COUNTING PAIRS OF WORDS ACCORDING TO THE NUMBER OF COMMON RISES, LEVELS, AND DESCENTS

TOUFIK MANSOUR AND MARK SHATTUCK

ABSTRACT. A *level* (L) is an occurrence of two consecutive equal entries in a word $w = w_1w_2\cdots$, while a *rise* (R) or *descent* (D) occurs when the right or left entry, respectively, is strictly larger. If $u = u_1u_2\cdots u_n$ and $v = v_1v_2\cdots v_n$ are k-ary words of the same given length and $1 \le i \le n-1$, then there is, for example, an occurrence of LR at index i if $u_i = u_{i+1}$ and $v_i < v_{i+1}$, and, likewise, for the other possibilities. Similar terminology may be used when discussing ordered d-tuples of k-ary words of length n (the set of which we'll often denote by $[k]^{nd}$).

In this paper, we consider the problem of enumerating the members of $[k]^{nd}$ according to the number of occurrences of the pattern ρ, where $d \ge 1$ and ρ is any word of length d in the alphabet $\{L,R,D\}$. In particular, we find an explicit formula for the generating function counting the members of $[k]^{nd}$ according to the number of occurrences of the patterns $\rho = L^iR^{d-i}$, $0 \le i \le d$, which, by symmetry, is seen to solve the aforementioned problem in its entirety. We also provide simple formulas for the average number of occurrences of ρ within all of the members of $[k]^{nd}$, providing both algebraic and combinatorial proofs. Finally, in the case $d = 2$, we solve the problem above where we also allow for *weak rises* (which we'll denote by R_w), i.e., indices i such that $w_i \le w_{i+1}$ in w. Enumerating the cases R_wR_w and RR_w seems to be more difficult, and to do so, we combine the kernel method with the simultaneous use of several recurrences.

1. INTRODUCTION

Let $[k] = \{1,2,\ldots,k\}$. A *k-ary word* is one all of whose letters are in $[k]$. If $w = w_1w_2\cdots w_n$ is a k-ary word and $i \in [n-1]$, then w is said to have a *level* (resp., *rise* or *descent*) at index i if $w_i = w_{i+1}$ (resp., $w_i < w_{i+1}$ and $w_i > w_{i+1}$). Throughout, we'll often represent levels, rises, and descents by L, R, and D, respectively. Concerning the pair of k-ary words $u = u_1u_2\cdots u_n$ and $v = v_1v_2\cdots v_n$, we'll say that there is an occurrence of LL (resp., LR or LD) at index i, where $i \in [n-1]$, if $u_i = u_{i+1}$ and $v_i = v_{i+1}$ (resp., $v_i < v_{i+1}$ or $v_i > v_{i+1}$), and similarly for the other six possibilities. Given a word ρ of length two in the alphabet $\{L,R,D\}$, let ν_ρ denote the statistic defined on ordered pairs of k-ary words of the same given length which counts the number of occurrences of ρ. By symmetry, and, if necessary, replacing a word $w = w_1w_2\cdots$ by its complement $w' = w'_1w'_2\cdots$, where $i' = k+1-i$ for $i \in [k]$, there are three distinct distributions for the various statistics ν_ρ, namely, those corresponding to $\rho = LL$, RR, or RL.

If $w = w_1w_2\cdots w_n$ is a word and $i \in [n-1]$, then we'll say that w has a *weak rise* (resp., *weak descent*) at index i if $w_i \le w_{i+1}$ (resp., $w_i \ge w_{i+1}$). Let us denote weak rises and weak descents by R_w and D_w, respectively. By symmetry, there are three additional distributions

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for patterns ρ when $d = 2$ if one allows for weak rises and weak descents, namely, LR_w, RR_w, and R_wR_w.

More generally, given $d \geq 1$, let $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)})$ be an ordered d-tuple of k-ary words of the same given length n (the set of which we will often denote by $[k]^{nd}$) and let $\rho = \rho_1\rho_2\cdots\rho_d$ be a word in the alphabet $\{L, R, D\}$. Then we'll say that α has an occurrence of ρ at index i if, for each $j \in [d]$, there is an occurrence of L, R, or D at index i of $\alpha^{(j)}$ as determined by letter ρ_j of ρ. In this context, one might call ρ a *pattern*. By symmetry, there are $d+1$ distinct distributions on $[k]^{nd}$ corresponding to patterns ρ, namely, those of the form L^iR^{d-i}, where $0 \leq i \leq d$.

This notation may be extended as follows. For example, if γ and δ are two words of the same length m and $i \in [n-m+1]$, then we say that the pair of words $u = u_1u_2\cdots u_n$ and $v = v_1v_2\cdots v_n$ has an occurrence of (γ, δ) at index i if the subword $u_iu_{i+1}\cdots u_{i+m-1}$ of u is order isomorphic to γ and the subword $v_iv_{i+1}\cdots v_{i+m-1}$ of v is order isomorphic to δ. For instance, let L_m denote the level of length m, which by definition corresponds to an index i for which $w_i = w_{i+1} = \cdots = w_{i+m}$ in a word w. Then there is an occurrence of $L_m^2 = (L_m, L_m)$ at index i in the word pair u and v if there is an occurrence of L_m at index i in both u and v. The notation may be easily extended when discussing the comparable problem on d-tuples of k-ary words.

The general enumeration problem concerning subword patterns dates to the 1970's when Carlitz and collaborators wrote several papers on rises, levels, and descents for various subclasses of words, including permutations and compositions (see, for example, [3, 4, 6]). Recently, the study has been extended to longer subword patterns on such structures as k-ary words [2], permutations [7], and finite set partitions [10, 11]. The enumeration of pairs of permutations according to the number of common rises was first considered by Carlitz, Scoville, and Vaughan [5]. Later, this result was q-generalized [8] and also extended to arbitrary m-tuples of permutations [8, 9] in various ways. See also the paper of Stanley [12], who considered a more general version of the problem related to binomial posets

Here, we study the analogous problem on k-ary words. In particular, we find explicit formulas for the generating functions that enumerate the members of $[k]^{nd}$ according to the number of occurrences of any pattern ρ of the form L^iR^{d-i}, $0 \leq i \leq d$. This then provides a complete solution to the problem of counting members of $[k]^{nd}$ according to the number of occurrences of any pattern ρ, where ρ has letters in $\{L, R, D\}$. For the case L^d, in addition, we are able to find a formula for the generating function that counts members of $[k]^{nd}$ according to the number of occurrences of $(L_m)^d$ for any $m \geq 1$ as well as an explicit formula for the number of members of $[k]^{nd}$ having exactly m occurrences of L^d for $0 \leq m \leq n-1$. We also give simple formulas for the average number of occurrences of L^iR^{d-i} for any i, providing both algebraic and combinatorial proofs.

In the third and fourth sections, we undergo the task of finding explicit formulas for the generating functions that count ordered pairs of k-ary words according to the number of occurrences of the patterns LR_w, R_wR_w, and RR_w. Combining this with the prior result provides a complete solution to the problem of counting ordered pairs of k-ary words according to the number of occurrences of any pattern ρ having letters in $\{L, R, D, R_w, D_w\}$. We remark that the cases R_wR_w and RR_w are apparently more difficult, and here we have used a technique which combines the use of several recurrences with the *kernel method* [1].

Up to equivalence, Table 1 below gives all the generating functions which count ordered pairs of k-ary words according to the number of occurrences of any pattern ρ having letters in $\{L, R, D, R_w, D_w\}$, where $k \geq 1$ is fixed. Taking $m = 1$ in the first entry gives the case of LL. Note that each generating function is of the form

$$\sum_{n\geq 0}\left(\sum_{\alpha} q^{\nu_\rho(\alpha)}\right)x^n,$$

where the inner sum is over all members α of $[k]^{2n}$ and $\nu_\rho(\alpha)$ counts the number of occurrences of the pattern ρ in α.

ρ	A formula for the generating function that counts members of $[k]^{2n}$ according to the number of occurrences of ρ	Reference
L_mL_m	$\dfrac{1}{1-k^2x\left(\frac{1-qx+(q-1)x^m}{1-qx+(q-1)x^{m+1}}\right)}$	Theorem 2.1
RR	$\dfrac{1}{1-kx+\sum_{j=1}^{k-1}\left(\frac{1-(1+jx(q-1))^k}{j(q-1)}\right)}$	Theorem 2.5
LR	$\dfrac{1}{1-\frac{k}{q-1}\left((1+(q-1)x)^k-1\right)}$	Theorem 2.5
LR_w	$\dfrac{1}{1-\frac{k}{q-1}\left(\frac{1}{(1-(q-1)x)^k}-1\right)}$	Theorem 3.1
R_wR_w	$\dfrac{1}{1-\frac{x(1-y^k)}{y^k(1-y)}-\frac{x}{y^{k+1}}\sum_{j=0}^{k-2}\frac{1}{y^j}\sum_{i=0}^{j}(-y)^i\binom{j}{i}\binom{k+j-i}{j+1}}$, where $y=1-(q-1)x$	Theorem 3.5
RR_w	$\dfrac{1}{1-\frac{x(1-z^k)}{1-z}-xz^{k-1}\sum_{j=0}^{k-2}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\binom{k+j-i}{j+1}z^{-i}}$, where $z=1+(q-1)x$	Theorem 4.4

TABLE 1. Formulas for all patterns ρ on ordered pairs of k-ary words, up to equivalence.

2. LEVELS AND RISES

In this section, we enumerate the members of $[k]^{nd}$ according to the number of occurrences of the pattern L^iR^{d-i}, where $d \geq 1$ and $0 \leq i \leq d$. We consider separately the cases where $i = d$ and $i < d$.

2.1. Counting L^d.

We may prove a more general result, wherein we allow for levels of arbitrary length within each word. Given $d, m \geq 1$, let $f = f_d^{(m)}(x;q)$ denote the generating function which counts d-tuples of k-ary words of length n according to the number of occurrences of $(L_m)^d$. That is,

$$f_d^{(m)}(x;q)=\sum_{n\geq 0}\left(\sum_{\alpha} q^{\nu(\alpha)}\right)x^n,$$

where the inner sum is over all members α of $[k]^{nd}$ and $\nu(\alpha)$ counts the number of common occurrences of the pattern L_m in α. Then f may be expressed explicitly as follows.

Theorem 2.1. *We have*

$$f_d^{(m)}(x;q)=\frac{1}{1-k^d x\left(\frac{1-qx+(q-1)x^m}{1-qx+(q-1)x^{m+1}}\right)}. \tag{1}$$

Proof. Suppose $\mathbf{v^{(1)}},\mathbf{v^{(2)}},\ldots,\mathbf{v^{(r)}}$ are members of $[k]^d$, with $\mathbf{v^{(j)}}=(v_1^{(j)},v_2^{(j)},\ldots,v_d^{(j)})$ for each $j\in[r]$. Let $f_{\mathbf{v^{(1)}}\cdots\mathbf{v^{(r)}}}$ denote the generating function counting the d-tuples of k-ary words of length n according to the number of occurrences of $(L_m)^d$ in which the first r entries of the i-th k-ary word in a d-tuple are $v_i^{(1)},v_i^{(2)},\ldots,v_i^{(r)}$ for all $i\in[d]$. Furthermore, given a vector $\mathbf{a}=(a_1,a_2,\ldots,a_d)$ in $[k]^d$, let $\mathbf{a}^r$ denote r copies of the vector $\mathbf{a}$.

From the definitions, we have for $1\leq j\leq m$,

$$\begin{aligned} f_{\mathbf{a}^j} &= x^j+\sum_{\mathbf{c}:\mathbf{c}\neq\mathbf{a}} f_{\mathbf{a}^j\mathbf{c}}+f_{\mathbf{a}^{j+1}}=x^j+x^j\sum_{\mathbf{c}:\mathbf{c}\neq\mathbf{a}} f_{\mathbf{c}}+f_{\mathbf{a}^{j+1}}=x^j+x^j(f-f_{\mathbf{a}}-1)+f_{\mathbf{a}^{j+1}}\\ &= x^j(f-f_{\mathbf{a}})+f_{\mathbf{a}^{j+1}}, \end{aligned} \tag{2}$$

with $f_{\mathbf{a}^{m+1}}=qxf_{\mathbf{a}^m}$. Taking $j=m$ in (2), and solving for $f_{\mathbf{a}^m}$, then gives

$$f_{\mathbf{a}^m}=\frac{x^m}{1-qx}(f-f_{\mathbf{a}}).$$

Applying (2), repeatedly, implies

$$f_{\mathbf{a}^j}=\left(x^j+x^{j+1}+\cdots+x^{m-1}+\frac{x^m}{1-qx}\right)(f-f_{\mathbf{a}}),\qquad 1\leq j\leq m. \tag{3}$$

Taking $j=1$ in (3), and solving for $f_{\mathbf{a}}$, gives

$$f_{\mathbf{a}}=\frac{\left(x+x^2+\cdots+x^{m-1}+\frac{x^m}{1-qx}\right)f}{1+x+\cdots+x^{m-1}+\frac{x^m}{1-qx}}=x\left(\frac{1-qx+(q-1)x^m}{1-qx+(q-1)x^{m+1}}\right)f,\qquad \mathbf{a}\in[k]^d.$$

Thus we have

$$f-1=\sum_{\mathbf{a}} f_{\mathbf{a}}=k^d x\left(\frac{1-qx+(q-1)x^m}{1-qx+(q-1)x^{m+1}}\right)f,$$

or

$$f=\frac{1}{1-k^d x\left(\frac{1-qx+(q-1)x^m}{1-qx+(q-1)x^{m+1}}\right)},$$

which completes the proof. □

Taking $m=1$ in (1) gives the following generating function for the number of common levels within d-tuples of k-ary words.

Corollary 2.2. *We have*

$$f_d^{(1)}(x;q)=\frac{1-(q-1)x}{1-(k^d-1+q)x}. \tag{4}$$

In this case, an explicit formula may be given.

Corollary 2.3. *If $n \geq 1$, then the number of d-tuples of k-ary words of length n having exactly j occurrences of L^d is given by $\binom{n-1}{j} k^d (k^d-1)^{n-1-j}$.*

Proof. From (4), we have

$$\begin{aligned}
\frac{1-(q-1)x}{1-(k^d-1+q)x} &= (1-(q-1)x)\sum_{n\geq 0}(k^d-1+q)^n x^n \\
&= 1+\sum_{n\geq 1}((k^d-1+q)^n-(q-1)(k^d-1+q)^{n-1})x^n \\
&= 1+\sum_{n\geq 1}k^d(k^d-1+q)^{n-1}x^n \\
&= 1+\sum_{n\geq 1}\left(\sum_{j=0}^{n-1}\binom{n-1}{j}k^d(k^d-1)^{n-1-j}q^j\right)x^n.
\end{aligned}$$

Extracting the coefficient of $x^n q^j$ gives the result. □

The formula in the prior corollary may be realized combinatorially as follows. Let α be a d-tuple of k-ary words of length n having exactly j occurrences of L^d. Pick any vector $\mathbf{a} = (a_1, a_2, \ldots, a_d)$ in $[k]^d$. Let S denote the set of indices $i \in [n-1]$ corresponding to occurrences of L^d. Note that there are $\binom{n-1}{j}$ choices regarding S. To form α, start by letting the entries of $\mathbf{a}$ comprise the first letters of the component words of α; note that there are k^d choices for $\mathbf{a}$. Then fill in the second entries of the component words of α as follows: if $1 \in S$, then the second letter within each word should be the same as the first, whereas if $1 \notin S$, then at least one of the second letters should differ from the first. Proceed similarly for the ℓ-th entry within each of the components of α going from left to right, using the very same entries as those for the $(\ell-1)$-st if $\ell-1 \in S$, or using a different vector for these entries if $\ell-1 \notin S$. Since there are $n-1-j$ members of $[n-1]-S$, there are $(k^d-1)^{n-1-j}$ choices regarding the ℓ-th entries of the components of α corresponding to those ℓ such that $\ell-1 \notin S$, where $2 \leq \ell \leq n$. □

Differentiating formula (1) with respect to q, and letting $q=1$, gives the following result.

Corollary 2.4. *If $n \geq m$, then the total number of occurrences of $(L_m)^d$ within all the members of $[k]^{nd}$ is given by $(n-m)k^{d(n-m)}$.*

This result may be proven directly as follows. First note that it suffices to show that the total number of occurrences of $(L_m)^d$ at index i, where $i \in [n-m]$ is fixed, is $k^{d(n-m)}$. This is equivalent to showing that the number of d-tuples α that have an occurrence of $(L_m)^d$ at index i is $k^{d(n-m)}$. Note that $a_i = a_{i+1} = \cdots = a_{i+m}$ for each component word within such α, and thus there are k^d choices for these entries within all of the component words. There are also $k^{d(n-m-1)}$ choices for all other entries within the components of α since they may be chosen freely. Thus, there are $k^d \cdot k^{d(n-m-1)} = k^{d(n-m)}$ possible α, and hence occurrences of $(L_m)^d$ at index i. □

2.2. **Counting $L^i R^{d-i}$, $i < d$.** Given $d \geq 2$ and $0 \leq i \leq d-1$, let $g = g_{d,i}(x;q)$ denote the generating function which counts d-tuples of k-ary words of length n according to the number of occurrences of $L^i R^{d-i}$. Then g may be expressed as follows.

Theorem 2.5. *We have*

$$g_{d,i}(x;q)=\frac{1}{1-k^i x\sum_{(b_1,\dots,b_{d-i-1})\in[k]^{d-i-1}}\left(\frac{1-\lambda^k}{1-\lambda}\right)},\qquad 0\le i\le d-2, \tag{5}$$

where

$$\lambda:=1+x(q-1)\prod_{j=1}^{d-i-1}(k-b_j),$$

with

$$g_{d,d-1}(x;q)=\frac{1}{1-\frac{k^{d-1}}{q-1}\left((1+(q-1)x)^k-1\right)}. \tag{6}$$

Proof. First assume $0\le i\le d-2$. Suppose $\mathbf{a}=(a_1,a_2,\dots,a_d)$ is a member of $[k]^d$. Let $g_{\mathbf{a}}$ be the generating function which counts the d-tuples α of k-ary words of length n according to the number of occurrences of L^iR^{d-i} in which the first entry in the j-th component of α is a_j for each $j\in[d]$. If $\mathbf{u}$ and $\mathbf{v}$ are two vectors, then we'll say that $\mathbf{u}>\mathbf{v}$ if each component of $\mathbf{u}$ is larger than each component of $\mathbf{v}$. Given any $\mathbf{a}\in[k]^d$, let us write $\mathbf{a}=(\widetilde{\mathbf{a}},\mathbf{a}')$, where $\widetilde{\mathbf{a}}\in[k]^i$ and $\mathbf{a}'\in[k]^{d-i}$. From the definitions, we have

$$g_{\mathbf{a}}=xg+x(q-1)\sum_{\substack{\mathbf{c}:\widetilde{\mathbf{c}}=\widetilde{\mathbf{a}},\\ \mathbf{c}'>\mathbf{a}'}}g_{\mathbf{c}},\qquad \mathbf{a}\in[k]^d. \tag{7}$$

Let us decompose $\mathbf{a}$ further as $(\widetilde{\mathbf{a}},\ell,\mathbf{a}'')$, where $\ell\in[k]$ and $\mathbf{a}''\in[k]^{d-i-1}$. If $\ell=k$, then

$$g_{\mathbf{a}}=g_{(\widetilde{\mathbf{a}},k,\mathbf{a}'')}=xg,$$

since the sum on the right-hand side of (7) is empty in this case. If $\ell=k-1$, then

$$g_{\mathbf{a}}=g_{(\widetilde{\mathbf{a}},k-1,\mathbf{a}'')}=xg+x(q-1)\sum_{\substack{\mathbf{c}=(\widetilde{\mathbf{c}},k,\mathbf{c}''):\\ \widetilde{\mathbf{c}}=\widetilde{\mathbf{a}},\\ \mathbf{c}''>\mathbf{a}''}}g_{\mathbf{c}}=xg+x(q-1)\cdot xg\prod_{j=i+2}^{d}(k-a_j)=xg\rho,$$

where $\rho=1+x(q-1)\prod_{j=i+2}^{d}(k-a_j)$. In general, if $\ell=k-m$, then by induction we have

$$g_{\mathbf{a}}=g_{(\widetilde{\mathbf{a}},k-m,\mathbf{a}'')}=xg\rho^m,\qquad 0\le m\le k-1, \tag{8}$$

since

$$\begin{aligned}
g_{(\widetilde{\mathbf{a}},k-m,\mathbf{a}'')}&=xg+x(q-1)\sum_{r=k-m+1}^{k}\sum_{\substack{\mathbf{c}=(\widetilde{\mathbf{c}},r,\mathbf{c}''):\\ \widetilde{\mathbf{c}}=\widetilde{\mathbf{a}},\\ \mathbf{c}''>\mathbf{a}''}}g_{\mathbf{c}}\\
&=xg+x(q-1)\sum_{r=k-m+1}^{k}xg\rho^{k-r}\cdot\prod_{j=i+2}^{d}(k-a_j)\\
&=xg\left(1+x(q-1)\prod_{j=i+2}^{d}(k-a_j)\left(\frac{1-\rho^m}{1-\rho}\right)\right)\\
&=xg\left(1+(\rho-1)\left(\frac{1-\rho^m}{1-\rho}\right)\right)\\
&=xg\rho^m.
\end{aligned}$$

Taking $m = k-\ell$ in (8), and writing $\mathbf{a} = (\tilde{a}, \ell, \mathbf{a}'')$, implies

$$g_{\mathbf{a}} = xg\rho^{k-\ell}, \qquad \mathbf{a} \in [k]^d.$$

Thus, if $1 \le i \le d-2$, we have

$$\begin{aligned} g-1 &= \sum_{\mathbf{a}} g_{\mathbf{a}} = \sum_{(\tilde{\mathbf{a}},\ell,\mathbf{a}'')\in[k]^d} g_{(\tilde{\mathbf{a}},\ell,\mathbf{a}'')} = \sum_{\tilde{\mathbf{a}}\in[k]^i} \sum_{\mathbf{a}''\in[k]^{d-i-1}} \sum_{\ell=1}^{k} xg\rho^{k-\ell} \\ &= k^i xg \sum_{\mathbf{a}''\in[k]^{d-i-1}} \sum_{\ell=1}^{k} \rho^{k-\ell} \\ &= k^i xg \sum_{(a_{i+2},\dots,a_d)\in[k]^{d-i-1}} \left(\frac{1-\rho^k}{1-\rho}\right), \end{aligned}$$

which gives formula (5), upon solving for g and renaming the variables. If $i = 0$, then there is no outer sum in $\tilde{\mathbf{a}}$ in the last calculation, but the same formula still holds. If $i = d-1$, then proceed similarly, noting that $\rho = 1 + x(q-1)$ in this case. The middle sum in $\mathbf{a}''$ in the last calculation would not occur, so that

$$g-1 = \left(\frac{k^{d-1}((1+(q-1)x)^k - 1)}{q-1}\right) g,$$

which gives (6). $\square$

Setting $q = 0$ in Theorem 2.5 gives an expression for the generating function which counts the members of $[k]^{nd}$ that avoid $L^i R^{d-i}$.

Taking $d = 2$ in the prior theorem gives formulas for the generating functions $u(x;q)$ and $v(x;q)$ counting ordered pairs of k-ary words according to the number of occurrences of RR or LR, respectively.

Corollary 2.6. *We have*

$$u(x;q) = \frac{1}{1 - kx + \frac{1}{q-1}\left(H_{k-1} - \sum_{b=1}^{k-1} \frac{(1+x(q-1)(k-b))^k}{k-b}\right)}, \tag{9}$$

where $H_{k-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}$ *denotes the* $(k-1)$*-st harmonic number, and*

$$v(x;q) = \frac{1}{1 - \frac{k}{q-1}\left((1+(q-1)x)^k - 1\right)}. \tag{10}$$

Proof. Taking $d = 2$ in (5) gives

$$u(x;q) = \frac{1}{1 + x\sum_{b=1}^{k} \frac{1-(1+x(q-1)(k-b))^k}{x(q-1)(k-b)}},$$

with

$$\begin{aligned}\sum_{b=1}^{k}\frac{1-(1+x(q-1)(k-b))^k}{(q-1)(k-b)} &= \lim_{b\to k}\left(\frac{1-(1+x(q-1)(k-b))^k}{(q-1)(k-b)}\right)\\ &\quad+\sum_{b=1}^{k-1}\frac{1-(1+x(q-1)(k-b))^k}{(q-1)(k-b)}\\ &= -kx+\frac{1}{q-1}\left(H_{k-1}-\sum_{b=1}^{k-1}\frac{(1+x(q-1)(k-b))^k}{k-b}\right).\end{aligned}$$

Taking $d=2$ in (6) gives (10). □

We have the following simple expression for the total number of occurrences of L^iR^{d-i}.

Corollary 2.7. *If $n\ge 1$, then the total number of occurrences of L^iR^{d-i}, $0\le i\le d-1$, within all the members of $[k]^{nd}$ is given by $(n-1)\binom{k}{2}^{d-i}k^{d(n-2)+i}$.*

Proof. Equivalently, we must show that there are $\binom{k}{2}^{d-i}k^{d(n-2)+i}$ members α of $[k]^{nd}$ that have an occurrence of L^iR^{d-i} at index j for each $j\in[n-1]$. To see this, note that there are k^i choices for the j-th and $(j+1)$-st entries within the first i component words of α, and $\binom{k}{2}^{d-i}$ choices for these entries within the final $d-i$ components of α. The remaining entries of α may be chosen in any of $k^{d(n-2)}$ ways. □

Dividing by k^{nd}, we see that the average number of occurrences of L^iR^{d-i} within the members of $[k]^{nd}$ is given by $\frac{(n-1)(k-1)^{d-i}}{2^{d-i}k^d}$.

As a consistency check, we will show how Corollary 2.7 can be obtained from Theorem 2.5. Let us first assume $0\le i\le d-2$. Note that

$$g_{d,i}(x;1)=\lim_{q\to1}(g_{d,i}(x;q))=\frac{1}{1-k^dx}$$

since

$$\lim_{q\to1}\left(\frac{1-\lambda^k}{1-\lambda}\right)=k$$

for all $(d-i-1)$-tuples $(b_1,\dots,b_{d-i-1})$, by L'Hôpital's rule. Let $s=\prod_{j=1}^{d-i-1}(k-b_j)$. Thus,

$$\frac{d}{dq}g_{d,i}(x;q)\mid_{q=1}=\lim_{q\to1}\left(\frac{d}{dq}g_{d,i}(x;q)\right)=\frac{k^ix\lim_{q\to1}\left(\sum_{(b_1,\dots,b_{d-i-1})\in[k]^{d-i-1}}\frac{d}{dq}\left(\frac{1-\lambda^k}{1-\lambda}\right)\right)}{(1-k^dx)^2},$$

with

$$\begin{aligned}\lim_{q\to1}\left(\frac{d}{dq}\left(\frac{1-\lambda^k}{1-\lambda}\right)\right) &= -\frac{1}{sx}\lim_{q\to1}\left(\frac{d}{dq}\left(\frac{1-(1+s(q-1)x)^k}{q-1}\right)\right)\\ &= \frac{1}{sx}\lim_{q\to1}\left(\frac{ks(q-1)\lambda^{k-1}x+(1-\lambda^k)}{(q-1)^2}\right)\\ &= \frac{1}{2sx}\lim_{q\to1}\left[(2k(k-1)s^2x^2-k(k-1)s^2x^2)\lambda^{k-2}\right]=\binom{k}{2}sx,\end{aligned}$$

by two applications of L′Hôpital's rule, so that

$$\frac{d}{dq}g_{d,i}(x;q)\,|_{q=1}=\frac{k^i\binom{k}{2}x^2}{(1-k^dx)^2}\sum_{(b_1,\ldots,b_{d-i-1})\in[k]^{d-i-1}}\prod_{j=1}^{d-i-1}(k-b_j).$$

Note the identity

$$\sum_{(b_1,\ldots,b_m)\in[k]^m}\prod_{j=1}^{m}(k-b_j)=\binom{k}{2}^m,\qquad m\geq 1,$$

which we cannot find in the literature, but can be shown easily by induction or by arguing that both sides count the m-tuples of doubleton subsets of $[k]$. The left-hand side achieves this by specifying the smaller elements $(b_1,\ldots,b_m)$ within an m-tuple of doubletons.

Thus, we have

$$\begin{aligned}
&\frac{d}{dq}g_{d,i}(x;q)\,|_{q=1}\\
&=\frac{k^i\binom{k}{2}^{d-i}x^2}{(1-k^dx)^2}=\left(\frac{k-1}{2}\right)^{d-i}x\sum_{n\geq 0}nk^{dn}x^n=\left(\frac{k-1}{2}\right)^{d-i}\sum_{n\geq 1}(n-1)k^{d(n-1)}x^n\\
&=\binom{k}{2}^{d-i}\sum_{n\geq 1}(n-1)k^{d(n-2)+i}x^n,
\end{aligned}$$

from which Corollary 2.7 follows when $0\leq i\leq d-2$, upon extracting the coefficient of x^n. A similar, shorter calculation may be given when $i=d-1$.

3. WEAK RISES

We first consider the case LR_w, where we may prove more. Given $d\geq 2$, let $h=h_d(x;q)$ denote the generating function which counts d-tuples of k-ary words of length n according to the number of occurrences of $L^{d-1}R_w$. Then h may be expressed as follows.

Theorem 3.1. *We have*

$$h=\frac{1}{1-\frac{k^{d-1}}{q-1}\left(\frac{1}{(1-(q-1)x)^k}-1\right)}. \tag{11}$$

Proof. Given $\mathbf{a}=(a_1,a_2,\ldots,a_d)\in[k]^d$, let $h_{\mathbf{a}}$ be the generating function that counts the members α of $[k]^{nd}$ according to the number of occurrences of $L^{d-1}R_w$ in which the first entry in the i-th component of α is a_i for each $i\in[d]$. Let us write $\mathbf{a}=(\widetilde{\mathbf{a}},\ell)$, where $\widetilde{\mathbf{a}}\in[k]^{d-1}$ and $\ell\in[k]$. From the definitions, we have

$$h_{\mathbf{a}}=h_{(\widetilde{\mathbf{a}},\ell)}=xh+x(q-1)\sum_{j=\ell}^{k}h_{(\widetilde{\mathbf{a}},j)},\qquad \mathbf{a}\in[k]^d. \tag{12}$$

Taking $\ell=k$ in (12) implies

$$h_{(\widetilde{\mathbf{a}},k)}=\frac{xh}{1-(q-1)x},$$

and in general, by induction using (12),

$$h_{(\widetilde{\mathbf{a}},\ell)}=\frac{xh}{(1-(q-1)x)^{k-\ell+1}},\qquad 1\leq\ell\leq k.$$

Therefore, we have

$$\begin{aligned} h-1 &= \sum_{\mathbf{a}} h_{\mathbf{a}} = \sum_{(\tilde{\mathbf{a}},\ell)} h_{(\tilde{\mathbf{a}},\ell)} = \sum_{\tilde{\mathbf{a}}\in[k]^{d-1}} \sum_{\ell=1}^{k} \frac{xh}{(1-(q-1)x)^{k-\ell+1}} \\ &= k^{d-1}xh\sum_{\ell=1}^{k}\frac{1}{(1-(q-1)x)^{\ell}} = \frac{k^{d-1}h}{q-1}\left(\frac{1}{(1-(q-1)x)^{k}}-1\right), \end{aligned}$$

which gives (11). □

We now study the case R_wR_w. Let $R=R(x;q)$ denote the generating function which counts ordered pairs of k-ary words of length n according to the number of occurrences of R_wR_w. That is,

$$R(x;q)=\sum_{n\geq 0}\left(\sum_{\alpha} q^{\mu(\alpha)}\right)x^n,$$

where the inner sum is over all ordered pairs α of k-ary words of length n and $\mu(\alpha)$ counts the number of occurrences of R_wR_w in α. If $(a,b)\in[k]^2$, then let $R_{(a,b)}(x;q)$ denote the generating function counting the ordered pairs of k-ary words of length n according to the number of occurrences of R_wR_w in which the first entry of the first word is a and the first entry of the second word is b. From the definitions, we can state the relation

$$R_{(a,b)}(x;q)=xR(x;q)+x(q-1)\sum_{c=a}^{k}\sum_{d=b}^{k}R_{(c,d)}(x;q), \quad 1\leq a,b\leq k, \tag{13}$$

which implies

$$R_{(a,b)}(x;q)=R_{(a+1,b)}(x;q)+x(q-1)\sum_{d=b}^{k}R_{(a,d)}(x;q), \quad 1\leq b\leq k,\ 1\leq a\leq k-1. \tag{14}$$

Let $R_a(u)=R_a(x;q,u)=\sum_{b=1}^{k}R_{(a,b)}(x;q)u^b$ for all a. Multiplying (14) by u^b, and summing over $b=1,2,\ldots,k$, we obtain

$$\begin{aligned} R_a(u) &= R_{a+1}(u)+x(q-1)\sum_{d=1}^{k}\frac{u-u^{d+1}}{1-u}R_{(a,d)}(x;q) \\ &= R_{a+1}(u)+\frac{xu(q-1)}{1-u}(R_a(1)-R_a(u)), \qquad 1\leq a\leq k-1. \end{aligned} \tag{15}$$

We may express $R_k(u)$ explicitly.

Lemma 3.2. *We have*

$$R_k(u)=\frac{uxR(x;q)}{y^k}\cdot\frac{1-(yu)^k}{1-yu}, \tag{16}$$

where $y=1-(q-1)x$.

Proof. By (13), we have $R_{(k,k)}(x;q)=xR(x;q)+x(q-1)R_{(k,k)}(x;q)$, which implies that

$$R_{(k,k)}(x;q)=\frac{x}{y}R(x;q).$$

Assume by induction that $R_{(k,k-\ell)}(x;q) = \frac{x}{y^{\ell+1}}R(x;q)$ for all $\ell = 0,1,\dots,j-1$. Then solving for $R_{(k,k-j)}(x;q)$ in (13) and applying the induction hypothesis yields

$$\begin{aligned}
yR_{(k,k-j)}(x;q) &= xR(x;q) + x(q-1)\sum_{b=k-j+1}^{k} R_{(k,b)}(x;q)\\
&= xR(x;q)\left(1 + x(q-1)\sum_{b=k-j+1}^{k}\frac{1}{y^{k+1-b}}\right) = xR(x;q)\left(1+x(q-1)\sum_{b=1}^{j}\frac{1}{y^b}\right)\\
&= xR(x;q)\left(1+(1-y)\frac{1/y-(1/y)^{j+1}}{1-1/y}\right) = \frac{x}{y^j}R(x;q),
\end{aligned}$$

whence $R_{(k,k-j)}(x;q) = \frac{x}{y^{j+1}}R(x;q)$, which completes the induction step. Therefore,

$$R_k(u) = \frac{xR(x;q)}{y^{k+1}}\sum_{b=1}^{k}(uy)^b = \frac{uxR(x;q)}{y^k}\cdot\frac{1-(yu)^k}{1-yu},$$

as desired. □

Let $R'_a(u) = \frac{R_a(u)}{xR(x;q)}$ for all a. We have the following recurrence for $R'_a(u)$.

Lemma 3.3. *If* $0 \le j \le k-1$, *then*

$$R'_{k-j}(u) = \frac{u(1-u)^j(1-(yu)^k)}{y^k(1-yu)^{j+1}} + \frac{u(1-y)}{1-u}\sum_{i=0}^{j-1}\left(\frac{1-u}{1-yu}\right)^{j-i} R'_{k-i}\left(\frac{1}{y}\right). \tag{17}$$

Proof. Substituting $u = 1/y$ into (15) gives

$$R'_{a+1}\left(\frac{1}{y}\right) = R'_a(1), \quad 1 \le a \le k-1, \tag{18}$$

and, by (16), we have

$$R'_k\left(\frac{1}{y}\right) = \frac{k}{y^{k+1}}. \tag{19}$$

By (15), (16), and (18), we may write

$$R'_a(u) = \frac{1-u}{1-yu}R'_{a+1}(u) + \frac{u(1-y)}{1-yu}R'_{a+1}\left(\frac{1}{y}\right), \qquad 1 \le a \le k-1,$$

with $R'_k(u) = \frac{u(1-(yu)^k)}{y^k(1-yu)}$. Iterating the last recurrence yields (17). □

In the next lemma, we provide an explicit formula for $R'_{k-j}\left(\frac{1}{y}\right)$.

Lemma 3.4. *If* $0 \le j \le k-1$, *then*

$$R'_{k-j}\left(\frac{1}{y}\right) = \frac{1}{y^{k+1+j}}\sum_{i=0}^{j}(-y)^i\binom{j}{i}\binom{k+j-i}{j+1}. \tag{20}$$

Proof. We proceed by induction on j. By (19), we have $R'_k\left(\frac{1}{y}\right)=\frac{k}{y^{k+1}}$, which agrees with (20) when $j=0$. Let us assume that the lemma holds for $0,1,\ldots,j-1$ and prove it for j. Rewriting (17), we have

$$R'_{k-j}(u)=\frac{u(1-u)^j(1-(yu)^k)+y^k u(1-y)\sum_{i=0}^{j-1}\frac{(1-yu)^{i+1}}{(1-u)^{i+1-j}}R'_{k-i}\left(\frac{1}{y}\right)}{y^k(1-yu)^{j+1}},$$

which implies

$$\frac{y^kR'_{k-j}(u)}{u(1-u)^j}=\frac{1-(yu)^k+y^k(1-y)\sum_{i=0}^{j-1}\frac{(1-yu)^{i+1}}{(1-u)^{i+1}}R'_{k-i}\left(\frac{1}{y}\right)}{(1-yu)^{j+1}}. \tag{21}$$

In order to find $R'_{k-j}\left(\frac{1}{y}\right)$, we need to compute $\lim_{u\to 1/y}\frac{y^kR'_{k-j}(u)}{u(1-u)^j}$. To do so, we make use of L′Hôpital's rule as follows. Let

$$a_s=\frac{d^s}{du^s}\left[1-(yu)^k+y^k(1-y)\sum_{i=0}^{j-1}\frac{(1-yu)^{i+1}}{(1-u)^{i+1}}R'_{k-i}\left(\frac{1}{y}\right)\right]\Bigg|_{u=1/y},$$

where $0\le s\le j+1$. We show that $a_s=0$ for all $s=0,1,\ldots,j$. Clearly, $a_0=0$, so assume $s\ge 1$. Then we have

$$\begin{aligned}\frac{(y-1)^{s-1}a_s}{s!y^s}&=-\binom{k}{s}(y-1)^{s-1}\\&\quad-\frac{y^{k-s}(y-1)^s}{s!}\sum_{i=0}^{s-1}\binom{s}{i+1}(i+1)!(-y)^{i+1}\frac{d^{s-i-1}}{du^{s-i-1}}\left((1-u)^{-i-1}\right)|_{u=1/y}\,R'_{k-i}\left(\frac{1}{y}\right)\\&=-\binom{k}{s}(y-1)^{s-1}-y^k\sum_{i=0}^{s-1}\binom{s}{i+1}(i+1)!(-y)^{i+1}\frac{(s-1)!}{i!s!}R'_{k-i}\left(\frac{1}{y}\right)\\&=-\binom{k}{s}(y-1)^{s-1}-y^k\sum_{i=0}^{s-1}\binom{s-1}{i}(-y)^{i+1}R'_{k-i}\left(\frac{1}{y}\right).\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}&\frac{(y-1)^{s-1}a_s}{s!y^s}\\&=-\binom{k}{s}(y-1)^{s-1}-y^k\sum_{i=0}^{s-1}\binom{s-1}{i}(-y)^{i+1}\left(\frac{1}{y^{k+1+i}}\sum_{\ell=0}^{i}(-y)^{\ell}\binom{i}{\ell}\binom{k+i-\ell}{i+1}\right)\\&=-\binom{k}{s}(y-1)^{s-1}-\sum_{i=0}^{s-1}\sum_{\ell=0}^{i}(-1)^{i+1}\binom{s-1}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}(-y)^{\ell}.\end{aligned}$$

Extracting the coefficient of y^ℓ, where $0\le\ell\le s-1$, from both sides of the last equation yields

$$[y^\ell]\left(\frac{(y-1)^{s-1}a_s}{s!y^s}\right)=(-1)^{s-\ell}\binom{k}{s}\binom{s-1}{\ell}+\sum_{i=\ell}^{s-1}(-1)^{i-\ell}\binom{s-1}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}. \tag{22}$$

Using the binomial identity

$$\sum_{i=\ell}^{s-1}(-1)^{i-\ell}\binom{s-1}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}=(-1)^{s-1-\ell}\binom{k}{s}\binom{s-1}{\ell},\qquad s\ge 1,\ \ell,k\ge 0,\tag{23}$$

we obtain $\frac{(y-1)^{s-1}a_s}{s!y^s}=0$ and thus $a_s=0$ for $0\le s\le j$, as desired. (Note that (23) may be shown by applying trinomial revision to the first two factors in the sum on the left-hand side, followed by computing the generating function in $k\ge 0$ of both sides, where s and ℓ are fixed.)

Applying similar reasoning in the case when $s=j+1$ implies

$$[y^\ell]\left(\frac{(y-1)^j a_{j+1}}{(j+1)!y^{j+1}}\right)=(-1)^{j+1-\ell}\binom{k}{j+1}\binom{j}{\ell}+\sum_{i=\ell}^{j-1}(-1)^{i-\ell}\binom{j}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1},\tag{24}$$

where $0\le\ell\le j$. Note that (21) gives

$$\lim_{u\to 1/y}\frac{y^k R'_{k-j}(u)}{u(1-u)^j}=\frac{a_{j+1}}{(j+1)!(-y)^{j+1}},$$

which implies, by (24),

$$\begin{aligned}\lim_{u\to 1/y}&R'_{k-j}(u)\\&=\frac{(-1)^{j+1}}{y^{k+1+j}}\sum_{\ell=0}^{j}\left((-1)^{j+1-\ell}\binom{k}{j+1}\binom{j}{\ell}+\sum_{i=\ell}^{j-1}(-1)^{i-\ell}\binom{j}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}\right)y^\ell\\&=\frac{1}{y^{k+1+j}}\sum_{\ell=0}^{j}\left((-1)^{\ell}\binom{k}{j+1}\binom{j}{\ell}+\sum_{i=\ell}^{j-1}(-1)^{j-1+i-\ell}\binom{j}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}\right)y^\ell.\end{aligned}$$

Writing

$$\begin{aligned}\sum_{i=\ell}^{j-1}(-1)^{j-1+i-\ell}\binom{j}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}=&\sum_{i=\ell}^{j}(-1)^{j-1+i-\ell}\binom{j}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}\\&+(-1)^\ell\binom{j}{\ell}\binom{k+j-\ell}{j+1},\end{aligned}$$

and using the identity (let $s=j+1$ in (23))

$$\sum_{i=\ell}^{j}(-1)^{j-1+i-\ell}\binom{j}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}=(-1)^{\ell-1}\binom{k}{j+1}\binom{j}{\ell},$$

we obtain

$$\lim_{u\to 1/y}R'_{k-j}(u)=\frac{1}{y^{k+1+j}}\sum_{\ell=0}^{j}(-y)^\ell\binom{j}{\ell}\binom{k+j-\ell}{j+1},$$

which completes the induction step, as required. □

By Lemma 3.4, (16), and (18), we have

$$\begin{aligned}R_a(1)&=R_{a+1}(1/y)=xR(x;q)R'_{a+1}(1/y)\\&=\frac{xR(x;q)}{y^{2k-a}}\sum_{i=0}^{k-1-a}(-y)^i\binom{k-1-a}{i}\binom{2k-1-a-i}{k-a},\qquad 1\le a\le k-1,\end{aligned}$$

with

$$R_k(1) = \frac{xR(x;q)}{y^k} \cdot \frac{1-y^k}{1-y}.$$

Since $R(x;q) = 1 + \sum_{a=1}^{k} R_a(1)$, it follows that

$$\begin{aligned} R(x;q) &= \frac{1}{1 - \frac{x(1-y^k)}{y^k(1-y)} - x\sum_{a=1}^{k-1} \frac{1}{y^{2k-a}} \sum_{i=0}^{k-1-a} (-y)^i \binom{k-1-a}{i}\binom{2k-1-a-i}{k-a}} \\ &= \frac{1}{1 - \frac{x(1-y^k)}{y^k(1-y)} - \frac{x}{y^{k+1}} \sum_{b=0}^{k-2} \frac{1}{y^b} \sum_{i=0}^{b} (-y)^i \binom{b}{i}\binom{k+b-i}{b+1}}, \end{aligned}$$

which leads to the main result of the section.

Theorem 3.5. *Let $k \geq 1$. Then the generating function $R(x;q)$ is given by*

$$R(x;q) = \frac{1}{1 - \frac{x(1-y^k)}{y^k(1-y)} - \frac{x}{y^{k+1}} \sum_{b=0}^{k-2} \frac{1}{y^b} \sum_{i=0}^{b} (-y)^i \binom{b}{i}\binom{k+b-i}{b+1}}, \tag{25}$$

where $y = 1 - (q-1)x$.

Setting $q = 1$ in (25) gives

$$R(x;1) = \frac{1}{1 - kx - x\sum_{b=0}^{k-2} \sum_{i=0}^{b} (-1)^i \binom{b}{i}\binom{k+b-i}{b+1}},$$

and by the identity $\sum_{i=0}^{b} (-1)^i \binom{b}{i}\binom{k+b-i}{b+1} = k$, we see that

$$R(x;1) = \frac{1}{1 - kx - x(k-1)k} = \frac{1}{1-k^2x},$$

in accordance with the fact that the number of the ordered pairs of k-ary words of length n is given by k^{2n}. Setting $q = 0$ in (25) gives the generating function which counts the ordered pairs of k-ary words avoiding the pattern R_wR_w.

4. COUNTING RR_w

We study the case RR_w. Let $T = T(x;q)$ denote the generating function which counts ordered pairs of k-ary words of length n according to the number of occurrences of RR_w. If $(a,b) \in [k]^2$, then let $T_{(a,b)}(x;q)$ denote the generating function counting the ordered pairs of k-ary words of length n according to the number of occurrences of RR_w in which the first entry of the first word is a and the first entry of the second word is b. From the definitions, we can state the relation

$$T_{(a,b)}(x;q) = xT(x;q) + x(q-1)\sum_{c=a+1}^{k}\sum_{d=b}^{k} T_{(c,d)}(x;q), \quad 1 \leq a,b \leq k, \tag{26}$$

which implies

$$T_{(a,b)}(x;q) = T_{(a,b+1)}(x;q) + x(q-1)\sum_{c=a+1}^{k} T_{(c,b)}(x;q), \quad 1 \leq a \leq k, 1 \leq b \leq k-1. \tag{27}$$

Define $T_b(u) = T_b(x;q,u) = \sum_{a=1}^{k} T_{(a,b)}(x;q)u^a$ for all b. Multiplying (27) by u^a, and summing over $a = 1,2,\dots,k$, we obtain

$$\begin{aligned} T_b(u) &= T_{b+1}(u) + x(q-1)\sum_{c=1}^{k}\frac{u-u^c}{1-u}T_{(c,b)}(x;q) \\ &= T_{b+1}(u) + \frac{x(q-1)}{1-u}(uT_b(1) - T_b(u)), \qquad 1 \le b \le k-1. \end{aligned} \tag{28}$$

One may give an explicit formula for $T_k(u)$.

Lemma 4.1. *We have*

$$T_k(u) = uxT(x;q)\frac{z^k-u^k}{z-u}, \tag{29}$$

where $z = 1 + (q-1)x$.

Proof. Note that $T_{(k,k)}(x;q) = xT(x;q)$, by (26). Then by induction, we have

$$T_{(a,k)}(x;q) = xz^{k-a}T(x;q), \qquad 1 \le a \le k,$$

since

$$T_{(k-j,k)}(x;q) = xT(x;q) + x(q-1)\sum_{c=k-j+1}^{k}T_{(c,k)}(x;q),$$

by (26). Thus,

$$T_k(u) = xT(x;q)\sum_{a=1}^{k}z^{k-a}u^a = uxT(x;q)\frac{z^k-u^k}{z-u}.$$

□

Define $T'_b(u) = \frac{T_b(u)}{xT(x;q)}$ for all b. Then $T'_b(u)$ satisfies the following recurrence.

Lemma 4.2. *If* $0 \le j \le k-1$, *then*

$$T'_{k-j}(u) = \frac{(1-u)^j}{(z-u)^j}T'_k(u) + \frac{u(z-1)}{z(1-u)}\sum_{i=0}^{j-1}\left(\frac{1-u}{z-u}\right)^{j-i}T'_{k-i}(z). \tag{30}$$

Proof. Substituting $u = z$ into (28) gives

$$T'_{b+1}(z) = zT'_b(1), \quad 1 \le b \le k-1, \tag{31}$$

and, by (29), we have

$$T'_k(z) = kz^k. \tag{32}$$

By (28), (29), and (31), we may write

$$T'_b(u) = \frac{1-u}{z-u}T'_{b+1}(u) + \frac{u(z-1)}{z(z-u)}T'_{b+1}(z), \qquad 1 \le b \le k-1,$$

with $T'_k(u) = u\frac{z^k-u^k}{z-u}$. Iterating the last recurrence yields (30). □

In the next lemma, we provide an explicit formula for $T'_{k-j}(z)$.

Lemma 4.3. *If* $0 \le j \le k-1$, *then*

$$T'_{k-j}(z) = z^{k-j}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\binom{k+j-i}{j+1}z^{j-i}. \tag{33}$$

Proof. We proceed by induction on j. By (32), we have $T'_k(z) = kz^k$, which agrees with (33) when $j = 0$. Let us assume that the lemma holds for $0, 1, \ldots, j-1$ and prove it for j. Rewriting (30), we have

$$T'_{k-j}(u) = \frac{u(1-u)^j(z^k - u^k) + \frac{(z-1)u}{z(1-u)}\sum_{i=0}^{j-1}(1-u)^{j-i}(z-u)^{i+1}T'_{k-i}(z)}{(z-u)^{j+1}},$$

which implies

$$\frac{T'_{k-j}(u)}{u(1-u)^j} = \frac{z^k - u^k + \frac{z-1}{z}\sum_{i=0}^{j-1}\frac{(z-u)^{i+1}}{(1-u)^{i+1}}T'_{k-i}(z)}{(z-u)^{j+1}}. \tag{34}$$

In order to find $T'_{k-j}(z)$, we need to compute $\lim_{u\to z}\frac{T'_{k-j}(u)}{u(1-u)^j}$. To do so, we make use of L$'$Hôpital's rule as follows. Let

$$b_s = \frac{d^s}{du^s}\left[z^k - u^k + \frac{z-1}{z}\sum_{i=0}^{j-1}\frac{(z-u)^{i+1}}{(1-u)^{i+1}}T'_{k-i}(z)\right]\Bigg|_{u=z}.$$

We show that $b_s = 0$ for all $s = 0, 1, \ldots, j$. Clearly, $b_0 = 0$, so assume $s \geq 1$. Then we have

$$\begin{aligned}
\frac{(1-z)^{s-1}b_s}{s!z^{k-s}} &= -\binom{k}{s}(1-z)^{s-1} - \frac{(1-z)^s}{s!z^{k+1-s}}\sum_{i=0}^{j-1}\frac{d^s}{du^s}\frac{(z-u)^{i+1}}{(1-u)^{i+1}}\,|_{u=z}\, T'_{k-i}(z)\\
&= -\binom{k}{s}(1-z)^{s-1} - \frac{(1-z)^s}{s!z^{k+1-s}}\sum_{i=0}^{s-1}\frac{s!(-1)^{i+1}}{(1-z)^s}\binom{s-1}{i}T'_{k-i}(z)\\
&= -\binom{k}{s}(1-z)^{s-1} + \frac{1}{z^{k+1-s}}\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}T'_{k-i}(z).
\end{aligned}$$

By the induction hypothesis, we have

$$\frac{(1-z)^{s-1}b_s}{s!z^{k-s}} = -\binom{k}{s}(1-z)^{s-1} + \sum_{i=0}^{s-1}\sum_{\ell=0}^{i}(-1)^{\ell+i}\binom{s-1}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}z^{s-1-\ell}, \tag{35}$$

where $1 \leq s \leq j$. Note that the right side of the last expression is a polynomial in z of degree at most $s-1$. Extracting the coefficient of z^ℓ, where $0 \leq \ell \leq s-1$, from both sides of the last equation yields

$$\begin{aligned}
[z^\ell]\left(\frac{(1-z)^{s-1}b_s}{s!z^{k-s}}\right) &= (-1)^{\ell-1}\binom{k}{s}\binom{s-1}{\ell}\\
&\quad + \sum_{i=s-1-\ell}^{s-1}(-1)^{s-1-\ell+i}\binom{s-1}{i}\binom{i}{s-1-\ell}\binom{k+i-s+1+\ell}{i+1}.
\end{aligned} \tag{36}$$

Using the identity (replace ℓ with $s-1-\ell$ in (23))

$$\sum_{i=s-1-\ell}^{s-1}(-1)^{s-1-\ell+i}\binom{s-1}{i}\binom{i}{s-1-\ell}\binom{k+i-s+1+\ell}{i+1} = (-1)^\ell\binom{k}{s}\binom{s-1}{\ell}, \tag{37}$$

we obtain $\frac{(1-z)^{s-1}b_s}{s!z^{k-s}} = 0$ and hence $b_s = 0$ for all $s = 0, 1, \ldots, j$, as desired.

Thus, (34) gives

$$\lim_{u\to z}\frac{T'_{k-j}(u)}{u(1-u)^j}=\frac{b_{j+1}}{(j+1)!(-1)^{j+1}}.$$

We then have

$$\begin{aligned}T'_{k-j}(z)&=\lim_{u\to z}T'_{k-j}(u)=z(1-z)^j\frac{b_{j+1}}{(j+1)!(-1)^{j+1}}\\&=(-1)^{j+1}z^{k-j}\frac{(1-z)^j b_{j+1}}{z^{k-j-1}(j+1)!}\\&=(-1)^{j+1}z^{k-j}\left(-\binom{k}{j+1}(1-z)^j+\sum_{i=0}^{j-1}\sum_{\ell=0}^{i}(-1)^{\ell+i}\binom{j}{i}\binom{i}{\ell}\binom{k+i-\ell}{i+1}z^{j-\ell}\right).\end{aligned}$$

Combining like powers of z in the last expression, rewriting the second sum, and using (37) then gives

$$\begin{aligned}&T'_{k-j}(z)\\&=(-1)^{j+1}z^{k-j}\sum_{\ell=0}^{j}\left((-1)^{\ell-1}\binom{k}{j+1}\binom{j}{\ell}+\sum_{i=j-\ell}^{j}(-1)^{j-\ell+i}\binom{j}{i}\binom{i}{j-\ell}\binom{k+i-j+\ell}{i+1}\right)z^{\ell}\\&+(-1)^{j+1}z^{k-j}\sum_{\ell=0}^{j}(-1)^{\ell+1}\binom{j}{\ell}\binom{k+\ell}{j+1}z^{\ell}\\&=z^{k-j}\sum_{\ell=0}^{j}(-1)^{\ell}\binom{j}{\ell}\binom{k+j-\ell}{j+1}z^{j-\ell},\end{aligned}$$

which completes the induction step, as required. □

By Lemma 4.3, (29), and (31), we have

$$\begin{aligned}T_{k-b-1}(1)&=\frac{T_{k-b}(z)}{z}=\frac{x}{z}T(x;q)T'_{k-b}(z)\\&=xz^{k-1-b}T(x;q)\sum_{i=0}^{b}(-1)^i\binom{b}{i}\binom{k+b-i}{b+1}z^{b-i},\qquad 0\le b\le k-2,\end{aligned}$$

with

$$T_k(1)=xT(x;q)\frac{1-z^k}{1-z}.$$

Since $T(x;q)=1+T_k(1)+\sum_{b=0}^{k-2}T_{k-b-1}(1)$, we obtain

$$T(x;q)=\frac{1}{1-\frac{x(1-z^k)}{1-z}-xz^{k-1}\sum_{b=0}^{k-2}\sum_{i=0}^{b}(-1)^i\binom{b}{i}\binom{k+b-i}{b+1}z^{-i}},$$

which leads to the main result in this section.

Theorem 4.4. *Let $k\ge 1$. Then the generating function $T(x;q)$ is given by*

$$T(x;q)=\frac{1}{1-\frac{x(1-z^k)}{1-z}-xz^{k-1}\sum_{b=0}^{k-2}\sum_{i=0}^{b}(-1)^i\binom{b}{i}\binom{k+b-i}{b+1}z^{-i}},\tag{38}$$

where $z=1+(q-1)x$.

Setting $q = 1$ in (38) gives

$$T(x;1) = \frac{1}{1 - kx - x\sum_{b=0}^{k-2}\sum_{i=0}^{b}(-1)^i\binom{b}{i}\binom{k+b-i}{b+1}},$$

and by the identity $\sum_{i=0}^{b}(-1)^i\binom{b}{i}\binom{k+b-i}{b+1} = k$, we see that

$$T(x;1) = \frac{1}{1 - kx - x(k-1)k} = \frac{1}{1 - k^2 x},$$

as required.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps, Generating functions for generating trees, (*Formal Power Series and Algebraic Combinatorics*, Barcelona, 1999), *Discrete Math.* **246:1-3** (2002), 29–55.

[2] A. Burstein and T. Mansour, Counting occurrences of some subword patterns, *Discrete Math. Theor. Comput. Sci.* **6** (2003), 1–12.

[3] L. Carlitz, Enumeration of compositions by rises, falls and levels, *Math. Nachr.* **77** (1977), 361–371.

[4] L. Carlitz and R. Scoville, Enumeration of permutations by rises, falls, rising maxima and falling maxima, *Acta Math. Sci. Hungar.* **25** (1974), 269–277.

[5] L. Carlitz, R. Scoville, and T. Vaughan, Enumeration of pairs of permutations, *Discrete Math.* **14** (1976), 215–239.

[6] L. Carlitz and T. Vaughan, Enumeration of sequences of given specification according to rises, falls and maxima, *Discrete Math.* **8** (1974), 147–167.

[7] S. Elizalde and M. Noy, Consecutive patterns in permutations, *Adv. in Appl. Math.* **30** (2003), 110–125.

[8] J. M. Fedou and D. Rawlings, Statistics on pairs of permutations, *Discrete Math.* **143** (1995), 31–45.

[9] T. M. Langley and J. B. Remmel, Enumeration of m-tuples of permutations and a new class of power bases for the space of symmetric functions, *Adv. in Appl. Math.* **36** (2006), 30–66.

[10] T. Mansour and A. Munagi, Enumeration of partitions by levels, rises, and descents, *J. Integer Seq.* **12** (2009), Art. 09.1.8.

[11] T. Mansour, M. Shattuck, and S. H. Yan, Counting subwords in a partition of a set, *Electron. J. Combin.* **17** (2010), #R19.

[12] R. P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, *J. Combin. Theory Ser. A* **20** (1976), 336–356.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, 31905 HAIFA, ISRAEL
E-mail address: tmansour@univ.haifa.ac.il

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, 31905 HAIFA, ISRAEL
E-mail address: maarkons@excite.com