Combinatorial sequences arising from a rational integral

Victor H. Moll
Department of Mathematics
Tulane University
New Orleans, LA 70118
vhm@math.tulane.edu

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Abstract

We present analytical properties of a sequence of integers related to the evaluation of a rational integral. We also discuss an algorithm for the evaluation of the 2-adic valuation of these integers that has a combinatorial interpretation.

1 Introduction

The sequence of positive integers

$$b_{l,m} = \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l} \tag{1.1}$$

for $m \in \mathbb{N}$ and $0 \leq l \leq m$ appeared in the process of evaluating the definite integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}. \tag{1.2}$$

The author has shown in [7] that the polynomial

$$P_m(a) := 2^{-2m} \sum_{l=0}^{m} b_{l,m} a^l \tag{1.3}$$

satisfies

$$P_m(a) := \frac{1}{\pi} 2^{m+3/2} (a+1)^{m+1/2} N_{0,4}(a;m). \tag{1.4}$$

The coefficients $b_{l,m}$ do not have a natural combinatorial interpretation, but they have some combinatorial flavor. The goal of this work is to present several conjectures that illustrate this. For instance, in Section 3 we discuss a criteria developed in order to establish the unimodality of $\{b_{l,m} : 0 \leq l \leq m\}$. We have conjectured that $b_{l,m}$ are logconcave, that is, $b_{l,m}^2 \geq b_{l-1,m} b_{l+1,m}$. We present several of our attempts to establish this conjecture. In the last section we discuss arithmetical properties of $b_{l,m}$. In particular we describe an algorithm to evaluate their 2-adic valuation. Based on extensive numerical data, we have conjectured that this valuation can be determined in terms of two natural operators acting on sequences: the first one simply repeats the initial element of a sequence, that is,

$$F(\{a_1, a_2, a_3, \cdots\}) \ := \ \{a_1, a_1, a_2, a_3, \cdots\}, \tag{1.5}$$

and the second one picks every other term:

$$T(\{a_1,\, a_2,\, a_3,\, \cdots\}) \;:=\; \{a_1,\, a_3,\, a_5,\, a_7,\, \cdots\}. \tag{1.6}$$

The algorithm involves the operator

$$L_l := \prod_{j=1}^{\omega(l)} F\left[T^{n_j} - c\right] \tag{1.7}$$

where $\omega(l) \in \mathbb{N}$, c is defined by

$$c \;:=\; \{\nu_2(m): \quad m \geq 1\} = \{0,\, 1,\, 0,\, 2,\, 0,\, 1,\, 0,\, 3,\, 0, \cdots\}, \tag{1.8}$$

and the exponents n_j are (conjecturally) related to the distinct compositions of a binary sequence of fixed length. Conjecture 4.8 presents all the details. We conclude that the combinatorics of $b_{l,m}$ is hidden in their arithmetic properties.

Section 2 presents a hypergeometric evaluation of (1.2). Section 3 discusses the unimodality of the sequence $b_{l,m}$ and describes our work on the conjectured logconcavity. Section 4 presents an alternative expression for $b_{l,m}$ that is used to discuss their divisibility properties and to state our main conjecture.

2 A hypergeometric evaluation of the integral

The integral (1.2) is now evaluated by standard methods in terms of the hypergeometric function

$${}_2F_1[a, b; c; z] = \sum_{k=0}^{\infty} \frac{(a)_k\, (b)_k}{(c)_k\, k!} z^k. \tag{2.1}$$

Here $a \in \mathbb{R}$, $k \in \mathbb{N}$, and $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the Pochhammer symbol, with the usual convention $(a)_0 = 1$. The reader will find in [3] detailed information about this function. In particular, the integral representations

$${}_2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \tag{2.2}$$

and

$${}_2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^{\infty} s^{b-1}(1+s)^{a-c}(1+sz)^{-a}\, ds \tag{2.3}$$

appear there. The gamma function in (2.2) and (2.3) is given by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt \tag{2.4}$$

for $z > 0$. This function has a meromorphic extension to $\mathbb{C}$ with simple poles at the negative integers. The special values

$$\Gamma(n) = (n-1)! \quad \text{and } \Gamma(n + \tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n}} \frac{(2n)!}{n!} \tag{2.5}$$

for $n \in \mathbb{N}$, will be used throughout.

The reader will find in [7, 9] alternative proofs of the value of $N_{0,4}(a;m)$. The latter establishes a connection between $N_{0,4}(a;m)$, the Taylor expansion around $c=0$ of the function $\sqrt{a+\sqrt{1+c}}$ and some results of Ramanujan. This was quite a *tour de force*.

Theorem 2.1. *Let $a>-1$ and $m \in \mathbb{N}$. Define*

$$P_m(a) = \frac{1}{\pi} 2^{m+3/2}(a+1)^{m+1/2} N_{0,4}(a;m). \tag{2.6}$$

Then $P_m(a)$ is a polynomial in a given by

$$P_m(a) \quad = \quad 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k. \tag{2.7}$$

Proof. The change of variable $t=x^2$ yields

$$N_{0,4}(a;m) = \frac{1}{2} \int_0^\infty t^{-1/2}(t+t_+)^{-m-1}(t+t_-)^{-m-1}\, dt$$

where $t_\pm = -a \pm \sqrt{a^2-1}$ are the roots of $t^2+2at+1=0$. The representation (2.2) and the identity $t_+t_-=1$ show that $N_{0,4}(a;m)$ is given by

$$\begin{aligned} N_{0,4}(a;m) \quad &= \quad \frac{\pi}{2^{4m+3/2}} \binom{4m+2}{2m+1} \sqrt{a-\sqrt{a^2-1}} \times \\ & \qquad {}_2F_1\left[m+1,\, \tfrac{1}{2}; 2m+2;\, 2\left(1-a^2+a\sqrt{a^2-1}\right)\right]. \end{aligned}$$

This can now be simplified using Kummer's formula

$${}_2F_1\left[\alpha, \beta; 2\beta; \frac{4z}{(1+z)^2}\right] = (1+z)^{2\alpha} {}_2F_1\left[\alpha, \alpha+\tfrac{1}{2}-\beta; \beta+\tfrac{1}{2}; z^2\right],$$

described in [3]. Apply it with $z=\sqrt{a-1}/\sqrt{a+1}$, $\alpha=\frac{1}{2}$ and $\beta=m+1$ and use $\sqrt{a-\sqrt{a^2-1}} = \frac{1}{\sqrt{2}}\left(\sqrt{a+1}-\sqrt{a-1}\right)$ to obtain

$$N_{0,4}(a;m) = \frac{\pi}{2^{4m+5/2}} \binom{4m+2}{2m+1} \frac{1}{\sqrt{a+1}}\, {}_2F_1\left[\tfrac{1}{2}, -m; m+\tfrac{3}{2}; \tfrac{a-1}{a+1}\right].$$

This can be simplified further using the relation

$$\begin{aligned} {}_2F_1\left[a,b;c;z\right] \quad &= \quad \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1\left[a,b;a+b-c+1;1-z\right] + \\ &+ \quad (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1\left[c-a,c-b;c-a-b+1;1-z\right], \end{aligned}$$

that in the case $b=-m \in \mathbb{N}$ reduces to

$${}_2F_1\left[a,-m;c;z\right] = \frac{\Gamma(c)\Gamma(c-a+m)}{\Gamma(c-a)\Gamma(c+m)} {}_2F_1\left[a,-m;a-m-c+1;1-z\right],$$

in view of $1/\Gamma(-m) = 0$. The resulting expression for the quartic integral is

$$N_{0,4}(a;m) = \frac{\pi}{2^{2m+3/2}\sqrt{a+1}}\binom{2m}{m}{}_2F_1\left[\frac{1}{2}, -m; -2m; \frac{2}{a+1}\right]. \tag{2.8}$$

The second argument of the ${}_2F_1$ is a negative integer so the hypergeometric series terminates. This proves that P_m, defined by (2.6), is a polynomial given by

$$P_m(a) = \sum_{k=0}^{m}\binom{m}{k}\binom{2k}{k}\binom{2m}{m}\binom{2m}{k}^{-1} 2^{-m+k}(a+1)^k.$$

It is elementary to check that this form is equivalent to (2.7). The proof is complete. □

We now define $d_l(m)$ to be the coefficient of a^l in $P_m(a)$.

Corollary 2.2. *For $m \in \mathbb{N}$ and $0 \leq l \leq m$, we have $d_l(m) = 2^{-2m}b_{l,m}$, that is,*

$$P_m(a) = 2^{-2m}\sum_{l=0}^{m} b_{l,m}a^l. \tag{2.9}$$

Proof. Expand the term $(a+1)^k$ in (2.7). □

3 Unimodality and logconcavity

A finite sequence of numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be *unimodal* if there exists an index $0 \leq j \leq m$ such that $d_0 \leq d_1 \leq \cdots \leq d_j$ and $d_j \geq d_{j+1} \geq \cdots \geq d_m$. The sequence $\{d_0, d_1, \cdots, d_m\}$ with $d_j > 0$ is said to be *logarithmically concave* (or *logconcave* for short) if $d_{j-1}d_{j+1} \leq d_j^2$ for $1 \leq j \leq m-1$. It is easy to see that if a sequence is logconcave then it is unimodal [22]. We say that a polynomial is unimodal (logconcave) if the sequences of its coefficients is unimodal (logconcave).

Unimodal sequences arise often in combinatorics, geometry, and algebra, and have been the subject of considerable research. The reader is referred to [11, 18] for surveys of the diverse techniques employed to prove that specific sequences are unimodal. Aside from establishing the unimodality (or logconcavity) of a specific sequence, it is desirable to produce a combinatorial proof. The reader will find in [23] an account of Kathy Ohara's proof of the unimodality of gaussian polynomials. A combinatorial proof of the logconcavity of $i(n,k)$, the number of permutations of n letters with k inversions, appears in [4].

We first established the unimodality of the sequence $\{b_{l,m}\}$ in [8] by a complicated argument. The proof of Theorem 3.1 given in [6] is completely elementary. The identity (2.7) shows that the unimodality of $b_{l,m}$ follows from it.

Theorem 3.1. *If $P(x)$ is a polynomial with positive nondecreasing coefficients, then $P(x+1)$ is unimodal.*

The theorem can be improved to conclude the unimodality of the polynomial $P(x+d)$, with arbitrary $d > 0$. The case $d \in \mathbb{N}$ appears in [2] and [20] treats arbitrary $d \in \mathbb{R}^{+}$.

We now turn to the question of logconcavity of the sequence $\{b_{l,m}\}$. Based on extensive numerical evidence, we propose

Conjecture 3.2. *For each $m \in \mathbb{N}$, the sequence $\{b_{l,m} : 0 \leq l \leq m\}$ is logconcave.*

We now describe our (failed) attemps to setteled this question.

(A). The first attempt is based on a result of F. Brenti [11] that is in the same spirit as Theorem 3.1:

Theorem 3.3. *Let $Q(x)$ be a logconcave polynomial. Then so is $Q(x+1)$.*

The hypothesis of this theorem do not hold in our case. Define

$$Q(x) = 2^{-2m} \sum_{k=0}^{m} 2^{k} \binom{2m-2k}{m-k} \binom{m+k}{m} x^{k} \equiv \sum_{k=0}^{m} a_k x^k. \tag{3.1}$$

Then $P_m(a) = Q(a+1)$. But the polynomial $Q(x)$ *is not* unimodal. Indeed,

$$\begin{aligned} 2^{4m-2k}\left(a_k^2 - a_{k-1}a_{k+1}\right) \quad = \quad & \binom{2m}{m-k}^2 \binom{m+k}{m}^2 \times \\ & \left(1 - \frac{k(m-k)(2m-2k+1)(m+k+1)}{(k+1)(m+k)(2m-2k-1)(m-k+1)}\right) \end{aligned}$$

could be negative. Symbolic experiments show that roughly $\sqrt{m}$ terms are negative.

(B) The WZ-method developed by H. Wilf and D. Zeilberger can be used to produce a recurrence for the numbers $b_{l,m}$. Details on this procedure can be found in [17]. One finds that $b_{l,m}$ satisfies

$$b_{l+1}(m) = \frac{2m+1}{l+1} b_{l,m} - \frac{(m+l)(m+1-l)}{l(l+1)} b_{l-1,m}. \tag{3.2}$$

Therefore the sequence $\{b_{l,m}\}$ is logconcave provided

$$(m+l)(m+1-l)b_{l-1,m}^2 + l(l+1)b_{l,m}^2 - l(2m+1)b_{l-1,m}b_{l,m} \geq 0. \tag{3.3}$$

We have extensive numerical evidence to support the next conjecture:

Conjecture 3.4. *The left-hand side of (3.3) attains its minimum at $l = m$ with value $2^{2m} m(m+1)\binom{2m}{m}^2$.*

The inequality (3.3) can be written in terms of $u = b_{l,m}/b_{l-1,m}$ as

$$l(l+1)u^2 - l(2m+1)u + (m+l)(m+1-l) \geq 0. \tag{3.4}$$

Unfortunately, the discriminant of the quadratic form, is

$$\text{disc } = l(4l^3 - 4m^2 - 4m - 3l), \tag{3.5}$$

that is not strictly negative.

(C) A useful criterion to establish the logconcavity of a sequence $\{a_j\}$ is provided by the zeros of the polynomial $P(x) = a_0 + a_1x + \cdots + a_mx^m$.

Theorem 3.5. *If the polynomial P has only real roots, then it is logconcave.*

The reader will find in [5] a proof and several examples.

The analysis of the zeros of $P_m(a)$ was discussed in [7] and [8]. It turns out that $P_m(a)$ is part of the family of Jacobi polynomials $P_m^{(\alpha,\beta)}(z)$ defined by

$$P_m^{(\alpha,\beta)}(z) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left(\frac{z+1}{2} \right)^k. \tag{3.6}$$

The special values $\alpha = m+\frac{1}{2}$, $\beta = -(m+\frac{1}{2})$ produce $P_m(a)$. The zeros of Jacobi polynomials are studied in detail in [19] (see page 145ff). We concluded that $P_m(a)$ has at most one real zero.

The zeros of $P_m(a)$ have interesting properties. Dimitrov [12] recently established our conjecture that, when scaled appropriately, the limit curve of these zeros is the left half of Bernoulli's lemniscate

$$L = \{z \in \mathbb{C} : \quad |z^2 - 1| = 1, \ \operatorname{Re} z < 0\}. \tag{3.7}$$

A generalization. The coefficients $\{b_{l,m}\}$ seem to have a property much stronger than the logconcavity stated in Conjecture 3.2. Introduce the operator $\mathfrak{L}$ on the space of sequences, via

$$\mathfrak{L}\,(a_l) := a_l^2 - a_{l-1}a_{l+1}. \tag{3.8}$$

The finite sequence $\{a_1, \cdots, a_n\}$ is replaced by $\{\ldots, 0, 0, a_1, \cdots, a_n, 0, 0, \ldots\}$ before applying $\mathfrak{L}$. Thus, $\{a_l\}$ is logconcave if $\mathfrak{L}(a_l)$ is nonnegative. We say that $\{a_l\}$ is r-logconcave if $\mathfrak{L}^{(k)}\,(a_l) \geq 0$ for $0 \leq k \leq r$. The sequence $\{a_l\}$ is ∞-logconcave if it is r-logconcave for every $r \in \mathbb{N}$.

Conjecture 3.6. *For each $m \in \mathbb{N}$, the sequence $\{b_{l,m} : 0 \leq l \leq m\}$ is ∞-logconcave.*

The binomial coefficients is the canonical sequence on which these issues are tested. The solution of the next conjecture should provide guiding principles on how to approach Conjecture 3.6.

Conjecture 3.7. *For $m \in \mathbb{N}$ fixed, the sequence of binomial coefficients $\{\binom{m}{l}\}$ is ∞-logconcave.*

A direct calculation proves the existence of rational functions $R_r(m,l)$ such that

$$\mathfrak{L}^{(r)} \binom{m}{l} = \binom{m}{l}^{2^r} R_r(m,l). \tag{3.9}$$

Moreover R_r satisfy the recurrence

$$R_{r+1}(m,l) = R_r^2(m,l) - \left[\frac{l(m-l)}{(l+1)(m-l+1)}\right]^{2^r} \times R_r(m,l-1)R_r(m,l+1).$$

Therefore we *only* need to prove that $R_r(m,l) \geq 0$. This could be difficult.

Note. Conjecture 3.2 has been established by Manuel Kauers and Peter Paule. The preprint *A computer proof of Moll's logconcavity conjecture* established the conjecture using the RISC package MultiSum.

4 Divisibility properties of $b_{l,m}$

The original expression for $b_{l,m}$ (1.1), written in the form

$$b_{l,m} \quad = \quad 2^l \sum_{k=l}^{m} 2^{k-l} \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l} \tag{4.1}$$

shows that the power of 2 that divides $b_{l,m}$ is at least l. A more detailed study of this power requires the alternative representation of $b_{l,m}$ discussed in this section.

The evaluation of $b_{l,m}$ using (4.1) is efficient if l is close to m. Indeed,

$$b_{m,m} = 2^m \binom{2m}{m} \quad \text{and} \quad b_{m-1,m} = 2^{m-1}(2m+1)\binom{2m}{m}.$$

The formulas described below are efficient for l small.

A direct computation of the (finite) Taylor series of the polynomial P_m yields its coefficients in terms of definite integrals related to (1.2). The details of the next theorem appear in [10].

Theorem 4.1. *There exist polynomials $\alpha_l(m)$ and $\beta_l(m)$, with positive integer coefficients, such that*

$$b_{l,m} = \frac{2^{m-l}}{l!m!}\left(\alpha_l(m)\prod_{k=1}^{m}(4k-1) - \beta_l(m)\prod_{k=1}^{m}(4k+1)\right). \tag{4.2}$$

The degrees of α_l and β_l are l and $l-1$ respectively. For instance

$$b_{0,m} \quad = \quad \frac{2^m}{m!}\prod_{k=1}^{m}(4k-1) \tag{4.3}$$

$$b_{1,m} \quad = \quad \frac{2^{m-1}}{m!}\left((2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1)\right) \tag{4.4}$$

Numerical calculations on the roots of these polynomials, lead us to conjecture the location of these roots. The next theorem was established by J. Little in [15].

Theorem 4.2. *For every $m \in \mathbb{N}$, all the zeros of the polynomials α_l, β_l lie on the vertical line* $\mathrm{Re}\, m = -\frac{1}{2}$.

The proof is based on the fact that the auxiliary polynomials $\alpha_l(u)$, $\beta_l(u)$, with $u = (s-1)/2$, satisfy the three-term recurrence

$$p_{l+1}(s) = 2sp_l(s) - (s^2 - (2l-1)^2)p_{l-1}(s). \tag{4.5}$$

A generalization of the classical Sturm separation theorem (see [16] for proofs) is then used to establish the result.

The valuations. Arithmetic properties of numbers appearing in combinatorics have always been of interest. The reader will find in [1] information about the prime decomposition of Catalan numbers and [21] describes divisibility by 2 of the Stirling numbers of second kind.

We now describe divisibility properties of the sequence $\{b_{l,m}\}$. We recall first some basic definitions on valuations. Given a prime p and a rational number r, there exist unique integers a, b, m with a and b not divisible by p such that

$$r = \frac{a}{b}p^m \tag{4.6}$$

The integer m is the p-adic valuation of r and we denote it by $\nu_p(m)$. Observe that we depart from the usual convention $m = -\nu_p(m)$.

A basic result of number theory states that

$$\nu_p(m!) = \sum_{k=1}^{\infty} \lfloor \frac{m}{p^k} \rfloor. \tag{4.7}$$

Naturally the sum is finite and we can end it at $k = \lfloor \log_2 m \rfloor$. There is a famous result of Legendre [13, 14] for the p-adic valuation of $m!$. It states that

$$\nu_p(m!) = \frac{m - s_p(m)}{p-1}, \tag{4.8}$$

where $s_p(m)$ is the sum of the base-p digits of m. In particular

$$\nu_2(m!) = m - s_2(m). \tag{4.9}$$

The 2-adic value of $b_{0,m}$ follows directly from (4.3). It follows that

$$\nu_2(b_{0,m}) = m - \nu_2(m!)$$

and Legendre's result (4.9) reduces this to

$$\nu_2(b_{0,m}) = s_2(m). \tag{4.10}$$

The next coefficient $b_{1,m}$ given in (4.4) was analyzed in [10]. The main result there is:

Theorem 4.3. *The 2-adic valuation of $b_{1,m}$ is given by*

$$\nu_2(b_{1,m}) = s_2(m) + \nu_2(m(m+1)). \tag{4.11}$$

The key element of the proof is to express the products in the definition of $b_{1,m}$ in terms of the Stirling numbers of the first kind

$$x(x+1)(x+2)\cdots(x+r-1) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} x^k \tag{4.12}$$

and the representation

$$\begin{bmatrix} r \\ r-k \end{bmatrix} = \sum_{i=0}^{k-1} \binom{r}{2k-i} C_{k,i} \tag{4.13}$$

for some integers $C_{k,i}$.

We now present some conjectures about the functions $\nu_2(b_{l,m})$ for $l \geq 2$. Introduce the notation

$$A_{l,m} = \alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1), \tag{4.14}$$

so that Theorem 4.1 states that

$$A_{l,m} = l!m!2^{-m+l} b_{l,m}, \text{ for } m \geq l. \tag{4.15}$$

For example,

$$A_{1,m} = m!2^{-m+1} b_{1,m} \tag{4.16}$$

and Theorem 4.3 implies that

$$\nu_2(A_{1,m}) = \nu_2(2m(m+1)). \tag{4.17}$$

The first few values of $\nu_2(A_{1,m})$ are given by

$$\nu_2(A_{1,m}) = \{2,\, 2,\, 3,\, 3,\, 2,\, 2,\, 4,\, 4,\, 2,\, 2,\, \cdots\} \tag{4.18}$$

and we observe that this set consists of blocks of length 2, starting at odd integers, on which the function $A_{1,m}$ has the same value. The explicit formula (4.17) can be used to chek this property. Indeed, for m odd, we have

$$\nu_2(A_{1,m}) = \nu_2(2m(m+1)) = 1 + \nu_2(m+1),$$

and

$$\nu_2(A_{1,m+1}) = \nu_2(2(m+1)(m+2)) = 1 + \nu_2(m+1).$$

This type of *block structure* it is conjectured to remains valid for $l \geq 2$.

Definition 4.4. Let $s \in \mathbb{N}$, $s \geq 2$. We say that a sequence $\{a_j : j \in \mathbb{N}\}$ is *simple of length* s, or just s-simple if, for each $t \in \{0, 1, 2, \cdots\}$, we have

$$a_{st+1} = a_{st+2} = \cdots = a_{s(t+1)}. \tag{4.19}$$

For example, the sequence $\{\nu_2(A_{1,m}),\, m \in \mathbb{N}\}$ is 2-simple.

The function $A_{2,m}$ is given by

$$A_{2,m} = 2(2m^2+2m+1) \prod_{k=1}^{m} (4k-1) - 2(2m+1) \prod_{k=1}^{m} (4k+1). \tag{4.20}$$

and its 2-adic values are

$$\nu_2(A_{2,m}) = \{5,\, 5,\, 5,\, 5,\, 6,\, 6,\, 6,\, 6,\, 5,\, 5,\, 5,\, 5,\, \cdots\}.$$

Similarly

$$\nu_2(A_{3,m}) = \{7,\, 7,\, 9,\, 9,\, 8,\, 8,\, 9,\, 9,\, 7,\, 7,\, 10,\, 10,\, \cdots\}.$$

Therefore $\nu_2(A_{2,m})$ is 4-simple and $\nu_2(A_{3,m})$ is 2-simple.

Conjecture 4.5. *Let $l \in \mathbb{N}$ be fixed. Then the set $\{\nu_2(A_{l,m}) : m \geq l\}$ is an s-simple sequence with $s = 2^{1+\nu_2(l)}$.*

A recurrence. The recurrence (3.2) for the numbers $b_{l,m}$ and (4.15) yield

$$A_{l+1,m} = 2(2m+1)A_{l,m} - 4(m+l)(m+1-l)A_{l-1,m}. \tag{4.21}$$

Using the WZ-method, now in the variable m, automatically produces the recurrence

$$b_{l,m+2} = \frac{2(8m^2-4l^2+24m+19)}{(m+2)(m-l+2)}b_{l,m+1} - \frac{4(4m+5)(4m+3)(m+l+1)}{(m+2)(m+1)(m-l+2)}b_{l,m}$$

that yields

$$A_{l,m+2} = \frac{8m^2-4l^2+24m+19}{m-l+2}A_{l,m+1} - \frac{(4m+5)(4m+3)(m+l+1)}{m-l+2}A_{l,m}.$$

In particular, for $l = 1$ we obtain

$$A_{1,m+2} = \frac{8m^2+24m+15}{m+1}A_{l,m+1} - \frac{(4m+5)(4m+3)(m+2)}{m+1}A_{1,m}. \tag{4.22}$$

Some partial results on Conjecture 4.5 can be obtained from this recurrence. To illustrate this, let $m = 2k-1$ and as induction hypothesis we assume

$$\nu_2(A_{1,2k-1}) = \nu_2(A_{1,2k}) := t. \tag{4.23}$$

To establish the conjecture we need to prove that

$$\nu_2(A_{1,2k+1}) = \nu_2(A_{1,2k+2}). \tag{4.24}$$

We write $A_{1,2k-1} = 2^t x$ and $A_{1,2k} = 2^t y$, with x, y odd integers. Then $m = 2k-1$ in (4.22) yields

$$kA_{1,2k+1} = 2^{t-1}\left[(32k^2+16k-1)y - (128k^3+64k^2-2k-1)x\right].$$

Writing

$$A_{1,2k+1} = 2^w z$$

with z odd and using $m = 2k$ in (4.22) yields

$$(2k+1)A_{1,2k+2} = 2^w(32k^2+48k+15)z - 2^{t+1}(k+1)(8k+3)(8k+5)y.$$

In particular, if $w < t+1$ we have

$$(2k+1)A_{1,2k+2} = 2^w\left[(32k^2+48k+15)z - 2^{t+1-w}(k+1)(8k+3)(8k+5)y\right],$$

so that $\nu_2(A_{1,2k+2}) = \nu_2(A_{1,2k+1})(= w)$ as required. *It is unlikely that these elementary arguments will produce a full proof of the conjecture.*

A geometrical interpretation. The graphs of the function $\nu_2(A_{l,m})$, where we reduce the repeating blocks to a single value, are shown in the next figures.

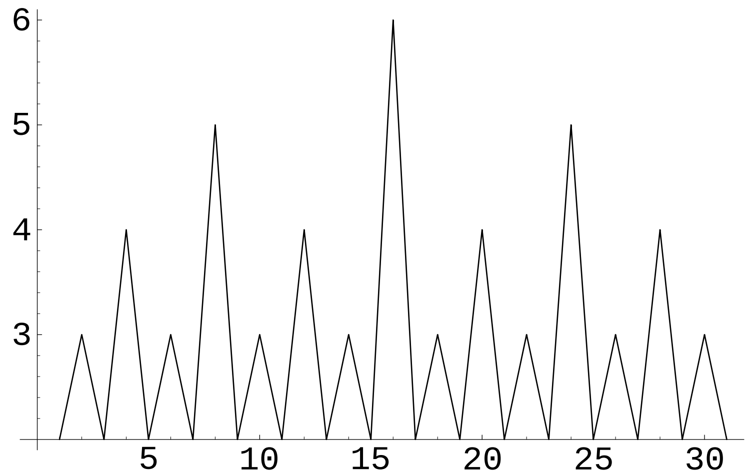

Figure 1: The 2-adic valuation of $d_1(m)$

The main experimental result is that these graphs have an *initial segment* from which the rest is determined by adding a *central piece* followed by a *folding rule*. For example, in the case $l = 1$, the first few values of the reduced table are

$$\{2,\ 3,\ 2,\ 4,\ 2,\ 4,\ 2,\ 3,\ 2,\ 5, \ldots\}.$$

The ingredients are:

initial segment: $\{2,\ 3,\ 2\}$,

central piece: the value at the center of the initial segment, namely 3.

rules of formation: start with the initial segment and add 1 to the central piece and reflect.

This produces the sequence

$$\{2,3,2\} \to \{2,3,2,4\} \to \{2,3,2,4,2,3,2\} \to \{2,3,2,4,2,3,2,5\} \to$$

$$\to \{2,3,2,4,2,3,2,5,2,3,2,4,2,3,2\}.$$

The details are shown in Figure 1.

The difficulty with this procedure is that, at the moment, we have no form of determining the initial segment nor the central piece. Figure 2 shows the beginning of the case $l = 9$. From here one could be tempted to predict that this graph extends as in the case $l = 1$. This is not correct as it can be seen in Figure 3. The new pattern described seems to be the correct one as shown in Figure 4.

The initial pattern could be quite elaborate. Figure 5 illustrates the case $l = 53$.

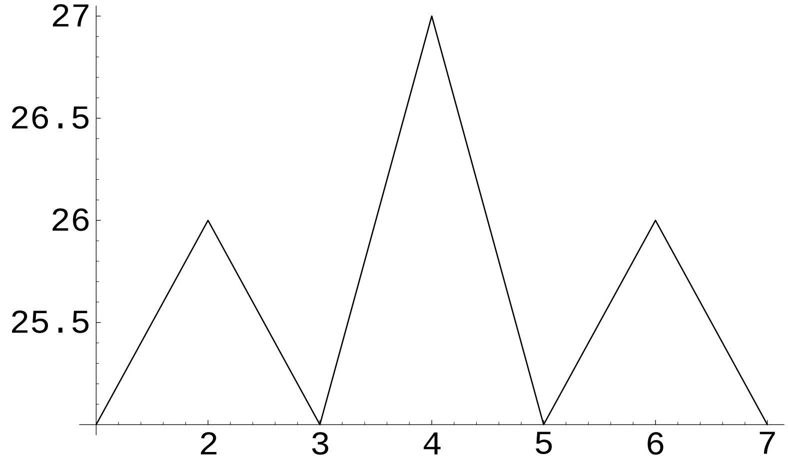

Figure 2: The beginning for $l = 9$

An algorithm and the main conjecture. We now present an algorithm that we hope will lead to the derivation of an analytic expression for $\nu_2(A_{l,m})$. The algorithm requires two operators defined on sequences:

$$F(\{a_1,\, a_2,\, a_3,\, \cdots\}) \;\; := \;\; \{a_1,\, a_1,\, a_2,\, a_3,\, \cdots\},$$

and

$$T(\{a_1,\, a_2,\, a_3,\, \cdots\}) \;\; := \;\; \{a_1,\, a_3,\, a_5,\, a_7,\, \cdots\}.$$

Now recall the sequence c defined in (1.8):

$$c \;\; := \;\; \{\nu_2(m): \quad m \geq 1\} = \{0,\, 1,\, 0,\, 2,\, 0,\, 1,\, 0,\, 3,\, 0, \cdots\}.$$

We begin the description of the algorithm with the example $l = 12$. The sequence

$$X_1(12) \;\; := \;\; \{\nu_2(A_{12}(m)): \quad m \geq 1\}$$

begins with

$$\begin{aligned} X_1(12) \;\; = \{ \;\; & 34, 34, 34, 34, 34, 34, 34, 34, 36, 36, 36, 36, 36, 36, 36, 36, \\ & 35, 35, 35, 35, 35, 35, 35, 35, 36, 36, 36, 36, 36, 36, 36, 36, \\ & 34, 34, 34, 34, 34, 34, 34, 34, 37, 37, 37, 37, 37, 37, 37, 37, \cdots\}. \end{aligned}$$

This is 8-simple, illustrating Conjecture 4.5, in view of $2^{1+\nu_2(12)} = 8$.

The first step in the algorithm is to replace $X_1(12)$ by $Y_1(12) = T^3(X_1(12))$. This sequence contains every eight element of $X_1(12)$ and it begins with

$$\begin{aligned} Y_1(12) \;\; = \{ \;\; & 34, 36, 35, 36, 34, 37, 36, 37, 34, 36, 35, 36, 34, 38, 37, 38, \\ & 34, 36, 35, 36, 34, 37, 36, 37, 34, 36, 35, 36, 34, 39, 38, 39, \\ & 34, 36, 35, 36, 34, 37, 36, 37, 34, 36, 35, 36, 34, 38, 37, 38, \cdots\}. \end{aligned}$$

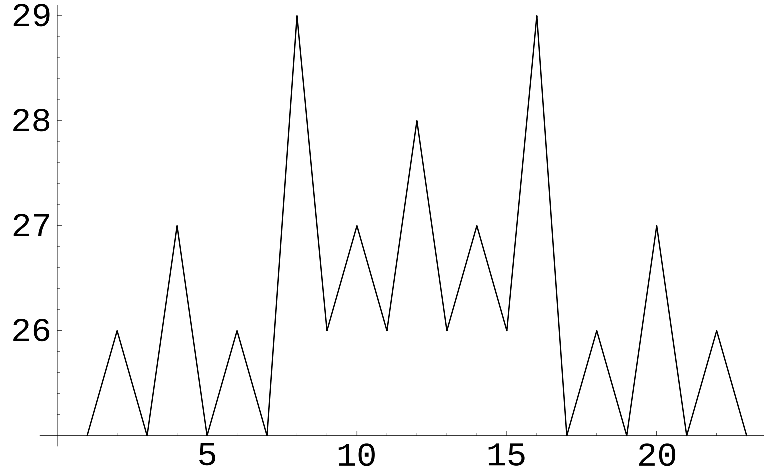

Figure 3: The continuation of $l = 9$

The next step is to define $Z_1(12) := Y_1(12) - c$, with c as above. This sequence begins with

$$\begin{aligned} Z_1(12) \;=\{ \;& 30, 31, 31, 30, 30, 33, 33, 31, 31, 32, 32, 31, 31, 33, 33, 30, \\ & 30, 31, 31, 30, 30, 34, 34, 32, 32, 33, 33, 32, 32, 34, 34, 30, \\ & 30, 31, 31, 30, 30, 33, 33, 31, 31, 32, 32, 31, 31, 33, 33, 30, \cdots \}, \end{aligned}$$

that is *almost* 2-simple, except that the first element appears only once. This motivates the map F. The last step is to define $W_1(12) := F\,(Z_1(12))$, that produces

$$\begin{aligned} W_1(12) \;=\{ \;& 30, 30, 31, 31, 30, 30, 33, 33, 31, 31, 32, 32, 31, 31, 33, 33, \\ & 30, 30, 31, 31, 30, 30, 34, 34, 32, 32, 33, 33, 32, 32, 34, 34, \\ & 30, 30, 31, 31, 30, 30, 33, 33, 31, 31, 32, 32, 31, 31, 33, 33, \cdots \}. \end{aligned}$$

This new sequence is 2-simple and the first loop is completed.

Algorithm.

1) Start with the sequence $X_1(l) := \{\nu_2(A_l(m)) : \quad m \geq 1\}$.

2) Find $n_1 \in \mathbb{N}$ so that the sequence $X_1(l)$ is 2^{n_1}-simple. Define $Y_1(l) := T^{n_1}\,(X_1(l))$. We conjecture that $n_1 = 1 + \nu_2(l)$.

3) Introduce the constant shift $Z_1(l) := Y_1(l) - c$.

4) Define $W_1(l) := F\,(Z_1(l))$.

The sequence W_1 is 2^{n_2}-simple. Then return to step 1) with W_1 instead of X_1.

Conjecture 4.6. *After a finite number of steps, the algorithm yields a constant sequence of the form* $\{a, a, a, a, a, a, \cdots\}$. *The constant term will be denoted by* $a_\infty(l)$.

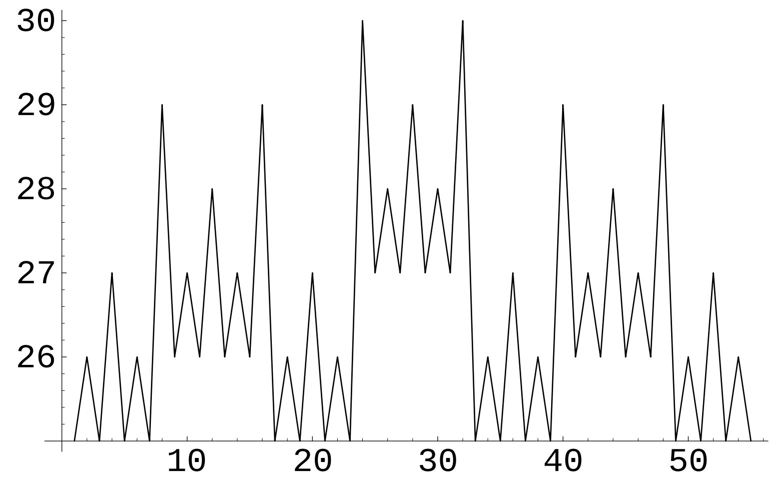

Figure 4: The pattern for $l = 9$ persists

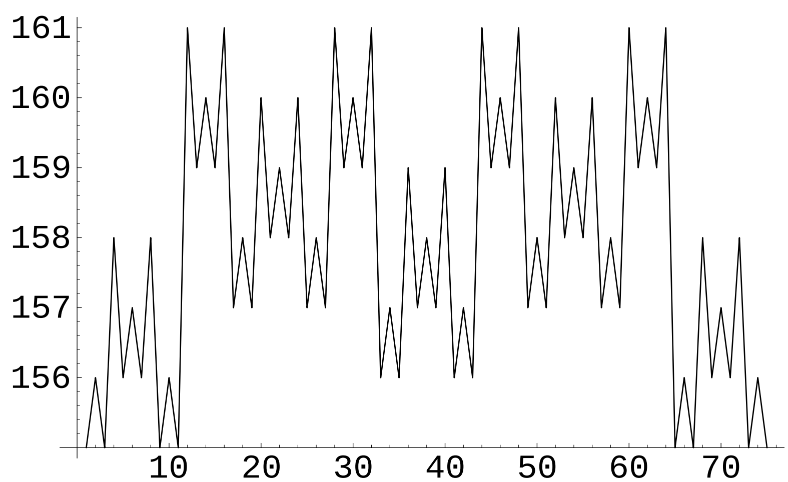

Figure 5: The initial pattern for $l = 53$

Definition 4.7. Let $\omega(l)$ be the number of steps required in the previous conjecture. The sequence of integers

$$\Omega(l) \quad := \quad \{n_1, \, n_2, \, n_3, \cdots, n_{\omega(l)}\} \tag{4.25}$$

is called the *reduction sequence* of l.

We now present the values of the limiting constant $a_\infty(l)$ and the sets $\Omega(l)$ for $1 \leq l \leq 32$. The data presented below was checked with tables of size 200 and validated with size 400.

l	$a_\infty(l)$	$\Omega(l)$
1	2	1
2	5	2
3	7	1, 1
4	11	3
5	13	1, 2
6	16	2, 1
7	18	1, 1, 1
8	23	4
9	25	1, 3
10	28	2, 2
11	30	1, 1, 2
12	34	3, 1
13	36	1, 2, 1
14	39	2, 1, 1
15	41	1, 1, 1, 1
16	47	5

l	$a_\infty(l)$	$\Omega(l)$
17	49	1, 4
18	52	2, 3
19	54	1, 1, 3
20	58	3, 2
21	60	1, 2, 2
22	63	2, 1, 2
23	65	1, 1, 1, 2
24	70	4, 1
25	72	1, 4, 1
26	75	2, 2, 1
27	77	1, 1, 2, 1
28	81	3, 1, 1
29	83	1, 2, 1, 1
30	86	2, 1, 1, 1
31	88	1, 1, 1, 1, 1
32	95	6

The **main conjecture** stated below provides a combinatorial interpretation of the sets $\Omega(l)$.

Conjecture 4.8. *The sets $\Omega(l)$ associated to the integers l from $2^j + 1$ to 2^{j+1} are the 2^j distinct compositions of $j + 1$.*

To obtain the order in which the compositions appear, write all the binary sequences of length j in lexicographic order and then preprend a 1 to each of these. For instance, for $j = 3$ we obtain

$$1000, \, 1001, \, 1010, \, 1011, \, 1100, \, 1101, \, 1110, \, 1111.$$

Read these sequences from right to left. The first part of the set associated to l is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on. In the case $j = 3$, this yields

$$4; \; 1,3; \; 2,2; \; 1,1,2; \; 3,1; \; 1,2,1; \; 2,1,1; \; 1,1,1,1$$

as desired.

One last experimental conjecture. Based on observations of the values of $\nu_2(A_{l,m})$, Dante Manna has conjectured an analytic expression for this function:

$$\nu_2(A_{l,m}) = \nu_2((m-l+1)_{2l}) + l. \tag{4.26}$$

This is a generalization of Theorem 4.3. Expressing the Pochhammer symbol as quotients of factorials, we can write (4.26) as

$$\nu_2(A_{l,m}) = 3l + s_2(m-l) - s_2(m+l). \tag{4.27}$$

It follows that the asymptotic value in Conjecture 4.6 is

$$a_\infty(l) = 3l - s_2(l). \tag{4.28}$$

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References

[1] R. Alter and K. Kubota. Prime and prime power divisibility of Catalan numbers. *J. Comb. Theory Ser. A*, 15:243–256, 1973.

[2] J. Alvarez, M. Amadis, G. Boros, D. Karp, V. Moll, and L. Rosales. An extension of a criterion for unimodality. *Elec. Jour. Comb.*, 8:1–7, 2001.

[3] G. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 1999.

[4] M. Bona. A combinatorial proof of the logconcavity of a famous sequence counting permutations. *Elec. Jour. Comb.*, 11:1–4, 2004.

[5] M. Bona. *Combinatorics of Permutations*. Chapman and Hall, Boca Raton, 1st edition, 2004.

[6] G. Boros and V. Moll. A criterion for unimodality. *Elec. Jour. Comb.*, 6:1–6, 1999.

[7] G. Boros and V. Moll. An integral hidden in Gradshteyn and Ryzhik. *Jour. Comp. Applied Math.*, 106:361–368, 1999.

[8] G. Boros and V. Moll. A sequence of unimodal polynomials. *Jour. Math. Anal. Appl.*, 237:272–287, 1999.

[9] G. Boros and V. Moll. The double square root, Jacobi polynomials and Ramanujan's master theorem. *Jour. Comp. Applied Math.*, 130:337–344, 2001.

[10] G. Boros, V. Moll, and J. Shallit. The 2-adic valuation of the coefficients of a polynomial. *Scientia*, 7:37–50, 2001.

[11] F. Brenti. Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry: an update. *Contemporary Mathematics*, 178:71–89, 1994.

[12] D. Dimitrov. Asymptotics of zeros of polynomials arising from rational integrals. *Jour. Math. Anal. Appl.*, 299:127–132, 2004.

[13] R. Graham, D. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison Wesley, Boston, 2nd edition, 1994.

[14] A. M. Legendre. *Theorie des Nombres*. Firmin Didot Freres, Paris, 1830.

[15] J. Little. On the zeroes of two families of polynomials arising from certain rational integrals. *Rocky Mountain Journal*, 35:1205–1216, 2005.

[16] M. Mignotte. *Mathematics for Computer Algebra*. Springer-Verlag, New York, NY., 1991.

[17] M. Petkovsek, H. Wilf, and D. Zeilberger. *A=B*. A. K. Peters, Ltd., 1st edition, 1996.

[18] R. Stanley. Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry. graph theory and its applications: East and West (Jinan, 1986). *Ann. New York Acad. Sci.*, 576:500–535, 1989.

[19] G. Szego. *Orthogonal polynomials*. American Mathematical Society, 4th edition, 1939.

[20] Yi Wang and Yeong-Nan Yeh. Proof of a conjecture on unimodality. *Europ. Journal of Comb.*, 26:617–627, 2005.

[21] S. De Wannemacker. On the 2-adic orders of Stirling numbers of the second kind. *INTEGERS*, 5(1):A–21, 2005.

[22] H. S. Wilf. *generatingfunctionology*. Academic Press, 1st edition, 1990.

[23] D. Zeilberger. Kathy O'Hara's constructive proof of the unimodality of the gaussian polynomials. *Amer. Math. Monthly*, 96:590–602, 1990.