ENUMERATING GRAPH DEPLETIONS

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ABSTRACT. Let G be a simple, connected graph with finite vertex set V and edge set E. A *depletion* of G is a permutation $v_1 v_2 v_3 \ldots v_n$ of the elements of V with the property that v_i is adjacent to some member of $\{v_1, v_2, \cdots, v_{i-1}\}$ for each $i \geq 2$. Depletions model the spread of a rumor or a disease through a population and are related to heaps. In this paper we develop techniques for enumerating the depletions of a graph.

1. INTRODUCTION

Let G be a simple, connected graph with finite vertex set V and edge set E. A *depletion* of G is a permutation $v_1 v_2 v_3 \ldots v_n$ of the elements of V with the property that v_i is adjacent to some member of $\{v_1, v_2, \ldots, v_{i-1}\}$ for each $i \geq 2$. In other words, each vertex in the list, with the exception of the first, must be adjacent to some vertex to its left in the list. If the nodes of a graph represent a population and its edges model contact between the nodes, then a depletion represents the order in which a rumor or a disease could spread through this population. While there is, quite naturally, a wealth of research pertaining to the rate with which a rumor or disease could diffuse through a system of nodes (see, for example, [9], [11], [10], [4], [3]) , the motivation for this research is to enumerate the ways that this rumor or disease could spread, that is, to count the number of depletions.

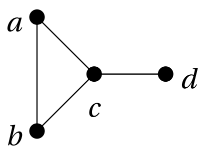

FIGURE 1. Starting from vertex a, there are three ways to deplete this graph: $abcd$, $acbd$, and $acdb$.

Given $a \in V$, we will denote by $\text{dep}(G, a)$ the set of all depletions of G starting at vertex a and we will denote by $\text{dep}(G)$ the set of depletions of G without regard to the starting vertex. To fix this idea, consider the graph pictured in Figure 1 and observe that

$$\text{dep}(G, a) = \{abcd, acbd, acdb\} \quad \text{and} \quad \text{dep}(G, d) = \{dcab, dcba\}.$$

It is easy to see that $|\text{dep}(G, b)| = 3$ and $|\text{dep}(G, c)| = 6$; hence, $|\text{dep}(G)| = 14$. Thus, in this notation, the problem under consideration in this paper is this: given a graph G and a vertex a, what are $|\text{dep}(G, a)|$ and $|\text{dep}(G)|$? Since $|\text{dep}(G)|$ can be found by summing

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$|\operatorname{dep}(G,a)|$ over all vertices a of G, we will concentrate most of our attention on methods for finding $|\operatorname{dep}(G,a)|$.

The inspiration for this study came from a paper of Golomb [6] that described an enumeration problem for a billiards game. In this game there are 15 balls, numbered 1 through 15. At first any of the balls may be pocketed, but thereafter only a ball bearing a number consecutive with a previously pocketed ball may be pocketed. For example, if we suppose that the 4 ball is pocketed first, then either the 3 or the 5 ball may be pocketed next; and, if the 5 ball is pocketed next, then either the 3 ball or the 6 ball may be pocketed next. This process continues until all of the balls have been pocketed. Golomb's objective was to demonstrate a variety of approaches to answer the question: in how many ways can this game be played? Of course the 15 billiard ball problem can be recast into a problem of counting the depletions of a graph: plays of the billiard ball game correspond to depletions of the path given in Figure 2. Golomb shows there are $\binom{14}{i-1}$ ways to complete this game if ball i is pocketed first.

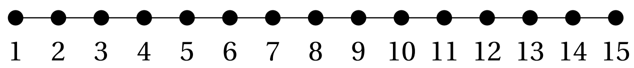

FIGURE 2. The 15 billiard balls represented as a graph.

Graph depletions are related to heaps. A sequence of distinct numbers $\{a_n : 1 \le n \le N\}$ forms a (binary) *heap* if $a_i < a_{2i}$ for $1 \le i \le N/2$ and $a_i < a_{2i+1}$ for $1 \le i \le (N-1)/2$. For example, the list $\ell = \{1,7,3,13,8,5,9,16,20,14\}$ is a heap. Heaps are employed in Heapsort, a rapid algorithm for sorting a list of numbers. Some of the pioneering work in this area was carried out by Williams [13] and Floyd [5]. Recently Bernstein [2] has used heaps to efficiently search for integer solutions of certain polynomial equations. Heaps are also an efficient way to implement a priority queue, a data structure used to identify and extract the element of highest priority from a queue. The recent paper of Navarro and Paredes [8], for example, employs heaps to solve the incremental sorting problem, a problem in priority queueing.

There is a natural correspondence between heaps and labeled binary trees and therefore between heaps and depletions. Consider a binary tree opening downward from the root. Attach labels to the binary tree by starting from the root and labeling its vertices with the elements of the heap by moving through the tree from top to bottom and from left to right. This corresponds to placing the labels on the tree according to a level-order transversal; see Figure 3. In this setting, the heap property assumes the following simple form: a labeled binary tree corresponds to a heap if each parent bears a smaller label than those of its children. This labeling gives us a simple recipe for depleting the tree from its root: select the vertices in the order of their labels from least to greatest. Thus the number of heaps of a set of distinct numbers corresponds to the number of depletions from the root of the corresponding binary tree. Skiena [12, p. 36] gives the number of complete heaps of k levels (that is $2^k - 1$ vertices) S_k by the recursive formula

$$S_k = \binom{2^k - 2}{2^{k-1} - 1} S_{k-1}^2, \quad S_1 = 1. \tag{1}$$

While the concept of a heap is naturally associated with binary trees, it can be extended to general rooted trees. Knuth gives a simple formula for the number of heaps on a rooted

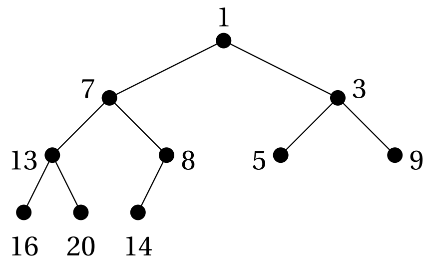

FIGURE 3. A binary tree labeled by the heap $\ell = \{1, 7, 3, 13, 8, 5, 9, 16, 20, 14\}$. The heap ℓ corresponds to a depletion of the tree.

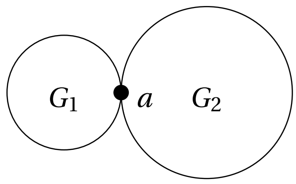

FIGURE 4. A graph G rendered as two subgraphs G_1 and G_2 with cut-point a.

tree. Let a tree T have vertex set $\{1, 2, \ldots, n\}$ with root 1. For each i, let s_i denote the size of the sub-tree rooted at i. Given a list ℓ of n distinct numbers, the number of ways to build a heap on T from ℓ is

$$n!/\prod_{i=1}^{n} s_i. \tag{2}$$

See exercise 20 of [7, p. 70] and equation (16) of [7, p. 154]. For example, the number of ways to build a heap from the list ℓ on the tree pictured in Figure 3 is $10!/(10 \cdot 6 \cdot 3 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1) = 3360$.

Throughout this paper we will observe the following conventions. P_n will denote the path with vertex set $\{0, 1, 2, \cdots, n-1\}$ and edge set $\{(i, i+1) : 0 \le i \le n-2\}$. For completeness, P_1 will denote the trivial graph with vertex set $\{0\}$ and no edges. C_n will denote the cycle graph with vertex set $\{0, 1, \cdots, n-1\}$ and edge set $\{(n-1, 0)\} \cup \{(i, i+1) : 0 \le i \le n-2\}$. K_n will denote a complete graph on the vertex set $\{0, 1, \ldots, n-1\}$ and $K_{m,n} = (V_1 + V_2, E)$ will denote a complete bipartite graph with disjoint parts $V_1 = \{v_0, 1, \ldots, v_{m-1}\}$ and $V_2 = \{w_0, w_1, \ldots, w_{n-1}\}$. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we will denote the Cartesian graph product of G_1 and G_2 by $G_1 \Box G_2$. The vertex set of $G_1 \Box G_2$ is $V_1 \times V_2$, the ordinary Cartesian product of the vertex sets of G_1 and G_2. There is an edge between (x_1, y_1) and (x_2, y_2) in $G_1 \times G_2$ provided that either $x_1 = x_2$ and $(y_1, y_2) \in E_2$ or $y_1 = y_2$ and $(x_1, x_2) \in E_1$.

Ad hoc procedures can be employed with some success in counting depletions. We will treat a few simple examples here and give two additional examples in §2. In forming a depletion of a complete graph, any vertex of the graph can be listed in any order; thus,

$$|\operatorname{dep}(K_n, i)| = (n-1)!, \quad 0 \le i \le n-1.$$

In forming a depletion of a cycle C_n, once the initial vertex has been selected, there will be two choices for each of the next $n-2$ vertices in the depletion list; thus,

$$|\operatorname{dep}(C_n, i)| = 2^{n-2}, \quad 0 \le i \le n-1. \tag{3}$$

In section §5 we will introduce graph reductions, a more elaborate method for counting depletions. At present, however, let us describe a simple version of this method. Suppose that a graph G can be rendered as two subgraphs G_1 and G_2 that share only a common vertex a. We suppose, in addition, that any path from a vertex of G_1 to a vertex of G_2 must contain a. This situation is depicted symbolically in Figure 4. The vertex a is customarily called a *cut-point* of G, a vertex whose removal separates the graph into more than 1 component. Now the simple idea is this: depletions of G starting from a are obtained by blending depletions of G_1 from a and depletions of G_2 from a. If G_1 has m vertices and G_2 has n vertices, then there will be $(n+m-2)!/((n-1)!(m-1)!)$ ways to blend a depletion of G_1 from a and a depletion of G_2 from a. This leads to the formula:

$$|\operatorname{dep}(G,a)| = \frac{(n+m-2)!}{(m-1)!(n-1)!}|\operatorname{dep}(G_1,a)||\operatorname{dep}(G_2,a)|.$$

Here is another simple but useful observation. Suppose that a is a leaf of the graph $G=(V,E)$ with $(a,b)\in E$. Let G' denote the subgraph of G induced by the vertex set $V\setminus\{a\}$. Then, since a is adjacent only to b,

$$|\operatorname{dep}(G,a)| = |\operatorname{dep}(G',b)|.$$

Armed with these two observations, we can produce the formula

$$\operatorname{dep}(P_{n+1},i) = \binom{n}{i}, \quad 0\le i\le n,$$

for counting the depletions of a path from a given vertex. These same techniques are sufficient to reproduce formulas (1) and (2), the results of Skiena and Knuth for trees.

Here is a brief overview of the content of this paper. In §2 we present two additional examples using ad hoc techniques. In §3 we introduce a polynomial function for the set of depletions of a graph relative to a path and the linear operator $\mathscr{D}$, which acts as an inverse to these depletion polynomials. In §4 we present some properties of $\mathscr{D}$. We are greatly aided at this point in our analysis by Gauss's hypergeometric function and some of its related identities. In §5 we introduce graph reductions and blends. This technique allows us to analyze the depletions of a graph through blending depletions of subgraphs. The process of blending the depletions of subgraphs is made algebraic through the depletion polynomials. In §6 we use the method of reductions and blends to count the depletions of two different families of graphs. First is the family of what we call *butterfly graphs*. A butterfly graph is essentially a sewing together of two cycles along an edge. Second is the family of Cartesian graph products of paths and complete graphs. An example from each of these families is given in Figure 5. Finally, in §7, we present a general method for counting the depletions of any unicyclic graph. Essentially we show that for any unicyclic graph G there exists a set of trees whose depletions partition the set of depletions of G. Since the depletions of trees can be calculated through Knuth's formula (2), the depletions of any unicyclic graph can be enumerated as well.

We will observe the following conventions with regard to notation for the falling and rising factorials. Given $x\in\mathbb{R}$ and an integer $n\ge 0$, we will write

$$(x)_n = (x)(x-1)(\cdots)(x-n+1)$$

for the *falling factorial* and

$$(x)^{(n)} = (x)(x+1)(\cdots)(x+n-1)$$

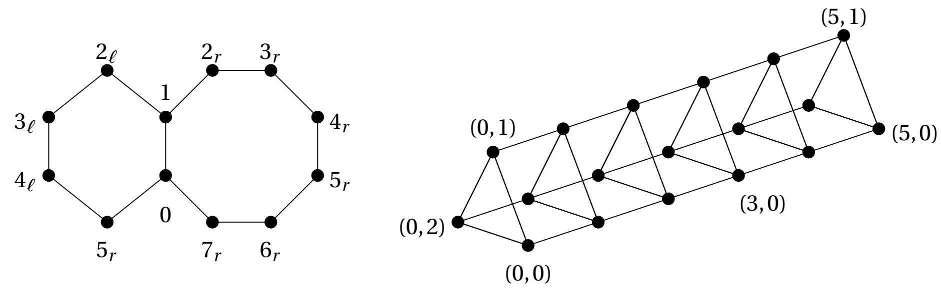

FIGURE 5. A $B_{6,8}$ butterfly graph and a $P_6 \square K_3$ graph.

for the *rising factorial.*

We end this section with some open problems:

(1) Enumerate the depletions (from an arbitrary vertex) of $P_m \square P_n$, a grid graph. The special case $m = 2$ is covered by our Theorem 6.2.
(2) Enumerate the depletions (from an arbitrary vertex) of the n-cube. The special cases $n = 2$ and 3 are covered by equation (3) and our Theorem 2.2.

2. SOME AD HOC TECHNIQUES

While we know of no general technique for enumerating the depletions of a given graph from a given vertex, in some cases special properties of the graph can come to our assistance.

Theorem 2.1. *For a complete bipartite graph,*

$$|\operatorname{dep}(K_{m,n}, i)| = \begin{cases} n(m+n-2)! & \text{if } i \in V_1 \\ m(m+n-2)! & \text{if } i \in V_2. \end{cases}$$

Proof. Given $i \in V_1$, there are n choices for the next vertex in the depletion sequence; namely, any element of V_2. Thereafter any of the remaining $m+n-2$ positions in the depletion sequence can be filled by any permutation of the remaining $m+n-2$ vertices, which gives the proposed formula. The case $i \in V_2$ can be treated similarly. □

Our next example treats a modification of a complete bipartite graph. Let $K'_{n,n}$ be the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

Theorem 2.2. *For any vertex i,*

$$|\operatorname{dep}(K'_{m+1,m+1}, i)| = (2m)!\frac{m(m-1)}{m+1}.$$

For example, the 3-cube is isomorphic to a $K'_{4,4}$; thus, starting from any vertex, there are $6! \cdot 3 \cdot 2/4 = 1080$ depletions of the 3-cube. The problem of enumerating the depletions of the n-cube is a challenging problem and appears to be beyond the techniques of this present work.

Proof. Color the two parts, V_1 and V_2, blue and red and let us assume that we begin our depletions from a red vertex. A depletion can thus be rendered as a sequence of red and blue vertices. We can partition the depletions according to the value of the index $k \geq 4$ at which for the first time in the depletion sequence both colors have appeared twice.

First, let us suppose that the kth element of our depletion sequence is a red vertex.

$$\underbrace{R\,\overbrace{B\cdots B}^{k-2}\,R}_{k}\cdots.$$

In this case, slots 2 through $k-1$ must be occupied by blue vertices, and these blue vertices must necessarily be neighbors of the leading red vertex. The red vertex in the kth position can be any of the remaining m red vertices. Thereafter, any permutation of the remaining $2m+2-k$ vertices can fill out the depletion sequence. Thus the count for such an array is

$$(m)_{k-2}\cdot(m)\cdot(2m+2-k)!.$$

Next, let us suppose that the kth element of our depletion sequence is a blue vertex.

$$\underbrace{R\,B\,\overbrace{R\cdots R}^{k-3}\,B}_{k}\cdots.$$

In this case slots 3 through $k-1$ must be occupied by red vertices, and these red vertices must be neighbors of the blue vertex in position 2. The blue vertex in position k can be any of the remaining m blue vertices. Thereafter any permutation of the remaining $2m+2-k$ vertices can fill out the depletion sequence. Thus the count for such an array is

$$(m)_{k-2}\cdot(m)\cdot(2m+2-k)!.$$

Summing over the range of k yields,

$$\begin{aligned}|\operatorname{dep}(K'_{m+1,m+1},i)| &= 2m\sum_{k=4}^{m+2}(m)_{k-2}(2m+2-k)!\\ &= 2m\sum_{j=2}^{m}(m)_j(2m-j)!\\ &= 2m(m!)^2\sum_{i=0}^{m-2}\binom{m+i}{m}\\ &= 2m(m!)^2\binom{2m-1}{m+1}\\ &= \frac{(2m)!m(m-1)}{m+1},\end{aligned}$$

as was to be shown. □

3. Polynomials associated with paths

In this section we develop a natural polynomial associated with the depletions of a graph relative to a path. In §5 we show how to exploit these polynomials to calculate the number of depletions of the graph through the techniques of graph reduction and blending.

Let $G=(V,E)$ be a connected, simple graph. Given a depletion and $x,y\in V$, we will write $x<y$ (relative to the depletion) if x appears before y in the depletion sequence. Let $\sigma=s_1s_2\cdots s_k$ be a path in G. A *σ-depletion* of G is a depletion of G that begins with s_1 and preserves the ordering of the vertices of σ; thus, $s_{i-1}<s_i$ for $2\le i\le k$. We will

write $\operatorname{dep}(G,\sigma)$ to denote the set of all σ-depletions of G. Notice that $\operatorname{dep}(G,\sigma)$ reduces to $\operatorname{dep}(G,a)$ if the path σ consists of the single vertex a.

Given the $\sigma : s_1 s_2 \cdots s_k$ in G and nonnegative integers $i_1, i_2, \cdots, i_k$ with $i_1 + i_2 + \cdots + i_k = |V| - k$, let $\Delta_\sigma(i_1, i_2, \ldots, i_k)$ be the set of σ-depletions of G of the form

$$s_1 \underbrace{\ldots}_{i_1} s_2 \underbrace{\ldots}_{i_2} s_3 \underbrace{\ldots}_{i_3} \ldots s_{k-1} \underbrace{\ldots}_{i_{k-1}} s_k \underbrace{\ldots}_{i_k}.$$

Thus the indices $i_1, i_2, \cdots, i_k$ measure the size of the *gaps* between successive occurrences of the elements of the path σ in the depletion. Let

$$N_\sigma(i_1, i_2, \ldots, i_k) = |\Delta_\sigma(i_1, i_2, \ldots, i_k)|$$

and define a polynomial for the σ-depletions of G as follows:

$$P_\sigma(x_1, x_2, \cdots, x_k) = \sum_{i_1+i_2+\cdots+i_k=|V|-k} \frac{N_\sigma(i_1, i_2, \ldots, i_k)}{i_1! i_2! \cdots i_k!} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}. \tag{4}$$

We will write Δ_σ^G, N_σ^G, and P_σ^G when we wish to emphasize the underlying graph.

We can recover $|\operatorname{dep}(G,\sigma)|$ from P_σ^G through differentiation. Given $m \geq 0$, we define the operator $\mathscr{D}^m$ on the space of polynomials in the independent variables $x_1, x_2, \ldots, x_k$ by

$$\mathscr{D}^m = \sum_{\substack{i_1+i_2+\cdots+i_k=m \\ i_1\geq 0, i_2\geq 0, \ldots, i_k \geq 0}} \frac{\partial^m}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_k^{i_k}}. \tag{5}$$

For example $\mathscr{D}^7 x_1^2 x_2 x_3^4 = 2!1!4!$. Our next lemma is an immediate consequence of the definition of $\mathscr{D}$ and equation (4).

Lemma 3.1. $\mathscr{D}^{|V|-k} P_\sigma^G(x_1, x_2, \ldots, x_k) = |\operatorname{dep}(G,\sigma)|$

We will close this section with two theorems concerning the σ-depletion polynomials for two family of graphs; we will investigate further properties of $\mathscr{D}$ in §4.

Theorem 3.2. *Let $\sigma = 01$ on C_n, the n-cycle on $\{0, 1, 2, \ldots, n-1\}$. Then*

$$P_\sigma(x_1, x_2) = \frac{1}{2(n-2)!}\left(x_1^{n-2} + (x_1 + 2x_2)^{n-2}\right).$$

Proof. For this graph, $i_1 + i_2 = n-2$. Given i_1 and i_2, our depletions conform to the pattern

$$0 \underbrace{\ldots}_{i_1} 1 \underbrace{\ldots}_{i_2}.$$

The i_1 vertices between 0 and 1 are fixed; there are 2 choices for each of the i_2 vertices following the 1, save for the last; thus,

$$N_\sigma(i_1, i_2) = \begin{cases} 2^{i_2-1} & \text{for } 1 \leq i_2 \leq n-2; \\ 1 & \text{for } i_2 = 0. \end{cases}$$

Thus

$$P_\sigma(x_1, x_2) = \frac{1}{(n-2)!0!} x_1^{n-2} x_2^0 + \sum_{i_2=1}^{n-2} \frac{2^{i_2-1}}{i_1! i_2!} x_1^{i_1} x_2^{i_2},$$

and, by the binomial theorem, we obtain

$$\begin{aligned} 2(n-2)!P_\sigma(x_1,x_2) &= \binom{n-2}{0} x_1^{n-2}x_2^0 + \sum_{i_2=0}^{n-2}\binom{n-2}{i_2} x_1^{i_1}(2x_2)^{i_2} \\ &= x_1^{n-2} + (x_1+2x_2)^{n-2}, \end{aligned}$$

as was to be shown. □

Hereafter, given $n, m \geq 1$, let

$$f(n,m) = |\operatorname{dep}(P_n \square K_m, (0,0))| \quad \text{and} \quad \phi(n,m) = \frac{f(n,m)}{(nm)!}. \tag{6}$$

We will present a formula for $f(n,m)$ in Theorem 4.4. At present, this notation is useful in the statement of our next theorem.

Theorem 3.3. *Let $\sigma = (0,0)\,(0,j_1)\cdots(0,j_{m-1})$ be a path on $P_{n+1}\square K_m$. Then*

$$P_\sigma^{P_{n+1}\square K_m}(x_1,x_2,\ldots,x_m) = \phi(n,m)\sum_{\ell=1}^{m}(x_\ell + x_{\ell+1}+\cdots+x_m)^{nm},$$

where $\phi(n,m)$ is given by equation (6).

Proof. Let P'_n be the subgraph of P_{n+1} induced by the vertex set $\{1,2,\ldots,n\}$. A key observation is that if we take a σ-depletion of $P_{n+1}\square K_m$ and strip out the vertices of the path σ, the result will be a depletion of $P'_n\square K_m$. The number of these depletions is determined by the number of elements of σ that sit at the head of the σ-depletion. To this end, let

$$I = \{(i_1,i_2,\ldots,i_m) : i_1+i_2+\cdots+i_m = nm; i_1,i_2,\ldots,i_m \geq 0\}$$

We will partition I according to which of the m indices is first positive. Thus, for $1 \leq \ell \leq m$, let I_ℓ be the subset of I subject to the following conditions:

$$i_1 = 0, i_2 = 0, \ldots, i_{\ell-1} = 0, i_\ell > 0, i_{\ell+1} \geq 0, \ldots, i_m \geq 0.$$

Fix $1 \leq \ell \leq m$ and consider a σ-depletion of $P_{n+1}\square K_m$ with $(i_1,i_2,\ldots,i_m) \in I_\ell$. This depletion must have the form

$$(0,0)(0,j_1)\cdots(0,j_{\ell-1})\underbrace{(1,x)\cdots}_{i_\ell}(0,j_\ell)\cdots.$$

The vertex $(1,x)$ is the first vertex of $P_{n+1}\square K_m$ that is not part of the path σ. There are two key things to note:

(1) If we strip out the vertices of σ from this depletion, then we are left with a depletion of $P'_n\square K_m$ starting from $(1,x)$.
(2) Since there are ℓ vertices of σ preceding $(1,x)$, there are ℓ choices for x.

This shows that $N_\sigma(i_1, i_2, \ldots, i_m) = \ell f(n,m)$ for $(i_1, i_2, \ldots, i_m) \in I_\ell$. With this observation in place, we may conclude that

$$\begin{aligned}
\frac{(nm)!}{f(n,m)} P_\sigma^{P_{n+1}\square K_m}(x_1, x_2, \ldots, x_m) &= \sum_{\ell=1}^{m} \sum_{(i_1,i_2,\ldots,i_m)\in I_\ell} \ell \frac{(nm)!}{i_1! i_2! \cdots i_m!} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \\
&= \sum_{\ell=1}^{m} \sum_{\substack{(i_1,i_2,\ldots,i_m)\in \\ I_\ell \cup I_{\ell+1} \cup \cdots \cup I_m}} \frac{(nm)!}{i_1! i_2! \cdots i_m!} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \\
&= \sum_{\ell=1}^{m} (x_\ell + \cdots + x_m)^{nm},
\end{aligned}$$

as was to be shown. □

4. THE OPERATOR $\mathscr{D}$

In this section we present some calculations involving the operator $\mathscr{D}$ that will be useful in the remainder of this paper.

Lemma 4.1. *For a set of independent variables $x_1, x_2, \ldots, x_k$,*

$$\mathscr{D}^\ell (x_1 + x_2 + \cdots + x_k)^\ell = \frac{(\ell + k - 1)!}{(k-1)!}.$$

for each $\ell \geq 0$.

Proof. Let $\ell \geq 0$ be given. If $i_1, i_2, \ldots, i_k$ is any set of nonnegative integers with $i_1 + i_2 + \cdots + i_k = \ell$, then

$$\frac{\partial^\ell}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_k^{i_k}} (x_1 + x_2 + \cdots + x_k)^\ell = \ell!.$$

Since there are $\binom{\ell+k-1}{k-1}$ ways to choose these indices $i_1, i_2, \cdots, i_k$ in this manner, it follows that

$$\mathscr{D}^\ell (x_1 + x_2 + \ldots + x_k)^\ell = \ell! \binom{\ell + k - 1}{k-1} = \frac{(\ell + k - 1)!}{(k-1)!},$$

as was to be shown. □

The statement of our next lemma is expressed in terms of Gauss's hypergeometric function, which is defined as

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)^{(k)} (b)^{(k)}}{(c)^{(k)} k!} z^k.$$

This is but one of a family of hypergeometric functions; see Chapters 15 and 16 of [1]. There are numerous identities related to the hypergeometric functions at specific values of z; significantly, the Chu-Vandermonde identity asserts that

$$ {}_2F_1(-n, b; c; 1) = \frac{(c-b)^{(n)}}{(c)^{(n)}}, \tag{7}$$

whenever n is a nonnegative integer; see, for example, (15.4.24) of [1].

Lemma 4.2. *Let $x_1, x_2, \dots, x_\nu, y_1, y_2, \dots, y_\mu$ be a collection of independent variables, let $X = x_1 + x_2 + \cdots + x_\nu$, and $Y = y_1 + y_2 + \cdots + y_\mu$. Let z be a parameter. Then for $\ell + r = n$,*

$$\mathscr{D}^n X^\ell (X+zY)^r = \frac{(n+\nu-1)!}{(\nu-1)!}\, {}_2F_1(-r,\mu;-n-\nu+1;z).$$

Proof. By the binomial theorem,

$$\mathscr{D}^n X^\ell (X+zY)^r = \sum_{i=0}^{r} \binom{r}{i} \mathscr{D}^n Y^i X^{n-i} z^i.$$

Since X and Y contain independent variables,

$$\mathscr{D}^n Y^i X^{n-i} = \mathscr{D}^i Y^i \mathscr{D}^{n-i} X^{n-i}.$$

Thus, by Lemma 4.1 and some algebra, we obtain

$$\begin{aligned}
\mathscr{D}^n X^\ell (X+zY)^r &= \sum_{i=0}^{r} \binom{r}{i} \mathscr{D}^i Y^i \mathscr{D}^{n-i} X^{n-i} z^i \\
&= \sum_{i=0}^{r} \binom{r}{i} \frac{(i+\mu-1)!}{(\mu-1)!} \frac{(n+\nu-1-i)!}{(\nu-1)!} z^i \\
&= \sum_{i=0}^{r} \frac{(-1)^i(-r)^{(i)}}{i!} (\mu)^{(i)} \frac{(-1)^i (n+\nu-1)!}{(\nu-1)!(-n-\nu+1)^{(i)}} z^i \\
&= \frac{(n+\nu-1)!}{(\nu-1)!} \sum_{i=0}^{r} \frac{(\mu)^{(i)}(-r)^{(i)}}{(-n-\nu+1)^{(i)} i!} z^i \\
&= \frac{(n+\nu-1)!}{(\nu-1)!}\, {}_2F_1(-r,\mu;-n-\nu+1;z),
\end{aligned}$$

as was to be shown. □

Our next result can be obtained by specializing Lemma 4.2 to $z = 1$ and applying equation (7), the Chu-Vandermonde identity.

Corollary 4.3. *In the setting of Lemma 4.2,*

$$\mathscr{D}^n X^\ell (X+Y)^r = \ell! r! \binom{\ell+\nu-1}{\ell} \binom{\ell+r+\nu+\mu-1}{r}.$$

Recall that $f(n,m)$ counts the number of depletions of $P_n \square K_m$ from the distinguished vertex $(0,0)$; see (6). We can use the methods of this section to give a simple formula for $f(n,m)$.

Theorem 4.4. *The function $f(1,m) = (m-1)!$ and, for $n \ge 1$,*

$$\frac{f(n+1,m)}{f(n,m)} = \prod_{k=2}^{m} (nm+k) = (mn+2)^{(m-1)},$$

In closed form, for $n \ge 1$,

$$f(n,m) = (m-1)! \prod_{k=1}^{n-1} (mk+2)^{(m-1)}.$$

Proof. Let S be the set of all paths in $P_{n+1}\square K_m$ of the form

$$\sigma = (0,0)\,(0,j_1)\cdots(0,j_{m-1}).$$

The collection $\{\mathrm{dep}(P_{n+1}\square K_m,\tau) : \tau \in S\}$ partitions $\mathrm{dep}(P_{n+1}\square K_m,(0,0))$. Thus, by symmetry, for any $\sigma \in S$,

$$f(n+1,m) = (m-1)!|\,\mathrm{dep}(P_{n+1}\square K_m,\sigma)|.$$

By Theorem 3.3 and Lemma 3.1, we obtain

$$\frac{f(n+1,m)}{f(n,m)} = \frac{(m-1)!}{(nm)!}\sum_{\ell=1}^{m}\mathscr{D}^{nm}(x_\ell + x_{\ell+1}+\cdots+x_m)^{nm}.$$

Finally, by Lemma 4.1 and some algebra,

$$\begin{aligned}\frac{f(n+1,m)}{f(n,m)} &= \frac{(m-1)!}{(nm)!}\sum_{\ell=1}^{m}\frac{(m-\ell+nm)!}{(m-\ell)!}\\ &= (m-1)!\sum_{\ell=1}^{m}\binom{nm+m-\ell}{m-\ell}\\ &= (m-1)!\sum_{j=0}^{m-1}\binom{nm+j}{j}\\ &= (m-1)!\binom{nm+m}{m-1}\\ &= \prod_{k=2}^{m}(nm+k),\end{aligned}$$

as was to be shown. The closed-form expression for $f(n,m)$ can be obtained from this by induction. □

5. Reductions and blends

Let σ be a path in G with vertex set S. We say that the subgraphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ form a *σ-reduction* of G provided that:

(1) $V_1 \cup V_2 = V$;
(2) $V_1 \cap V_2 = S$;
(3) G_1 and G_2 are the subgraphs in G induced by the vertex sets V_1 and V_2 respectively; and
(4) if $x_1 \in V_1$, $x_2 \in V_2$, and $(x_1,x_2) \in E$ in G, then either $x_1 \in S$ or $x_2 \in S$.

A σ-reduction of a graph G is pictured in Figure 6.

Let p_1 and p_2 be σ-depletions of the graphs G_1 and G_2 respectively. A *blend* q of p_1 and p_2 is an ordering of the vertices of G that preserves the ordering of the vertices in p_1 and p_2. Thus if $x<y$ in p_1, then $x<y$ in q; and, if $x<y$ in p_2, then $x<y$ in q. It is clear that a blend is a σ-depletion of G. Our next theorem asserts the converse.

Theorem 5.1. *Let $\{G_1,G_2\}$ be a σ-reduction of G. Every σ-depletion of G is a blend of σ-depletions from G_1 and G_2.*

For example, for the graph G and for the path $\sigma = abc$ given in Figure 6, the σ-depletion $aedbgcf$ of G is a blend of the σ-depletions $aebcf$ and $adbgc$ of G_1 and G_2 respectively.

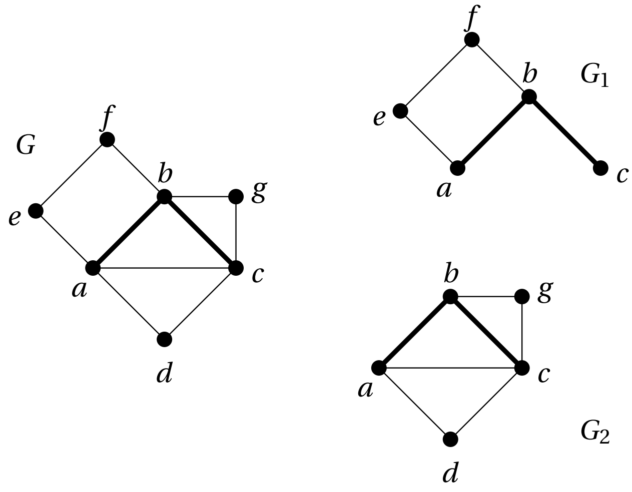

FIGURE 6. G_1 and G_2 form a σ-reduction of G for the path $\sigma = abc$.

Proof. Let p be a σ-depletion of G. Let p_1 be the list obtained by keeping in p those vertices from V_1 and let p_2 be the list obtained by keeping in p those vertices from V_2.

First we will show that p_1 is a σ-depletion of G_1. Note that p_1 contains all of the vertices of V_1 and only the vertices of V_1. Let x be member of p_1 other than its leading element. Then x is an element of p as well. But this means that there is some element y to the right of x in the listing of p with $(x, y) \in V$. Since $x \in V_1$ and since $(x, y) \in V$, property (4) of the definition of a σ-reduction shows that $y \in V_1$ as well. Since x and y are in V_1 and $(x, y) \in V$, it follows from property (3) of the definition of a σ-reduction that $(x, y) \in V_1$. Finally the elements of σ are members in order of the list p_1, since they are not removed in forming p_1. This shows that p_1 is a σ-depletion of G_1. The same proof shows that p_2 is a σ-depletion of G_2.

Finally since the orderings of the vertices in p_1 and p_2 are inherited from the ordering in p, it is clear that p is a blend of p_1 and p_2. □

Theorem 5.2. *Let $\{G_1, G_2\}$ be a σ-reduction of G. Then*

$$P_\sigma^G(x_1, x_2, \cdots, x_k) = P_\sigma^{G_1}(x_1, x_2, \cdots, x_k) P_\sigma^{G_2}(x_1, x_2, \cdots, x_k).$$

Proof. Set nonnegative integers $v_1, v_2, \cdots, v_k$ with $v_1 + v_2 + \cdots + v_k = |V| - k$ and consider the term $N_\sigma^G(v_1, v_2, \cdots, v_k)$. By Theorem 5.1, we can count the σ-depletions of G by counting the blends of the σ-depletions of G_1 and G_2. Let I denote the set of nonnegative integers $i_1, i_2, \cdots, i_k, j_1, j_2, \cdots, j_k$ satisfying

$$\begin{aligned} i_1 + i_2 + \cdots + i_k &= |V_1| - k, \\ j_1 + j_2 + \cdots + j_k &= |V_2| - k \end{aligned}$$

and

$$i_1 + j_1 = v_1, \quad i_2 + j_2 = v_2, \quad \ldots \quad i_k + j_k = v_k,$$

Given a σ-depletion of G_1 from $\Delta_\sigma^{G_1}(i_1, i_2, \ldots, i_k)$ and a σ-depletion of G_2 from $\Delta_\sigma^{G_2}(j_1, j_2, \ldots, j_k)$, we can create

$$\frac{(i_1 + j_1)!}{i_1! j_1!} \frac{(i_2 + j_2)!}{i_2! j_2!} \cdots \frac{(i_k + j_k)!}{i_k! j_k!} = \frac{v_1!}{i_1! j_1!} \frac{v_2!}{i_2! j_2!} \cdots \frac{v_k!}{i_k! j_k!}.$$

blends from these σ-depletions, each one a member of $\Delta^G_\sigma(v_1, v_2, \dots, v_k)$. This leads to

$$\begin{aligned} &N^G_\sigma(v_1, v_2, \dots, v_k) \\ &\quad = \sum_I N^{G_1}_\sigma(i_1, i_2, \dots, i_k) N^{G_2}_\sigma(j_1, j_2, \dots, j_k) \frac{v_1!}{i_1! j_1!} \frac{v_2!}{i_2! j_2!} \cdots \frac{v_k!}{i_k! j_k!} \end{aligned}$$

or, equivalently,

$$\frac{N^G_\sigma(v_1, v_2, \dots, v_k)}{v_1! v_2! \cdots v_k!} = \sum_I \frac{N^{G_1}_\sigma(i_1, i_2, \dots, i_k)}{i_1! i_2! \cdots i_k!} \frac{N^{G_2}_\sigma(j_1, j_2, \dots, j_k)}{j_1! j_2! \cdots j_k!}.$$

The result follows by comparing coefficients. □

6. Some examples using reductions and blends

In this section we apply the techniques of the previous sections to two different families of graphs. Our first family is the class $\{B_{m,n} : m, n \geq 3\}$, the so-called butterfly graphs; our second family is the class $\{P_n \square K_m : n, m \geq 1\}$. A representative from each of these families is given in Figure 5. While Theorem 4.4 already gives a formula for $|\operatorname{dep}(P_n \square K_m, (0,0))|$, it only treats depletions from the distinguished vertex $(0,0)$. In this section, we will use the technique of graph reductions to treat the general case.

We begin by defining a butterfly graph, $B_{m,n}$. For $m, n \geq 3$, let $V_\ell = \{0, 1, 2_\ell, \dots, (m-1)_\ell\}$ and $V_r = \{0, 1, 2_r, \dots, (n-1)_r\}$. Let E_ℓ and E_r be the edges of the cycle graphs, taken in their natural order, on V_ℓ and V_r respectively. Let $E = E_1 \cup E_2$ and define $B_{m,n} = (V, E)$. We should think of $B_{m,n}$ as being composed of a left cycle on m vertices and a right cycle on n vertices that share the common edge $(0,1)$. A $B_{6,8}$ butterfly graph is pictured in Figure 5.

Theorem 6.1. *Given $p, q \geq 1$ and $n = p + q$, $|\operatorname{dep}(B_{p+2,q+2}, 0)|$ is equal to*

$$\frac{1}{4} \frac{n!}{p! q!} \left(2^{n+1} + {}_2F_1(-q, 1; -n, 2) + {}_2F_1(-p, 1; -n, 2) \right).$$

Proof. Let $G = B_{p+2,q+2}$, let $\sigma = 01$, and observe that $|\operatorname{dep}(G, 0)| = |\operatorname{dep}(G, \sigma)|$. The path 01 produces a σ-reduction of G. Since each of the reduced sub-graphs is a cycle, we can readily obtain their polynomials from Theorem 3.2; thus, in accord with Theorem 5.2, $P^G_\sigma(x_1, x_2)$, the polynomial for G relative to σ, is given by their product, which assumes the form

$$\frac{1}{4 p! q!} \left(x_1^{p+q} + x_1^p (x_1 + 2x_2)^q + x_1^q (x_1 + 2x_2)^p + (x_1 + 2x_2)^{p+q} \right).$$

Recalling that $n = p + q$, we have

$$\mathscr{D}^n x_1^{p+q} = n!$$

and, by the binomial theorem,

$$\mathscr{D}^n (x_1 + 2x_2)^n = n! \sum_{i=0}^n 2^i = n!(2^{n+1} - 1).$$

Likewise, by Lemma 4.2,

$$\mathscr{D}^n x_1^p (x_1 + 2x_2)^q = n!\, {}_2F_1(-q, 1; -n, 2)$$

and

$$\mathscr{D}^n x_1^q (x_1 + 2x_2)^p = n!\, {}_2F_1(-p, 1; -n, 2).$$

Thus

$$\mathscr{D}^n P_\sigma(x_1,x_2) = \frac{1}{4}\frac{n!}{p!q!}\left(2^{n+1} + {}_2F_1(-q,1;-n,2) + {}_2F_1(-p,1;-n,2)\right).$$

as was to be shown. □

Our next theorem concerns the number of depletions of $P_{n+1}\square K_m$ from a generic vertex. To state this theorem, it will be helpful to recall that $f(n,m) = |\operatorname{dep}(P_n\square K_m,(0,0))|$ and that $\phi(n,m) = f(n,m)/(nm)!$; see display (6). A formula for $f(n,m)$ is given in Theorem 4.4. For completeness, it is convenient to attach a meaning to $\phi(0,m)$. Setting $f(0,m) = 1/m$ is consistent with Theorem 4.4 and gives $\phi(0,m) = 1/m$.

Theorem 6.2. *Let $n \ge 0$ and $m \ge 1$. For each $p_1, p_2 \ge 0$ with $p_1 + p_2 = n$,*

$$|\operatorname{dep}(P_{n+1}\square K_m,(p_1,0))| = f(n+1,m)\frac{\phi(p_1,m)\phi(p_2,m)}{\phi(n,m)}\left(1+\frac{(m-1)(nm+1)}{(p_1m+1)(p_2m+1)}\right).$$

Proof. For ease of notation, let $G = P_{n+1}\square K_m$. Since $\phi(0,m) = 1/m$, observe that the result is true if either p_1 or $p_2 = 0$. Thus we can assume hereafter that $p_1, p_2 \ge 1$. Let σ be any path in G from $(p_1,0)$ of the form $\sigma : (p_1,0)(p_1,j_1)(p_1,j_2)\cdots(p_1,j_{m-1})$. We will concentrate our efforts on calculating $|\operatorname{dep}(G,\sigma)|$; this is not in vain, since, by symmetry,

$$|\operatorname{dep}(G,(p_1,0))| = (m-1)!|\operatorname{dep}(G,\sigma)|.$$

The path σ creates a reduction, splitting G into two subgraphs that are isomorphic to $G_1 = P_{p_1+1}\square K_m$ and $G_2 = P_{p_2+1}\square K_m$ respectively. For $i \in \{1,2\}$, we can arrange it so that the isomorphism maps the path σ to $\sigma_i : (0,0)(0,j_1)(0,j_2)\cdots(0,j_{m-1})$ in G_i.

Let $X_i = x_i + x_2 + \cdots + x_m$, then, by Theorem 3.3,

$$P^{G_i}_{\sigma_i}(x_1,x_2,\cdots,x_m) = \phi(p_i,m)\sum_{j=1}^m (X_j)^{p_i m}, \quad i = 1,2.$$

Thus, by Theorem 5.2,

$$\begin{aligned}\frac{1}{\phi(p_1,m)\phi(p_2,m)}P^G_\sigma(x_1,x_2,\ldots,x_m) &= \sum_{1\le i,j\le m}(X_i)^{p_1m}(X_j)^{p_2m}\\ &= \sum_{i=1}^m (X_i)^{nm} + \sum_{1\le i<j\le m}(X_i)^{p_1m}(X_j)^{p_2m} + \sum_{1\le j<i\le m}(X_i)^{p_1m}(X_j)^{p_2m}.\end{aligned}$$

The plan of attack for the rest of the proof is clear: we will find $|\operatorname{dep}(G,\sigma)|$ by applying $\mathscr{D}^{nm}$ to each of the terms on the right. Some work is required in each case to achieve a simple form.

By Lemma 4.1 and some algebra,

$$\begin{aligned}\mathscr{D}^{nm}\sum_{i=1}^m (X_i)^{nm} = \sum_{i=1}^m \mathscr{D}^{nm}(X_i)^{nm} = \sum_{i=1}^m \frac{(nm+m-i)!}{(m-i)!} &= (nm)!\sum_{j=0}^{m-1}\binom{nm+j}{j}\\ &= (nm)!\binom{nm+m}{m-1}.\end{aligned}$$

If we let $Y_{ij} = x_i + x_{i+1} + \cdots x_{j-1}$, then we have, by Corollary 4.3,

$$\begin{aligned}
\mathscr{D}^{nm} \sum_{1\le i<j\le m} (X_i)^{p_1 m}(X_j)^{p_2 m} &= \sum_{1\le i<j\le m} \mathscr{D}^{nm}(X_j)^{p_2 m}(X_j + Y_{ij})^{p_1 m} \\
&= (p_2 m)!(p_1 m)! \sum_{1\le i<j\le m} \binom{p_2 m + m - j}{p_2 m}\binom{p_1 m + p_2 m + m - i}{p_1 m} \\
&= (p_2 m)!(p_1 m)! \sum_{0\le \ell<k\le m-1} \binom{p_2 m + \ell}{\ell}\binom{p_1 m + p_2 m + k}{p_2 m + k}.
\end{aligned}$$

We can simplify the sum as follows:

$$\begin{aligned}
\sum_{0\le \ell<k\le m-1} \binom{p_2 m + \ell}{\ell}\binom{p_1 m + p_2 m + k}{p_2 m + k} &= \sum_{k=1}^{m-1} \binom{p_1 m + p_2 m + k}{p_2 m + k} \sum_{\ell=0}^{k-1} \binom{p_2 m + \ell}{\ell} \\
&= \sum_{k=1}^{m-1} \binom{p_1 m + p_2 m + k}{p_2 m + k}\binom{p_2 m + k}{k-1}
\end{aligned}$$

However

$$\binom{p_1 m + p_2 m + k}{p_2 m + k}\binom{p_2 m + k}{k-1} = \binom{nm+1}{p_1 m}\binom{nm+1+(k-1)}{k-1}.$$

Summing this last term from $k = 1$ to $k = m-1$ yields

$$\binom{nm+1}{p_1 m}\binom{nm+m}{m-2} = \binom{nm}{p_1 m}\binom{nm+m}{m-1}\frac{nm+1}{p_2 m+1}\frac{m-1}{nm+2}.$$

Since $(p_1 m)!(p_2 m)!\binom{nm}{p_1 m} = (nm)!$, we may conclude that

$$\mathscr{D}^{nm} \sum_{1\le i<j\le m} (X_i)^{p_1 m}(X_j)^{p_2 m} = (nm)!\binom{nm+m}{m-1}\frac{nm+1}{p_2 m+1}\frac{m-1}{nm+2}.$$

Likewise

$$\mathscr{D}^{nm} \sum_{1\le j<i\le m} (X_i)^{p_1 m}(X_j)^{p_2 m} = (nm)!\binom{nm+m}{m-1}\frac{nm+1}{p_1 m+1}\frac{m-1}{nm+2}.$$

Upon combining terms and simplifying, we obtain

$$\frac{1}{\phi(p_1,m)\phi(p_2,m)}|\operatorname{dep}(G,\sigma)| = (nm)!\binom{nm+m}{m-1}\left(1+\frac{(m-1)(nm+1)}{(p_1 m+1)(p_2 m+1)}\right).$$

By symmetry $|\operatorname{dep}(G,(p_1,0))| = (m-1)!|\operatorname{dep}(G,\sigma)|$ and, by Theorem 4.4,

$$(m-1)!(nm)!\binom{nm+m}{m-1} = (nm)!\prod_{k=2}^{m}(nm+k) = \frac{f(n+1,m)}{\phi(n,m)},$$

which brings our proof to its conclusion. □

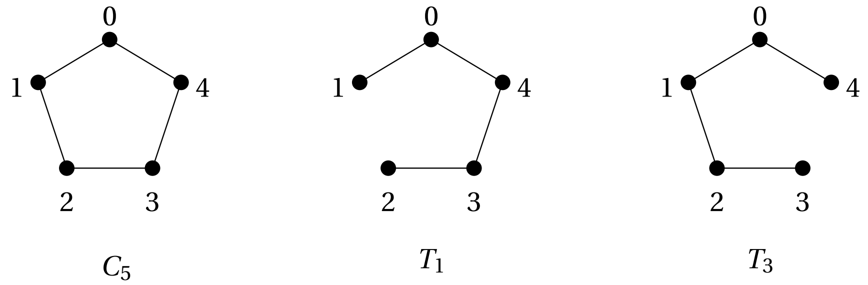

FIGURE 7. C_5 and two of its spanning trees, T_1 and T_3.

7. SPANNING TREES THAT SPAN THE SET OF DEPLETIONS

For each vertex i of the n-cycle C_n, let T_i denote the sub-tree of C_n obtained by deleting edge $(i, i+1)$, where the arithmetic is taken modulo n.

Theorem 7.1. *Let $m = n-2$ or $n-1$, whichever is odd. Then the sets*

$$\operatorname{dep}(T_1, 0), \quad \operatorname{dep}(T_3, 0), \quad \operatorname{dep}(T_5, 0), \quad \ldots \quad \operatorname{dep}(T_m, 0)$$

partition the set $\operatorname{dep}(C_n, 0)$.

C_5 and its spanning trees T_1 and T_3 are pictured in Figure 7. Since there are 4 ways to deplete each of the spanning trees, there are 8 ways to deplete the cycle.

Proof. Given a depletion $p \in \operatorname{dep}(C_n, 0)$, let v denote the last element in the list p. We note that p is a blend of the increasing sub-chain $0\,1\,2\ldots v-1\,v$ and the decreasing sub-chain $n\,n-1\,n-2\ldots v+1\,v$, where we have identified n with 0, modulo n. Thus, if v is odd, then $p \in \operatorname{dep}(T_v, 0)$, and, if $v(p)$ is even, then $p \in \operatorname{dep}(T_{v-1}, 0)$.

Let $i, j \in \{0, 1, \ldots, n-1\}$ be odd with $i < j$, and consider the corresponding trees, T_i and T_j. Any depletion of T_i starting from 0 must end with either i or $i+1$ and any depletion of T_j starting from 0 must end with either j or $j+1$. Since i and j are both odd, it follows that a depletion of C_n starting from 0 cannot be a member of both $\operatorname{dep}(T_i, 0)$ and $\operatorname{dep}(T_j, 0)$, which completes our proof. $\square$

The real force of this observation is that it gives us a strategy whereby we can enumerate the depletions of any unicyclic graph. Consider a cycle C_n for which each vertex i is the root of a tree T_i. The depletion of the resulting necklace of trees can be found by partitioning the cycle in accord with Theorem 7.1 and then enumerating the depletions of the resulting trees by means of Knuth's formula, equation (2).

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