ON THE r-DERANGEMENTS OF TYPE B

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ABSTRACT. Extensions of a set partition obtained by imposing bounds on the size of the parts and the coloring of some of the elements are examined. Combinatorial properties and the generating functions of some counting sequences associated with these partitions are established. Connections with Riordan arrays are presented.

1. INTRODUCTION

Consider two sets of n symbols $[n] = \{1, 2, \ldots, n\}$ and $[\overline{n}] = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$, with $i \neq \overline{j}$, for any i, j. Define $X_n = [n] \cup [\overline{n}]$. The symbol $\overline{j}$ is called the `colored version` of the symbol j. Naturally there are $(2n)!$ permutations of X_n. Some of these permutations respect the sign, that is, satisfy $\sigma(\overline{j}) = \overline{\sigma(j)}$. These are called `signed permutations` or `permutations of type` B. General information about them and their relations to Coxeter groups appears in Section 8.1 of [5].

The number of signed permutations on $[n]$ is $2^n n!$, since each one of them is formed by a permutation of $[n]$ and a choice of sign. An example is

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \overline{1} & \overline{2} & \overline{3} & \overline{4} \\ \overline{2} & 1 & 3 & \overline{4} & 2 & \overline{1} & \overline{3} & 4 \end{pmatrix}. \tag{1.1}$$

Observe that the complete permutation is determined by the values of $\pi(1)$, $\pi(2)$, $\pi(3)$, $\pi(4)$ and these must be a permutation of $[4]$ with some choices of overlines. The remaining images are determined from the sign rule. In order to simplify notation, only the first half of the bottom row in (1.1) is retained and π is now written simply (in the so-called `line notation`) as

$$\pi = \overline{2}\,1\,3\,\overline{4}. \tag{1.2}$$

As in the classical case, it is possible to express a signed permutation as a product of disjoint cycles [6]. The notation for the cycles is explained with the example. $\pi = \overline{4}6\overline{3}51\overline{2}\,\overline{9}87$. In order to compute the cycle of π containing 1, start by ignoring the coloring to produce the cycle $1 \to 4 \to 5$. Now insert the color back as they appear in the one-line notation for π. This produces the cycle $(1\overline{4}5)$. Continuing this process gives the final expression for π as $(1\overline{4}5)(\overline{2}6)(\overline{3})(7\overline{9})(8)$. A second example illustrates a

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point that `could lead to confusion`. The permutations considered here have no fixed points, for instance

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \overline{1} & \overline{2} & \overline{3} \\ 2 & 1 & \overline{3} & \overline{2} & \overline{1} & 3 \end{pmatrix}. \tag{1.3}$$

The short hand notation for π is $1\,2\,\overline{3}$ and its cycle decomposition, as explained above, is written as $(12)(\overline{3})$. The interpretation of the last term, $(\overline{3})$, is not that $\overline{3}$ is a fixed point of π, but that $\pi(3) = \overline{3}$ and (necessarily) $\pi(\overline{3}) = 3$.

The goal of the present work is to study a variety of functions for signed permutations in terms of their cycle structure.

Recall that $\left[{n \atop k} \right]$, the (unsigned) Stirling number of the first kind, counts the number of permutations of n elements with k disjoint cycles. The recurrence for $\left[{n \atop k} \right]$ is

$$\left[{n \atop k} \right] = \left[{n-1 \atop k-1} \right] + (n-1) \left[{n-1 \atop k} \right], \quad \text{for } n > k > 1, \tag{1.4}$$

with the initial/boundary conditions $\left[{n \atop n} \right] = 1$ and $\left[{n \atop 1} \right] = (n-1)!$ for $n > 0$ and naturally $\left[{n \atop k} \right] = 0$ for $n < k$. Other functions include some restrictions on the length of the cycles. For example, if the cycles are restricted to have length at least 2, one obtains the *derangement numbers* d_n, given by

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}. \tag{1.5}$$

They satisfy the recurrence $d_n = nd_{n-1} + (-1)^n$, with $d_0 = 1$. It follows from (1.5) that

$$\lim_{n \to \infty} \frac{d_n}{n!} = \frac{1}{e}. \tag{1.6}$$

Broder [7] introduced the notion of *r-permutations*. For $r \in \mathbb{N}$ an r-permutation of $n + r$ is a permutation where the first r elements, called *special*, are in distinct cycles. The number of r-permutations of $[n + r]$ into $k + r$ cycles are counted by the *r-Stirling numbers of the first kind*, denoted by $\left[{n \atop k} \right]_r$. An *r-derangement* on $[n + r]$ is an r-permutation without fixed points. Information about these concepts appears in [18, 23]. These concepts are now extended to signed permutations.

Definition 1.1. A *derangement of type B* on $[n]$ is a signed permutation σ such that $\sigma(i) \neq i$ for every $i \in [n]$. The set of all such permutations is denoted by $\mathcal{D}_n^B$ and its cardinality by d_n^B.

Chow [9] (see also Assaf [1]) proved that

$$d_n^B = n! \sum_{k=0}^{n} \frac{(-1)^k 2^{n-k}}{k!}, \tag{1.7}$$

and the analog of (1.6) is

$$\lim_{n\to\infty} \frac{d_n^B}{n!\,2^n} = \frac{1}{\sqrt{e}}. \tag{1.8}$$

This sequence appears as $A000354$ in OEIS and its first few values (starting at $n = 0$) are

$$1,\ 1,\ 5,\ 29,\ 233,\ 2329,\ 27949,\ 391285,\ \cdots. \tag{1.9}$$

Formula (1.7) is equivalent to

$$\sum_{n=0}^{\infty} d_n^B \frac{x^n}{n!} = \frac{e^{-x}}{1-2x}. \tag{1.10}$$

The main object of the present work is introduced next.

Definition 1.2. Let $n, r \in \mathbb{N}$. A *type* B *r-derangement* on the set $[n+r]$ is a signed permutation on $[n+r]$, without fixed points and with r elements (called *special*) restricted to be in distinct cycles. The set of all r-derangements of type B on $[n+r]$ is denoted by $\mathcal{D}_{n,r}^B$. Its cardinality is denoted by $d_{n,r}^B$. The case $r = 0$ recovers d_n^B in Definition 1.1.

The number of elements of $\mathcal{D}_{n,r}^B$ with $k+r$ cycles is called the *r-Stirling number of type* B and is denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_{\geq 2,r}^B$. Counting over all possible cycles gives the relation

$$d_{n,r}^B = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\geq 2,r}^B. \tag{1.11}$$

Example 1.3. The permutation $\sigma = (1\bar{7}4)(\bar{2})(3\bar{6}5)$ is a type B r-derangement for $0 \leq r \leq 3$ on the set $[7]$.

Note 1.1. The case $r = 0$ has been discussed in [18]. The recurrence

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\geq 2,0}^B = 2n \begin{bmatrix} n \\ k \end{bmatrix}_{\geq 2,0}^B + 2n \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\geq 2,0}^B + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{\geq 2,0}^B, \tag{1.12}$$

with the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{\geq 2,0}^B = \delta_{n,0} \text{ for } n > 0 \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{\geq 2,0}^B = 0 \text{ for } n, k < 0, \tag{1.13}$$

is established there.

2. A recurrence for the r-Stirling numbers of type B

This section presents a recurrence for the r-Stirling numbers of type B. The initial condition involve the *Lah numbers* $L(n,k)$, defined as the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets [20].

Theorem 2.1. *For $n \geq 0$ and k, $r \geq 1$ the recursion*

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\geq 2,r}^{B} &= \begin{bmatrix} n \\ k-1 \end{bmatrix}_{\geq 2,r}^{B} + 2n! \sum_{j=1}^{n} \frac{2^j}{(n-j)!} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{\geq 2,r}^{B} \\
&\qquad + 4rn! \sum_{j=0}^{n} \frac{(j+1)2^j}{(n-j)!} \begin{bmatrix} n-j \\ k \end{bmatrix}_{\geq 2,r-1}^{B} \\
&= \begin{bmatrix} n \\ k-1 \end{bmatrix}_{\geq 2,r}^{B} + 4r \begin{bmatrix} n \\ k \end{bmatrix}_{\geq 2,r-1}^{B} \\
&\qquad + 4n! \sum_{j=1}^{n} \frac{2^{j-1}}{(n-j)!} \left(\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{\geq 2,r}^{B} + 2r(j+1) \begin{bmatrix} n-j \\ k \end{bmatrix}_{\geq 2,r-1}^{B} \right),
\end{aligned}$$

holds. The initial conditions are given by

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{\geq 2,r}^{B} = 2^n n! \sum_{j=0}^{r} \binom{r}{j} \binom{n-1}{r-j-1} 2^{r-j} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\geq 2,r}^{B} = 1.$$

Proof. The initial condition is established first. Recall that $\begin{bmatrix} n \\ 0 \end{bmatrix}_{\geq 2,r}^{B}$ counts the number of signed permutations of $[n+r]$, with r cycles, without fixed points and r special elements restricted to be in distinct cycles. Therefore each cycle contains a single special point and, since there are no fixed points, the special points appearing in a cycle of length 1 must be colored. The next step is to place the n non-special points in the r cycles. Let j be the number of cycles that receive no new points ($0 \leq j \leq r$). Then

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{\geq 2,r}^{B} = \sum_{j=0}^{r} \binom{r}{j} \cdot 2^{n+r-j} (r-j)!\, L(n, r-j). \tag{2.1}$$

The binomial coefficient $\binom{r}{j}$ takes into account the choices of cycles without new points, the factor $(r-j)!$ takes care of the ordering of the cycles with new points, the Lah number counts the ways to partition the n non-special points in order to place them in the $r-j$ cycles and finally the power of 2 describes whether to color the remaining elements or not. The form of the initial condition stated above is obtained by using the expression

$$L(n,k) = \binom{n}{k} \binom{n-1}{k-1} (n-k)! = \frac{n!}{k!} \binom{n-1}{k-1}, \tag{2.2}$$

appearing in [20].

To prove the general case of the recurrence, recall that there are $r+k$ cycles of size at least 2, r containing the special points and k containing no special points. Now consider cases for the image of non-special element $n+1$ under π.

`Option 1`. The element $\overline{n+1}$ is fixed by π. Recall the warning: this means that $\pi(n+1) = \overline{n+1}$ and $\pi(\overline{n+1}) = n+1$. Removing the cycle $\{n+1, \overline{n+1}\}$ leaves $\begin{bmatrix} n \\ k-1 \end{bmatrix}_{\geq 2,r}^{B}$ choices. This gives the first term in the recurrence.

`Option 2`. Assume $n+1$ is the image under a signed permutation of an uncolored symbols. There are two cases to consider:
Case 1: $n+1$ belongs to one the first r cycles. Then the cycle $\mathfrak{b}$ containing $n+1$ has, in addition to $n+1$, one special element and j additional ones $(0 \leq j \leq n)$ from $\{1, 2, \ldots, n\}$, for a total of $j+2$ elements. Since $n+1$ is not colored, there are 2^{j+1} ways to color or not the other elements in $\mathfrak{b}$. The $j+2$ elements of the cycle $\mathfrak{b}$ can now be permuted cyclically in $(j+1)!$ ways. The count is now

$$r \sum_{j=0}^{n} \binom{n}{j} (j+1)! 2^{j+1} \begin{bmatrix} n-j \\ k \end{bmatrix}_{\geq 2, r-1}^{B}. \tag{2.3}$$

In this count, the number of special cycles has been reduced by 1, since $n+1$ occupies one of them.
Case 2. If $n+1$ does not belong to one of the special cycles, then the number in Case 1 becomes

$$\sum_{j=1}^{n} \binom{n}{j} j! 2^{j} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{\geq 2, r}^{B}. \tag{2.4}$$

The index j counts the number of additional elements in the cycle containing $n+1$ and the remaining terms are interpreted as before.

`Option 3`. Assume $\overline{n+1}$ belongs to one of the cycles. Then the count is similar as in Option 2. The numbers of cases where $\overline{n+1}$ is in one of r special cycles is

$$2r \sum_{j=0}^{n} \binom{n}{j} (j+1)! 2^{j} \begin{bmatrix} n-j \\ k \end{bmatrix}_{\geq 2, r-1}^{B}. \tag{2.5}$$

Otherwise, $\overline{n+1}$ is in a cycle without special points and this contributes

$$\sum_{j=0}^{n} \binom{n}{j} j! 2^{j} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{\geq 2, r}^{B} \tag{2.6}$$

to the count. This completes the proof. □

Note 2.1. The recurrence is now used to generate the values of $\begin{bmatrix} n \\ k \end{bmatrix}_{\geq 2,r}^{B}$. The matrix below gives the values for $r = 3$ in the range $0 \leq n, k \leq 6$:

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}^B_{\geq 2,3} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 1 & 0 & 0 & 0 & 0 & 0 \\ 144 & 28 & 1 & 0 & 0 & 0 & 0 \\ 1824 & 592 & 48 & 1 & 0 & 0 & 0 \\ 25344 & 11232 & 1552 & 72 & 1 & 0 & 0 \\ 391680 & 213888 & 41824 & 3280 & 100 & 1 & 0 \\ 6727680 & 4267008 & 1061248 & 119520 & 6080 & 132 & 1 \end{pmatrix}. \tag{2.7}$$

Note 2.2. The special value $d_{r,0} = 1$ comes directly from the definition. The recurrence in Theorem 2.1 is now used to check that $d^B_{1,r} = 1 + 4r$. Indeed,

$$d^B_{r,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^B_{\geq 2,r} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}^B_{\geq 2,r}, \tag{2.8}$$

and the initial condition in Theorem 2.1 shows that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^B_{\geq 2,r} = 2^1 \cdot 1! \times \sum_{j=0}^{r-1} \binom{r}{j} \binom{0}{r-j-1} 2^{r-j} = 4r, \tag{2.9}$$

and the second term in the recurrence yields

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^B_{\geq 2,r} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^B_{\geq 2,r} + 4r \begin{bmatrix} 0 \\ 1 \end{bmatrix}^B_{\geq 2,r} + \text{an empty sum} = 1, \tag{2.10}$$

for a total of $d^B_{1,r} = 1 + 4r$.

3. An approach from the theory of Riordan arrays

This section considers the matrix $\mathcal{C}_{\geq 2,r} := \left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]^B_{\geq 2,r} \right)_{n,k \geq 0}$ as a Riordan array. Enumerative arguments for some combinatorial identities are presented. The unsigned entries of the matrix $\mathcal{C}^{-1}_{\geq 2,r}$ are discussed first.

Recall that an infinite lower triangular matrix $L = [l_{n,k}]_{n,k \geq 0}$ is called an *exponential Riordan array* [3] if its k^{th}-column has generating function of the form $g(z)\,(f(z))^k/k!, k = 0, 1, 2, \ldots g(z)$, where $g(z)$ and $f(z)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $f(z), g(z)$ is denoted by $(g(z), f(z))$.

Multiplying (g, f) by a column vector $(c_0, c_1, \ldots)^t$ with exponential generating function $h(z)$, results in a column vector with exponential generating function $g \cdot h \circ f$. This property is known as the fundamental theorem of exponential Riordan arrays or *summation property* [12]. The product of two exponential Riordan arrays $(g(z), f(z))$ and $(h(z), \ell(z))$ is defined by:

$$(g(z), f(z)) * (h(z), \ell(z)) = (g(z)h(f(z)), \ell(f(z))).$$

The set of all exponential Riordan matrices is a group under this operation [3], [21]. The identity is $I=(1,z)$ and

$$(g(z),f(z))^{-1}=\left(\frac{1}{\left(g\circ\overline{f}\right)(z)},\overline{f}(z)\right), \tag{3.1}$$

here $\overline{f}(z)$ denotes the compositional inverse of $f(z)$; that is, $(\overline{f}\circ f)(z)=z$.

Deutsch et al. [12] gave an algorithm to calculate the entry $l_{n,k}$ of an exponential Riordan array. They proved that every element $l_{n+1,k}$ of an exponential Riordan array can be expressed as a linear combination of the elements in the preceding row. In particular, for $L=[l_{n,k}]_{n,k\geq 0}=(g(z),f(z))$ there are sequences (a_n) and (z_n) such that

$$l_{n+1,0}=\sum_{i\geq 0} i!a_i l_{n,i}, \tag{3.2}$$

$$l_{n+1,k}=a_0 l_{n,k-1}+\frac{1}{k!}\sum_{i\geq k} i!(z_{i-k}+ka_{i-k+1})l_{n,i}, \tag{3.3}$$

where

$$A(t)=a_0+a_1t+a_2t^2+\cdots,\quad Z(t)=z_0+z_1t+z_2t^2+\cdots$$

satisfy the functional relations

$$A(t)=f'(\bar{f}(t)),\quad Z(t)=\frac{g'(\bar{f}(t))}{g(\bar{f}(t))}. \tag{3.4}$$

Conversely, (3.4) implies (3.2) and (3.3).

The next results states that the matrix defined by $\mathcal{C}_{\geq 2,r}:=\left(\left[{n \atop k}\right]^B_{\geq 2,r}\right)_{n,k\geq 0}$ is an exponential Riordan array.

Theorem 3.1. *The matrix $\mathcal{C}_{\geq 2,r}$ is the exponential Riordan array given by*

$$\left(\left(\frac{1+2z}{1-2z}\right)^r,-\ln(1-2z)-z\right).$$

Proof. Let $C_k(x):=\sum_{n\geq 0}\left[{n+k \atop k}\right]^B_{\geq 2,r}\frac{x^n}{n!}$ be the generating function of the columns of the matrix $\mathcal{C}_{\geq 2,r}$. The initial values in Theorem 2.1 shows that the generating function of the first column is given by

$$C_0(x)=\sum_{n\geq 0}2^n n!\sum_{j=0}^{r}\binom{r}{j}\binom{n-1}{r-j-1}2^{r-j}\frac{x^n}{n!}=\sum_{j=0}^{r}\binom{r}{j}2^{r-j}\sum_{n\geq 0}\binom{n-1}{r-j-1}(2x)^n.$$

The identity

$$\frac{1}{(1-x)^t}=\sum_{n\geq 0}\binom{t+n-1}{t-1}x^n,$$

gives

$$\left(\frac{2x}{1-2x}\right)^{r-j} = \sum_{n\geq 0} \binom{n-1}{r-j-1} x^n,$$

Therefore

$$C_0(x) = \sum_{j=0}^{r} \binom{r}{j} 2^{r-j} \left(\frac{2x}{1-2x}\right)^{r-j} = \left(1+\frac{4x}{1-2x}\right)^r = \left(\frac{1+2x}{1-2x}\right)^r.$$

The generating function for the column $C_1(x)$ is presented next. The relation

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_{\geq 2,r}^{B} = \sum_{k=2}^{n} \binom{n}{k} (k-1)! 2^k \begin{bmatrix} n-k \\ 0 \end{bmatrix}_{\geq 2,r}^{B} + n \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_{\geq 2,r}^{B}. \tag{3.5}$$

follows from the following combinatorial argument. Assume a non-special cycle has k elements ($2 \leq k \leq n$). These can be chosen and colored in $\binom{n}{k}(k-1)!$ ways. The coloring of these these elements produce 2^k options. The remaining elements can be organized in $\left[\begin{smallmatrix} n-k \\ 0 \end{smallmatrix}\right]_{\geq 2,r}^{B}$ ways. Summing over k gives the first expression in (3.5). On the other hand, in the case where a non-special block has only one element gives $n\left[\begin{smallmatrix} n-1 \\ 0 \end{smallmatrix}\right]_{\geq 2,r}^{B}$ options. Therefore

$$C_1(x) = C_0(x) \sum_{n\geq 2} \frac{(2x)^n}{n} + xC_0(x) = C_0(x)(-x-\log(1-2x)).$$

Similarly

$$C_k(x) = C_0(x) \frac{(-x-\log(1-2x))^k}{k!}, \tag{3.6}$$

since $C_0(x)$ takes into account the r special cycles and $(-x-\log(1-2x))^k/k!$ are sequences of unordered k cycles. The result now follows from (3.6) and the definition of a Riordan array. □

4. Counting distances on graphs

Given a graph G the sequence $S_G(n)$ is defined as the number of vertices which have a distance n from a given vertex in G. This sequence was discussed in [2] and [10]. The example $S_{\mathbb{Z}^2}(3) = 12$ is pictured in Figure 1. In the case of the r-dimension lattice $\mathbb{Z}^r$, it follows that

$$\sum_{n\geq 0} S_{\mathbb{Z}^r}(n) x^n = \left(\frac{1+x}{1-x}\right)^r.$$

The first column in the Riordan array given in Theorem 3.1 gives the next statement. A combinatorial proof is presented.

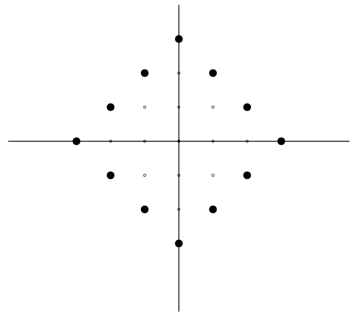

FIGURE 1. Points at a distance 3 from a fixed point.

Theorem 4.1. *For $r, n \in \mathbb{N}$*

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{\geq 2,r}^{B} = 2^n n! S_{\mathbb{Z}^r}(n). \tag{4.1}$$

Proof. A permutation σ with no non-special elements is written as $\sigma = c_1 c_2 \cdots c_r$, where c_i are the special cycles containing one of the $[r]$ elements. Define $x_\sigma = (x_1, \dots, x_r) \in \mathbb{Z}^r$ by

$$x_i = \begin{cases} 0, & \text{if } \sigma_i = \bar{i}; \\ -(ord(c_i) - 1), & \text{if } \bar{i} \in c_i; \\ ord(c_i) - 1, & \text{otherwise.} \end{cases}$$

where $ord(c_i)$ is the length of the cycle c_i. It is clear that $\sum_{i=1}^{r} |x_i| = n$, so x_σ is at distance n from the origin. This construction does not take into account the sign in the $[n]$ elements. This can be done in 2^n ways. Finally, the $n!$ term count the ordering of position in the cycles. The proposition follows. $\square$

Lengyel [15] presented the interesting expression:

$$\frac{1+x}{1-x} = \sum_{n=1}^{\infty} x^{\lfloor \frac{n}{\phi} \rfloor} + \sum_{n=1}^{\infty} x^{\lfloor n\phi^2 \rfloor}, \quad \text{for } |x| < 1.$$

Here ϕ is the golden ratio.

The next statement characterizes the two diagonals below the main one.

Theorem 4.2. *For $n \geq 0$, the identities*

$$\begin{bmatrix} n+1 \\ n \end{bmatrix}_{\geq 2,r}^{B} = 2(n+1)(n+2r),$$

$$\begin{bmatrix} n+2 \\ n \end{bmatrix}_{\geq 2,r}^{B} = \frac{4}{3}\binom{n+2}{2}(3n^2 + n + 12nr + 12r^2),$$

hold.

Proof. There are two cases to consider, either

- There is an element that belongs to one of the r special cycles. This happens in $(n+1)r2^2$ ways; or,
- Two elements form a non special cycle. This can be done in $2^2\binom{n+1}{2}$ ways.

The first identity follows from here. The proof of the second identity is similar. $\square$

4.1. **Combinatorial interpretation of the inverse matrix.** The matrix $\mathcal{C}_{\geq 2,r}$ is a Riordan array and since the set of all Riordan matrices is a group, its inverse exists. This subsection presents a combinatorial interpretation of the unsigned inverse matrix of $\mathcal{C}_{\geq 2,r}$ in terms of plane increasing trees. The particular case $r=0$ was presented in [18]. The notation $\mathcal{T}_{\geq 2,r} := [T_{\geq 2,r}(n,k)]_{n,k\geq 0}$ is used for the inverse of $\mathcal{C}_{\geq 2,r}$. Then (3.1) shows that $\mathcal{T}_{\geq 2,r}$ is the exponential Riordan array given by

$$\mathcal{T}_{\geq 2,r} = \left(\left(-1+\frac{1}{1+W\left(-\frac{1}{2}e^{-\frac{1}{2}-z}\right)}\right)^r, \frac{1}{2}+W\left(-\frac{1}{2}e^{-\frac{1}{2}-z}\right)\right),$$

with $W(z)$ is the Lambert W function. This is defined by

$$W(x)e^{W(x)} = x, \tag{4.2}$$

for all real (or complex) x. More information on this function appears in [11].

Let $\overline{\mathcal{T}}_{\geq 2,r} := \left[\overline{T}_{\geq 2,r}(n,k)\right]_{n,k\geq 0}$ be the unsigned matrix of $\mathcal{T}_{\geq 2,r}$, that is

$$\begin{aligned}\overline{\mathcal{T}}_{\geq 2,r} := \left[\overline{T}_{\geq 2,r}(n,k)\right]_{n,k\geq 0} &= \left[(-1)^{n+k}T_{\geq 2,r}(n,k)\right]_{n,k\geq 0}\\ &= \left(\left(-1+\frac{1}{1+W\left(-\frac{1}{2}e^{-\frac{1}{2}+z}\right)}\right)^r, -\frac{1}{2}-W\left(-\frac{1}{2}e^{-\frac{1}{2}+z}\right)\right).\end{aligned}$$

The first few rows of the matrix $\overline{\mathcal{T}}_{\geq 2,3} = \left[\overline{T}_{\geq 2,3}(n,k)\right]_{n,k\geq 0}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 12 & 1 & 0 & 0 & 0 & 0 & 0\\ 192 & 28 & 1 & 0 & 0 & 0 & 0\\ 3936 & 752 & 48 & 1 & 0 & 0 & 0\\ 99456 & 22304 & 1904 & 72 & 1 & 0 & 0\\ 3001344 & 748672 & 76320 & 3920 & 100 & 1 & 0\\ 105544704 & 28412416 & 3265792 & 203040 & 7120 & 132 & 1 \end{pmatrix}.$$

Now let $\overline{T}_k(z)$ be the generating function of the k^{th} column of $\overline{\mathcal{T}}_{\geq 2,r}$. Then

$$\overline{T}_0(z) = \left(-1+\frac{1}{1+W\left(-\frac{1}{2}e^{-\frac{1}{2}+z}\right)}\right)^r = \left(F'(z)\right)^r,$$

where

$$F(z) := -\tfrac{1}{2} - W\left(-\tfrac{1}{2}e^{-\frac{1}{2}+z}\right). \tag{4.3}$$

The generating function $F(z)$ counts the total number of labeled rooted trees of subsets of an n-set [16]. In the sequence A005172 of OEIS, Peter Bala states that $F(z)$ satisfies the functional equation

$$x = \int_0^{F(x)} \frac{1}{\Phi(t)} dt, \tag{4.4}$$

where $\Phi(t) = (1+2t)/(1-2t)$. Then [4, Theorem 1] provides an alternative combinatorial interpretation for this sequence. It turns out that this sequence also counts the plane increasing trees on n vertices, such that each vertex of out-degree $k \geq 1$ is colored with one of 2^{k+1} colors. Therefore $F'(z)$, the derivative of $F(z)$, counts plane increasing trees on $n+1$ vertices.

The power series expansion of $F'(z)$ starts as

$$F'(z) = 1 + 4\frac{z}{1!} + 32\frac{z^2}{2!} + 416\frac{z^3}{3!} + 7552\frac{z^4}{4!} + 176128\frac{z^5}{5!} + \cdots$$

Figure 2 shows the corresponding trees with 3 vertices. The parenthesis denotes the possible colorings for a vertex.

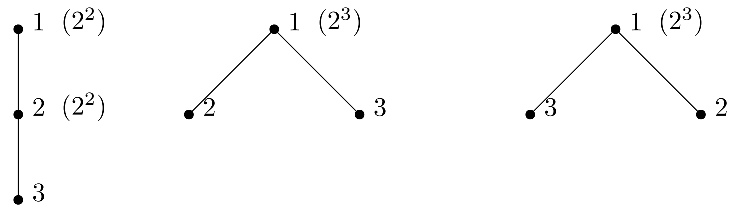

FIGURE 2. Colored plane increasing trees on 3 vertices.

The identity

$$\overline{T}_k(z) = \left(F'(z)\right)^r \frac{1}{k!} \left(F(z)\right)^k, \tag{4.5}$$

leads to the next result.

Theorem 4.3. *The sequence $\overline{T}_{\geq 2,r}(n,k)$ counts the number of r-ordered and k-unordered elements in a $r+k$-forest of increasing trees on n vertices, such that each vertex of out-degree $k \geq 1$ is colored with one of 2^{k+1} colors and the r ordered connected components are rooted.*

The next recurrence now follows from (3.2) and (3.3).

Theorem 4.4. *The sequence $\overline{T}_{\geq 2,r}(n,k)$ for $n \geq k \geq 0$ is determined by the recurrence*

$$\overline{T}_{\geq 2,r}(n+1,k) = \overline{T}_{\geq 2,r}(n,k-1) + \frac{1}{k!}\sum_{i=k}^{n} i!2^{i-k+2}((i-k+1)r+k)\overline{T}_{\geq 2,r}(n,i),$$

and the initial conditions $\overline{T}_{\geq 2,r}(n,0) = \delta_{n,0}$ for $n \geq 0$, and $\overline{T}_{\geq 2,r}(n,k) = 0$ for $n,k < 0$.

5. Properties of the numbers $d_{r,n}^B$

This section discusses the expression

$$d_{r,n}^B = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\geq 2,r}^B, \tag{5.1}$$

for the total number of r-derangements of type B on $[n+r]$. This is the row sum of the matrix in (2.7). The case $r=0$ appears in (1.7).

Theorem 5.1. *The recurrence*

$$d_{r,n}^B = d_{r-1,n}^B + 2nd_{r,n-1}^B + 2nd_{r-1,n-1}^B, \tag{5.2}$$

holds. It is supplemented by the initial conditions $d_{r,0}^B = 1$ *and* $d_{0,n}^B = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} 2^{n-k}$.

Proof. Consider a permutation π in $\mathcal{D}_{n,r}^B$. There are r special elements and assume that 1 is the first of them. The recurrence is obtained by distinguishing cases according to the type of cycle $\mathfrak{c}$ containing 1. The initial conditions appeared in Note 1.1.

`Case 1`. Assume $\mathfrak{c}$ is of length 1. Then the cycle must be $(\overline{1})$, since there are no fixed points in π. The rest of the permutation is in $\mathcal{D}_{n,r-1}^B$, counting for the term $d_{n,r-1}^B$.

`Case 2`. If $\mathfrak{c}$ is of length 2, then $\mathfrak{c}$ is of the form $(1\,x)$ with x non-special. Then 1 and x can be colored or not, for a total of n choices. The rest of the permutation is in $\mathcal{D}_{n-1,r-1}^B$, for a total of $4nd_{n-1,r-1}^B$ choices.

`Case 3`. Finally consider the case when the cycle $\mathfrak{c}$ has at least 3 elements. To produce such a permutation, choose k to be non-special and drop it from the list of non-special elements. Then form an arbitrary B-r-derangement on $n-1+r$ elements in $d_{n-1,r}^B$ ways. After that, insert k to the right of 1 and optionally color it. This accounts for $2nd_{n-1,r}^B$ choices. In this process, there are certain permutations that have been counted twice. Namely, those for which 1 is in a cycle of length 1 for a permutation in $\mathcal{D}_{n-1,r}^B$. Inserting k then produces a cycle of length 2, already counted in Case 2. Therefore these must be excluded. Thus, the total count is $2n\left(d_{n-1,r}^B - d_{n-1,r-1}^B\right)$.

The proof is complete. □

Corollary 5.1. For fixed $n \in \mathbb{N}$, the function $d_{n,r}^B$ is a polynomial in r of degree n.

Proof. This follows from $d_{r,0} = 1$ and (5.2), written in the form

$$d_{r,n}^B - d_{r-1,n}^B = 2n(d_{r,n-1}^B + d_{r-1,n-1}^B). \tag{5.3}$$

□

Example 5.2. The value $d_{r,1}^B = 1 + 4r$ and the difference equation (5.3) give

$$d_{r,2}^B - d_{r-1,2}^B = 4(1 + 4r + 1 + 4(r-1)) = 32r - 8. \tag{5.4}$$

Therefore $d_{r,2}^B$ is a quadratic polynomial in r. The ansatz $d_{r,2}^B = a_0r^2 + a_1r + a_2$ in (5.4) gives $a_2 = 16$ and $a_1 = 8$. The remaining coefficient comes from (1.7); so that

$$d_{r,2}^B = 16r^2 + 8r + 5. \tag{5.5}$$

Similarly,

(5.6)

$$\begin{aligned} d_{r,3}^B &= 64r^3 + 48r^2 + 92r + 29, \\ d_{r,4}^B &= 256r^4 + 256r^3 + 992r^2 + 592r + 233, \\ d_{r,5}^B &= 1024r^5 + 1280r^4 + 8320r^3 + 7200r^2 + 7796r + 2329, \\ d_{r,6}^B &= 4096r^6 + 6144r^5 + 60160r^4 + 67840r^3 + 141424r^2 + 83672r + 27949. \end{aligned}$$

An explicit formula for $d_{r,n}^B$ is presented next.

Theorem 5.3. *For all n, $r \geq 0$,*

$$d_{r,n}^B = 2^n \sum_{i=0}^{r} \binom{r}{i} n^{\underline{i}} 2^i \sum_{k=0}^{n-i} \binom{n-i}{k} (-1)^k (i+1)^{\overline{n-i-k}} 2^{-k}. \tag{5.7}$$

Here $n^{\underline{i}} = n(n-1)\cdots(n-i+1)$ and $n^{\overline{j}} = n(n+1)\cdots(n+j-1)$.

Proof. We consider cases with $r - i$ special elements ($0 \leq i \leq r$) are in cycles of length one (and therefore necessarily barred). The other i special elements are in cycles of length greater than one. Choose first these special elements (in $\binom{r}{i}$ ways), and then place the i non-special elements into their cycles. This gives $n(n-1)\cdots(n-i+1) = n^{\underline{i}}$ choices. Then mark both the i special elements and the non-special ones with a bar, for a total of $2^i \cdot 2^i$ options. The other $n - i$ non-special elements are placed into the i special cycles or outside them. This yields $(i+1)^{\overline{n-i}}$ options. There are 2^{n-i} ways to mark these elements or not. Up to now, the count is

$$\binom{r}{i} n^{\underline{i}} 2^{2i} (i+1)^{\overline{n-i}} 2^{n-i}. \tag{5.8}$$

Observe that in the non-special part of the permutation, the placement above might have introduced some fixed points and these must be removed. Among the above permutations, the number of those type B derangements which have exactly k fixed points among the $n - i$ elements is

$$\binom{n-i}{k} n^{\underline{i}} 2^{2i} (i+1)^{\overline{n-i-k}} 2^{n-i-k}.$$

The inclusion-exclusion principle now completes the proof. □

5.1. **The generating function and asymptotics of $d^B_{r,n}$.** Theorem 5.3 is now used to produce the exponential generating function of $\{d^B_{r,n}\}$.

Theorem 5.4. *The formula*

$$\sum_{n=0}^{\infty} d^B_{r,n}\frac{x^n}{n!} = \frac{e^{-x}}{1-2x}\left(\frac{1+2x}{1-2x}\right)^r, \tag{5.9}$$

holds.

Proof. Start with

$$\sum_{n=0}^{\infty}\frac{d^B_{r,n}}{2^n}\frac{x^n}{n!} = \sum_{i=0}^{r}\binom{r}{i}2^i\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k\sum_{n=i+k}^{\infty}\binom{n-i}{k}n^{\underline{i}}(i+1)^{\overline{n-i-k}}\frac{x^n}{n!}.$$

The innermost sum is

$$\sum_{n=i+k}^{\infty}\binom{n-i}{k}n^{\underline{i}}(i+1)^{\overline{n-i-k}}\frac{x^n}{n!} = \frac{x^i}{(1-x)^{i+1}}\frac{x^k}{k!}.$$

Multiplying the right hand side by $\left(-\frac{1}{2}\right)^k$ and summing over k produces $e^{-x/2}$. Thus

$$\sum_{n=0}^{\infty}\frac{d^B_{r,n}}{2^n}\frac{x^n}{n!} = \sum_{i=0}^{r}\binom{r}{i}2^i\frac{e^{-x/2}x^i}{(1-x)^{i+1}} = \frac{e^{-x/2}}{1-x}\left(\frac{1+x}{1-x}\right)^r.$$

Substituting $2x$ in place of x produces the result. □

The asymptotics of the type B r-derangements, when r is fixed and n tends to infinity, are presented next. Standard methods of the analysis of the principal part of the generating function are used. Details appeared in [24, Theorem 5.2.1].

Corollary 5.5. For any fixed $r \geq 0$ and large n

$$\frac{d^B_{r,n}}{n!} \sim \frac{(-2)^n}{\sqrt{e}}\sum_{i=0}^{r}\binom{r}{i}2^i\left[\binom{-i-1}{n} - \frac{2i-1}{2}\binom{-i}{n}\right].$$

6. A generalization of the r-Stirling numbers

Let $\sigma = c_1c_2\cdots c_s$ be the usual cycle representation of a permutation into s disjoint cycles. The orders of the cycles are denoted by $ord(c_j)$, $1 \leq j \leq s$. For example, $\overline{4}\,3\,\overline{2}\,\overline{7}\,5\,\overline{8}\,1\,9\,6$ is written as $(1\overline{4}\overline{7})(\overline{2}3)(5)(6\overline{8}9)$. Then $ord(c_1) = 3$, $ord(c_2) = 2$, $ord(c_3) = 1$, $ord(c_4) = 3$. The derangements of type B can be characterized by

$$\mathcal{D}^B_n = \{\sigma = c_1\cdots c_s : \text{for all } i\in[s] \text{ either } ord(c_i)\geq 2 \text{ or } c_i\subseteq[\bar{n}] = \{\bar{1},\bar{2},\ldots,\bar{n}\}\}.$$

This leads to the following

Definition 6.1. Let $\sigma = c_1 \cdots c_s$ be a permutation of type B of length n with s cycles. The permutation σ is a *m-restricted permutation of type B* if for all $1 \leq i \leq s$ either the order of each cycle satisfies $ord(c_i) \leq m$ or each item in c_i is signed, that is $c_i \subseteq [\bar{n}]$. The permutation σ is an *m-associated permutation of type B* if for all $1 \leq i \leq s$ either the order of each cycle satisfies that $ord(c_i) \geq m$ or $c_i \subseteq [\bar{n}]$.

Denote by $\mathcal{A}^B_{n,\leq m}(\mathcal{A}^B_{n,\geq m})$ the set of all m-restricted (associated) permutations of type B in $\mathcal{D}^B_n$. The *m-restricted (associated) factorial numbers of type B*, $A^B_{n,\leq m}$ is the corresponding cardinalities. Note that $\mathcal{D}^B_n = \mathcal{A}^B_{n,\geq 2}$. The *associated (restricted) Stirling numbers of the first kind of type B*, ${n \brack k}^B_{\geq m}({n \brack k}^B_{\leq m})$, is the number of permutations of $A^B_{n,\geq(\leq)m}$ with k cycles. The basic relations

$$A^B_{n,\geq m} := \sum_{k=0}^{n} {n \brack k}^B_{\geq m}, \qquad A^B_{n,\leq m} := \sum_{k=0}^{n} {n \brack k}^B_{\leq m},$$

hold.

The *restricted Stirling numbers of the first kind (of type A)* ${n \brack k}_{\leq m}$ enumerate the number of permutations on n elements with k cycles with the restriction that none of the cycles contain more than m items. Similarly, the *associated Stirling numbers of the first kind* ${n \brack k}_{\geq m}$ equals the number that each cycle contains at most m items. Komatsu et al. [14] presented a variety of combinatorial properties for these sequences. Recently, Moll et al. [19] obtained new combinatorial and arithmetical properties for them.

The restricted and associated Stirling numbers of the first kind satisfy the recurrence relations

$$\text{(6.1)} \qquad {n \brack k}_{\leq m} = \sum_{i=0}^{m-1} \frac{(n-1)!}{(n-1-i)!} {n-i-1 \brack k-1}_{\leq m},$$

$$\text{(6.2)} \qquad {n \brack k}_{\geq m} = \sum_{i=m-1}^{n-1} \frac{(n-1)!}{(n-1-i)!} {n-i-1 \brack k-1}_{\geq m}.$$

with initial values

$${0 \brack 0}_{\leq m} = 1 \quad \text{and} \quad {n \brack 0}_{\leq m} = {0 \brack n}_{\leq m} = 0,$$
$${0 \brack 0}_{\geq m} = 1 \quad \text{and} \quad {n \brack 0}_{\geq m} = {0 \brack n}_{\geq m} = 0,$$

for $n > 0$. Introduce now the *incomplete factorial numbers* $A_{i,\leq m}$ and $A_{i,\geq m}$, as the total number of incomplete permutations, that is

$$A_{n,\leq m} := \sum_{k=0}^{n} {n \brack k}_{\leq m} \quad A_{n,\geq m} := \sum_{k=0}^{n} {n \brack k}_{\geq m}.$$

Theorem 6.2. *For $n \geq 0$, the identities*

$$A^B_{n,\leq m} = \sum_{i=0}^{n} \binom{n}{i} 2^i A_{i,\leq m} A_{n-i,\geq m+1},$$

$$A^B_{n,\geq m} = \sum_{i=0}^{n} \binom{n}{i} 2^i A_{i,\geq m} A_{n-i,\leq m-1},$$

hold.

Proof. Let $\sigma \in \mathcal{A}^B_{n,\geq k}$ and consider $A_1 = \{s \in [n] : \text{there is } i \in [k] \text{ such that } \sigma_s^i = s\}$, and $A_2 = A_1^c$. It is clear that $A_2 \subseteq [n]$ and so there is no coloring on them. Consider now the the function

$$\varphi : \mathcal{A}^B_{n,\leq k} \longrightarrow \bigcup_{i=0}^{n} \binom{[n]}{n-i} \times [2]^{[i]} \times \mathcal{A}_{i,\leq k} \times \mathcal{A}_{n-1,\geq k+1},$$

given by $\varphi(\sigma) = (A_2, f, \sigma|_{A_1}, \sigma|_{A_2})$, where $f(x) = \chi_{[\bar{n}]}(x)$. It is easy to check that this is a bijection, completing the proof. □

6.1. **The r-version.** This subsection consider the analog of the results presented above for the case of r-permutations.

Definition 6.3. Let $n, r \in \mathbb{N}$ and $m \geq 1$. A *type* B *m-associated r-permutation* on the set $[n+r]$ is a signed permutation on $[n+r]$, with the restriction that the order of each cycle is at least m, and the first r elements (called *special*) are restricted to be in distinct cycles. The set of all m-associated r-permutation of type B on $[n+r]$ is denoted by $\mathcal{A}^B_{n,\geq m,r}$ and its cardinality by $A^B_{n,\geq m,r}$.

The number of elements of $\mathcal{A}^B_{n,\geq m,r}$ with $k+r$ cycles is called the *m-associated r-Stirling number of type B* and is denoted by $\begin{bmatrix} n \\ k \end{bmatrix}^B_{\geq m,r}$. Counting over all possible cycles gives the relation

$$A^B_{n,\geq m,r} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^B_{\geq m,r}. \tag{6.3}$$

In the cases $m = 0, 1$, there are no restrictions on the sign, so one can color any element in 2^{n+r} ways and then choose the blocks as the normal r-Stirling numbers of the first kind. It follows that

$$\begin{bmatrix} n \\ 0 \end{bmatrix}^B_{\geq m,r} = 2^{n+r} \begin{bmatrix} n \\ 0 \end{bmatrix}_r = 2^{n+r} n! \binom{n+r-1}{r-1}.$$

The case $m = 2$ was described in Theorem 2.1. The next result gives a general recurrence relation. This can be used to analyze the situation for $m > 2$.

The symbol $Par_{\le c}(a,b)$ denotes the number of composition of a into b positive parts, each part of size at most c, then

$$Par_{\le c}(a,b) = \begin{cases} 0, & c \le 0 \text{ and } (a \ne 0 \text{ or } b \ne 0); \\ \sum_{i=0}^{b} (-1)^i \binom{b}{i} \binom{a-ci-1}{b-1}, & \text{otherwise.} \end{cases}$$

Introduce the function

$$\tau_{m,n}(j) = \begin{cases} 2^{j+1} & \text{if } m-1 \le j \le n \\ 1 & \text{otherwise.} \end{cases} \tag{6.4}$$

Theorem 6.4. *For $n \ge 0$, $k, r \ge 1$ and $m > 2$, the recursion*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}^B_{\ge m,r} = \sum_{j=0}^{n} j!\tau_{m,n}(j) \binom{n}{j} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}^B_{\ge m,r} + r \sum_{j=0}^{n} (j+1)!\tau_{m,n+1}(j+1) \binom{n}{j} \begin{bmatrix} n-j \\ k \end{bmatrix}^B_{\ge m,r-1},$$

holds. The initial conditions are given by

$$\begin{bmatrix} n \\ 0 \end{bmatrix}^B_{\ge m,r} = n! \sum_{p=0}^{r} \sum_{j=0}^{p} \binom{r}{p} \binom{p}{j} \sum_{k=0}^{n} 2^{n+p-k-j} Par_{\le m-2}(k,j) Par_{\ge m-1}(n-k, p-j).$$

Proof. The initial conditions are discussed first. For $m > 2$, choose $r-p$ elements from the r special elements and assign them to its own bar; that is, assign each of them to themselves but with a bar. This can be done in $\binom{r}{p}$ ways. From the remaining p special elements, choose j of them which will contain cycles of size less than m so the have to have all sign. This can be done in $\binom{p}{j}$ ways. Then choose k elements out of n to put in these j cycles in $\binom{n}{k} Par_{\le m-2}(k,j) k!$ ways. The remaining $n-k$ elements will be in the $p-j$ cycles with length greater or equal to m, in $Par_{\ge m-1}(n-k, p-j)$ ways. Note that

$$Par_{\ge c}(a,b) = Par_{\ge 1}(a-(c-1)b, b) = \binom{a-(c-1)b-1}{b-1}.$$

The inclusion-exclusion principle gives the expression for $Par_{\le c}(a,b)$. Adding over all possibilities for p, j, k gives the result.

The recursion is discussed next. The discussion is divided into cases:

(1) Either $n+1$ is in a cycle without special elements and of length $< m$. This can be done by selecting the j elements which are in these cycles and taking care of the cyclic order, to produce

$$\binom{n}{j} j! \begin{bmatrix} n-j \\ k-1 \end{bmatrix}^B_{\ge m,r}.$$

(2) $n+1$ is in a cycle without special elements of length $\ge m$. As before, select the j elements and choose their signs in 2^{j+1} ways.

(3) $n+1$ is in a cycle with a special element and of length $< m$. Select j elements and take care of the cyclic order to obtain
$$r\binom{n}{j}(j+1)!\begin{bmatrix} n-j \\ k \end{bmatrix}_{\geq m, r-1}.$$
(4) $n+1$ is in a cycle with a special element and of length $\geq m$. Select j elements as before and now colored them in 2^{j+2} ways.

Summing over these options gives the result. □

6.2. **General case of Howard's identity of type** B**.** Howard [13] established several combinatorial identities involving binomial coefficients and Stirling number of both kinds. Caicedo et al. [8] gave combinatorial proofs and generalization for some of them. For example, the identity
$$\begin{bmatrix} n \\ n-k \end{bmatrix} = \sum_{\ell=0}^{k} \binom{n}{2k-\ell}\begin{bmatrix} 2k-\ell \\ k-\ell \end{bmatrix}_{\geq 2}$$
is such an example. The next statement provides a B-analogue.

Theorem 6.5. *The identity*
$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}^B_{\geq m, r} &= \sum_{p=0}^{r}\sum_{l=0}^{k}\binom{n}{ml}\binom{r}{p}\binom{n-ml}{(m-1)p}\frac{(2^m-1)^{l+p}(ml)!((m-1)p)!}{m^l l!} \\ &\quad \cdot \begin{bmatrix} n-ml-(m-1)p \\ k-l \end{bmatrix}^B_{\geq m+1, r-p} \\ &= n!\sum_{p=0}^{r}\sum_{l=0}^{k}\binom{r}{p}\frac{(2^m-1)^{l+p}}{m^l l!(n-m(l+p)+p)!}\begin{bmatrix} n-m(l+p)+p \\ k-l \end{bmatrix}^B_{\geq m+1, r-p}\end{aligned}$$
holds.

Proof. Consider a type B r-permutation $\sigma \in \mathcal{A}^B_{n,\geq m, r}$ with $k+r$ cycles and let l be the number of cycles $\mathfrak{b}$ of size m such that $\mathfrak{b} \cap ([r] \cup [\bar{r}]) = \emptyset$ and $|\mathfrak{b} \cap \overline{[n+r]}| < m$. Let p be the number of cycles with the above property with $\mathfrak{b} \cap ([r] \cup [\bar{r}]) \neq \emptyset$. The number of permutations with this two statistics are counted by the expression on the left-hand side. Adding for all cases of p and l, yields the result. □

The special case $r=0$ is stated next.

Corollary 6.6. The identity
$$\begin{bmatrix} n \\ k \end{bmatrix}^B_{\geq m} = \sum_{l=0}^{k}\binom{n}{ml}\frac{(ml)!}{m^l l!}(2^m-1)^l\begin{bmatrix} n-ml \\ k-l \end{bmatrix}^B_{\geq m+1}$$
holds.

The case $m = 1$ is similar.

Corollary 6.7. For $n, k \geq 0$, the identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^B = \sum_{p=0}^{r} \sum_{\ell=0}^{k} \binom{r}{p} \binom{n}{\ell} \begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}_{\geq 2, r-p},$$

holds. Moreover, if $r = 0$ one obtains

$$\begin{bmatrix} n \\ k \end{bmatrix}^B = \sum_{\ell=0}^{k} \binom{n}{\ell} \begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}_{\geq 2}^B.$$

6.3. **The Riordan matrices.** It turns out that the sequence $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\geq m,r}^B$ can be encoded by a Riordan matrix.

Theorem 6.8. *The matrix* $\mathcal{C}_{\geq m,r} := \left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\geq m,r}^B\right)_{n,k\geq 0}$ *is an exponential Riordan array given by*

$$\mathcal{C}_{\geq m,r} = \left(\left(\frac{1-x^{m-1}}{1-x} - \frac{2^m x^{m-1}}{1-2x}\right)^r, -\ln(1-2x) - \sum_{\ell=1}^{m-1} \frac{2^k - 1}{k} x^k\right).$$

Theorem 4.2 can be generalized as follows. The proof is left to the reader.

Theorem 6.9. *If $m > 0$, the two diagonals below the main diagonal are*

$$\begin{aligned}
\begin{bmatrix} n+1 \\ n \end{bmatrix}_{\geq m,r}^B &= 2^{(n+r+1)\delta_{m,1} + 2\delta_{m,2} - 1}(n+1)(n+2r), \\
\begin{bmatrix} n+2 \\ n \end{bmatrix}_{\geq m,r}^B &= \frac{2^{(n+r+2)\delta_{m,1}}}{12} \binom{n+2}{2} \\
&\quad \cdot \left(3 \cdot 2^{4\delta_{m,2}}(4r(r+n-1) + n(n-1)) + 2^{3(\delta_{m,2}+\delta_{m,3})+3}(n+3r)\right)
\end{aligned}$$

where $\delta_{a,b}$ is the Kronecker delta function.

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