 6\}$.

Thus $\overline{Z}(M) = \{1, 3, 5, 7\}$.
Now we can see that the induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$ are K_4 each. Thus, we have $\gamma(Z(\Gamma(M))) = \gamma(\overline{Z}(\Gamma(M))) = 1$.
Also, we observe that the total graph $T(\Gamma(M))$ is the union of two disjoint K_4's with $\{4, 5\}$ is one of the γ-sets of $T(\Gamma(M))$. Hence, we have $\gamma(T(\Gamma(M))) = 2$.

Theorem 3.3:[9]
Let R be a ring and M be an R-module. Then the following hold:

(1) $Z(\Gamma(M))$ is a complete (induced) subgraph of $T(\Gamma(M))$ and $Z(\Gamma(M))$ is disjoint from $\overline{Z}(\Gamma(M))$.
(2) If N is a submodule of M, then $T(\Gamma(N))$ is an (induced) subgraph of $T(\Gamma(M))$.

Proposition 3.4:
Let R be a ring and M be an R-module. Then the following hold:

(1) $\gamma(Z(\Gamma(M))) = 1$.
(2) $\gamma(T(\Gamma(M))) = 1$ if M is singular.

Proof.

(1) As $Z(\Gamma(M))$ is a complete by Theorem 3.3(1), so we have $\gamma(Z(\Gamma(M))) = 1$.
(2) If M is singular then $Z(M) = M$ and so $T(\Gamma(M)) = Z(\Gamma(M))$ which yields $T(\Gamma(M))$ is complete. Thus we have $\gamma(T(\Gamma(M))) = 1$.

The next theorem gives a complete description of $T(\Gamma(M))$. If we allow α and β to be infinite, then of course $\beta - 1 = \dfrac{(\beta - 1)}{2} = \beta$.

Theorem 3.5:[9]
Let R be a ring and M be an R-module such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$.

(1) If $2 = 1_R + 1_R \in Z(R)$ then $\overline{Z}(\Gamma(M))$ is the union $\beta - 1$ disjoint K_α's.
(2) If $2 = 1_R + 1_R \notin Z(R)$ then $\overline{Z}(\Gamma(M))$ is the union of $\dfrac{\beta - 1}{2}$ disjoint $K_{\alpha,\alpha}$'s.

Proposition 3.6:
Let R be a ring and M be an R-module such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$, then $\gamma(T(\Gamma(M))) = \beta$.

Proof.
Let us consider the following two cases for $Z(R)$.
Case 1: Suppose that $2 = 1_R + 1_R \in Z(R)$. Then we have from Theorem 3.5(1) that the graph $\overline{Z}(\Gamma(M))$ is the union $\beta - 1$ disjoint K_α's and we know that $\gamma(K_\alpha) = 1$. Thus

$\gamma(\overline{Z}(\Gamma(M))) = \beta - 1$ and by Proposition 3.4 we have $\gamma(Z(\Gamma(M))) = 1$. consequently $\gamma(T(\Gamma(M))) = \gamma(Z(\Gamma(M))) + \gamma(\overline{Z}(\Gamma(M))) = 1 + \beta - 1 = \beta$.
Case 2: Suppose that $2 = 1_R + 1_R \notin Z(R)$.Then again we have from Theorem 3.5(2) that the graph $\overline{Z}(\Gamma(M))$ is the union of $\dfrac{\beta-1}{2}$ disjoint $K_{\alpha,\alpha}$'s and we know that $\gamma(K_{\alpha,\alpha}) = 2$. Thus $\gamma(\overline{Z}(\Gamma(M))) = \dfrac{\beta-1}{2} \times 2 = \beta - 1$ and by Proposition 3.4 we have $\gamma(Z(\Gamma(M))) = 1$. So, $\gamma(T(\Gamma(M))) = \gamma(Z(\Gamma(M))) + \gamma(\overline{Z}(\Gamma(M))) = 1 + \beta - 1 = \beta$.

Proposition 3.7:
Let R be a ring and M be a non-zero and non-singular R-module such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$, then $\gamma(T(\Gamma(M))) = \dfrac{\beta+1}{2}$.

Proof.
Since M is non-singular, we have $Z(M) = 0$. Therefore, $\left|\dfrac{M}{Z(M)}\right| = |M| = \beta$. Now, we show that $Z(R) = 0$. Let $0 \neq x \in Z(R)$, then there exist $0 \neq y \in R$ such that $xy = 0$. Let us consider an element $0 \neq m \in M$, and we have $(xy)m = 0$ which yields $x(ym) = 0$. Then $ym = 0$ as $x \neq 0$ which yields either $y = 0$ or $m = 0$, a contradiction. Therefore, $Z(R) = 0$. So, $2 = 1_R + 1_R \notin Z(R)$ and from Theorem 3.5(2) we have the graph $\overline{Z}(\Gamma(M))$ is the union of $\dfrac{\beta-1}{2}$ disjoint $K_{1,1}$'s and by Proposition 3.4 we have $\gamma(Z(\Gamma(M))) = 1$. Therefore we will have $\gamma(T(\Gamma(M))) = \gamma(Z(\Gamma(M))) + \gamma(\overline{Z}(\Gamma(M))) = 1 + (\dfrac{\beta-1}{2}) \times 1 = \dfrac{\beta+1}{2}$.

4. Some domination parameters of $T(\Gamma(M))$

In this section, some domination parameters of $T(\Gamma(M))$ has been studied. The bondage number of $T(\Gamma(M))$ has also been determined. Finally, it has been proved that $T(\Gamma(M))$ is excellent, domatically full and well covered under some conditions.

We begin with the following proposition.

Proposition 4.1:
Let R be a ring and M be an R-module such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$. A set $S = \{x_1, x_2, ..., x_\beta\} \subset V(T(\Gamma(M)))$ is a γ-set of $T(\Gamma(M))$ if and only if $x_j \notin x_i + Z(M)$ for all $1 \leq i, j \leq \beta$ and $i \neq j$.

Proof.
The if part follows directly from Proposition 3.6 as $\gamma(T(\Gamma(M))) = \beta$.
Conversely, let S be a γ-set of $T(\Gamma(M))$. Let us assume that there exist $j, k \in \{1, 2, ..., \beta\}$

such that $x_j \in x_i + Z(M)$. Since $|S| = \beta$, there exist a coset $x + Z(M)$ such that $x_i \notin x + Z(M)$ for all $x_i \in S$. Now, the vertices in $-x + Z(M)$ cannot be dominated by S, a contradiction.

Theorem 4.2:[9]
Let R be a ring and M be an R-module. Let x be a vertex of the graph $T(\Gamma(M))$. Then

$$deg(x) = \begin{cases} |Z(M)| - 1, & \text{if } 2 \in Z(R) \text{ and } x \in Z(M) \\ |Z(M)|, & \text{otherwise.} \end{cases}$$

Proposition 4.3:
Let R be a ring and M be an R-module such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$, then

(1) $T(\Gamma(M))$ is excellent.
(2) the domatic number $d(T(\Gamma(M))) = \alpha$.
(3) $T(\Gamma(M))$ is domatically full.

Proof.
By Proposition 3.6 we have $\gamma(T(\Gamma(M))) = \beta$.
The proof of (1) and (2) are trivial.
(3) By (2) we have $d(T(\Gamma(M))) = \alpha = |Z(M)|$. Also, we have by Theorem 4.2 that $\delta(T(\Gamma(M))) = |Z(M)| - 1 = \alpha - 1$. Therefore, we have $d(T(\Gamma(M))) = \delta(T(\Gamma(M))) + 1$. Hence, $T(\Gamma(M))$ is domatically full.

We now find the bondage number of the graph $T(\Gamma(M))$. We begin with the following example.

Example 2:

(i) If G is a simple graph of order n, then $1 \leq b(G) \leq n - 1$.
(ii) $b(K_n) = n - 1$ for a complete graph K_n, but the converse is not true, in general and $b(\overline{K_n}) = 0$ for a null graph $\overline{K_n}$.
(iii) Let G be a complete r-partite graph with partite sets $V_1, V_2, ..., V_r$. Then $b(G) = min\{|V_1|, |V_2|, ..., |V_r|\}$. In particular, $b(K_{m,n}) = min\{m, n\}$.
(iv) If G is a partition of disjoint subgraphs $G_1, G_2, ..., G_k$, then

$$b(G) = min\{b(G_1), b(G_2), ..., b(G_k)\}.$$

(v) Let C_n and P_n be a n-cycle and a path with n vertices, respectively. Then $b(P_n) = 1$ and $b(C_n) = 2$.

Proposition 4.4:
Let R be a ring and M be an R-module such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$, then $b(T(\Gamma(M))) = \alpha - 1$.

Proof.
Suppose that $2 = 1_R + 1_R \in Z(R)$. Then, by Theorem 3.5(1), the graph $\overline{Z}(\Gamma(M))$ is the union of $\beta - 1$ disjoint K_α's and we know that $b(K_\alpha) = \alpha - 1$. Hence $b(\overline{Z}(\Gamma(M))) = \alpha - 1$. Also $Z(\Gamma(M))$ is complete, by Theorem 3.3(1) . Thus, $b(Z(\Gamma(M))) = \alpha - 1$. On the other hand, $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$ are disjoint, by Theorem 3.3(1) . Therefore, $b(T(\Gamma(M))) = \alpha - 1$.

Now, suppose that $2 = 1_R + 1_R \notin Z(R)$. Then, by Theorem 3.5(2), $\overline{Z}(\Gamma(M))$ is the union of $\dfrac{\beta-1}{2}$ disjoint $K_{\alpha,\alpha}$'s and we know that $b(K_{\alpha,\alpha}) = \alpha$. Thus $b(\overline{Z}(\Gamma(M))) = \alpha$. But $Z(\Gamma(M))$ is complete and disjoint from $\overline{Z}(\Gamma(M))$, by Theorem 3.3(1) . So, $b(Z(\Gamma(M)))$ and hence $b(T(\Gamma(M)))$ is equal to $\alpha - 1$.

Lemma 4.5:
Let M be a finite module over a ring R such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$. Then

$$T(\Gamma(M)) = \begin{cases} K_\alpha \cup \underbrace{K_\alpha \cup K_\alpha \cup \ldots \cup K_\alpha}_{(\beta-1)\text{ copies}}, & \text{if } 2 \in Z(R) \\ K_\alpha \cup \underbrace{K_{\alpha,\alpha} \cup K_{\alpha,\alpha} \cup \ldots \cup K_{\alpha,\alpha}}_{\left(\frac{\beta-1}{2}\right)\text{ copies}}, & \text{if } 2 \notin Z(R). \end{cases}$$

Proof.

It follows from Theorem 3.5 directly.

Proposition 4.6:
Let M be a finite module over a ring R such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$. Then $T(\Gamma(M))$ is well covered.

Proof.
If $2 \in Z(R)$, then by Lemma 4.5 we have $i(T(\Gamma(M))) = \beta$.
If $2 \notin Z(R)$, then all the vertices in one partition of $K_{\alpha,\alpha}$ together with a vertex of $Z(M)$, form an i-set of $T(\Gamma(M))$ and so $i(T(\Gamma(M))) = \left(\dfrac{\beta-1}{2}\right)\alpha + 1$. Similarly $\beta_0(T(\Gamma(M)))$ is same as $i(T(\Gamma(M)))$. Thus

$$i(T(\Gamma(M))) = \beta_0(T(\Gamma(M))) = \begin{cases} \beta, & \text{if } 2 \in Z(R) \\ \left(\dfrac{\beta-1}{2}\right)\alpha + 1, & \text{otherwise.} \end{cases}$$

Hence, $T(\Gamma(M))$ is well covered.

Corollary 4.7:
Let M be a finite module over a ring R such that $|Z(M)| = \alpha$, then $\omega(T(\Gamma(M))) = \alpha$.

As proved above, we can prove the following.

Proposition 4.8:
Let M be a finite module over a ring R such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$. Then

(1)
$$\gamma_t(T(\Gamma(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Z(R) \\ \beta + 1, & \text{otherwise.} \end{cases}$$
.
(2) $\gamma_s(T(\Gamma(M))) = \gamma_w(T(\Gamma(M))) = \beta$.
(3) $\gamma_p(T(\Gamma(M))) = \beta$.

Proposition 4.9:
Let M be a finite module over a ring R such that $|Z(M)| = \alpha$ and $\left|\dfrac{M}{Z(M)}\right| = \beta$. Then

(1) $\gamma\gamma(T(\Gamma(M))) = 2\beta$.
(2)
$$\gamma i(T(\Gamma(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Z(R) \\ \beta + \left(\dfrac{\beta-1}{2}\right)\alpha + 1, & \text{otherwise.} \end{cases}$$
.
(3)
$$ii(T(\Gamma(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Z(R) \\ (\beta-1)\alpha + 2, & \text{otherwise.} \end{cases}$$
.
(4)
$$tt(T(\Gamma(M))) = \begin{cases} 4\beta, & \text{if } 2 \in Z(R) \text{ and } \alpha \geq 4 \\ 2(\beta+1), & \text{if } 2 \notin Z(R) \\ \text{does not exist}, & \text{otherwise.} \end{cases}$$
.

ACKNOWLEDGEMENT :
The author would like to express his deep gratitude to the referee for a very careful reading of the article, and many valuable suggestions to improve the article.

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