A NEW CLASS OF SYMMETRIC FUNCTIONS OF BINARY PRODUCTS OF TRIBONACCI NUMBERS AND OTHER WELL-KNOWN NUMBERS

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Abstract. In this paper, we will recover the generating functions of Tribonacci numbers and Chebychev polynomials of first and second kind. By making use of the operator defined in this paper, we give some new generating functions for the binary products of Tribonacci with some remarkable numbers and polynomials. The technique used here is based on the theory of the so-called symmetric functions.

1. Introduction

Recently, Fibonacci and Lucas numbers have been investigated very largely and authors tried to develop and give some directions to mathematical calculations using these type of special numbers [16, 22, 29, 32]. One of these directions goes through to the Tribonacci and the Tribonacci-Lucas numbers. In fact Tribonaccinumbers have been firstly defined by M. Feinberg in 1963 and then some important properties for these numbers have been investigated by [18, 21, 25, 28, 33]. On the other hand, Tribonacci-Lucas numbers have been introduced and investigated by Elia in [17]. In addition, there exists another term, namely incomplete, on Fibonacci, Lucas and Tribonacci numbers. As a brief background, the incomplete Fibonacci, Lucas and Tribonacci numbers were introduced by authors [19, 26, 27], and further the generating functions of these numbers were presented.

Definition 1. *For $n \geq 2$, it is known that while the Tribonacci sequence $\{T_n\}_{n\in\mathbb{N}}$ is defined by*

$$\left\{ \begin{array}{c} T_n = T_{n-1} + T_{n-2} + T_{n-3}, n \geq 3 \\ T_0 = T_1 = 1, T_2 = 2 \end{array} \right. . \tag{1.1}$$

It is also well-known that each of the Tribonacci numbers is actually a linear combination of

$$T_n = -\frac{(r_2 + r_3 - r_2 r_3 - 2)}{(r_1 - r_2)(r_1 - r_3)} r_1^n + \frac{(r_1 + r_3 - r_1 r_3 - 2)}{(r_1 - r_2)(r_2 - r_3)} r_2^n - \frac{(r_1 + r_2 - r_1 r_2 - 2)}{(r_1 - r_3)(r_2 - r_3)} r_3^n, \tag{1.2}$$

where r_1, r_2, and r_3 are roots of the characteristic equations of (1.1) such that

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$$r_1 = \frac{1}{3}\left[1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}\right] = 1.8393,$$
$$r_2 = \frac{1}{3}\left[1+w\sqrt[3]{19+3\sqrt{33}}+w^2\sqrt[3]{19-3\sqrt{33}}\right] = -0.41964+0.60629i,$$
$$r_3 = \frac{1}{3}\left[1+w^2\sqrt[3]{19+3\sqrt{33}}+w\sqrt[3]{19-3\sqrt{33}}\right] = -0.41964-0.60629i,$$

with $w = \frac{-1+i\sqrt{3}}{2}$. Meanwhile we note that equations in (1.2) are called the Binet's formulas for Tribonacci numbers.

Definition 2. *Let k be a positive real number. Then, the recurrence relation of generalized k-Fibonacci numbers $\{G_{k,n}\}_{n\in\mathbb{N}}$ is defined by*

$$G_{k,n+1} = kG_{k,n} + G_{k,n-1},\ n \geq 1$$

with initial conditions $G_{k,0} = a$, $G_{k,1} = b$ $(a, b \in \mathbb{R})$.

Generalized k-Fibonacci number is called to each element of Generalized k-Fibonacci sequence. Taking $a = 1$, $b = k$ and $a = 2$, $b = k$ gives k-Fibonacci numbers and k-Lucas numbers, respectively. A few special values for Generalized k-Fibonacci sequence$\{G_{k,n}\}_{n\in\mathbb{N}}$ are listed below:

i. If $k = 1$, then we have generalized Fibonacci numbers$\{G_{1,n}\}$

$$\{G_{1,n}\} = \{a,\ b,\ a+b,\ a+2b,\ 2a+3b, ...\}.$$

- Putting $a = 1, b = 1$ reduces to Fibonacci numbers known as $\{F_n\} = \{1, 1, 2, 3, 5, ...\}$.
- Substituting $a = 2, b = 1$ yields Lucas numbers given by $\{L_n\} = \{2, 1, 3, 4, 7, 11, ...\}$.

ii. If $k = 2$, then we have generalized Pell numbers

$$\{G_{2,n}\} = \{a,\ b,\ a+2b,\ 2a+5b,\ 5a+12b,\ 12a+29b, ...\}.$$

- Taking $a = 0$, $b = 1$ gives Pell numbers given by $\{P_n\} = \{0, 1, 2, 5, 12, 29, ...\}$.
- In the case when $a = 2$ and $b = 2$, it reuces to Pell-Lucas numbers known as $\{Q_n\} = \{2, 2, 6, 14, 34, 82, ...\}$.

In this part, we define k-Pell and k-Pell Lucas numbers, Mersenne numbers and k-Jacobsthal-Lucas numbers.

Definition 3. [30] *For $n \in \mathbb{N}$, the k-Mersenne numbers, say $\{M_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by:*

$$\begin{cases} M_{k,n+1} = 3kM_{k,n} - 2M_{k,n-1} \text{ for all } n \geq 1, \\ M_0 = 0,\ M_1 = 1 \end{cases}.$$

Definition 4. [22] *For $n \in \mathbb{N}$, the k-Pell numbers, denoted by $\{P_{k,n}\}_{n\in\mathbb{N}}$ defined recursively by*

$$\{.P_{k,n} = 2P_{k,n-1} + kP_{k,n-2},\ n \geq 2P_{k,0} = 0,\ P_{k,1} = k.$$

Definition 5. [22] *We define k-Pell-Lucas numbers* $\{Q_{k,n}\}_{n\in\mathbb{N}}$ *by the following recurrence relation as*

$$\{.Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2},\ n \geq 2\ Q_{k,0} = 2,\ Q_{k,1} = 2.$$

Definition 6. *We define k-Jacobsthal numbers* $\{J_{k,n}\}_{n\in\mathbb{N}}$ *by the following recurrence relation as*

$$\{.J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2},\ n \geq 2 J_{k,0} = 0,\ J_{k,1} = 1\ .$$

Definition 7. *For* $n \in \mathbb{N}$*, the k-Jacobsthal Lucas numbers, denoted by* $\{j_{k,n}\}_{n\in\mathbb{N}}$ *defined recursively by*

$$\{.j_{k,n} = kj_{k,n-1} + 2j_{k,n-2},\ n \geq 2\ j_{k,0} = 2,\ j_{k,1} = k.$$

The further contents of this paper are as follows: Section 1 gives the preliminary, in Section 2, we introduce new symmetric functions and some of its properties. We also give some more useful definitions which are used in the following sections. In Section 3, we prove our main result which relates the symmetric function defined in the previous section with the symmetrizing operator. This main theorem unifies several previously known results about the generating functions**.**

2. Definitions and some properties

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition 8. [24] *Let k and n be two positive integers and* $\{a_1, a_2, ..., a_n\}$ *are set of given variables the k-th elementary symmetric function* $e_k(a_1, a_{2,}..., a_n)$ *is defined by*

$$e_k(a_1, a_{2,}..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2}...a_n^{i_n} \qquad (0 \leq k \leq n)$$

with $i_1, i_2, ..., i_n = 0$ *or* 1.

Definition 9. [24] *Let k and n be two positive integers and* $A = \{a_1, a_2, ..., a_n\}$ *are set of given variables the k-th complete homogeneous symmetric function* $h_k(a_1, a_2, ..., a_n)$ *is defined by*

$$h_k(a_1, a_{2,}..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2}...a_n^{i_n} \qquad (0 \leq k \leq n)$$

with $i_1, i_2, ..., i_n \geq 0$.

Remark 1. *Set* $e_0(a_1, a_{2,}..., a_n) = 1$ *and* $h_0(a_1, a_{2,}..., a_n) = 1$*, by usual convention. For* $k < 0$*, we set* $e_k(a_1, a_{2,}..., a_n) = 0$ *and* $h_k(a_1, a_{2,}..., a_n) = 0$.

Definition 10. [11] *Let A and P be any two alphabets. We define* $S_n(A - P)$ *by the following form*

$$\frac{\Pi_{p\epsilon P}(1 - pt)}{\Pi_{a\epsilon A}(1 - at)} = \sum_{n=0}^{\infty} S_n(A - P)t^n, \tag{2.1}$$

with the condition $S_n(A - P) = 0$ *for* $n < 0$.

Equation (2.1) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A-P)t^n = \left(\sum_{n=0}^{\infty} S_n(A)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right), \tag{2.2}$$

where

$$S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(A).$$

Definition 11. [9] *Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows*

$$\partial_{p_i p_{i+1}}(f) = \frac{f(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) - f(p_1, \cdots p_{i-1}, p_{i+1}, p_i, p_{i+2} \cdots p_n)}{p_i - p_{i+1}}.$$

Definition 12. *Let n be positive integer and $P = \{p_1, p_2\}$ are set of given variables, then, the n-th symmetric function $S_n(p_1 + p_2)$ is defined by*

$$S_n(P) = S_n(p_1 + p_2) = \frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2},$$

with

$$\begin{aligned} S_0(P) &= S_0(p_1 + p_2) = 1, \\ S_1(P) &= S_1(p_1 + p_2) = p_1 + p_2, \\ S_2(P) &= S_2(p_1 + p_2) = p_1^2 + p_1 p_2 + p_2^2, \\ &\vdots \end{aligned}$$

Definition 13. *Given an alphabet $P = \{p_1, p_2\}$, the symmetrizing operator $\delta^k_{p_1 p_2}$ is defined by*

$$\delta^k_{p_1 p_2}(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2}, \textit{ for all } k \in \mathbb{N}_0.$$

3. Generating functions of the products of Tribonacci numbers and some well-known numbers

The following Proposition allows us to obtain many new generating functions of binary products of Tribonacci numbers, some well-known numbers and Chebychev polynomials.

Proposition 1. [11] *Given two alphabets $P = \{p_1, p_2\}$ and $A = \{a_1, a_2, \ldots\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A) S_{k+n-1}(p_1 + p_2) t^n = \frac{\sum\limits_{n=0}^{k-1} S_n(-A) p_1^n p_2^n S_{k-n-1}(p_1 + p_2) t^n}{\prod\limits_{a \in A}(1 - a p_1 t) \prod\limits_{a \in A}(1 - a p_2 t)} - \frac{p_1^k p_2^k t^{k+1} \sum\limits_{n=0}^{\infty} S_{n+k+1}(-A) S_n(p_1 + p}{\prod\limits_{a \in A}(1 - a p_1 z) \prod\limits_{a \in A}(1 - a p_2 z)}$$

- For $P=\{0,0\}$ and $A=\{a_1,a_2,a_3\}$ using the formula (2.1), we have

$$\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{(1-a_1t)(1-a_2t)(1-a_3t)}, \tag{3.1}$$

with

$$(1-a_1t)(1-a_2t)(1-a_3t) = 1+S_1(-A)t+S_2(-A)t^2+S_3(-A)t^3.$$

Setting $S_1(-A)=S_2(-A)=S_3(-A)=-1$ in (3.1) and this gives

$$\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{1-t-t^2-t^3}, \tag{3.2}$$

which represents a generating function for Tribonacci numbers, such that $T_n=S_n(A)$.

- The case of $A=\{a_1,a_2,a_3\}$ and $P=\{p_1,p_2\}$

If $k=0,1$ in Proposition 3.1 we deduce the following lemmas.

Lemma 1. *Given two alphabets $A=\{a_1,a_2,a_3\}$ and $P=\{p_1,p_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_n(P)t^n = \frac{1-p_1p_2S_2(-A)t^2-p_1p_2(p_1+p_2)S_3(-A)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1-a_1p_2t)(1-a_2p_2t)(1-a_3p_2t)}. \tag{3.3}$$

Lemma 2. *Given two alphabets $A=\{a_1,a_2,a_3\}$ and $P=\{p_1,p_2\}$, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n-1}(P)t^n = \frac{-S_1(-A)t-(p_1+p_2)S_2(-A)t^2-S_3(-A)((p_1+p_2)^2-p_1p_2)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1-a_1p_2t)(1-a_2p_2t)(1-a_3p_2t)}. \tag{3.4}$$

Replacing p_2 by $(-p_2)$ in (3.3) and (3.4), we obtain

$$\sum_{n=0}^{\infty} S_n(A)S_n(p_1+[-p_2])t^n = \frac{1+p_1p_2S_2(-A)t^2+p_1p_2(p_1-p_2)S_3(-A)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1+a_1p_2t)(1+a_2p_2t)(1+a_3p_2t)}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} S_n(A)S_{n-1}(p_1+[-p_2])t^n = \frac{-S_1(-A)t-(p_1-p_2)S_2(-A)t^2-S_3(-A)((p_1-p_2)^2+p_1p_2)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1+a_1p_2t)(1+a_2p_2t)(1+a_3p_2t)}, \tag{3.6}$$

with

$$\begin{aligned} & (1-a_1p_1t)\,(1-a_2p_1t)\,(1-a_3p_1t)\,(1+a_1p_2t)\,(1+a_2p_2t)\,(1+a_3p_2t) \\ = \; & 1+(p_1-p_2)\,S_1(-A)t+(S_2(-A)(p_1-p_2)^2-p_1p_2S_2(-A)(S_2(-A)-2))t^2 \\ & +[S_3(-A)(p_1-p_2)^3-p_1p_2(p_1-p_2)(S_2(-A)S_1(-A)-3S_3(-A))]t^3 \\ & +(-p_1p_2(p_1-p_2)^2S_3(-A)S_1(-A)+p_1^2p_2^2(S_2(-A)^2-2S_3(-A)S_1(-A)))t^4 \\ & +S_3(-A)p_1^2p_2^2\,(p_1-p_2)\,S_2(-A)t^5-S_3(-A)^2p_1^3p_2^3t^6. \end{aligned}$$

This case consists of four related parts.

Firstly, the substitutions of $S_1(-A)=S_2(-A)=S_3(-A)=-1$ and $\begin{cases} p_1-p_2=k, \\ p_1p_2=1, \end{cases}$ in (3.5), we have the following theorems and corollary.

Theorem 1. *For $n\in\mathbb{N}$, the new generating function of the binary product of k-Fibonacci numbers with Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_nF_{k,n}t^n=\frac{1-t^2-kt^3}{1-kt-(k^2+3)t^2-k(k^2+4)t^3-(k^2+1)t^4+kt^5-t^6}. \tag{3.7}$$

We can state the following corollary.

Corollary 1. *The following identity holds true:*

$$T_nF_{k,n}=S_n(A)S_n(p_1+[-p_2]).$$

Theorem 2. *For $n\in\mathbb{N}$, the new generating function for the combined Tribonacci numbers and k-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} L_{k,n}T_nt^n=\frac{2-kt-(k^2+2)t^2-k(k^2+3)t^3}{1-kt-(k^2+3)t^2-k(k^2+4)t^3-(k^2+1)t^4+kt^5-t^6}. \tag{3.8}$$

Proof. We have

$$L_{k,n}=2S_n(p_1+[-p_2])-kS_{n-1}(p_1+[-p_2]).\text{(see [3])}$$

We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{k,n}T_n t^n &= \sum_{n=0}^{\infty}(2S_n(p_1+[-p_2]) - kS_{n-1}(p_1+[-p_2]))S_n(A)t^n \\
&= 2\sum_{n=0}^{\infty} S_n(p_1+[-p_2])S_n(A)t^n - k\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n \\
&= 2\sum_{n=0}^{\infty} F_{k,n}T_n t^n - k\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n \\
&= 2\sum_{n=0}^{\infty} F_{k,n}T_n t^n - \frac{k}{p_1+p_2}\sum_{n=0}^{\infty}(p_1^n - (-p_2)^n)S_n(A)t^n \\
&= 2\sum_{n=0}^{\infty} F_{k,n}T_n t^n - \frac{k}{p_1+p_2}\left(\sum_{n=0}^{\infty} S_n(A)(p_1t)^n - \sum_{n=0}^{\infty} S_n(A)(-p_2t)^n\right) \\
&= 2\sum_{n=0}^{\infty} F_{k,n}T_n t^n - \frac{k}{p_1+p_2}\left(\frac{1}{1-p_1t-p_1^2t^2-p_1^3t^3} - \frac{1}{1+p_2t-p_2^2t^2+p_2^3t^3}\right) \\
&= 2\sum_{n=0}^{\infty} F_{k,n}T_n t^n - \\
&\quad k\left(\frac{t+(p_1-p_2)\,t^2+\left((p_1-p_2)^2+p_1p_2\right)t^3}{\begin{array}{c}1-(p_1-p_2)t-((p_1-p_2)^2+3p_1p_2)t^2-((p_1-p_2)^3+4p_1p_2(p_1-p_2))t^3\\ -p_1p_2((p_1-p_2)^2+p_1p_2)t^4+p_1^2p_2^2(p_1-p_2)t^5-p_1^3p_2^3t^6\end{array}}\right)
\end{aligned}$$

Let's remember that for k-Lucas numbers $\begin{cases} p_1 - p_2 = k, \\ p_1p_2 = 1, \end{cases}$ and then

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{k,n}T_n t^n &= \frac{2-2t^2-2kt^3}{1-kt-(k^2+3)t^2-k(k^2+4)t^3-(k^2+1)t^4+kt^5-t^6} \\
&\quad -\frac{kt+k^2t^2+k(k^2+1)t^3}{1-kt-(k^2+3)t^2-k(k^2+4)t^3-(k^2+1)t^4+kt^5-t^6} \\
&= \frac{2-kt-(k^2+2)t^2-k(k^2+3)t^3}{1-kt-(k^2+3)t^2-k(k^2+4)t^3-(k^2+1)t^4+kt^5-t^6}
\end{aligned}$$

This completes the proof. □

- Put $k = 1$ in the relationships (3.7) and (3.8), we obtain the following corollaries.

Corollary 2. *For $n \in \mathbb{N}$, the new generating function of the product of Fibonacci numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_nF_n t^n = \frac{1-t^2-t^3}{1-t-4t^2-5t^3-2t^4+t^5-t^6}.$$

Corollary 3. *For $n \in \mathbb{N}$, the new generating function of the product of Fibonacci numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n L_n t^n = \frac{1-t-3t^2-4t^3}{1-t-4t^2-5t^3-2t^4+t^5-t^6}.$$

Secondly, the substitution of $S_1(-A) = S_2(-A) = S_3(-A) = -1$ and $\begin{cases} p_1 - p_2 = 2, \\ p_1 p_2 = k, \end{cases}$ in (3.6), we deduce the following theorems and corollary.

Theorem 3. *For $n \in \mathbb{N}$, the new generating function of the binary product of k-Pell numbers with Tribonacci numbers*

$$\sum_{n=0}^{\infty} T_n P_{k,n} t^n = \frac{t+2t^2+(k+4)t^3}{1-2t-(4+3k)t^2-8(1+k)t^3-k(4+k)t^4+2k^2t^5-k^3t^6}. \tag{3.9}$$

We can state the following corollary.

Corollary 4. *The following identity holds true:*

$$T_n P_{k,n} = S_n(A) S_{n-1}(p_1 + [-p_2]).$$

Theorem 4. *The generating function for the combined Tribonacci numbers and k-Pell-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} Q_{k,n} T_n t^n = \frac{2-2t-2\,(k+2)\,t^2-2\,(3k+4)\,t^3}{1-2t-(4+3k)t^2-8(1+k)t^3-k(4+k)t^4+2k^2t^5-k^3t^6}. \tag{3.10}$$

Proof. We have

$$Q_{k,n} = 2S_n(p_1 + [-p_2]) - 2S_{n-1}(p_1 + [-p_2]), \text{ (see [3]).}$$

We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{k,n} T_n t^n &= \sum_{n=0}^{\infty} (2S_n(p_1+[-p_2]) - 2S_{n-1}(p_1+[-p_2]))S_n(A)t^n \\
&= 2\sum_{n=0}^{\infty} S_n(p_1+[-p_2])S_n(A)t^n - 2\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n \\
&= \frac{2}{p_1+p_2}\sum_{n=0}^{\infty} (p_1^{n+1}-(-p_2)^{n+1})S_n(A)t^n - 2\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n \\
&= \frac{2}{p_1+p_2}\left(p_1\sum_{n=0}^{\infty} S_n(A)(p_1t)^n + p_2\sum_{n=0}^{\infty} S_n(A)(-p_2t)^n\right) \\
&\quad -2\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n \\
&= \frac{2}{p_1+p_2}\left(\frac{p_1}{1-p_1t-p_1^2t^2-p_1^3t^3} + \frac{p_2}{1+p_2t-p_2^2t^2+p_2^3t^3}\right) \\
&\quad -2\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n
\end{aligned}$$

We have

$$\frac{p_1}{1-p_1t-p_1^2t^2-p_1^3t^3} + \frac{p_2}{1+p_2t-p_2^2t^2+p_2^3t^3} =$$

$$\frac{p_1+p_2-p_1p_2(p_1+p_2)t^2 - p_1p_2(p_1^2-p_2^2)t^3}{\begin{array}{c}1-(p_1-p_2)t-(p_1^2-p_2^2+p_1p_2)t^2-((p_1^3-p_2^3+p_1^2p_2-p_1p_2^2)t^3 \\ -p_1p_2(p_1^2-p_2^2-p_1p_2)t^4+p_1^2p_2^2(p_1-p_2)t^5-p_1^3p_2^3t^6\end{array}}$$

So

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{k,n} T_n t^n &= 2\left(\frac{1-p_1p_2t^2-p_1p_2(p_1-p_2)t^3}{\begin{array}{c}1-(p_1-p_2)t-((p_1-p_2)^2+3p_1p_2)t^2-((p_1-p_2)^3+4p_1p_2(p_1-p_2))t^3 \\ -p_1p_2((p_1-p_2)^2+p_1p_2)t^4+p_1^2p_2^2(p_1-p_2)t^5-p_1^3p_2^3t^6\end{array}}\right) \\
&\quad -2\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n
\end{aligned}$$

Let's remember that for k-Lucas numbers $\begin{cases} p_1 - p_2 = 2, \\ p_1 p_2 = k, \end{cases}$ and then

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{k,n} T_n t^n &= 2\frac{1-kt^2-2kt^3}{1-2t-(4+3k)t^2-8(1+k)t^3-k(4+k)t^4+2k^2t^5-k^3t^6} \\ &\quad -2\frac{t+2t^2+(k+4)t^3}{1-2t-(4+3k)t^2-8(1+k)t^3-k(4+k)t^4+2k^2t^5-k^3t^6} \\ &= \frac{2-2t-2(k+2)t^2-2(3k+4)t^3}{1-2t-(4+3k)t^2-8(1+k)t^3-k(4+k)t^4+2k^2t^5-k^3t^6}.\end{aligned}$$

This completes the proof. □

- Put $k = 1$ in the relationships (3.9) and (3.10), we obtain the following corollaries.

Corollary 5. *For $n \in \mathbb{N}$, the new generating function of the binary product of Pell numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n P_n t^n = \frac{t+2t^2+5t^3}{1-2t-7t^2-16t^3-5t^4+2t^5-t^6}.$$

Corollary 6. *For $n \in \mathbb{N}$, the new generating function of the binary product of Pell numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n Q_n t^n = \frac{2-2t-6t^2-17t^3}{1-2t-7t^2-16t^3-5t^4+2t^5-t^6}.$$

Third, the substitution, $S_1(-A) = S_2(-A) = S_3(-A) = -1$ and $\begin{cases} p_1 - p_2 = 3k, \\ p_1 p_2 = -2, \end{cases}$ in (3.6) we deduce the following theorem.

Theorem 5. *For $n \in \mathbb{N}$, the new generating function of the binary product of k-Mersenne numbers and Tribonacci numbers is given by*

(3.11)
$$\sum_{n=0}^{\infty} T_n M_{k,n} t^n = \frac{t+3kt^2+(9k^2-2)t^3}{1-3kt-3(3k^2-2)t^2-3k(9k^2-8)t^3+2(9k^2-2)t^4+12kt^5+8t^6}.$$

We can state the following corollary.

Corollary 7. *The following identity holds true:*

$$T_n M_{k,n} = S_n(A) S_{n-1}(p_1 + [-p_2]).$$

- Put $k = 1$ in the relationship (3.11), we obtain the following corollaries.

Corollary 8. *For $n \in \mathbb{N}$, the new generating function of the binary product of Mersenne numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n M_n t^n = \frac{t+3t^2+7t^3}{1-3t-3t^2-3t^3+14t^4+12t^5+8t^6}.$$

Fourthly, the substitution of $S_1(-A) = S_2(-A) = S_3(-A) = -1$ and $\left\{\begin{array}{l} p_1 - p_2 = 2, \\ p_1 p_2 = k, \end{array}\right.$ in (3.6), we deduce the following theorems and corollary.

For $n \in \mathbb{N}$, the new generating function of the binary product of k-Jacobsthal numbers with Tribonacci numbers

$$\sum_{n=0}^{\infty} T_n J_{k,n} t^n = \frac{t + kt^2 + (k^2+2)t^3}{1 - kt - (k^2+6)t^2 - k(k^2+8)t^3 - 2(k^2+2)t^4 + 4kt^5 - 8t^6}, \tag{3.12}$$

We can state the following corollary.

Corollary 9. *The following identity holds true:*

$$T_n J_{k,n} = S_n(A) S_{n-1}(p_1 + [-p_2]).$$

Theorem 6. *For $n \in \mathbb{N}$, the new generating function for the combined Tribonacci numbers and k-Jacobsthal-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} T_n\ j_{k,n} t^n = \frac{2 - kt - \left(k^2+4\right) t^2 - k\left(k^2+6\right) t^3}{1 - kt - (k^2+6)t^2 - k(k^2+8)t^3 - 2(k^2+2)t^4 + 4kt^5 - 8t^6}. \tag{3.13}$$

Proof. We have

$$j_{k,n} = 2S_n(p_1 + [-p_2]) - kS_{n-1}(p_1 + [-p_2]), \text{ (see [3])}.$$

We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n j_{k,n} t^n &= \sum_{n=0}^{\infty} (2S_n(p_1+[-p_2]) - kS_{n-1}(p_1+[-p_2]))S_n(A)t^n \\
&= 2\sum_{n=0}^{\infty} S_n(p_1+[-p_2])S_n(A)t^n - k\sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])S_n(A)t^n \\
&= \frac{2}{p_1+p_2}\sum_{n=0}^{\infty}(p_1^{n+1}-(-p_2)^{n+1})S_n(A)t^n - k\sum_{n=0}^{\infty} T_n J_{k,n} t^n \\
&= \frac{2}{p_1+p_2}\left(p_1\sum_{n=0}^{\infty} S_n(A)(p_1t)^n + p_2\sum_{n=0}^{\infty} S_n(A)(-p_2t)^n\right) - k\sum_{n=0}^{\infty} T_n J_{k,n} t^n \\
&= \frac{2}{p_1+p_2}\left(\frac{p_1}{1-p_1t-p_1^2t^2-p_1^3t^3} + \frac{p_2}{1+p_2t-p_2^2t^2+p_2^3t^3}\right) - k\sum_{n=0}^{\infty} T_n J_{k,n} t^n \\
&= 2\left(\frac{1-p_1p_2t^2-p_1p_2(p_1-p_2)t^3}{\begin{array}{c}1-(p_1-p_2)t-((p_1-p_2)^2+3p_1p_2)t^2-((p_1-p_2)^3+4p_1p_2(p_1-p_2))t^3 \\ -p_1p_2((p_1-p_2)^2+p_1p_2)t^4+p_1^2p_2^2(p_1-p_2)t^5-p_1^3p_2^3t^6\end{array}}\right) \\
&\quad -k\sum_{n=0}^{\infty} J_n P_n t^n \\
&= \frac{2-4t^2-4kt^3}{1-kt-(k^2+6)t^2-k(k^2+8)t^3-2(k^2+2)t^4+4kt^5-8t^6} \\
&\quad - \frac{kt+k^2t^2+k(k^2+2)t^3}{1-kt-(k^2+6)t^2-k(k^2+8)t^3-2(k^2+2)t^4+4kt^5-8t^6} \\
&= \frac{2-kt-\left(k^2+4\right)t^2-k\left(k^2+6\right)t^3}{1-kt-(k^2+6)t^2-k(k^2+8)t^3-2(k^2+2)t^4+4kt^5-8t^6}.
\end{aligned}$$

This completes the proof. □

- Put $k=1$ in the relationships (3.12) and (3.13), we obtain the following corollaries.

Corollary 10. *For $n \in \mathbb{N}$, the new generating function of the binary product of Jacobsthal numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n J_n t^n = \frac{t+t^2+3t^3}{1-t-7t^2-9t^3-6t^4+4t^5-8t^6}.$$

Corollary 11. *For $n \in \mathbb{N}$, the new generating function of the binary product of Jacobsthal-Lucas numbers and Tribonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n j_n t^n = \frac{2-t-5t^2-7t^3}{1-t-7t^2-9t^3-6t^4+4t^5-8t^6}.$$

4.Generating Functions of Binary Products of Tribonacci Numbers and Chebychev Polynomials

In this part, we now derive the new generating functions of binary product of Tribonacci numbers with Chebychev polynomials.

Replacing $p_1$by $2p_1$ and p_2 by $(-2p_2)$ in (3.3) yields

$$\sum_{n=0}^{\infty} S_n(A)S_n(2p_1+[-2p_2])t^n = \frac{1+4p_1p_2S_2(-A)t^2+8p_1p_2(p_1-p_2)S_3(-A)t^3}{P_{AP}}, \tag{4.1}$$

with

$$\begin{aligned} P_{AP} &= (1-2a_1p_1t)(1-2a_2p_1t)(1-2a_3p_1t)(1+2a_1p_2t)(1+2a_2p_2t)(1+2a_3p_2t) \\ &= 1+2(p_1-p_2)S_1(-A)t+4\left[S_2(-A)(p_1-p_2)^2-p_1p_2(S_1(-A)^2-2S_2(-A))\right]t^2 \\ &\quad -8\left[-S_3(-A)(p_1-p_2)^3+p_1p_2(p_1-p_2)(S_2(-A)S_1(-A)-3S_3(-A)\right]t^3 \\ &\quad +16\left[-p_1p_2(p_1-p_2)^2S_3(-A)S_1(-A)+p_1^2p_2^2(S_2(-A)^2-2S_3(-A)S_1(-A))\right]t^4 \\ &\quad +32S_3(-A)p_1^2p_2^2(p_1-p_2)S_2(-A)t^5-64S_3(-A)^2p_1^3p_2^3t^6. \end{aligned}$$

Remark 2. *In what follows,*$\{T_n(x)\}_{n\in\mathbb{N}}$ *(respectively* $\{U_n(x)\}_{n\in\mathbb{N}}$ *represents the Chebychev polynomial sequence of the first (respectively second) order and* $\{T_n\}_{n\in\mathbb{N}}$ *the Tribonacci sequence.* $P=\{p_1,\ p_2\}$ *(respectively* $A=\{a_1,a_2,a_3\}$*) is the set of roots of caracteristic equation of Chebychev polynomial of first and second order (respectively of Tribonacci numbers).*

By making the following restrictions: $S_1(-A)=S_2(-A)=S_3(-A)=-1$ and $\begin{cases} p_1-p_2=x, \\ 4p_1p_2=-1, \end{cases}$, in (4.1), we get following Theorems:

Theorem 7. *For* $n\in\mathbb{N}$ *, the new generating function for the combined Tribonacci numbers and Chebychev polynomials of second kind is given by*

$$\sum_{n=0}^{\infty} T_nU_n(x)t^n = \frac{1+t^2+2xt^3}{1-2xt-(4x^2-3)t^2-8x(x^2-1)t^3+(4x^2-1)t^4+2xt^5+t^6}, \tag{4.2}$$

Corollary 12. *The following identity holds true:*

$$T_nU_n(x) = S_n(A)S_n(2p_1+[-2p_2]).$$

Theorem 8. *For* $n\in\mathbb{N}$*, the new generating function of the binary product of Tribonacci numbers and Chebychev polynomials of first kind is given by*

$$\sum_{n=0}^{\infty} T_nT_n(x)t^n = \frac{1-xt-\left(2x^2-1\right)t^2-x\left(4x^2-3\right)t^3}{1-2xt-(4x^2-3)t^2-8x(x^2-1)t^3+(4x^2-1)t^4+2xt^5+t^6}. \tag{4.3}$$

Proof. We have

$$T_n(x) = S_n(2p_1+[-2p_2]) - xS_{n-1}(2p_1+[-2p_2]),\ \text{(see [13])}.$$

We see that

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n T_n(x) t^n &= \sum_{n=0}^{\infty} S_n(A)\left(S_n\left(2p_1+[-2p_2]\right)-xS_{n-1}\left(2p_1+[-2p_2]\right)\right)t^n \\
&= \sum_{n=0}^{\infty} S_n(A)S_n\left(2p_1+[-2p_2]\right)t^n - x\sum_{n=0}^{\infty} S_n(A)S_{n-1}\left(2p_1+[-2p_2]\right)t^n \\
&= \sum_{n=0}^{\infty} T_n U_n(x)t^n - \frac{x}{2(p_1+p_2)}\sum_{n=0}^{\infty} S_n(A)((2p_1)^n-(-2p_2)^n)t^n \\
&= \sum_{n=0}^{\infty} T_n U_n(x)t^n \\
&\quad -\frac{x}{2(p_1+p_2)}\left(\sum_{n=0}^{\infty} S_n(A)\left(2p_1t\right)^n - \sum_{n=0}^{\infty} S_n(A)\left(-2p_2t\right)^n\right).
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{1-t-t^2-t^3},$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n T_n(x) t^n &= \sum_{n=0}^{\infty} T_n U_n(x)t^n \\
&\quad -\frac{x}{2(p_1+p_2)}\left(\frac{1}{1-2p_1t-4p_1^2t^2-8p_1^3t^3}-\frac{1}{1+2p_2t-4p_2^2t^2+8p_2^3t^3}\right) \\
&= \sum_{n=0}^{\infty} T_n U_n(x)t^n \\
&\quad -\frac{x}{2}\left(\frac{2t+4(p_1-p_2)t^2+2(4(p_1-p_2)^2+4p_1p_2)t^3}{1-2xt-(4x^2-3)t^2-8x(x^2-1)t^3+(4x^2-1)t^4+2xt^5+t^6}\right) \\
&= \sum_{n=0}^{\infty} T_n U_n(x)t^n \\
&\quad -x\left(\frac{t+2xt^2+(4x^2-1)t^3}{1-2xt-(4x^2-3)t^2-8x(x^2-1)t^3+(4x^2-1)t^4+2xt^5+t^6}\right),
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n T_n(x) t^n &= \frac{1+t^2+2xt^3}{1-2xt-(4x^2-3)t^2-8x(x^2-1)t^3+(4x^2-1)t^4+2xt^5+t^6} \\
&\quad -\frac{xt+2x^2t^2+x(4x^2-1)t^3}{1-2xt-(4x^2-3)t^2-8x(x^2-1)t^3+(4x^2-1)t^4+2xt^5+t^6},
\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} T_n T_n(x)\, t^n = \frac{1 - xt - \left(2x^2 - 1\right) t^2 - x\left(4x^2 - 3\right) t^3}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.$$

This completes the proof. □

5. Conclusion

In this paper, by making use of Eq. (3.1), we have derived some new generating functions of binary products of Tribonacci numbers and well-known Numbers and Chebychev polynomials. The derived lemmas and corollaries are based on symmetric functions and products of these numbers and polynomials.

References

[1] A. Boussayoud, S. Boughaba, M. Kerada, S. Araci and M. Acikgoz, Generating functions of binary products of k-Fibonacci and orthogonal polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.113, (2019), 2575–2586.

[2] A. Boussayoud, A. Abderrezzak and S. Araci, A new symmetric endomorphism operator for some generalizations of certain generating functions, Notes Number Theory Discrete Math.24, (2018), 45–58.

[3] Kh. Boubellota, A. Boussayoud, M. Kerada, Symmetric Functions for k-Fibonacci Numbers and Orthogonal Polynomials, Turkish Journal of Analysis and Number Theory.6(3), (2018), 98–102.

[4] A. Boussayoud, A. Abderrezzak and P.B. Zhang, Symmetric functions for families of generating functions, Neural, Parallel and Scientific Computations.26, (2018), 53–64.

[5] A. Boussayoud, M. Kerada and N. Harrouche, On the k-Lucas numbers and Lucas Polynomials, Turkish Journal of Analysis and Number.5(3), (2017), 121–125.

[6] A. Boussayoud, M. Bolyer and M. Kerada, On some identities and symmetric functions for Lucas and Pell numbers, Electron. J. Math. Analysis Appl. 5(1), (2017), 202–207.

[7] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, Online.J. Anal. Comb.12, (2017), 1–10.

[8] A. Boussayoud, M. Chelgham, S. Boughaba, On some identities and generating functions for Mersenne numbers and polynomials, Turkish Journal of Analysis and Number Theory.6(3), (2018), 93–97.

[9] A. Boussayoud and N. Harrouche, Complete symmetric functions and k- fibonacci numbers, Commun. Appl. Anal. 20, (2016), 457–467.

[10] A. Boussayoud, M. Boulyer and M. Kerada, A simple and accurate method for determination of some generalized sequence of numbers, Int. J. Pure Appl. Math.108, (2016), 503–511.

[11] A. Boussayoud, A. Abderrezzak and M. Kerada, Some applications of symmetric functions, Integers. 15, A#48, (2015), 1–7.

[12] A. Boussayoud, M. Kerada, R. Sahali and W. Rouibah, Some applications on generating functions, J. Concr. Appl. Math. 12, (2014), 321–330.

[13] A. Boussayoud and M. Kerada, Symmetric and generating functions, Int. Electron. J. Pure Appl. Math. 7, (2014), 195–203.

[14] M. Chelgham, A. Boussayoud, Construction of Symmetric functions of generalized Tribonacci numbers, Journal of Science and Arts. 50 (2020), 75–84.

[15] P. Catarino, H. Campos and P. Vasco, On the Mersenne sequence, Ann. Math. Inform.46, (2016), 37–53.

[16] A. Dil and I. Mezo, A symmetric algorithm for hyperharmonic and Fibonacci numbers, Appl. Math. Comput. 206, (2008), 942–951.

[17] M. Elia, Derived sequences, the Tribonacci recurrence and cubic forms, Fibonacci Q. 39(2), (2001), 107–109.

[18] M. Feinberg, Fibonacci–Tribonacci, Fibonacci Q. 1(3) , (1963), 70–74.

[19] P. Filipponi, Incomplete Fibonacci and Lucas numbers, Rend. Circ. Mat. Palermo (Serie II). 45, (1996), 37–56.

[20] V. E. Hoggatt, Fibonacci and Lucas numbers, A publication of the Fibonacci Association. University of Santa Clara, Santa Clara, Houghton Mifflin Company,1969.

[21] E. Kilic, Tribonacci sequences with certain indices and their sums, Ars Comb. 86, (2008), 13–22.

[22] T. Koshy, Fibonacci and Lucas numbers with applications, John Wileyand Sons Inc, NY, 2001.

[23] T. Koshy and Z. Gao, Catalan numbers with Mersenne subscripts, Math. Sci. 38, (2013), 86—-91.

[24] M.Merca, A generalization of the symmetry between complete and elementary symmetric functions, Indian J.Pure Appl. Math.,45, (2014) 75–89.

[25] A.N. Philippou and A.A. Muwafi, Waiting for the Kth consecutive success and the Fibonacci sequence of order K, Fibonacci Q. 20(1), (1982), 28–32.

[26] A. Pinter and H.M. Srivastava, Generating functions of the incomplete Fibonacci and Lucas numbers, Rend. Circ. Mat. Palermo (Serie II).48, (1999), 591–596.

[27] J.S. Ramirez and V.F. Sirvent, Incomplete Tribonacci numbers and polynomials, J. Integer Seq. 17, (14.4.2), (2014), 1–13.

[28] W.R. Spickerman, Binet's formula for the Tribonacci sequence, Fibonacci Q. 20(2), (1982), 118–120.

[29] N. Taskara, K. Uslu and H.H. Gulec, On the properties of Lucas numbers with binomial coefficients, Appl. Math. Lett. 23, (2010), 68–72.

[30] K. Uslo and V. Denis, Somme identities of k–Mersenne numbers, Advances and Applications in Discrete Mathematics, 18(4), 2017, 413–423.

[31] N.N. Vorobiov, Numeros de Fibonacci, Editora MIR, URSS, 1974.

[32] Y.Yazlik and N. Taskara, A note on generalized k–Horadam sequence, Comput. Math. Appl. 63(1), (2012), 36–41.

[33] N. Yilmaz and N. Taskara, Tribonacci and Tribonacci–Lucas numbers via the determinants of special matrices, Appl. Math. Sci. 8(39), (2014), 1947–1955.

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