# GENERALIZATION OF A STATISTIC ON LINEAR DOMINO ARRANGEMENTS

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ABSTRACT. In this paper, we generalize an earlier statistic on square-and-domino tilings by considering only those squares covering a multiple of $k$, where $k$ is a fixed positive integer. We consider the distribution of this statistic jointly with the one that records the number of dominos in a tiling. We derive both finite and infinite sum expressions for the corresponding joint distribution polynomials, the first of which reduces when $k = 1$ to a prior result. The cases $q = 0$ and $q = -1$ are noted for general $k$. Finally, the case $k = 2$ is considered specifically, where further results may be given, including a combinatorial proof when $q = -1$.

## 1. INTRODUCTION

Let $F_n$ be the Fibonacci number defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. See, for example, sequence A000045 in [12]. Let $G_n = G_n(t)$ be the Fibonacci polynomial defined by $G_n = G_{n-1} + tG_{n-2}$ if $n \geq 2$, with $G_0 = 0$ and $G_1 = 1$; note that $G_n(1) = F_n$ for all $n$. See, for example, [10]. Finally, the $q$-binomial coefficient $\binom{x}{k}_q$ is defined by

$$\binom{x}{k}_q = \begin{cases} \prod_{i=1}^{k} \frac{1-q^{x-i+1}}{1-q^i}, & \text{if } k \geq 0; \\ 0, & \text{if } k < 0. \end{cases}$$

Polynomial generalizations of $F_n$ have arisen in connection with statistics on binary words [3], Morse code sequences [4], lattice paths [5], and linear domino arrangements [10, 11]. Let us recall now a statistic related to domino arrangements. If $n \geq 1$, then let $\mathscr{F}_n$ denote the set of coverings of the numbers $1, 2, \ldots, n$, arranged in a row by indistinguishable dominos and indistinguishable squares, where pieces do not overlap, a domino is a rectangular piece covering two numbers, and a square is a piece covering a single number. The members of $\mathscr{F}_n$ are also called (linear) *tilings* or *domino arrangements*. (If $n = 0$, then $\mathscr{F}_0$ consists of the empty tiling having length zero.)

Note that members of $\mathscr{F}_n$ correspond uniquely to words in the alphabet $\{d, s\}$ comprising $i$ $d$'s and $n-2i$ $s$'s for some $i$, $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. In what follows, we will frequently identify tilings $c$ by such words $c_1c_2\cdots$. For example, if $n = 4$, then $\mathscr{F}_4 = \{dd, dss, sds, ssd, ssss\}$. Note that $|\mathscr{F}_n| = F_{n+1}$ for all $n$. Given $\pi \in \mathscr{F}_n$, let $\rho(\pi)$ denote the sum of the numbers covered by squares in $\pi$. For example, if $n = 15$ and $\pi = sds^2d^2sd^2s \in \mathscr{F}_{15}$ (see Figure 1 below), then $\rho(\pi) = 1+4+5+10+15 = 35$.

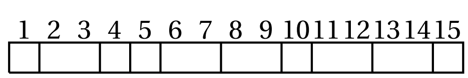


FIGURE 1. The tiling $\pi = sds^2d^2sd^2s \in \mathscr{F}_{15}$ has $\rho(\pi) = 35$.

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*Date*: 2013-06-14.

2000 *Mathematics Subject Classification.* 11B39, 05A15.

*Key words and phrases.* tilings, Fibonacci numbers, polynomial generalization.

The statistic $\rho$ was introduced in [11], where its distribution was studied on $r$-mino arrangements. Let $\nu(\pi)$ denote the number of dominos in the tiling $\pi$. Then the joint distribution for the $\rho$ and $\nu$ statistics on $\mathscr{F}_n$ is given by

$$\sum_{\pi\in\mathscr{F}_n} q^{\rho(\pi)} t^{\nu(\pi)} = \sum_{j=0}^{n} q^{\binom{n-2j+1}{2}} \binom{n-j}{j}_{q^2} t^j, \qquad n \geq 0, \tag{1}$$

where $q$ and $t$ are indeterminates. Equation (1) is the $r = 2$ case (corresponding to square-and-domino tilings) of [11, Theorem 2.1], which is a result on more general $r$-mino arrangements. Here, we will provide a different generalization of (1). Note that (1) reduces to the well-known formula $F_{n+1} = \sum_{j=0}^{n} \binom{n-j}{j}$ when $q = t = 1$.

Recently, generalizations of the Fibonacci sequence have been studied which specify the recurrence for each value of the index mod $k$, where $k$ is a fixed positive integer. For example, the recurrence

$$Q_m = a_j Q_{m-1} + b_j Q_{m-2}, \qquad m \equiv j \pmod{k}, \tag{2}$$

with $Q_0 = 0$ and $Q_1 = 1$, was considered in [8], where a Binet-like formula is derived. See also [6] for the case when $b_j = 1$ for all $j$ and [13] for the case $k = 2$. These generalizations so far have been studied primarily from an algebraic standpoint such as through the use of generating functions [6] or orthogonal polynomials [8]. In [7], a special case of (2) and a closely related sequence are studied from a more combinatorial viewpoint in terms of statistics on linear tilings and new generalizations of $F_n$ are obtained which extend prior ones.

In this paper, we continue this study by considering a generalization of the $\rho$ statistic defined above, where one looks only at squares that cover multiples of $k$. More precisely, let $\rho_k$ record the sum divided by $k$ of all the multiples of $k$ which are covered by squares within a member of $\mathscr{F}_n$. Note that $\rho_k$ reduces to $\rho$ when $k = 1$.

In the next section, we obtain an explicit formula for all $k$ (see Theorem 2.2 below) for the joint distribution

$$a_n^{(k)}(q,t) := \sum_{\pi\in\mathscr{F}_n} q^{\rho_k(\pi)} t^{\nu(\pi)}.$$

This yields an infinite family of $q$-generalizations for the numbers $G_n(t)$ defined above, and setting $q = 1$ yields seemingly new expressions for $G_n(t)$. When $k = 1$ in our formula, we obtain the explicit expression (1) above, but with a different proof than that given in [11]. We also note some special cases of $q$ and provide an infinite expansion for $a_n^{(k)}(q,t)$ (see Theorem 2.7 below). In the third section, we consider specifically the case $k = 2$, where further combinatorial results may be given. In particular, we provide a combinatorial proof explaining the values of $a_n^{(2)}(-1,1)$ as well as an explicit expression for the sum of the $\rho_2$ values taken over all the members of $\mathscr{F}_n$. Note that $\rho_2$ records half the sum of the even numbers covered by squares within a tiling.

## 2. General formulas

Suppose $k$ is a fixed positive integer. Given $\pi \in \mathscr{F}_n$, let $\nu(\pi)$ denote the number of dominos of $\pi$ and let $\rho_k(\pi)$ denote the sum divided by $k$ of all the multiples of $k$ covered by squares of $\pi$. For example, if $\pi = s^2d^3sdsdsd^2s^2d^2 \in \mathscr{F}_{25}$ (see Figure 2 below), then $\nu(\pi) = 9$ and

$$\rho_3(\pi) = \frac{9+12+15+21}{3} = 19.$$

If $q$ and $t$ are indeterminates, then define the distribution polynomial $a_n^{(k)}(q,t)$ by

$$a_n^{(k)}(q,t) := \sum_{\pi\in\mathscr{F}_n} q^{\rho_k(\pi)} t^{\nu(\pi)}, \qquad n \geq 1,$$

with $a_n^{(0)}(q,t) := 1$. For example, if $n = 6$ and $k = 3$, then

$$a_6^{(3)}(q,t) = 2t^2 + t^3 + q(1+t)(t + 2qt + q^2 + q^2 t).$$

Note that $a_n^{(k)}(1,t) = G_{n+1}$ for all $k$ and $n$.

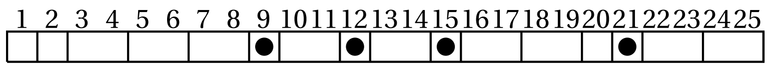


FIGURE 2. The tiling $\pi = s^2d^3sdsdsd^2s^2d^2 \in \mathscr{F}_{25}$ has $\rho_3(\pi) = 19$.

In what follows, we will often suppress arguments and write $a_n$ for $a_n^{(k)}(q,t)$. Considering whether the last piece within a member of $\mathscr{F}_n$ is a square or a domino yields the recurrence

$$a_n = q^{\frac{n}{k}} a_{n-1} + t a_{n-2}, \qquad n \geq 2, \tag{3}$$

if $n$ is divisible by $k$, and the recurrence

$$a_n = a_{n-1} + t a_{n-2}, \qquad n \geq 2, \tag{4}$$

if $n$ is not, with the initial conditions $a_0 = 1$ and

$$a_1 = \begin{cases} q, & \text{if } k = 1; \\ 1, & \text{if } k > 1. \end{cases}$$

To solve recurrences (3) and (4), we first ascertain an explicit formula for the generating function of the numbers $a_n$.

**Theorem 2.1.** *We have*

(5)

$$\sum_{n\geq 0} a_n x^n = \left(\sum_{r=0}^{k-1} x^r G_{r+1} - t x^k \sum_{r=0}^{k-1} (-tx)^r G_{k-1-r}\right) \sum_{j\geq 0} \frac{G_k^j q^{\binom{j+1}{2}} x^{jk}}{\prod_{i=0}^{j}(1 - 2tq^i x^k G_{k-1} + (-t)^k q^{2i} x^{2k})}.$$

*Proof.* It is more convenient to first consider the generating function for the numbers $a'_n := a_{n-1}^{(k)}(q,t)$. Then the sequence $a'_n$ has initial values $a'_0 = 0$ and $a'_1 = 1$ and satisfies the recurrences

$$a'_{mk+r} = a'_{mk+r-1} + t a'_{mk+r-2}, \qquad 2 \leq r \leq k \qquad \text{and} \qquad m \geq 0, \tag{6}$$

with

$$a'_{mk+1} = q^m a'_{mk} + t a'_{mk-1}, \qquad m \geq 1. \tag{7}$$

Let

$$a_r(x) = \sum_{m\geq 0} a'_{mk+r} x^m,$$

where $r \in [k]$. Then multiplying the recurrences (6) and (7) by $x^m$, and summing the first over $m \geq 0$ and the second over $m \geq 1$, gives

$$\begin{aligned} a_r(x) &= a_{r-1}(x) + t a_{r-2}(x), \qquad 3 \leq r \leq k, \\ a_2(x) &= a_1(x) + t x a_k(x), \\ a_1(x) &= 1 + q x a_k(qx) + t x a_{k-1}(x). \end{aligned}$$

By induction on $r$, we obtain

$$a_r(x) = G_{r-1}a_2(x) + tG_{r-2}a_1(x), \qquad 2 \le r \le k.$$

Therefore,

$$a_r(x) = G_{r-1}(a_1(x) + txa_k(x)) + tG_{r-2}a_1(x),$$

which implies

$$a_r(x) = G_r a_1(x) + txG_{r-1}a_k(x), \qquad 2 \le r \le k. \tag{8}$$

Taking $r = k$ in (8) gives

$$a_1(x) = \frac{1 - txG_{k-1}}{G_k}a_k(x).$$

By induction on $r$, we obtain

$$a_r(x) = \frac{G_r + (-t)^r xG_{k-r}}{G_k}a_k(x), \qquad 1 \le r \le k.$$

Since $a_1(x) = 1 + qxa_k(qx) + txa_{k-1}(x)$, the last relation may be rewritten as

$$a_k(x) = \frac{G_k}{1 - 2txG_{k-1} + (-t)^k x^2} + \frac{qxG_k}{1 - 2txG_{k-1} + (-t)^k x^2}a_k(qx). \tag{9}$$

Iterating (9) yields

$$a_k(x) = \sum_{j\ge0} \frac{G_k^{j+1} q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j}(1 - 2tq^i xG_{k-1} + (-t)^k q^{2i} x^2)}.$$

Thus, we have

$$a_r(x) = (G_r + (-t)^r xG_{k-r}) \sum_{j\ge0} \frac{G_k^{j} q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j}(1 - 2tq^i xG_{k-1} + (-t)^k q^{2i} x^2)}, \qquad 1 \le r \le k,$$

which implies

$$\begin{aligned}\sum_{n\ge0} a'_n x^n &= \sum_{r=1}^{k}\sum_{m\ge0} a'_{mk+r}x^{mk+r} = \sum_{r=1}^{k} x^r a_r(x^k)\\ &= \left(\sum_{r=1}^{k} x^r G_r + x^k \sum_{r=1}^{k}(-tx)^r G_{k-r}\right)\sum_{j\ge0}\frac{G_k^j q^{\binom{j+1}{2}} x^{jk}}{\prod_{i=0}^{j}(1 - 2tq^i x^k G_{k-1} + (-t)^k q^{2i} x^{2k})}.\end{aligned}$$

The result now follows since

$$\sum_{n\ge0} a_n x^n = \sum_{n\ge0} a'_{n+1}x^n = \frac{1}{x}\sum_{n\ge0} a'_n x^n.$$

□

We now derive an explicit formula for the polynomials $a_n^{(k)}(q,t)$.

**Theorem 2.2.** *If $n = km + r$, where $m \ge 0$ and $0 \le r \le k-1$, then*

$$a_n = G_{r+1}S(m) + (-t)^{r+1}G_{k-1-r}S(m-1), \tag{10}$$

*where*

$$S(m) = \sum_{j=0}^{m}(-1)^{kj} G_k^j q^{\binom{j+1}{2}} t^{m-(k-1)j} \sum_{a=j}^{m} d_+^a d_-^{m+j-a} \binom{a}{j}_q \binom{m+j-a}{j}_q, \qquad m \ge 0,$$

*with $S(-1) = 0$ and*

$$d_\pm = G_{k-1} \pm \sqrt{G_k G_{k-2}}.$$

*Proof.* Note first that

$$d_{\pm} = G_{k-1} \pm \sqrt{G_{k-1}^2 - (-t)^{k-2}},$$

by the identity $G_m^2 - G_{m+1}G_{m-1} = (-t)^{m-1}$, which can be shown by induction (see, e.g., [2, Identity 8] for the $t = 1$ case). Then

$$1 - 2tq^i xG_{k-1} + (-t)^k q^{2i} x^2 = (1 - d_+ tq^i x)(1 - d_- tq^i x).$$

Let $n = mk + r$, where $m \geq 0$ and $0 \leq r \leq k-1$. By Theorem 2.1, we have

$$a_n = G_{r+1}[x^m](a(x)) + (-t)^{r+1} G_{k-1-r}[x^{m-1}](a(x)),$$

where

$$a(x) = \sum_{j\geq 0} \frac{G_k^j q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j}(1 - 2tq^i xG_{k-1} + (-t)^k q^{2i} x^2)}.$$

Using the expansion [1]

$$\frac{y^j}{\prod_{i=0}^{j}(1-q^i y)} = \sum_{a\geq j} \binom{a}{j}_q y^a$$

and the fact $d_+ d_- = (-t)^{k-2}$, we have

$$\begin{aligned}
[x^m](a(x)) &= \sum_{j\geq 0} [x^m] \left( \frac{G_k^j q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j}(1 - d_+ tq^i x)(1 - d_- tq^i x)} \right) \\
&= \sum_{j\geq 0} \frac{G_k^j q^{\binom{j+1}{2}}}{d_+^j d_-^j t^{2j}} [x^{m+j}] \left( \frac{(d_+ tx)^j}{\prod_{i=0}^{j}(1 - d_+ tq^i x)} \cdot \frac{(d_- tx)^j}{\prod_{i=0}^{j}(1 - d_- tq^i x)} \right) \\
&= \sum_{j=0}^{m} \frac{G_k^j q^{\binom{j+1}{2}}}{(-1)^{kj} t^{kj}} \sum_{a=j}^{m} \binom{a}{j}_q (d_+ t)^a \cdot \binom{m+j-a}{j}_q (d_- t)^{m+j-a} \\
&= \sum_{j=0}^{m} (-1)^{kj} G_k^j q^{\binom{j+1}{2}} t^{m-(k-1)j} \sum_{a=j}^{m} d_+^a d_-^{m+j-a} \binom{a}{j}_q \binom{m+j-a}{j}_q,
\end{aligned}$$

which completes the proof. □

Letting $k = 1$ in Theorem 2.2 gives the following expression for $a_n^{(1)}(q,t)$.

**Corollary 2.3.** *If* $n \geq 0$*, then*

$$a_n^{(1)}(q,t) = \sum_{j=0}^{n} q^{\binom{n-2j+1}{2}} \binom{n-j}{j}_{q^2} t^j. \tag{11}$$

*Proof.* When $k=1$, we have $d_\pm = \pm\frac{1}{\sqrt{t}}$ since $G_0=0$ and $G_{-1}=\frac{1}{t}$. Taking $k=1$ in (10) then gives

$$\begin{aligned} a_n^{(1)}(q,t) = S(n) &= \sum_{j=0}^{n}(-1)^j q^{\binom{j+1}{2}} t^n \sum_{a=j}^{n}\left(\frac{1}{\sqrt{t}}\right)^a \binom{a}{j}_q \cdot (-1)^{n+j-a}\left(\frac{1}{\sqrt{t}}\right)^{n+j-a}\binom{n+j-a}{j}_q \\ &= \sum_{j=0}^{n} q^{\binom{j+1}{2}} t^{\frac{n-j}{2}} \sum_{a=j}^{n}(-1)^{n-a}\binom{a}{j}_q\binom{n+j-a}{j}_q \\ &= \sum_{j=0}^{n} q^{\binom{n-j+1}{2}} t^{\frac{j}{2}} \sum_{a=n-j}^{n}(-1)^{n-a}\binom{a}{n-j}_q\binom{2n-j-a}{n-j}_q \\ &= \sum_{j=0}^{n}(-1)^j q^{\binom{n-j+1}{2}} t^{\frac{j}{2}} \sum_{a=0}^{j}(-1)^a\binom{a+n-j}{n-j}_q\binom{n-a}{n-j}_q \\ &= \sum_{\substack{j=0\\ j\text{ even}}}^{n}(-1)^j q^{\binom{n-j+1}{2}}\binom{n-j/2}{n-j}_{q^2} t^{\frac{j}{2}} = \sum_{j=0}^{n} q^{\binom{n-2j+1}{2}}\binom{n-j}{j}_{q^2} t^j, \end{aligned}$$

where we have used the identity

$$\sum_{a=0}^{n-m}(-1)^a\binom{a+m}{m}_q\binom{n-a}{m}_q = \begin{cases}\binom{\frac{n+m}{2}}{m}_{q^2}, & \text{if } n\equiv m \pmod 2;\\ 0, & \text{otherwise,}\end{cases} \qquad (0\le m\le n). \tag{12}$$

Note that (12) may be obtained by writing

$$\begin{aligned}\sum_{a\ge0}(-1)^a\binom{a+m}{m}_q x^a\cdot\sum_{a\ge0}\binom{a}{m}_q x^a &= \frac{1}{\prod_{i=0}^m(1+q^ix)}\cdot\frac{x^m}{\prod_{i=0}^m(1-q^ix)} = \frac{x^m}{\prod_{i=0}^m(1-q^{2i}x^2)} \\ &= \sum_{a\ge0}\binom{a+m}{m}_{q^2}x^{2a+m},\end{aligned}$$

and extracting the coefficient of $x^n$ from both sides. □

*Remark:* Formula (11) corresponds to the $r=2$ case of [11, Theorem 2.1], which is a result on more general $r$-mino arrangements where no restriction is placed on the positions of $r$-minos or squares. The proof there was combinatorial, though it does not seem that it can be extended to prove Theorem 2.2 above.

Taking $q=1$ and $r=k-1$ in (10), and noting $a_n^{(k)}(1,t)=G_{n+1}$, yields the following identity.

**Corollary 2.4.** *If $m \geq 0$ and $k \geq 1$, then*

$$G_{(m+1)k} = G_k \sum_{j=0}^{k} (-1)^{kj} G_k^j t^{m-(k-1)j} \sum_{a=j}^{m} d_+^a d_-^{m+j-a} \binom{a}{j} \binom{m+j-a}{j}, \tag{13}$$

*where $d_\pm = G_{k-1} \pm \sqrt{G_k G_{k-2}}$.*

We have the following explicit formula for the number of members of $\mathscr{F}_n$ (weighted according to the value of $v$) in which no square covers a multiple of $k$.

**Corollary 2.5.** *If $n = km + r$, where $m \geq 0$ and $0 \leq r \leq k-1$, then*

$$a_n^{(k)}(0,t) = t^m G_{r+1} T(m) + (-1)^{r+1} t^{m+r} G_{k-1-r} T(m-1), \tag{14}$$

*where*

$$T(m) = \sum_{i=0}^{m} \binom{m+1}{2i+1} G_{k-1}^{m-2i} (G_k G_{k-2})^i.$$

*Proof.* Setting $q = 0$ in (10) implies

$$a_{mk+r}^{(k)}(0,t) = t^m G_{r+1} \sum_{a=0}^{m} d_+^a d_-^{m-a} + (-1)^{r+1} t^{m+r} G_{k-1-r} \sum_{a=0}^{m-1} d_+^a d_-^{m-1-a},$$

with

$$\sum_{a=0}^{m} d_+^a d_-^{m-a} = \frac{d_+^{m+1} - d_-^{m+1}}{d_+ - d_-} = \frac{1}{2\sqrt{G_k G_{k-2}}} \sum_{i=0}^{m} 2 \binom{m+1}{2i+1} G_{k-1}^{m-2i} (\sqrt{G_k G_{k-2}})^{2i+1}.$$

□

For example, when $k = 1$ in (14), we see that $a_n^{(1)}(0,t)$ equals $t^{\frac{n}{2}}$ for $n$ even and zero for $n$ odd. Taking $k = 2$ in (14) gives $a_{2m}^{(2)}(0,t) = t^m$ and $a_{2m+1}^{(2)}(0,t) = (m+1)t^m$ for $m \geq 0$. These formulas are readily seen directly.

We next consider the case $q = -1$. Recall that for any generating function in $q$, the evaluation at $q = -1$ gives the difference in cardinalities between those members of a structure having an even value for the statistic counted by $q$ with those having an odd value. Letting $q = -1$ and $t = 1$ in (5) gives the following formulas, where $f_i := \sum_{n \geq 0} a_n^{(i)}(-1,1) x^n$:

$$\begin{aligned}
f_1 &= \frac{(1 - x - x^3 - x^4)(1 - x^6)}{1 - x^{12}}, \\
f_2 &= \frac{(1 + x + x^3 + x^4 + 2x^5 - x^6 + x^7 + x^9 - x^{10})(1 - x^{12})}{1 - x^{24}}, \\
f_3 &= \frac{1 + x + 2x^2 - x^3 + x^4}{1 - x^6}, \\
f_4 &= \frac{(1 + x + 2x^2 + 3x^3 - 2x^4 + x^5 - x^6)(1 + x^4 + x^8)}{1 - 5x^8 + x^{16}}.
\end{aligned}$$

The first three generating functions show that the sequences $a_n^{(i)}(-1,1)$, $i = 1,2,3$, are periodic with periods 12, 24, and 6, respectively. The sequences $a_n^{(1)}(-1,1)$ and $a_n^{(2)}(-1,1)$ are seen to satisfy the stronger conditions $p_{n+6} = -p_n$ and $p_{n+12} = -p_n$ for all $n \geq 0$. From the appearance of the generating function $f_4$, it seems that the sequence $a_n^{(4)}(-1,1)$ would not be periodic, which is indeed the case. It turns out that there are no other values of $k$ for which the sequence $a_n^{(k)}(-1,1)$ is periodic.

**Proposition 2.6.** *The sequence $a_n^{(k)}(-1,1)$ is never periodic (or eventually periodic) when $k \geq 4$.*

*Proof.* Substituting $q=-1$ and $t=1$ into the infinite part of (5) gives

$$\begin{aligned}
\sum_{j\geq 0} \frac{F_k^j(-1)^{\binom{j+1}{2}}x^{jk}}{\prod_{i=0}^{j}(1-2F_{k-1}(-1)^i x^k+(-1)^k x^{2k})}
&= \sum_{m\geq 0} \frac{F_k^{2m}(-1)^m x^{2mk}}{(1-2F_{k-1}x^k+(-1)^k x^{2k})((1+(-1)^k x^{2k})^2-4F_{k-1}^2 x^{2k})^m}\\
&+ \sum_{m\geq 0} \frac{F_k^{2m+1}(-1)^{m+1}x^{(2m+1)k}}{((1+(-1)^k x^{2k})^2-4F_{k-1}^2 x^{2k})^{m+1}}\\
&= \frac{1+F_{k-3}x^k+(-1)^k x^{2k}}{1+(F_k^2-4F_{k-1}^2+2(-1)^k)x^{2k}+x^{4k}},
\end{aligned}$$

and thus

$$\sum_{n\geq 0} a_n^{(k)}(-1,1)x^n = \frac{(\sum_{r=0}^{k-1}F_{r+1}x^r - x^k\sum_{r=0}^{k-1}(-1)^r F_{k-1-r}x^r)(1+F_{k-3}x^k+(-1)^k x^{2k})}{1+(F_k^2-4F_{k-1}^2+2(-1)^k)x^{2k}+x^{4k}}.$$

Let $a(x)$ and $b(x)$ denote the numerator and the denominator in the (unsimplified) expression above for $\sum_{n\geq 0} a_n^{(k)}(-1,1)x^n$. Suppose now that

$$\frac{a(x)}{b(x)} = c(x) + \frac{d(x)}{1-x^\ell},$$

where $\ell$ is a positive integer, $c(x)$ is any polynomial (possibly zero), and $d(x)$ is of the form $d(x) = x^{m+1}e(x)$, with $m$ denoting the degree of $c(x)$ (we take $m$ to be $-1$ if $c(x)$ is the zero polynomial) and $e(x)$ being a polynomial of degree at most $\ell-1$. Then $(1-x^\ell)(a(x)-b(x)c(x)) = b(x)d(x)$ implies that the equation $b(x)=0$ must have at least one root of unity among its roots since $e(x) = \frac{d(x)}{x^{m+1}}$ is of degree at most $\ell-1$, with $e(x)$ not identically zero. Then the equation $b(u)=0$, where $u = x^{\frac{1}{2k}}$, must also have at least one root of unity among its roots, since $r$ a root of unity implies $r^{2k}$ is as well.

The equation $b(u)=0$ is given by

$$1+(F_k^2-4F_{k-1}^2+2(-1)^k)u+u^2=0. \tag{15}$$

If $k\geq 4$, then

$$F_k^2-4F_{k-1}^2+2(-1)^k \leq -5,$$

since

$$F_k^2-4F_{k-1}^2 = (F_k-2F_{k-1})(F_k+2F_{k-1}) = -F_{k-3}(F_k+2F_{k-1}) \leq -7.$$

Note that an equation of the form

$$1-au+u^2=0, \qquad a\geq 5,$$

has (real) roots $\frac{a}{2}\pm\frac{\sqrt{a^2-4}}{2}$. So the only possible roots of unity that are also roots to such an equation are $\pm 1$. However, the equations $\frac{a}{2}+\frac{\sqrt{a^2-4}}{2}=\pm 1$ and $\frac{a}{2}-\frac{\sqrt{a^2-4}}{2}=\pm 1$ have solutions $a=\pm 2$ in each case, but $a\geq 5$. Thus no roots of unity satisfy equation (15) when $k\geq 4$, which implies the result. □

*Remark:* When $k = 1,2,3$, the equation (15) is satisfied by roots of unity and it works out that the sequences $a_n^{(k)}(-1,1)$ are periodic in these cases.

Let $(x:q)_s = \prod_{i=0}^{s-1}(1-q^i x)$. We conclude this section with the following infinite expansion for the numbers $a_n^{(k)}(q,t)$ for all $k \geq 1$.

**Theorem 2.7.** *If $n = km + r$, where $m \geq 1$ and $0 \leq r \leq k-1$, then*

$$a_n = t^m G_{r+1} \sum_{s\geq 0} q^{sm}\left(d_+^m b_s + d_-^m c_s\right) + (-1)^{r+1} t^{m+r} G_{k-1-r} \sum_{s\geq 0} q^{s(m-1)}\left(d_+^{m-1} b_s + d_-^{m-1} c_s\right), \tag{16}$$

*where*

$$b_s = \sum_{j\geq s} \frac{(-1)^s G_k^j q^{\binom{j+1}{2}+\binom{s+1}{2}+s} d_+}{t^j (q:q)_s (q:q)_{j-s} \prod_{i=0}^{j}(q^s d_+ - q^i d_-)},$$

$$c_s = \sum_{j\geq s} \frac{(-1)^s G_k^j q^{\binom{j+1}{2}+\binom{s+1}{2}+s} d_-}{t^j (q:q)_s (q:q)_{j-s} \prod_{i=0}^{j}(q^s d_- - q^i d_+)},$$

*and*

$$d_\pm = G_{k-1} \pm \sqrt{G_k G_{k-2}}.$$

*Proof.* Note first that $d_\pm = G_{k-1} \pm \sqrt{G_{k-1}^2 - (-t)^{k-2}}$, as in the proof of Theorem 2.2, and thus

$$1 - 2tq^s x G_{k-1} + (-t)^k q^{2s} x^2 = (1-\rho_s x)(1-\theta_s x),$$

where $\rho_s = d_+ t q^s$ and $\theta_s = d_- t q^s$.

Let $n = mk + r$, where $m \geq 1$ and $0 \leq r \leq k-1$. By partial fractions, let us write

$$\sum_{j\geq 0} \frac{G_k^j q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j}(1 - 2tq^i x G_{k-1} + (-t)^k q^{2i} x^2)} = \sum_{s\geq 0} \frac{b_s}{1-\rho_s x} + \sum_{s\geq 0} \frac{c_s}{1-\theta_s x},$$

where $b_s$ and $c_s$ are constants to be determined. By Theorem 2.1,

$$\begin{aligned}
a_n &= G_{r+1}[x^m]\left(\sum_{j\geq 0} \frac{G_k^j q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j}(1 - 2tq^i x G_{k-1} + (-t)^k q^{2i} x^2)}\right) \\
&\quad + (-t)^{r+1} G_{k-1-r}[x^{m-1}]\left(\sum_{j\geq 0} \frac{G_k^j q^{\binom{j+1}{2}} x^j}{\prod_{i=0}^{j}(1 - 2tq^i x G_{k-1} + (-t)^k q^{2i} x^2)}\right) \\
&= G_{r+1}[x^m]\left(\sum_{s\geq 0} \frac{b_s}{1-\rho_s x} + \sum_{s\geq 0} \frac{c_s}{1-\theta_s x}\right) \\
&\quad + (-t)^{r+1} G_{k-1-r}[x^{m-1}]\left(\sum_{s\geq 0} \frac{b_s}{1-\rho_s x} + \sum_{s\geq 0} \frac{c_s}{1-\theta_s x}\right) \\
&= G_{r+1} \sum_{s\geq 0} (b_s \rho_s^m + c_s \theta_s^m) + (-t)^{r+1} G_{k-1-r} \sum_{s\geq 0} (b_s \rho_s^{m-1} + c_s \theta_s^{m-1}).
\end{aligned}$$

We also have

$$\begin{aligned} b_s &= \sum_{j\geq s} \frac{G_k^j q^{\binom{j+1}{2}}}{\rho_s^j \prod_{i=0}^{s-1}(1-\rho_i/\rho_s)\prod_{i=s+1}^{j}(1-\rho_i/\rho_s)\prod_{i=0}^{j}(1-\theta_i/\rho_s)} \\ &= \sum_{j\geq s} \frac{(-1)^s G_k^j q^{\binom{j+1}{2}}\rho_s^{j+1}}{d_+^j t^j \prod_{i=0}^{s-1}(q^i-q^s)\prod_{i=s+1}^{j}(q^s-q^i)\prod_{i=0}^{j}(d_+tq^s-d_-tq^i)} \\ &= \sum_{j\geq s} \frac{(-1)^s G_k^j q^{\binom{j+1}{2}+\binom{s+1}{2}}\rho_s}{t^{j+1}(q:q)_s(q:q)_{j-s}\prod_{i=0}^{j}(d_+q^s-d_-q^i)} \\ &= \sum_{j\geq s} \frac{(-1)^s G_k^j q^{\binom{j+1}{2}+\binom{s+1}{2}+s}d_+}{t^{j}(q:q)_s(q:q)_{j-s}\prod_{i=0}^{j}(q^s d_+-q^i d_-)} \end{aligned}$$

and, similarly,

$$c_s = \sum_{j\geq 0} \frac{(-1)^s G_k^j q^{\binom{j+1}{2}+\binom{s+1}{2}+s}d_-}{t^{j}(q:q)_s(q:q)_{j-s}\prod_{i=0}^{j}(q^s d_--q^i d_+)},$$

which gives (16). □

## 3. The case $k=2$

In this section, we consider further results concerning the polynomial sequence $a_n^{(2)} = a_n^{(2)}(q,t)$. Taking $k=2$ in (10), and noting $d_+ = d_- = 1$ in this case, gives the explicit formulas

$$\begin{aligned} a_{2m}^{(2)} &= \sum_{j=0}^{m} q^{\binom{j+1}{2}} t^{m-j} \sum_{a=j}^{m} \binom{a}{j}_q \binom{m+j-a}{j}_q \\ &\quad - t\sum_{j=0}^{m-1} q^{\binom{j+1}{2}} t^{m-1-j} \sum_{a=j}^{m-1} \binom{a}{j}_q \binom{m+j-1-a}{j}_q, \qquad m\geq 0, \end{aligned} \tag{17}$$

and

$$a_{2m+1}^{(2)} = \sum_{j=0}^{m} q^{\binom{j+1}{2}} t^{m-j} \sum_{a=j}^{m} \binom{a}{j}_q \binom{m+j-a}{j}_q, \qquad m\geq 0. \tag{18}$$

Though we are unable to give simpler expressions for the polynomials (17) and (18), they are seen to be solutions to the following relatively simple recurrences.

**Proposition 3.1.** *If $m\geq 2$, then*

$$a_{2m}^{(2)} = (q^m+qt+t)a_{2m-2}^{(2)} - qt^2 a_{2m-4}^{(2)}, \tag{19}$$

*with $a_0^{(2)}=1$ and $a_2^{(2)} = q+t$, and*

$$a_{2m+1}^{(2)} = (q^m+2t)a_{2m-1}^{(2)} - t^2 a_{2m-3}^{(2)}, \tag{20}$$

*with $a_1^{(2)}=1$ and $a_3^{(2)} = q+2t$.*

*Proof.* We provide a combinatorial argument, the initial values being clear. To show (19), first note that if $m\geq 2$, then the total weight of all the members of $\mathscr{F}_{2m}$ ending in $ss$ is $q^m a_{2m-2}^{(2)}$, while the weight of those ending in $d$ is $ta_{2m-2}^{(2)}$. To determine the weight of the members of $\mathscr{F}_{2m}$ ending in $ds$, first insert a domino before the final square within any member of $\mathscr{F}_{2m-2}$ ending in $s$. By subtraction, the total weight of all the members of

$\mathscr{F}_{2m-2}$ ending in $s$ is $a_{2m-2}^{(2)}-ta_{2m-4}^{(2)}$, and the inserted domino increases both the $v$ and $\rho_2$ values by 1 (note that the final square moves from position $2m-2$ to $2m$). Thus, the total weight of all members of $\mathscr{F}_{2m}$ ending in $ds$ is $qt(a_{2m-2}^{(2)}-ta_{2m-4}^{(2)})$, which gives (19). By similar reasoning, the total weight of all members of $\mathscr{F}_{2m+1}$ ending in $ss$, $d$ and $ds$ is $q^m a_{2m-1}^{(2)}$, $ta_{2m-1}^{(2)}$ and $t(a_{2m-1}^{(2)}-ta_{2m-3}^{(2)})$, respectively, which gives (20). □

We were unable to find, in general, two-term recurrences comparable to (19) and (20) for the sequences $a_{mk+r}^{(k)}(q,t)$, where $k$ and $r$ are fixed and $m\ge 0$. Let

$$f(x;q,t)=\sum_{n\ge 0}a_n^{(2)}(q,t)x^n,$$

which we'll also denote by $f(x)$.

**Proposition 3.2.** *We have*

$$f(x;q,t)=(1+x-tx^2)\sum_{j\ge 0}\frac{q^{\binom{j+1}{2}}x^{2j}}{\prod_{i=0}^{j}(1-tq^ix^2)^2}. \tag{21}$$

*Proof.* This follows from setting $k=2$ in (5) above, but we give an alternative derivation using Proposition 3.1 as follows. Let $b(x)=\sum_{m\ge 0}a_{2m}^{(2)}x^m$. Multiplying (19) by $x^m$, and summing over $m\ge 2$, implies

$$b(x)-1-(t+q)x=qx(b(qx)-1)+tx(1+q)(b(x)-1)-qt^2x^2b(x),$$

or

$$b(x)=\frac{1}{1-tx}+\frac{qx}{(1-tx)(1-qtx)}b(qx).$$

Iterating the last equation gives

$$\begin{aligned}b(x)&=\sum_{j\ge 0}\frac{q^{\binom{j+1}{2}}x^j}{(1-tx)\prod_{i=1}^{j}(1-tq^ix)^2}\\&=(1-tx)\sum_{j\ge 0}\frac{q^{\binom{j+1}{2}}x^j}{\prod_{i=0}^{j}(1-tq^ix)^2}.\end{aligned}$$

Similarly, if $c(x)=\sum_{m\ge 0}a_{2m+1}^{(2)}x^m$, then we have

$$c(x)=\sum_{j\ge 0}\frac{q^{\binom{j+1}{2}}x^j}{\prod_{i=0}^{j}(1-tq^ix)^2},$$

and thus

$$\begin{aligned}f(x)&=b(x^2)+xc(x^2)\\&=(1-tx^2)\sum_{j\ge 0}\frac{q^{\binom{j+1}{2}}x^{2j}}{\prod_{i=0}^{j}(1-tq^ix^2)^2}+x\sum_{j\ge 0}\frac{q^{\binom{j+1}{2}}x^{2j}}{\prod_{i=0}^{j}(1-tq^ix^2)^2},\end{aligned}$$

as desired. □

Substituting $q=-1$ in (21) yields the following result.

**Corollary 3.3.** *We have*

$$\sum_{n\ge 0}a_n^{(2)}(-1,t)x^n=\frac{(1+x+tx^2)(1+x-tx^2)(1-x+tx^2)}{1-(2t^2-1)x^4+t^4x^8}. \tag{22}$$

**Corollary 3.4.** *The sequence $a_n^{(2)}(-1,1)$ is determined by the condition*

$$f(n+12) = -f(n), \qquad n \geq 0,$$

*with the values of $a_n^{(2)}(-1,1)$ for $0 \leq n \leq 11$ given by* $1,1,0,1,1,2,-1,1,0,1,-1,0$.

*Proof.* Letting $t=1$ in (22), we have

$$\begin{aligned}\sum_{n\geq 0} a_n^{(2)}(-1,1)x^n &= \frac{(1+x+x^2)(1+x-x^2)(1-x+x^2)}{1-x^4+x^8}\\ &= \frac{(1+x+x^3+x^5-x^6)(1+x^4)(1-x^{12})}{(1-x^4+x^8)(1+x^4)(1-x^{12})}\\ &= \frac{(1+x+x^3+x^4+2x^5-x^6+x^7+x^9-x^{10})(1-x^{12})}{1-x^{24}},\end{aligned}$$

which implies the result. □

**Combinatorial proof of Corollary 3.4.**

Let $\mathscr{F}_n^e$ and $\mathscr{F}_n^o$ denote the subsets of $\mathscr{F}_n$ having even and odd $\rho_2$ values, respectively. We first define an involution of $\mathscr{F}_n$ off of a set $\mathscr{F}_n'$ which pairs members of $\mathscr{F}_n^e$ and $\mathscr{F}_n^o$. Let $\mathscr{F}_n' \subseteq \mathscr{F}_n$ consist of those tilings of the form

$$\pi = d^i(sd^{2i_1}s)(sd^{2i_2}s)\cdots(sd^{2i_\ell}s), \tag{23}$$

if $n$ is even, and of the form

$$\pi = d^i(sd^{2i_1}s)(sd^{2i_2}s)\cdots(sd^{2i_\ell}s)sd^j, \tag{24}$$

if $n$ is odd, for some $\ell$ where $i,j,i_1,i_2,\ldots,i_\ell \geq 0$. We define an involution of $\mathscr{F}_n - \mathscr{F}_n'$ as follows. Given $\lambda \in \mathscr{F}_n - \mathscr{F}_n'$, let $j_o$ denote the smallest index $j \geq 1$ such that either

(i) an odd number of dominos occurs between the $(2j-1)$-st and $(2j)$-th squares, or
(ii) an even number of dominos occurs between the $(2j-1)$-st and $(2j)$-th squares with at least one domino between the $(2j)$-th and $(2j+1)$-st squares (or between the $(2j)$-th square and the end of the tiling, if the $(2j)$-th square is right-most).

Now exchange positions of the $(2j_o)$-th square and the domino that precedes it if (i) occurs, or exchange the positions of the $(2j_o)$-th square and the domino that directly follows it if (ii) occurs. Let $\lambda'$ denote the resulting member of $\mathscr{F}_n'$. Then $\lambda$ and $\lambda'$ have opposite $\rho_2$-parity (since their $\rho_2$ values differ by one), and the mapping $\lambda \mapsto \lambda'$ is an involution of $\mathscr{F}_n - \mathscr{F}_n'$. For example, if $n=28$ and $\lambda = d^2sd^2s^4d^3sdsd^2s \in \mathscr{F}_{28}$, then $j_o = 3$ and $\lambda' = d^2sd^2s^4d^2sd^2sd^2s$. See Figure 3 below, where the $(2j_o-1)$-st and $(2j_o)$-th squares are shaded in each tiling.

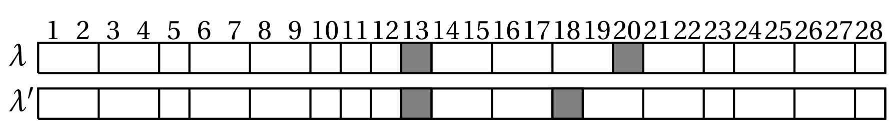


FIGURE 3. The tiling $\lambda$ has $\rho_2(\lambda)=35$, while $\rho_2(\lambda')=34$.

We now consider the signed sum of members of $\mathscr{F}_n'$, i.e., $\sum_{\pi\in F_n'}(-1)^{\rho_2(\pi)}$. First observe that if $i$ is even in (23) and (24) above, then one may verify that

$$\rho_2(\pi) \equiv \binom{\ell+1}{2} \pmod 2,$$

whereas if $i$ is odd, then

$$\rho_2(\pi) \equiv \binom{\ell}{2} \pmod 2.$$

For the remainder of the proof, we will assume that $n$ is even, the proof in the odd case being similar. Assume further that $n = 2m$, where $m$ is odd, as the argument for the case of even $m$ is basically the same.

First suppose that $\pi \in \mathscr{F}'_n$ is of the form in (23) above, with $i$ even. Note that $m$ odd implies $\ell$ is odd. Let $\bar{\pi}$ be the tiling of length $m$ given by

$$\bar{\pi} = d^{\frac{i}{2}} s d^{i_1} s d^{i_2} \cdots s d^{i_\ell};$$

note that all members of $\mathscr{F}_m$ arise uniquely as $\pi$ ranges over all members of $\mathscr{F}'_n$ for which $i$ is even. Let $s(\sigma)$ denote the number of squares in a tiling $\sigma$. Then we have

$$\rho_2(\pi) \equiv \binom{\ell+1}{2} \equiv \frac{\ell+1}{2} = \frac{s(\bar{\pi})+1}{2} \pmod 2.$$

If $\pi \in \mathscr{F}'_n$ is of the form in (23) with $i$ odd, then $m$ odd implies $\ell$ is even. Let $\pi^*$ be the tiling of length $m-1$ given by

$$\pi^* = d^{\frac{i-1}{2}} s d^{i_1} s d^{i_2} \cdots s d^{i_\ell};$$

note that all members of $\mathscr{F}_{m-1}$ arise uniquely in this manner. Observe that in this case

$$\rho_2(\pi) \equiv \binom{\ell}{2} \equiv \frac{\ell}{2} = \frac{s(\pi^*)}{2} \pmod 2.$$

Therefore, we have

$$\begin{aligned} \sum_{\pi \in \mathscr{F}'_n} (-1)^{\rho_2(\pi)} &= \sum_{\substack{\pi \in \mathscr{F}'_n \\ i \text{ even}}} (-1)^{\rho_2(\pi)} + \sum_{\substack{\pi \in \mathscr{F}'_n \\ i \text{ odd}}} (-1)^{\rho_2(\pi)} \\ &= \sum_{\sigma \in \mathscr{F}_m} (-1)^{(s(\sigma)+1)/2} + \sum_{\sigma \in \mathscr{F}_{m-1}} (-1)^{s(\sigma)/2}. \end{aligned} \tag{25}$$

To evaluate the last two sums, we consider the statistic $\lceil s(\sigma)/2 \rceil$ on $\mathscr{F}_r$ where $r \geq 1$ and pair members of $\mathscr{F}_r$ of opposite parity with respect to this statistic. Given $\sigma = \sigma_1\sigma_2 \cdots \in \mathscr{F}_r$, let $a_o$ denote the smallest index $a \geq 1$ such that either

(i) $\sigma_{2a-1} = d$, or

(ii) $\sigma_{2a-1}\sigma_{2a} = ss$.

Define an involution of $\mathscr{F}_r$ by replacing $\sigma_{2a_o-1} = d$ with $ss$ if (i) occurs or by replacing $\sigma_{2a_o-1}\sigma_{2a_o} = ss$ with $d$ if (ii) occurs. Note that this mapping changes the value of $\lceil s(\sigma)/2 \rceil$ by one, whence it changes its parity. If $r \equiv 0 \pmod 3$, then there is a single unpaired tiling in $\mathscr{F}_r$, namely, $(sd)^{r/3}$, which has sign $(-1)^{\lceil r/6 \rceil}$. If $r \equiv 1 \pmod 3$, then the single unpaired tiling $(sd)^{(r-1)/3}s$ has sign $(-1)^{\lceil (r+2)/6 \rceil}$. If $r \equiv 2 \pmod 3$, then each member of $\mathscr{F}_r$ is paired with another of opposite parity, whence the resulting sum is zero.

Applying the preceding to (25) shows that if $m \equiv 0 \pmod 3$, i.e., if $m = 6p+3$ for some $p$ (since $m$ was assumed odd) and $n = 12p+6$, then

$$\begin{aligned} a_n^{(2)}(-1,1) &= \sum_{\pi\in\mathscr{F}'_n} (-1)^{\rho_2(\pi)} \\ &= \sum_{\sigma\in\mathscr{F}_{6p+3}} (-1)^{\lceil s(\sigma)/2\rceil} + \sum_{\sigma\in\mathscr{F}_{6p+2}} (-1)^{\lceil s(\sigma)/2\rceil} \\ &= (-1)^{\lceil (6p+3)/6\rceil} + 0 = (-1)^{p+1}. \end{aligned}$$

Similarly, if $n = 12p+2$, then $a_n^{(2)}(-1,1) = (-1)^{p+1} + (-1)^p = 0$, and if $n = 12p+10$, then $a_n^{(2)}(-1,1) = 0 + (-1)^{p+1} = (-1)^{p+1}$. This yields the values of $a_n^{(2)}(-1,1)$ given in Corollary 3.4 above in the case when $n = 2m$, where $m$ is odd. The other cases are obtained similarly. □

*Remark:* Comparable proofs may be given to explain the periodic nature of the $a_n^{(1)}(-1,1)$ and $a_n^{(3)}(-1,1)$ values witnessed above.

Let $U_n(t)$ denote the $n$-th Chebyshev polynomial of the second kind defined by $U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t)$, with $U_0(t) = 1$ and $U_1(t) = 2t$ (see, e.g., [9]).

**Theorem 3.5.** *The coefficient of $x^n$ for $n \ge 0$ in $\frac{d}{dq} f(x;q,t)\mid_{q=1}$ is given by*

$$\frac{(i\sqrt{t})^{n+1}}{8(4t+1)}\Big(\frac{(2n+1)(4t+1)(-1)^n + 2n(n+1) - 4t - 1}{2i\sqrt{t}}U_n(y) \\ + ((4t+1)(-1)^n + 4t - 1 - 2n(n+2))U_{n-1}(y)\Big),$$

*where $y = \frac{1}{2i\sqrt{t}}$ and $i = \sqrt{-1}$.*

*Proof.* Differentiating the generating function $f(x;q,t)$ in (21) with respect to $q$, and substituting $q = 1$, yields

$$g(x;t) := \frac{d}{dq} f(x,q)\mid_{q=1} = \frac{x^2(1-tx^2)(1+tx^2)}{(1-x-tx^2)^3(1+x-tx^2)^2}.$$

By partial fractions, we may rewrite this as

$$\begin{aligned} g(x;t) = &-\frac{3-2tx}{16(1+x-tx^2)} + \frac{2+x}{8(1+x-tx^2)^2} - \frac{1+2tx}{16(1-x-tx^2)} \\ &+ \frac{1-tx}{4t(1-x-tx^2)^2} - \frac{1-2tx-x}{4t(1-x-tx^2)^3}. \end{aligned}$$

By the fact that $\sum_{n\ge0} U_n(t)x^n = \frac{1}{1-2tx+x^2}$, we obtain

$$\sum_{n\ge1} nU_n(t)x^{n-1} = \frac{2t-2x}{(1-2tx+x^2)^2}$$

and

$$\sum_{n\ge2} n(n-1)U_n(t)x^{n-2} = \frac{8t^2-2-12tx+6x^2}{(1-2tx+x^2)^3}.$$

Let $y = \frac{1}{2i\sqrt{t}}$, where $i = \sqrt{-1}$. Extracting the coefficient of $x^n$ from each summand then gives

$$[x^n]\left(-\frac{3-2tx}{16(1+x-tx^2)}\right) = -\frac{(-i\sqrt{t})^n}{16}\left(3U_n(y) - 2i\sqrt{t}U_{n-1}(y)\right),$$

$$[x^n]\left(\frac{2+x}{8(1+x-tx^2)^2}\right) = \frac{(2+n)(-i\sqrt{t})^n}{8}U_n(y),$$

$$[x^n]\left(-\frac{1+2tx}{16(1-x-tx^2)}\right) = -\frac{(i\sqrt{t})^n}{16}\left(U_n(y) - 2i\sqrt{t}U_{n-1}(y)\right),$$

$$[x^n]\left(\frac{1-tx}{4t(1-x-tx^2)^2}\right) = \frac{(1+4t+(t+1)n)(i\sqrt{t})^n}{4t(1+4t)}U_n(y) - \frac{(1+n)(2t-1)(i\sqrt{t})^{n-1}}{4(1+4t)}U_{n-1}(y),$$

$$[x^n]\left(-\frac{1-2tx-x}{4t(1-x-tx^2)^3}\right) = \frac{(tn^2-(t+2)n-2(1+4t))(i\sqrt{t})^n}{8t(1+4t)}U_n(y) + \frac{(tn^2+(4t-1)n-1+3t)(i\sqrt{t})^n}{4(1+4t)}U_{n-1}(y).$$

Adding all of these expressions yields the desired result. □

Let $t_n(\rho_2)$ denote the sum of the $\rho_2$ values of all the members of $\mathscr{F}_n$. Letting $t = 1$ in the prior theorem, and noting $i^n U_n(-i/2) = F_{n+1}$, gives the following expression for $t_n(\rho_2)$.

**Corollary 3.6.** *If* $n \geq 0$, *then*

$$t_n(\rho_2) = (-1)^n \frac{(2n+1)F_{n+1} - 2F_n}{16} + \frac{(2n^2+2n-5)F_{n+1} + (4n^2+8n-6)F_n}{80}. \tag{26}$$

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