

GERM ORDER FOR ONE-DIMENSIONAL PACKINGS

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ABSTRACT. Every set of natural numbers determines a generating function convergent for $q \in (-1, 1)$ whose behavior as $q \rightarrow 1^-$ determines a germ. These germs admit a natural partial ordering that can be used to compare sets of natural numbers in a manner that generalizes both cardinality of finite sets and density of infinite sets. For any finite set D of positive integers, call a set S “ D -avoiding” if no two elements of S differ by an element of D . We study the problem of determining, for fixed D , all D -avoiding sets that are maximal in the germ order. In many cases, we can show that there is exactly one such set. We apply this to the study of one-dimensional packing problems.

1. INTRODUCTION

This article is concerned with *packing problems* and *distance-avoiding set problems* in $\mathbb{N} = \{0, 1, 2, \dots\}$. A collection of nonempty finite subsets of \mathbb{N} is called a *packing* if all sets in the collection are disjoint. We restrict to the case where all sets in the collection are translates of some fixed finite set $B \subset \mathbb{N}$ (a *packing body*) that has 0 as its smallest element; we call such a packing a B -packing. The challenge is to find B -packings that cover as much of \mathbb{N} as possible. Meanwhile, given some fixed finite set D of positive integers (a set of *forbidden distances*), a set $S \subseteq \mathbb{N}$ is called *D -avoiding* if no two elements of S differ by an element of D . The challenge is to find D -avoiding sets that are as large as possible. In both situations (B -packings and D -avoiding sets), our notion of “as large as possible” involves the germ-order mentioned in the title, which refines both cardinality of finite sets and density of infinite sets.

The B -packing problem can be seen as a special case of the D -avoiding set problem for suitable D : since the equations $b_1 + s_1 = b_2 + s_2$ and $b_1 - b_2 = s_2 - s_1$ are equivalent, disjointness of the translates $B + s$ ($s \in S$) is equivalent to S being D -avoiding for $D = \{|b - b'| : b, b' \in B, b \neq b'\}$. (See also basic fact (2) in the next section.) Though we were originally motivated by the packing problem, our results apply to the general D -avoiding set problem.

The primary issue is to define what it means for the set S to be “as big as possible.” For instance, consider the D -avoiding set problem with $D = \{3, 5\}$. Three D -avoiding sets are

$$S = \{0, 2, 4, 6, \dots\}, \quad S' = \{1, 3, 5, 7, \dots\}, \quad \text{and} \quad S'' = \{0, 1, 2, 8, 9, 10, \dots\}.$$

(The third of these sets is obtained via a greedy algorithm for constructing D -avoiding sets that considers elements of \mathbb{N} in increasing order, including each element in the set

if it does not introduce a forbidden distance with previous elements. Alternatively, one can describe S'' as the lexicographically first infinite D -avoiding set.) With respect to subset-inclusion, all three sets are maximal: none of them can be augmented without creating a distance belonging to D . However, using a different partial order, we will find that S is “bigger” than S' which is in turn “bigger” than S'' . We define the partial order \preceq , called the *germ order*, using the generating function

$$S_q := \sum_{n \in S} q^n.$$

Definition 1. For subsets S, S' of \mathbb{N} write

$$S' \preceq S$$

iff there exists $\epsilon > 0$ such that $S'_q \leq S_q$ for all q in the interval $(1 - \epsilon, 1)$. In this case we say that S dominates S' (and that S' is dominated by S); we sometimes say that S is bigger than or better than S' . We also write

$$S' \prec S$$

to mean $S' \preceq S$ and $S' \neq S$.

That is, we compare sizes of sets $S \subseteq \mathbb{N}$ by examining the germs “at 1^- ” of the corresponding generating functions S_q . In particular the three $\{3, 5\}$ -avoiding sets S, S', S'' defined above satisfy $S \succeq S' \succeq S''$.

This definition is reminiscent of Abel’s method of evaluating divergent series, in which one assigns to the series $\sum a_n$ the value $\lim_{q \rightarrow 1^-} \sum a_n q^n$ if this limit exists. Its application to measuring sets of natural numbers is (apparently) new, but it is likely to hold little novelty for analytic number theorists, who have long used the philosophically similar but technically more recondite notion of Dirichlet density to measure sets of primes. Our definition also has thematic links to work from the earliest days in the study of infinite series. For instance, Grandi’s formula $1 - 1 + 1 - 1 + 1 - 1 + \dots = 1/2$ corresponds to the fact that the germ of $(2\mathbb{N})_q$ exceeds the germ of $(2\mathbb{N} + 1)_q$ by $1/2 + O(1 - q)$, while Callet’s formula $1 + 0 - 1 + 1 + 0 - 1 + \dots = 2/3$ corresponds to the fact that the germ of $(3\mathbb{N})_q$ exceeds the germ of $(3\mathbb{N} + 2)_q$ by $2/3 + O(1 - q)$.

Our approach resembles the sort of “tame nonstandard analysis” in which \mathbb{R} is replaced by the ordered ring $\mathbb{R}(x)$ where $1/x$ is a formal infinitesimal (also known as “the ring of rational functions ordered at infinity”); our ordering of rational functions corresponds to that of $\mathbb{R}(x)$ if one identifies $1/x$ with $1 - q$.

This paper documents our search for germ-optimal solutions to distance-avoiding set problems. We call a D -avoiding set S a (the) *winner* for D if S dominates S' for every D -avoiding set S' . A winner for D , if it exists, is necessarily unique, as it is the maximum element of the poset of D -avoiding sets ordered by their germs at 1^- . It should however be borne in mind that a general poset can have multiple maximal elements without having a maximum element, and that some posets (like \mathbb{R} with its usual ordering) have no maximal elements at all.

We will prove (Theorem 7) that for all D , any D -avoiding set that is germ-maximal must be eventually periodic. Since the germs of the eventually periodic sets are totally ordered, this implies that there can be at most one germ-maximal D -avoiding set. It follows that if every D -avoiding set is dominated by a maximal D -avoiding set, then there is a (unique) winner.

In some cases (Sections 3, 5) we can both prove existence of winners and give explicit constructions. We do not know if a winner exists for every D , but in Section 6 we describe some strategies for constructing winners, one of which shows that a sequence of local improvements of a certain kind must converge. We do not know that the limit is a winner, however.

Conjecture 2. *For every finite set D of positive integers, there exists a winner for D .*

Regarding periodicity, some of the winners we find in Section 3 are not just eventually periodic but actually periodic. Theorem 3 gives a sufficient condition for winners to be periodic. In other cases, however, “boundary effects” contribute to winners not being periodic from the start.

This contrasts with distance-avoiding set problems in the integers, where there are no boundary effects. The existing literature on D -avoiding problems in dimension 1 is primarily focused on the boundaryless case, and the typical problem is to optimize the density of D -avoiding sets. It is known in this case that for any finite D there is a periodic D -avoiding set realizing the optimal density. See the survey by Liu [Li] for a description of many results in this direction, including the determination of optimal densities for some D .

The motivation for our work was the study of disk packings. It is our hope that the approach taken here will ultimately lead to results establishing a strong kind of uniqueness for optimal sphere-packings in dimensions 2, 8, and 24. (See [Co] for a survey of the recent breakthroughs in the study of 8- and 24-dimensional sphere-packing.) We also hope that the germ approach will have relevance to the study of densest packings in other dimensions.

For alternative approaches to measuring efficiency of packings, see [Ku]. The most sophisticated of these approaches is that of Bowen and Radin [Bo]; their ergodic theory approach has attractive features (for instance, it works in spaces with nonamenable symmetry groups), but it does not seem to work so well when the region being packed is not the entire space. Packings in \mathbb{N} could be viewed as special packings of $\mathbb{R}^{\geq 0}$; the lack of symmetry makes it hard to apply the constructions of Bowen and Radin.

See also [Be], [Bl], [Ch], and [Ka] for work on measuring sizes of sets bearing some philosophical similarity to ours.

2. BASIC FACTS

We begin by recording some basic facts about the generating functions S_q .

- (1) The boundedness of the coefficients of S_q implies that S_q converges for all complex q inside the unit circle, though we will only care about q in the interval $(0, 1)$.
- (2) If the sets $B + s$ ($s \in S$) are disjoint, then $(B + S)_q = B_q S_q$, so maximizing the union of the S -translates of B (with respect to germ-order) is equivalent to maximizing the translation-set S .
- (3) If S is finite, $S_q = |S| + o(1)$, or equivalently, $S_q \rightarrow |S|$ as $q \rightarrow 1^-$; if S is infinite, S_q diverges as $q \rightarrow 1^-$.
- (4) If $S = \{a, a + d, a + 2d, \dots\}$ with $a \geq 0$ and $d > 0$, then

$$S_q = \left(\frac{1}{d}\right) \frac{1}{1-q} + \left(\frac{d-1-2a}{2d}\right) + O(1-q).$$

- (5) More generally, if S is infinite with density α , then

$$S_q = \alpha \frac{1}{1-q} + o\left(\frac{1}{1-q}\right).$$

One can prove this by considering the series $S_q - \alpha \frac{1}{1-q}$; the assumption that S has density α implies that the series is Cesaro summable, hence Abel summable, which implies that the series is $o\left(\frac{1}{1-q}\right)$.

- (6) The set S is *eventually periodic* iff there exist $N \in \mathbb{N}$ and $d \geq 1$ such that for all $n \geq N$, $n \in S$ iff $n + d \in S$. In this case $S_q := \sum_{n \in S} q^n$ is a rational function of q , and indeed is of the form $P(q)/(1 - q^d)$ for some polynomial P . The converse is also true: if S_q is rational, then the Skolem-Mahler-Lech theorem tells us that the set of indices n such that the coefficient of q^n in S_q vanishes is the union of a finite set and a union of finitely many arithmetic progressions, which implies that S is eventually periodic. We call such sets S **rational**. Note that this usage coincides with the notion of rationality for subsets of a monoid in automata theory, specialized to the monoid \mathbb{N} . When S is rational, S_q has a simple pole at 1, and letting $t = 1 - q$ we can expand S_q as a Laurent series $\sum_{n \geq -1} a_n t^n$ where a_{-1} is the density of S .
- (7) If S is not eventually periodic, then as we noted above S_q is not rational. This fact, combined with the fact that the coefficients of S_q belong to the finite set $\{0, 1\}$, allows us to apply a classic result of Szegő (see e.g. [TW]) to conclude that S_q is transcendental and has the unit circle as its natural boundary.

We now turn to some basic facts about the germ order \preceq .

- (8) It is easy to see that \preceq is indeed a partial order. In the case where S and S' are finite, the germ-ordering refines ordering by cardinality. If S and S' have finite but nonempty symmetric difference, then S_q and S'_q differ by a nonzero polynomial, and since a polynomial can change sign only finitely often, one of

the sets must strictly dominate the other in the germ-ordering. Likewise, if S and S' are both eventually periodic and distinct, S_q and S'_q differ by a nonzero rational function, and once again one set must strictly dominate the other in the germ-ordering. Thus \preceq is a total order on the rational subsets of \mathbb{N} , and by (5) above it refines the preorder given by comparing densities.

- (9) Although \preceq is a total ordering for rational subsets of \mathbb{N} , the same is not true for unrestricted subsets of \mathbb{N} ; for instance, if S is the set of natural numbers whose base ten expansion has an even number of digits and S' is its complement, then it can be shown that S and S' are \preceq -incomparable.
- (10) For every nonempty S , $(S + 1)_q = qS_q \prec S_q$, so germs are emphatically not translation-invariant.

3. EXAMPLES OF PERIODIC WINNERS

In this section we give a sufficient condition for a winner to exist. Under this condition the winner is always periodic. This condition is not necessary, however, as shown by the examples in Section 5 of sets D for which there exists a winner that is not periodic. We show in Section 4 that winners are always eventually periodic.

For D a fixed finite set, we write $\|D\|$ for the largest element of D . In this section we will often find it convenient to refer to sets $S \subset \mathbb{N}$ in terms of their indicator functions; thus we will often view S as an element of $\{0, 1\}^{\mathbb{N}}$ and will correspondingly write S and subsets of S of the form $S \cap [a, b]$ as bit strings (of length $b - a + 1$ in the latter case). We denote the concatenation of bit strings A and B (with A finite) by AB . This slight abuse of notation should cause no confusion.

3.1. Some periodic winners, including the symmetric case. A D -avoiding bit string R that is finite and has length greater than $\|D\|$ is called **repeatable** if the concatenation RR is D -avoiding. Note that in this case the infinite string $RRR \cdots$ is also D -avoiding.

Theorem 3. *Fix D . If there is an integer $m > \|D\|$ such that the (germ-)maximal D -avoiding string of length m is repeatable, then the infinite string obtained by repeating this string is the winner. In particular the winner exists and is periodic.*

An example will demonstrate the idea of the proof. With $D = \{3, 5\}$ and window size $m = 8$, the best D -avoiding string is 10101010, which is repeatable. Theorem 3 asserts that as a consequence of this, the infinite periodic string S with repetend 10101010, i.e., the set of even integers, is the winner. To justify this, consider any challenger S' and compare S' to S in positions 1–8, then in positions 9–16, then 17–24, etc. The set S wins (or ties) every time, hence S dominates S' . Note that in this example choosing $m = 6$ doesn't work, as the best D -avoiding string of length 6 is 111000 which is not repeatable.

Proof of Theorem 3: Let m be as in the statement of the theorem, and let S be the set corresponding to the periodic infinite string whose repetend is the germ-maximal D -avoiding string of length m . Let S' be the set corresponding to any other D -avoiding

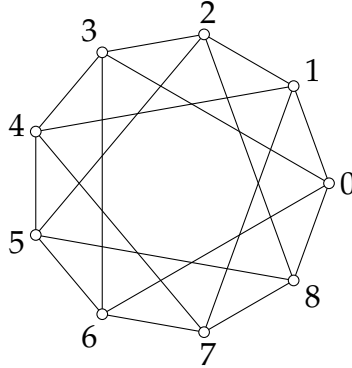


FIGURE 1. The germ-maximal independent set of vertices in this circulant graph is $\{0, 2, 4\}$.

infinite string. Then $|S|_q - |S'|_q$ can be written as the sum $\sum_{i=0}^{\infty} q^{im}(p(q) - p'_i(q))$ where $p(q)$ and $p'_i(q)$ ($i \geq 0$) are polynomials of degree at most $m - 1$ with all coefficients equal to 0 or 1. Since there are only finitely many possibilities for the coefficients of $p(q) - p'_i(q)$, and since the coefficients of $p(q)$ form the germ-maximal D -avoiding string of length m , there exists a fixed $\epsilon > 0$ such that for all i , the polynomial $p(q) - p'_i(q)$ is either identically zero or else positive on the interval $(1 - \epsilon, 1)$. The claim follows. \square

Theorem 3 applies to many, but definitely not all, sets D . For instance we call D **symmetric** if there is an integer $k > \|D\|$ such that $i \in D$ iff $k - i \in D$. (In other words D is symmetric if $-D$ is a translate of D , specifically $-D = D - k$.) We call the number k the **offset of symmetry** for D . Corollary 5 below shows that Theorem 3 applies to all symmetric sets.

Example 4. Consider $D = \{1, 3, 6, 8\}$, which is symmetric with offset of symmetry $k = 9$. A union of congruence classes modulo 9 is D -avoiding if and only if the corresponding vertices form an independent set in the *circulant graph* shown in Figure 1. Here the edges join classes which differ by $\pm 1 (= \pm 8)$ or $\pm 3 (= \pm 6)$. The germ-maximal independent set in this graph is $\{0, 2, 4\}$, whose corresponding bit string is both optimal among length 9 strings and repeatable, so by Theorem 3 the set $\{n \in \mathbb{N} : n \equiv 0, 2, 4 \pmod{9}\}$ is the winner.

Corollary 5. *If D is symmetric then there is a periodic winner, with period dividing the offset of symmetry.*

Proof. Fix symmetric D and let k be the offset of symmetry. We claim that every D -avoiding string of length k is repeatable, so in particular the best D -avoiding string of length k is repeatable. Thus Theorem 3 applies with $m = k$.

To see why this is true, let s be a D -avoiding string of length k and consider the string ss , whose two halves correspond to the sets S and $S + k$. Suppose this is not D -avoiding. Then there must be elements x in S and y in $S + k$ with $y - x \in D$. By

symmetry we have $k - (y - x) = x - (y - k) \in D$. But $y \in S + k$ so $y - k \in S$, and since $x \in S$, this contradicts that S is D -avoiding. \square

For example, if $D = \{1, 2, \dots, k - 1\}$, then Corollary 5 implies that there is a winner, which is easily seen to consist of the multiples of k .

Example 6. For non-symmetric D , it is still sometimes possible to apply Theorem 3. For instance suppose $D = \{1, 2, n\}$. If $n \not\equiv 0 \pmod{3}$, then the winner is clearly the periodic sequence with repetend 100. For $D = \{1, 2, 3n\}$ with $n > 1$, one can show that the best D -avoiding string of length $3n + 1$ is $(100)^n 0$. This is repeatable, so Theorem 3 tells us we have the winner. Note however that because D is not symmetric, D -avoiding strings s of this (or any) length are not guaranteed to be repeatable. So, some analysis is required (a) to determine exactly which one is best and (b) to verify that it happens to be repeatable.

Some other D 's for which we can manually determine that the optimal sequence (of some length m) is repeatable:

$$D = \{1, 3, 4\} \quad (m = 7, s = 1010000)$$

$$D = \{2, 3, 5\} \quad (m = 7, s = 1100000)$$

$$D = \{2, 3, 6\} \quad (m = 9, s = 110001000)$$

$$D = \{2, 3, 7\} \quad (m = 10, s = 1100011000)$$

$$D = \{3, 4, 7\} \quad (m = 10, s = 1110000000)$$

Interestingly, in each of these cases the optimum is obtained by the greedy algorithm.

Clearly it would be helpful to have a bound on the window size. In each example above we have $m \in D + D$, but we do not know if this must always be the case, even for sets of the form $D = \{a, b, a + b\}$ corresponding to packings of the body $B = \{0, a, a + b\}$.

In Section 5 we show that there exist 3-element D 's for which the optimal D -avoiding set is not periodic (but rather only eventually periodic). Thus Theorem 3 cannot be used, and in particular, for such a D , there can be no (finite) window size m such that the optimal D -avoiding string of length m is repeatable.

We remark that if the packing body B is symmetric, then the associated distance set D need not be symmetric; e.g., consider $B = \{0, 1, 4, 5\}$ and $D = \{1, 3, 4, 5\}$. Also, the reverse is possible: consider $B = \{0, 1, 3\}$ (nonsymmetric) and $D = \{1, 2, 3\}$ (symmetric).

4. STRUCTURE OF WINNERS: EVENTUAL PERIODICITY

Although we believe (see Conjecture 2) that for every D there is a maximal D -avoiding set, it is a priori conceivable that there is no germ-maximal D -avoiding set; that is, there might exist D -avoiding sets $T_1 \prec T_2 \prec T_3 \prec \dots$ but no D -avoiding set that dominates them all.

The closest we have come to proving Conjecture 2 is the following result:

Theorem 7. *For every finite set D of positive integers, every germ-maximal D -avoiding set is rational.*

Note that Example 11 shows that the theorem cannot be strengthened to assert that the germ-maximal sets must be periodic.

Theorem 7, in combination with the fact that the rational sets are totally ordered under the germ ordering, implies

Corollary 8. *For every finite set D of positive integers, there is at most one germ-maximal D -avoiding set.*

Our approach to proving Theorem 7 uses a block coding of the kind often employed in dynamical systems theory. We prepare for the proof by introducing this coding and proving two helpful lemmas.

Let $m = \max(D) + 1$ and replace the indicator sequence of S (an element of $\{0, 1\}^{\mathbb{N}}$) by a symbolic sequence using a block code of block length m , with an alphabet containing (at most) 2^m symbols, which we will call **letters**. More concretely, if the indicator sequence of S is written as (b_0, b_1, b_2, \dots) (where b_n is 1 or 0 according to whether $n \in S$ or $n \notin S$), then we define the m -**block encoding** of (b_0, b_1, b_2, \dots) to be (w_0, w_1, w_2, \dots) where the letter w_n is the m -tuple $(b_n, b_{n+1}, \dots, b_{n+m-1})$; we call w_n a **consonant** or a **vowel** according to whether $b_n = 1$ or $b_n = 0$ (conditions that align with the respective cases $n \in S$ and $n \notin S$). Say that a letter $\alpha = (b_1, \dots, b_m)$ in $\{0, 1\}^m$ is **legal** if the set $\{i : b_i = 1\}$ is D -avoiding; we let \mathcal{A} be the set of legal letters. Given two letters α and α' in \mathcal{A} , say that $\alpha' = (b'_1, \dots, b'_m)$ is a **successor** of $\alpha = (b_1, \dots, b_m)$ iff $b'_i = b_{i+1}$ for $1 \leq i \leq m-1$. For every set $S \subseteq \mathbb{N}$, the associated block-encoding $w = (w_0, w_1, w_2, \dots)$ has the property that for all $n \geq 1$, w_n is a successor of w_{n-1} ; S is D -avoiding if and only if w has the additional property that every letter w_n is legal. Call such an infinite word (w_0, w_1, w_2, \dots) **D -legal**; there is a one-to-one correspondence between D -avoiding sets and D -legal infinite words, and finding a germ-maximal D -avoiding set is equivalent to finding a D -legal infinite word for which the set of locations of consonants is germ-maximal. We write $w \preceq w'$ iff the associated sets S, S' satisfy $S \preceq S'$.

Let S be a D -avoiding subset of \mathbb{N} , and let $w = (w_0, w_1, \dots)$ be the associated infinite word in $\mathcal{A}^{\mathbb{N}}$.

Let α be a letter that occurs infinitely often in the word w , and let $K = \{k \in \mathbb{N} : w_k = \alpha\} = \{k_0, k_1, k_2, \dots\}$, where $k_0 < k_1 < k_2 < \dots$. After possibly deleting a prefix, we divide the infinite word w into infinitely many subwords $(w_{k_0}, w_{k_0+1}, \dots, w_{k_1-1})$, $(w_{k_1}, w_{k_1+1}, \dots, w_{k_2-1})$, $(w_{k_2}, w_{k_2+1}, \dots, w_{k_3-1})$, \dots . Each of these finite words is associated with the word $c_k := (w_{k_{i-1}}, w_{k_{i-1}+1}, \dots, w_{k_i-1}, w_{k_i})$ (for $k \geq 1$) that both begins and ends with the letter α ; define a **circular word** as a word whose first and last letters are the same. (Note that we are not modding out by cyclic shift of such words.) Let \mathcal{C} be the set of all circular words beginning and ending with α . We define the **length** of a circular word to be the number of letters it contains, counting its first and last letter as a single letter. (Thus, if α, β , and γ are letters, the circular word $(\alpha, \beta, \gamma, \alpha)$, which for brevity we may also write as $\alpha\beta\gamma\alpha$, has length 3.) If $c \in \mathcal{C}$ has length a and $c' \in \mathcal{C}$ has

length a' , let $c : c'$ denote the circular word of length $a + a'$ in \mathcal{C} obtained by deleting the final α from c and then concatenating with c' . The operation $:$ is associative, and indeed, the word w itself can be written as $p : c_1 : c_2 : c_3 : \dots$, where p is a possibly empty prefix and the circular words c_i are **primitive** (i.e., each c_i contains α only at the beginning and at the end).

Every circular word $c \in \mathcal{C}$ is associated with a polynomial $P_c = P_c(q)$ (sometimes we will omit the subscript or will write P_i to mean P_{c_i}) whose degree is at most the length a of the circular word c and whose coefficients are 0's and 1's according to whether the respective letters in the circular word are vowels or consonants; we call P_c the **generating function** of c . So if $w = c_1 : c_2 : c_3 : \dots$ is the D -legal infinite word representing the D -avoiding set S , S_q can be written as $P_1 + q^{a_1}P_2 + q^{a_1+a_2}P_3 + \dots = P_1 + A_1P_2 + A_1A_2P_3 + \dots$ where a_i is the length of c_i and A_i is q^{a_i} .

For any circular word c with length a , we define $|c| := P_c(q)/(1 - q^a)$; $|c|$ can equivalently be defined as the generating function of the infinite periodic word $c : c : c : c : \dots$. Given two periodic words c, c' in \mathcal{C} (possibly of different lengths), write $c \preceq c'$ iff $|c| \preceq |c'|$; call this the germ-ordering on circular words. It is clear from the second way of defining $|c|$ that $|c| = |c : c| = |c : c : c| = \dots$. We have $|c| = |c'|$ iff $c : c : c : \dots = c' : c' : c' : \dots$.

The following two lemmas are the linchpins of the proof of Theorem 7.

Lemma 9. *If $c \preceq c'$, then $c \preceq c : c' \preceq c' : c \preceq c'$.*

Proof: Write $|c| = P/(1 - A)$ and $|c'| = P'/(1 - A')$; we also have $|c : c'| = (P + AP')/(1 - AA')$ and $|c' : c| = (P' + A'P)/(1 - AA')$. The stipulated relation $c \preceq c'$ is equivalent to $P/(1 - A) \preceq P'/(1 - A')$, or

$$(1) \quad P(1 - A') \preceq P'(1 - A);$$

the desired relations $c \preceq c : c'$, $c : c' \preceq c' : c$, and $c' : c \preceq c'$ are respectively equivalent to

$$(2) \quad P/(1 - A) \preceq (P + AP')/(1 - AA'),$$

$$(3) \quad (P + AP')/(1 - AA') \preceq (P' + A'P)/(1 - AA'), \text{ and}$$

$$(4) \quad (P' + A'P)/(1 - AA') \preceq P'/(1 - A').$$

To prove (2), note that (by cross-multiplying, expanding, and cancelling terms) we can write it equivalently as $-AA'P \preceq AP' - AP - AAP'$, which is just (1) multiplied by A . The two denominators in (3) are identical, so (3) is equivalent to $P + AP' \preceq P' + A'P$, which in turn is equivalent to (1). The proof of (4) is similar to the proof of (2). \square

Note that the proof also tells us that if $c \prec c'$, then $c \prec c : c' \prec c' : c \prec c'$.

Lemma 10. *If the concatenation $w = c_1 : c_2 : c_3 : \dots$ is germ-maximal in the set of D -legal words, then we must have $c_1 \succeq c_2 \succeq c_3 \succeq \dots$ in the germ-ordering.*

Proof: We will show that $c_1 \succeq c_2$ since that contains the idea of the general argument. If $c_1 = c_2$ there is nothing to prove, so assume $c_1 \neq c_2$, and let $w' = c_2 : c_1 : c_3 : \dots$, which must be D -legal if w is (indeed, the whole reason for the block coding was to make this claim true). The sets S and S' respectively associated with w and w' have finite symmetric difference, so w and w' must be comparable. Since we are assuming w is germ-maximal, we must have $w \succeq w'$ in the germ ordering. That is, we must have

$$P_1 + A_1 P_2 \succeq P_2 + A_2 P_1$$

(all the later terms match up and cancel). But this is equivalent to $P_1/(1 - A_1) \succeq P_2/(1 - A_2)$, so $c_1 \succeq c_2$ as claimed. \square

Proof of Theorem 7: Let S be a D -avoiding subset of \mathbb{N} that is germ-maximal. Let $w = (w_0, w_1, \dots)$ be the associated infinite word in $\mathcal{A}^{\mathbb{N}}$. Suppose first that the letter $w_0 = \alpha$ occurs infinitely often in w , and let c_1, c_2, \dots be the primitive circular words starting and ending with α with $w = c_1 : c_2 : c_3 : \dots$.

By an easy pigeonhole argument, for all N there must exist $i, j \geq N$ with $i < j$ such that the sum of the lengths of the words c_i, c_{i+1}, \dots, c_j is a multiple of the length of c_1 , say r times the length of c_1 . Let w' be the word obtained from w by replacing the $j - i + 1$ letters c_i, c_{i+1}, \dots, c_j by r occurrences of the letter c_1 . Let S and S' be the sets associated with w and w' , respectively. Lemma 10 tells us that $c_1 \succeq c_i \succeq c_{i+1} \succeq \dots \succeq c_j$, so repeated application of Lemma 9 gives $|c_1 : c_1 : \dots : c_1| \succeq |c_i : c_{i+1} : \dots : c_j|$. If strict inequality holds, then $w' \succ w$, contradicting maximality of w . (Here we use the fact that the difference $S'_q - S_q$ can be expressed as $1 - q^n$ times $|c_1 : c_1 : \dots : c_1| - |c_i : c_{i+1} : \dots : c_j|$, where n is the common value of ra_1 and $a_i + a_{i+1} + \dots + a_j$.) So we must have $|c_1 : c_1 : \dots : c_1| = |c_i : c_{i+1} : \dots : c_j|$, implying that c_i, c_{i+1}, \dots, c_j are all the circular word c_1 . Since the circular words c_i are in germ-decreasing order, this means that c_1, c_2, \dots, c_N are all equal. Since this is true for all N , we must have $w = c_1 : c_1 : \dots : c_1$; that is, w is periodic.

If the letter w_0 does not occur infinitely many times in w , then we instead find the smallest i for which the letter w_i does occur infinitely often in w . Ignoring the prefix w_0, \dots, w_{i-1} , we may apply the preceding argument to the letters $w_i, w_{i+1}, w_{i+2}, \dots$. The conclusion now is that the word w is eventually periodic, with the periodicity beginning at the letter w_i . \square

5. SOME NON-PERIODIC WINNERS

We now demonstrate another technique for proving that strings are winners, different from the method of Section 3. It is not clear how broadly this approach can be applied. The following infinite family of examples was discovered by Abrams' student Eric Zhang.

Proposition 11. *Let $D = \{2, 4, 6k + 1\}$ for a positive integer k . The winner for D is*

$$S^* = (110000)^k 100100 100100 \dots$$

In particular, the winner is not periodic.

Proof. We split all strings into *bytes* of length 6. Define two D -avoiding strings of k bytes as follows:

$$\begin{aligned} A &= (110000)^k \\ B &= (100100)^k. \end{aligned}$$

We begin by proving the following facts:

Fact 1. A, AB are the germ-maximal D -avoiding strings for their respective lengths.

Fact 2. Let Q, R be strings of length $6k$ such that QR is D -avoiding. Then either $R \preceq B$ or $QR \preceq BB$.

We first prove Fact 2. Suppose we fix a D -avoiding string R of length $6k$ such that $B \prec R$. We will show that $QR \preceq BB$ as long as QR is a D -avoiding string of length $2 \times 6k$.

Let $D' = \{2, 4\}$.

We first observe that the assumption $B \prec R$ restricts the structure of R significantly. Recall that any D -avoiding string is also D' -avoiding. In particular, any byte (of length 6) of any D -avoiding string contains at most two 1's. As each byte of B already has exactly two 1's, $B \prec R$ means the same is true for R . Write R in bytes of length 6 as $R = R_1 R_2 \cdots R_k$. Note that the germ-maximal D -avoiding byte is 110000, and that the runner-up is 100100. To satisfy $B \prec R$, therefore, at least one R_i must equal 110000. Further, to avoid distances 2 and 4, we must fill in four consecutive 0's right before R_i . That is, if $R_i = 110000$, then $R_{i-1} = 110000$. It follows that $R_1 = 110000$.

Now, still with the assumption $B \prec R$, consider a D -avoiding string QR of length $2 \times 6k$. Suppose $BB \prec QR$. Write $QR = Q_1 \dots Q_k R_1 \dots R_k$ in bytes. By the argument above, $R_1 = 110000$. Again to have $BB \prec QR$ each Q_i must have two 1's, so as above working from Q_k back to Q_1 we see that each $Q_j = 110000$, i.e., $Q = A$. Yet this is a contradiction, since the distance between the 1 in the first position of Q_1 and the 1 in the second position of R_1 is the forbidden $6k + 1$. So $QR \preceq BB$ and Fact 2 is proved.

Next we prove Fact 1. It is straightforward to show that A is the germ-maximal D -avoiding string of length $6k$. With $D' = \{2, 4\}$, we have of course that any D -avoiding string is also D' -avoiding. For strings of length $6k$, the forbidden distance $6k + 1$ doesn't come into play and so any D' -avoiding string is also D -avoiding. The germ-maximal D' -avoiding string of length $6k$ is easily seen to be A .

To prove AB is the germ-maximal D -avoiding string of its length, we can use Fact 2. Suppose there is a D -avoiding string P of length $2 \times 6k$ such that $P \succ AB$. Write $P = QR$ where Q, R each have length $6k$. The argument of the preceding paragraph shows $Q \preceq A$. If in addition $R \preceq B$, we would have the contradiction $QR \preceq AB$, so it must instead be the case that $R \succ B$. Now Fact 2 implies $P = QR \preceq BB \preceq AB$, a contradiction. So Fact 1 is proved.

Using Facts 1 and 2, we may now complete the proof of the proposition. Let S be any D -avoiding infinite string. We will show $S \preceq S^* = ABBBB \cdots$.

Henceforth we write $S = S_1 S_2 S_3 \cdots$ where each S_i is a string of length $6k$, which we refer to as *block i* . (So a block is made of k bytes.) We use the word *span* to mean a collection of indices of consecutive blocks.

By Fact 1 above, S^* beats or ties S on block 1 and also on the span $\{1, 2\}$, i.e., on the union of blocks 1 and 2.

For each k , we inductively construct a partition P_k of the set $\{1, 2, \dots, k\}$ into spans such that S^* beats or ties S on each of these spans. Each span in the partition P_k will have size 1 or 2. For $k = 1, 2$ let P_k be the singletons $\{\{1\}\}$ and $\{\{1, 2\}\}$, respectively. For $k \geq 3$, assume we have partitions P_n for all $n < k$. Let $QR = S_{k-1} S_k$. Recall S^* has BB in the corresponding blocks. We define P_k as follows:

- (1) if $R \preceq B$, let P_k be $P_{k-1} \cup \{\{k\}\}$;
- (2) otherwise, by Fact 2, $QR \preceq BB$; in this case let P_k be $P_{k-2} \cup \{\{k-1, k\}\}$.

To show $S \preceq S^*$, it is sufficient to partition the positive integers into spans such that S^* beats or ties S on the blocks indexed by each span. We use König's Lemma to produce such a partition from the sequence of partitions P_k . Specifically we construct an acyclic graph with vertex set equal to the positive integers by joining each $k \geq 3$ to exactly one of $k-1$ or $k-2$, depending on which partition was used to construct P_k . By König's Lemma this graph contains an infinite path $\{j_1, j_2, \dots\}$, which must be increasing since each vertex only connects to one smaller vertex. By construction, the partitions P_{j_1}, P_{j_2}, \dots are nested, so their union is the desired partition of the positive integers. \square

A similar analysis can be applied to certain other examples, such as $D = \{4, 7, 11\}$ and $D = \{5, 8, 13\}$. Note that these both correspond to packing problems. In each case the appropriate versions of Facts 1 and 2 are finite checks and can be verified e.g. using Algorithm 12 (Section 6). However A and B can become rather long; for $D = \{5, 8, 13\}$, for instance, we use blocks of length 18 to prove Fact 2, while Fact 1 requires going a little further out. We do not know which or how many (non-symmetric) D 's, even of size 3, will yield to this approach.

6. SEARCH FOR WINNERS: SOME ALGORITHMS

6.1. A finite set of candidates for an optimal sequence. We next present a fast algorithm for finding the germ-optimal D -avoiding bit string of any fixed (finite) length.

We note two properties of our setup:

- The condition for being a D -avoiding string is a local condition, i.e. to check whether the string S is D -avoiding can be done locally by checking the condition on contiguous substrings of S of length no more than $\|D\| + 1$.
- The germ order has the property that given two strings of the same length A and B , if A is bigger than B in germ order then AX is bigger than BX for all strings X .

These two properties are enough to allow for a standard dynamic programming algorithm on a line to compute, in linear time in the length l , a finite list of sequences of length l each of which is optimal conditioned on the values of its final $\|D\|$ bits. The list has one sequence ending with each legal string of length $\|D\|$, so the size of the list is constant in l . The optimal sequence of length l can then be found by comparing the sequences on the list.

Algorithm 12. *Given a finite set $D \subset \mathbb{Z}_{\geq 0}$, fix any integer $m > \|D\|$. Let $\{\sigma_1, \dots, \sigma_r\}$ denote the set of all D -avoiding strings of length m . Then there is an efficient algorithm to compute, for any $k \geq 1$, the set $S_k = \{s_{k,1}, \dots, s_{k,r}\}$, where $s_{k,i}$ is the optimal D -avoiding string of length km that ends in the substring σ_i . The optimal D -avoiding string of length km is then the best element of the finite set S_k .*

We briefly spell out the dynamic program. Initially define $s_{1,i} = \sigma_i$, so that $S_1 = \{\sigma_1, \dots, \sigma_r\}$. Assuming we have $S_k = \{s_{k,1}, \dots, s_{k,r}\}$ where $s_{k,i}$ is the optimal D -avoiding string of length km that ends in the substring σ_i , we efficiently generate S_{k+1} as follows. For each i , define $s_{k+1,i}$ to be the D -avoiding string of biggest germ order from the set $S_k \sigma_i$ (that has r elements) consisting of the elements of S_k concatenated with σ_i . Then define $S_{k+1} = \{s_{k+1,1}, \dots, s_{k+1,r}\}$. The optimality of $s_{k+1,i} = \sigma_{j_1} \sigma_{j_2} \dots \sigma_{j_k} \sigma_i$ follows from the guarantee that if $s_{k+1,i}$ is the optimal string of length $(k+1)m$ ending in σ_i then the substring $\sigma_{j_1} \sigma_{j_2} \dots \sigma_{j_k}$ must be optimal among strings that end in σ_{j_k} and hence was one of the elements considered in S_k .

In practice, these lists tend to stabilize fairly quickly.

6.2. Convergence and maximality of local optimization. Motivated by the algorithm above, we explore the result of a sequence of “local improvements,” each of which replaces a local patch of a string with the optimal substring that is consistent with the adjacent regions. Specifically, consider a finite difference set D and two “boundary” strings $A, B \in \{0, 1\}^{\|D\|}$. In light of the comments above, it follows that for any $\ell \geq \|D\|$ there is a unique maximum string $G \in \{0, 1\}^\ell$ for which

$$AZB \preceq AGB$$

for all strings Z of length ℓ . We introduce the notation $\Gamma_\ell(A, B)$ for this maximum string. It follows that if $w \in \{0, 1\}^*$ (or $\{0, 1\}^{\mathbb{N}}$) is a germ-maximal D -avoiding string then any appearance of the strings A and B in w separated by exactly ℓ symbols must enclose the string $\Gamma_\ell(A, B)$. (Note that it makes sense to define this notion for $\ell < \|D\|$, though in this case one must focus on *consistent* pairs (α, β) for which there exists at least one such x .)

In general, for two strings $w, w' \in \{0, 1\}^{\mathbb{N}}$, we write

$$w \vdash_\ell^{A,B} w'$$

if w can be written $XAYBZ$ for a string $Y \in \{0, 1\}^\ell$ so that $w' = XAGBZ$, where $G = \Gamma_\ell(A, B)$. We likewise define

$$w \vdash_\ell w'$$

if $w \vdash_\ell^{A,B} w'$ for some pair $A, B \in \{0, 1\}^{\|D\|}$. Observe that

$$w \vdash_\ell w' \quad \Rightarrow \quad w \preceq w'.$$

Theorem 13. *Let D be a finite subset of \mathbb{N} and $\ell \geq \|D\|$. Let $w = w^{(0)} \in \{0, 1\}^{\mathbb{N}}$ be a D -avoiding string and let $w^{(1)}, w^{(2)}, \dots$ be a sequence of elements of $\{0, 1\}^{\mathbb{N}}$ for which*

$$w^{(0)} \vdash_\ell w^{(1)} \vdash_\ell w^{(2)} \vdash_\ell \dots$$

Then this sequence converges in the sense that there is a string $w^ \in \{0, 1\}^{\mathbb{N}}$ so that for any position t , $w_t^* = w_t^{(k)}$ for all sufficiently large k .*

Proof. Define $g^{(i)}$ to be the power series associated with $w^{(i)}$. Then for each i we may write

$$g^{(i+1)} = g^{(i)} + X^t p(X)$$

where $p(X)$ is a polynomial of degree no more than $\ell - 1$ with coefficients in $\{-1, 0, 1\}$. When $w^{(i)} \neq w^{(i+1)}$, the value of t is determined by the length of the common prefix of the two strings. As $w^{(i)} \preceq w^{(i+1)}$, $X^t p(X) \geq 0$ and hence $p(X) \geq 0$ in the germ order.

Let $P_\ell = \{a_{\ell-1}X^{\ell-1} + \dots + a_0 \mid a_i \in \{-1, 0, 1\}\}$ denote the set of all polynomials of degree at most $\ell - 1$ with coefficients in $\{-1, 0, 1\}$, let

$$R_\ell = \left\{ x \in \mathbb{R} \mid p(x) = 0 \text{ for some } p(X) \in P_\ell \setminus \{0\} \right\},$$

and define

$$\epsilon_\ell = \max(\{x \in R_\ell \mid x < 1\}).$$

Observe that if $p(X) \in P_\ell$ exceeds 0 in the germ order, then $p(x) > 0$ for all $x \in (\epsilon_\ell, 1)$. The same can be said for any polynomial of the form $X^t p(X)$, and we conclude that for any point $x_0 \in (\epsilon_\ell, 1)$, the values $g^{(i)}(x_0)$ are monotonically increasing. As $g^{(i)}(x_0) \leq 1 + x_0 + x_0^2 + \dots = 1/(1 - x_0)$, the monotone sequence $g^{(i)}(x_0)$ is bounded and hence converges to a particular value $g^*(x_0)$.

Finally, for a fixed point $x_0 \in (\epsilon_\ell, 1)$, define

$$\epsilon_0 = \min \left(\left\{ |q(x_0)| \mid q(X) \in P_\ell \setminus \{0\} \right\} \right).$$

Considering two strings w and \tilde{w} for which $w \vdash_\ell \tilde{w}$ corresponding to a substring replacement starting at position t , the power series g and \tilde{g} associated with these strings satisfy $\tilde{g}(x_0) = g(x_0) + x_0^t q(x_0)$ for a nonzero polynomial $q(X) \in P_\ell$; hence $\tilde{g}(x_0) \geq g(x_0) + x_0^t \epsilon_0$. Then observe that if

$$\left| g^{(i)}(x_0) - g^*(x_0) \right| < x_0^t \epsilon_0$$

for all $i \geq k$ then no such replacement is possible at step i and, indeed, the t th bit of all strings $w^{(i)}$ must agree for $i \geq k$. It follows that the sequence $w^{(i)}$ converges pointwise to a particular string w^* . \square

Let D be a finite subset of \mathbb{N} and $\ell \geq \|D\|$. For a D -avoiding string $w \in \{0,1\}^*$ and a position $t > \ell$, let $r_t(w)$ be the string obtained by replacing bits $t, t+1, \dots, t+\ell-1$ with the best possible legal alternative, i.e. $r_t(w)$ is defined by $w \vdash_{\ell}^{A,B} r_t(w)$ with $A = w_{t-\|D\|} \dots w_{t-1}$ and $B = w_{t+\ell} \dots w_{t+\ell+\|D\|-1}$.

Corollary 14. *Let D be a finite subset of \mathbb{N} and $\ell \geq \|D\|$. Let $w \in \{0,1\}^{\mathbb{N}}$ be a D -avoiding string. Let t_1, t_2, \dots be a sequence of integers so that $t_i > \ell$ for each i and each integer in the set $\{\ell+1, \dots\}$ appears infinitely often in the sequence. Then the sequence*

$$\begin{aligned} w^{(0)} &= w, \\ w^{(i)} &= r_{t_i}(w^{(i-1)}) \text{ for } i \geq 1 \end{aligned}$$

converges to an ℓ -maximal element w^* , which is to say that $r_t(w^*) = w^*$ for all $t > \ell$.

7. FURTHER THOUGHTS

7.1. A topological aside. Our germ ordering is not well-behaved relative to the weak topology on the power set of \mathbb{N} , wherein a sequence of sets S_n converges if and only if $S_n \cap F$ is eventually constant for every finite $F \subset \mathbb{N}$. As an illustration of this (related to the famous Ross-Littlewood Paradox), consider the sequence of sets $S_n = \{n, n+1, \dots, 10n\}$; we have $S_1 \prec S_2 \prec S_3 \prec \dots$, but it is unclear what the limit of the S_n 's should be. Surely it is not the pointwise limit of the sets, since that is the null set! One way to understand what is going on here is to note that, even though for each n there exists $\epsilon_n > 0$ such that $(S_n)_q < (S_{n+1})_q$ for all q in $(1 - \epsilon_n, 1)$, we have $\inf \epsilon_n = 0$, so that the intersection of the intervals $(1 - \epsilon_n, 1)$ is empty.

This sort of situation comes into play when one tries to prove Conjecture 2 by showing that $c \succeq c_1, c_2, c_3, \dots$ implies $c : c : c : \dots \succeq c_1 : c_2 : c_3 : \dots$. If we take ϵ_n satisfying $|c| \geq |c_n|$ for all q in $(1 - \epsilon_n, 1)$, and the infimum of the ϵ_n is not known to be positive, then the obvious approach to proving the implication fails.

7.2. Truncated germs. In our work a rational set $S \subseteq \mathbb{N}$ is replaced by the power series $\sum_{n \in S} q^n$, which is rewritten as the Laurent series $\sum_{n \geq -1} a_n (1 - q)^n$, and the coefficients $a_{-1}, a_0, a_1, a_2, \dots$ are used to put a total ordering on the rational sets. The coefficients a_n carry finer and finer information as n increases, so it is natural to discard this information after some point. The classical theory of packings retains only a_{-1} (the density of S); we suggest that it is natural to retain both a_{-1} and a_0 . That is, we define a non-Archimedean valuation ν from the set of rational subsets of \mathbb{N} to $\mathbb{Q} \times \mathbb{Q}$, where we view $\mathbb{Q} \times \mathbb{Q}$ as the lexicographic product of the ordered ring \mathbb{Q} with itself. It can be shown that the pairs (a_{-1}, a_0) that occur are those of the form $(0, k)$ or $(1, -k)$ where k is a nonnegative integer, along with pairs of the form (p, q) where p is a rational number strictly between 0 and 1 and where q is an arbitrary rational number. This valuation is not translation-invariant; if $\nu(S) = (p, q)$, then $\nu(S+1) = (p, q-p)$. Note that under this valuation, the sets $\{3, 6, 9, 12, 15, 18\}$ and $\{1, 3, 6, 9, 15, 18\}$ discussed at the end of

Section 4 have the same size, since the germs of $\{1\}$ and $\{12\}$ differ by $o(1)$. The valuation is emphatically not countably additive, as can for instance be seen by viewing \mathbb{N} as a union of singleton sets.

One can try to extend this valuation to various classes of sets that include but are not limited to the rational subsets of \mathbb{N} . One way to do this without directly invoking the expansion of $\sum_{n \in S} q^n$ as a Laurent series in $1 - q$ is to define a partial preorder on the power set of \mathbb{N} (the *lim inf preorder*) such that S dominates S' in the lim inf preorder iff $\liminf_{q \rightarrow 1^-} (\sum_{n \in S} q^n - \sum_{n \in S'} q^n) \geq 0$. This partial preordering, restricted to the rational sets, coincides with the total preordering obtained by factoring the germ-ordering through the valuation ν .

7.3. Efficiency gaps. In the case of packing \mathbb{N} with translates of $B = \{0, 1, 2, \dots, k-1\}$, there is an appreciable *efficiency gap* between the best packing and all other packings (where an element x of a non-Archimedean ordered ring extending \mathbb{R} is said to be *appreciable* when there exist positive r, s in \mathbb{R} with $r < x < s$):

Theorem 15. For $k \geq 1$ and $D = \{1, 2, \dots, k-1\}$, if S^* is the D -avoiding set $\{0, k, 2k, 3k, \dots\}$ and S is any other D -avoiding set, $S_q \preceq (S^*)_q - \frac{1}{k} + O(1 - q)$.

Proof: We focus on the case $k = 2$ for clarity. Let $S^* = \{0, 2, 4, \dots\}$ and let S be some $\{1\}$ -avoiding set other than S^* . We can split S into two pieces, one of which looks like an initial segment of S^* and the other of which doesn't. In more detail, we write S as the disjoint union of two sets, one of the form $\{0, 2, \dots, 2(m-1)\}$ (empty if $m = 0$) and one of the form $\{t_1, t_2, t_3, \dots\}$ (with $t_1 < t_2 < t_3 < \dots$) satisfying $t_1 \geq 2m + 1$, $t_2 \geq 2m + 3$, $t_3 \geq 2m + 5$, etc. The germ of S is dominated by the germ of $\{0, 2, \dots, 2(m-1)\} \cup \{2m + 1, 2m + 3, 2m + 5, \dots\}$; but this germ is the same (up to $O(1 - q)$) as the germ of $\{1, 3, 5, \dots\}$, which falls short of the germ of $\{0, 2, 4, \dots\}$ by $\frac{1}{2} + O(1 - q)$. The case $k > 2$ is similar. \square

On the other hand, the non-periodic winners of Section 3 beat the corresponding periodic contenders by a non-appreciable amount. This includes the cases $D = \{4, 7, 11\}$ and $D = \{5, 8, 13\}$ arising from packing problems.

7.4. Choice of regularizer. The germ of $\sum_{n \in S} q^n$ as $q \rightarrow 1^-$ can also be thought of as the germ of $\sum_{n \in S} e^{-n/s}$ as $s \rightarrow +\infty$; we think of this as being associated with the function e^{-t} along with a rescaling factor s that measures "spread". The function e^{-t} is a natural regularizer to use for packing problems in \mathbb{N} or $[0, \infty)$; likewise the function e^{-t^2} would be a natural regularizer to use for packing problems in \mathbb{Z} or $(-\infty, \infty)$. Some aspects of the theory are sensitive to the choice of regularizer but others are not; e.g., numerical evidence suggests that basic fact (4) from Section 2 remains true for the regularizer q^{n^2} (corresponding to the Gaussian kernel e^{-t^2}).

7.5. Connection to sphere-packing. Packing problems and distance-avoiding set problems in \mathbb{N} were chosen as a testbed for ideas about analogous problems in \mathbb{R}^n , and more

specifically, sphere-packing problems. Note that the problem of packing spheres of radius 1 in \mathbb{R}^n is equivalent to the problem of packing points in \mathbb{R}^n so that no two are at distance less than 2 (the points are the centers of the spheres). We will not pursue the topic of sphere-packing in depth, but we will mention the conjectures that motivated this work.

Conjecture 16. *Let S be a subset of \mathbb{R}^2 , no two of whose points are at distance less than 2, and let S^* be the set of center-points in a hexagonal close-packing of disks of radius 1 in \mathbb{R}^2 . Let*

$$\delta(S) = \liminf_{s \rightarrow \infty} \left(\sum_{(x,y) \in S^*} e^{-(x^2+y^2)/s^2} - \sum_{(x,y) \in S} e^{-(x^2+y^2)/s^2} \right).$$

Then either S is related to S^ by an isometry of \mathbb{R}^2 , in which case $\delta(S) = 0$, or else S is not related to S^* by an isometry of \mathbb{R}^2 , in which case $\delta(S) > 0$.*

Remark: In private communication, Henry Cohn has shown that when S is related to S^* by an isometry of \mathbb{R}^2 , $\delta(S)$ is indeed 0.

Conjecture 17. *In the previous Conjecture, “ $\delta(S) > 0$ ” can be replaced by “ $\delta(S) \geq 1$ ” in the conclusion.*

That is, there is an appreciable efficiency-gap for 2-dimensional sphere-packing.

The dichotomy between $\delta(S) = 0$ and $\delta(S) \geq 1$ in Conjecture 17 might at first seem to contradict the continuity of the summands as a function of the positions of the points; if all the points move continuously, won't the lim inf also change continuously? The catch is that the lim inf can (and often does) diverge. For instance, if one obtains S from S^* by translating a half-plane's worth of points by $\epsilon > 0$, or dilating the configuration S^* by a factor of $c > 1$, then the lim inf diverges, no matter how close ϵ is to 0, or how close c is to 1.

Clearly the bound in Conjecture 17 cannot be improved, since removing a single point from S^* gives a set S for which the lim inf is exactly 1.

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