L-functions of symmetric powers of cubic exponential sums

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Abstract

For each positive integer k, we investigate the L-function attached to the k-th symmetric power of the F-crystal associated to the family of cubic exponential sums of $x^3 + \lambda x$ where λ runs over $\overline{\mathbb{F}}_p$. We explore its rationality, field of definition, degree, trivial factors, functional equation, and Newton polygon. The paper is essentially self-contained, due to the remarkable and attractive nature of Dwork's p-adic theory.

A novel feature of this paper is an extension of Dwork's effective decomposition theory when $k < p$. This allows for explicit computations in the associated p-adic cohomology. In particular, the action of Frobenius on the (primitive) cohomology spaces may be explicitly studied.

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Contents

1 Introduction

This paper represents a continuation in the study of L-functions attached to symmetric powers of families of exponential sums. Our central object of study will be the family of cubic exponential sums of $x^3 + \lambda x$. Similar studies have considered the Legendre family of elliptic curves [1] [8], and the n-variable Kloosterman family [10] [13].

Various approaches have been used to study these L-functions. Dwork [8] used the existence of a Tate-Deligne mapping (excellent lifting) of the elliptic family to study the symmetric powers; this line of investigation was continued by Adolphson [1] in his thesis. Another related method is the symmetric power of the associated F-crystal. This approach was explored by Robba [13] who used index theory to calculate the degrees of such functions coming from the family of one-variable Kloosterman sums. A third approach, developed by Fu and Wan [10], used ℓ-adic methods of Deligne and Katz to study Kloosterman sums in n-variables. Since we are interested in the p-adic properties of the zeros and poles, and we believe excellent lifting does not exist for our family when $p \equiv -1 \bmod(3)$, we take an approach similar to that of Robba's.

The L-function, denoted $M_k(T)$, attached to the k-th symmetric power of the cubic family is defined as follows. Fix a prime $p \geq 5$ and let $\zeta_p \in \mathbb{C}_p$ be a primitive p-th root of unity. Let λ be an element of $\overline{\mathbb{F}}_p$ and denote by $deg(\lambda) := [\mathbb{F}_p(\lambda) : \mathbb{F}_p]$ its degree. Define $q := p^{deg(\lambda)}$. The L-function associated to the cubic exponential sums

$$S_m(\lambda) := \sum_{x \in \mathbb{F}_{q^m}} \zeta_p^{Tr_{\mathbb{F}_{q^m}/\mathbb{F}_p}(x^3 + \lambda x)} \qquad m = 1, 2, 3, \ldots$$

is well-known to be a quadratic polynomial with coefficients in $\mathbb{Z}[\zeta_p]$:

$$L(\lambda, T) := \exp\left(\sum_{m\geq 1} S_m(\lambda)\frac{T^m}{m}\right) = (1-\pi_1(\lambda)T)(1-\pi_2(\lambda)T).$$

The L-function of the k-th symmetric power of this family is defined by:

$$M_k(T) := \prod_{\lambda\in|\mathbb{A}^1|}\prod_{i=0}^{k}(1-\pi_1(\lambda)^i\pi_2(\lambda)^{k-i}T^{deg(\lambda)})^{-1}$$

where $|\mathbb{A}^1|$ denotes the set of Zariski closed points of $\mathbb{A}^1$. The main theorem of this paper is the following.

Theorem 1.1. *Let k be a positive integer.*

1. *$M_k(T)$ is a rational function with coefficients in $\mathbb{Z}[\zeta_p]$. Further, if $p \equiv 1$ mod(3), then the coefficients of $M_k(T)$ lie in the ring of integers of the unique subfield L of $\mathbb{Q}(\zeta_p)$ with $[L:\mathbb{Q}] = 3$. However, if $p \equiv -1$ mod(3) then $M_k(T)$ has integer coefficients.*

2. *For k odd, $M_k(T)$ is a polynomial and satisfies the functional equation*

$$M_k(T) = cT^{\delta}M_k\left(p^{-(k+1)}T^{-1}\right) \tag{1.1}$$

where c is a nonzero constant and $\delta := deg\, M_k$. Furthermore, if k is odd and $k < p$, then $M_k(T)$ has degree $(k+1)/2$, and writing

$$M_k(T) = 1 + c_1T + c_2T + \cdots + c_{(k+1)/2}T^{(k+1)/2},$$

a quadratic lower bound for the Newton polygon is given by:

$$ord_p(c_m) \geq \frac{(p-1)^2}{3p^2}(m^2+(k+1)m)$$

for $m = 0, 1, \ldots, (k+1)/2$.

3. *For k even, we may factorize $M_k(T)$ as follows:*

$$M_k(T) = \begin{cases} N_k(T)\widetilde{M}_k(T) & \text{for } k \text{ even and } k < 2p \\ \frac{N_k(T)\widetilde{M}_k(T)}{Q_k(T)} & \text{for } k \text{ even and } k \geq 2p \end{cases}$$

where $\widetilde{M}_k(T)$ has degree $\leq k$ and satisfies the functional equation (1.1), and

$$\begin{aligned} N_k(T) &= (1-p^{k/2}T)^{m_k} && \text{if } p \equiv 1 \ mod(12) \\ &= (1-(-\bar{g})^{k/2}T)^{m_k} && \text{if } p \equiv 5 \ mod(12) \\ &= (1-p^{k/2}T)^{n_k}(1+p^{k/2}T)^{m_k-n_k} && \text{if } 4|k \text{ and } p \equiv 7 \ mod(12) \\ &= (1-p^{k/2}T)^{m_k-n_k}(1+p^{k/2}T)^{n_k} && \text{if } 4\nmid k \text{ and } p \equiv 7 \ mod(12) \\ &= (1-\bar{g}^{k/2}T)^{n_k}(1+\bar{g}^{k/2}T)^{m_k-n_k} && \text{if } 4|k \text{ and } p \equiv 11 \ mod(12) \\ &= (1-\bar{g}^{k/2}T)^{m_k-n_k}(1+\bar{g}^{k/2}T)^{n_k} && \text{if } 4\nmid k \text{ and } p \equiv 11 \ mod(12) \end{aligned}$$

where

$$m_k := \begin{cases} 1+\left\lfloor\frac{k}{2p}\right\rfloor & \text{if } 4|k \\ \left\lfloor\frac{k}{2p}\right\rfloor & \text{if } 4\nmid k \end{cases} \quad \text{and} \quad n_k := \begin{cases} 1+\left\lfloor\frac{k}{4p}\right\rfloor & \text{if } 4|k \\ \left\lfloor\frac{k}{4p}\right\rfloor & \text{if } 4\nmid k \end{cases}$$

and where $\bar{g}$ is the complex conjugate of the Gauss sum $g := g_2((p^2-1)/3)$ (see §4). Note, we conjecture $Q_k(T)$ always equals 1; see the conjecture below.

The lower bound for the Newton polygon is obtained using an extension of Dwork's "General Theory" (see [5]). This theory produces an explicit algorithm to compute elements in cohomology. More specifically, given an analytic element ξ on the level of the cochain ($\mathcal{M}_a^{(k)}(b', b)$ in our notation) with known growth rate, Theorem 6.3 describes the growth rate for the reduction $\bar{\xi}$ of this element in the cohomolgy H_k^1. Since we are able to easily understand the Frobenius action on the cochain level in terms of growth rates, we can translate this into growth rates in cohomology. The lower bound of the Newton polygon follows (see Theorem 8.1).

We expect the lower bound may be improved to $\frac{1}{3}(m^2+m+mk)$. This would be optimal since, for $k=3$ and $p=7$, Vasily Golyshev has computed $ord_p(c_1)=5/3$ which coincides with this predicted lower bound. This expected lower bound follows from the general philosophy that as $p\to\infty$, the Newton polygon equals the Hodge polygon (see [17, Conjecture 1.9] [20]). (Note, the $\frac{k}{3}m$ term is just an artifact leftover from the divisibility of the Frobenius matrix by $p^{k/3}$.) We believe this lower bound will also hold for k even and $k<p$.

Looking for lower bounds of this type are inspired by a reciprocity theorem of Wan's [16]. The Govêa-Mazur conjecture predicts a certain (essentially linear) upper bound related to the arithmetic variation of dimensions associated to classical modular forms of weight k on $\Gamma_1(N)\cap\Gamma_0(p)$, with $(N,p)=1$. To prove a quadratic upper bound for this conjecture, Wan [16] proves a reciprocity theorem which cleverly transforms a quadratic uniform lower bound for the Newton polygon associated to an Atkin's operator on a space of p-adic modular forms into a quadratic upper bound for the Govêa-Mazur conjecture.

At this time, we are only able to prove a lower bound for the Newton polygon when k is odd and $k<p$. The reason for the second restriction comes from certain rational numbers in the proof of our decomposition theorem (Theorem 6.3). Their denominators are p-adic units when $k<p$, but are often not units when $k\geq p$.

We believe the obstruction to proving a similar bound for k even and $k<p$ lies in the underlying variety, or motive, associated to the k-th symmetric power, being singular. Evidence for this is given in Livné [11]. In that paper, Livné relates the k-th moment of the cubic family ax^3+bx to $\mathbb{F}_p$-rational points on a hypersurface W_k in $\mathbb{P}^{k-2}$. When k is odd, W_k is nonsingular, but when k is even, W_k has ordinary double points. Livné overcomes this difficulty by embedding W_k in a non-singular family. Interestingly, Dwork proceeds in a similar manner when trying to generalize his effective decomposition theory [5, §3d], which works well for non-singular projective hypersurfaces, to singular ones [6]. We are curious whether one can adapt these methods to the current situation.

A first step in this direction may lie in a partial effective decomposition theorem (Theorem 6.12) proven when k is even and $k<p$. Even though the theorem does not give complete information about the cohomology, it does produce several non-trivial basis vectors of the cohomology and some information concerning the lower bound of the Newton polygon.

Another topic is the denominator of $M_k(T)$. As we shall see in §5 and §7.1, $M_k(T)$ is a quotient of characteristic polynomials of Frobenius matrices acting on cohomology spaces:

$$M_k(T)=\frac{N_k(T)det(1-\bar{\beta}_kT|PH_k^1)}{det(1-p\bar{\beta}_kT|H_k^0)}. \tag{1.2}$$

The polynomial $Q_k(T)$ is defined as $det(1-p\bar{\beta}_kT|H_k^0)$. For every odd k, or for every even k with $k<2p$, H_k^0 has dimension zero. We suspect this is always true:

Conjecture 1.2. $H_k^0=0$ *for all positive integers* k. *Consequently,* $M_k(T)$ *is a polynomial for all positive integers* k.

Our methods also allow us to study the L-function, denoted $M_k(d,T)$, attached to the symmetric powers of the family $x^d+\lambda x$, when d is not divisible by p. In particular, we are able to prove:

Theorem 1.3. $M_k(d,T)$ *is a rational function with coefficients in* $\mathbb{Z}[\zeta_p]$. *If* $r:=(d,p-1)$ *then the coefficients of* $M_k(d,T)$ *lie in the ring of integers of the unique subfield* L *of* $\mathbb{Q}(\zeta_p)$ *with* $[L:\mathbb{Q}]=r$. *In particular, if* $(d,p-1)=1$ *then* $M_k(d,T)$ *has integer coefficients.*

We begin the paper by defining a relative cohomology theory (the so-called d-Airy F-crystal) tailored specifically for the family x^d+ax, where a is the parameter. This will be a free module $\mathcal{M}_a(b',b)$ of rank d over a power series ring $L(b')$. It will carry an action of Frobenius $\alpha(a)$ and a connection ∂_a. As we shall see in §3.1, when we specialize the parameter a to a lifting in $\overline{\mathbb{Q}}_p$ of an element $\bar{z}\in\overline{\mathbb{F}}_p^*$, the relative cohomology reduces to a $\mathbb{C}_p$-vector space M_z, and the L-function of $x^d+\bar{z}x$ has the cohomological description as the characteristic polynomial of the Frobenius on M_z:

$$L(x^d+\bar{z}x,T)=det(1-\bar{\alpha}_{z,s}T|M_z)$$

We then prove explicitly the functional equation for this L-function via Dwork's theory. This means defining a relative dual space $\mathcal{R}'_{-\pi,a}(b',b)$ to $\mathcal{M}_{\pi,a}(b',b)$, and an isomorphism $\overline{\Theta}_{-\pi,a}$ between them which relates the Frobenius $\bar{\alpha}_\pi(a)$ to its conjugate dual operator $\bar{\alpha}^*_{-\pi}(a)$:

$$\overline{\Theta}_{-\pi,a^p}=p^{-1}\bar{\alpha}_\pi(a)\circ\overline{\Theta}_{-\pi,a}\circ\bar{\alpha}^*_{-\pi}(a).$$

Lastly, we find the exact p-adic order of the entries in the Frobenius matrix $\bar{\alpha}(a)$. This allows us to explicitly describe the Newton polygon of $L(x^d+\bar{z}x,T)$; see §3.1.

Next, for $d = 3$, the connection ∂_a on the relative cohomology produces a system of differential equations, the Airy differential system, which has only an irregular singular point at infinity. The (dual) Frobenius is an isomorphism on the local solutions of this system and so, locally, is given by an invertible constant matrix M. For local solutions near 0, M may be explicitly described in terms of Gauss sums. When the solutions are near infinity, M may be explicitly described in terms of the square roots of p or Gauss sums g; see §4.1 for details. The description of M depends on the congruence class of p modulo 12 and is the main reason for so many different forms of $N_k(T)$ in Theorem 1.1.

In §5, we present the general theory for a cohomological formula for $M_k(T)$ following the work of Robba. In short, the k-th symmetric power of the free module $\mathcal{M}_a(b', b)$ is another free module $\mathcal{M}_a^{(k)}(b', b)$ which carries a new action of Frobenius β_k which is built from the k-th symmetric power of the Frobenius $\bar{\alpha}(a)$, and a new connection which is also denoted ∂_a. This means we are able to take cohomology once again, creating the finite dimensional $\mathbb{C}_p$-vector spaces H_k^0 and H_k^1. The cohomological formula for $M_k(T)$ is equation (1.2) above.

In §7.1, we shall see that the dual space of H_k^1 splits into three parts: a constant subspace $\mathbb{C}_p^{k+1}$, a trivial subspace $\mathfrak{T}_k$, and a primitive part PH_k^{1*}. For the first two, the action of Frobenius is explicitly described. These descriptions lead to the polynomials $P_k(T)$ and $N_k(T)$ described in Theorem 1.1. The action of the Frobenius on PH_k^{1*} is more difficult to understand, however, we are able to present the theory for the functional equation of $\widetilde{M}_k(T)$. This is similar in nature to that of $L(x^d + \bar{z}x, T)$, yet different since the analogous operator to that of $\overline{\Theta}_{-\pi,a}$ has a kernel which must be dealt with.

In §6, we present an effective decomposition theory for the cohomology space H_k^1 when k is odd and $k < p$. More precisely, an explicit procedure is described which takes an element ξ of $\mathcal{M}_a^{(k)}(b)$ and produces its reduction $\bar{\xi}$ in $H_k^1 = \mathcal{M}_a^{(k)}(b)/\partial_a \mathcal{M}_a^{(k)}(b)$. As an application, we may compute a non-trivial lower bound for the entries of the Frobenius matrix $\bar{\beta}_k$ acting on PH_k^1. This produces the quadratic lower bound for the Newton polygon given in Theorem 1.1; see §8.

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2 Relative Dwork Theory

2.1 Relative Cohomology

In this section we define and study a cohomology theory specifically suited for the more general family $x^d + ax$ with $p \nmid d$. While the growth conditions on the Banach spaces $L(b'; \rho)$ and $\mathcal{K}(b', b; \rho)$ below may seem elaborate, they will allow us to obtain a detailed description of the associated cohomology, as well as, provide us with an efficient means of reduction modulo the operator D_a. As an arithmetic application, in §2.6 we are able to explicitly study the action of Frobenius on the relative cohomology space $\mathcal{H}_a(b', b)$ defined below.

Let d be a positive integer relatively prime to p. Let b and b' be two positive real numbers. We will assume throughout this section that $b \geq b' > 0$. With $\rho \in \mathbb{R}$, define the spaces

$$\begin{aligned}
L(b'; \rho) &:= \{\sum_{i=0}^{\infty} B_i a^i : ord(B_i) \geq b'(1 - 1/d)i + \rho, \forall i \geq 0\} \\
L(b') &:= \bigcup_{\rho \in \mathbb{R}} L(b'; \rho) \\
\mathcal{K}(b', b; \rho) &:= \{\sum_{i,j \geq 0} B_{ij} a^i x^j : ord(B_{ij}) \geq b'(1 - 1/d)i + bj/d + \rho, \forall i \geq 0\} \\
\mathcal{K}(b', b) &:= \bigcup_{\rho \in \mathbb{R}} \mathcal{K}(b', b; \rho) \\
\mathcal{V}(b', b; \rho) &:= \left(\bigoplus_{i=0}^{d-1} L(b'; \rho) x^i \right) \cap \mathcal{K}(b', b; \rho) \\
\mathcal{V}(b', b) &:= \bigcup_{\rho \in \mathbb{R}} \mathcal{V}(b', b; \rho).
\end{aligned}$$

Notice that $\mathcal{K}(b', b)$ is an $L(b')$-module.

Fix $\pi \in \mathbb{C}_p$ such that $\pi^{p-1} = -p$. Define D_a, an $L(b')$-module endomorphism of $\mathcal{K}(b', b)$, by

$$D_a := x\frac{\partial}{\partial x} + \pi(dx^d + ax).$$

It is useful to keep in mind that, formally,

$$D_a = e^{-\pi(x^d+ax)} \circ x\frac{\partial}{\partial x} \circ e^{\pi(x^d+ax)}. \tag{2.1}$$

Note, we need to use the word "formally" because multiplication by $e^{\pi(x^d+ax)}$ is not an endomorphism of $\mathcal{K}(b', b)$.

Using this, we may define a *relative cohomology space* as the $L(b')$-module

$$\mathcal{H}_a(b', b) := \mathcal{K}(b', b)/D_a\mathcal{K}(b', b).$$

As the following theorem demonstrates, $\mathcal{H}_a(b', b)$ is a free $L(b')$-module of rank d.

Theorem 2.1. *Let b be a positive real number such that $e := b - \frac{1}{p-1} > 0$. Then, for every $\rho \in \mathbb{R}$ we have*

$$\mathcal{K}(b', b; \rho) = \mathcal{V}(b', b; \rho) \oplus D_a\mathcal{K}(b', b; \rho + e)$$

and

$$\mathcal{K}(b', b) = \mathcal{V}(b', b) \oplus D_a\mathcal{K}(b', b)$$

Furthermore, D_a is an injective operator on $\mathcal{K}(b', b)$.

Remark 2.2. We will prove this theorem using Dwork's general method [5] which consists of the following six lemmas. Note that the last three lemmas follow automatically from the validity of the first three.

Lemma 2.3. *Define $H := \pi(dx^d + ax)$. Then for every $\rho \in \mathbb{R}$*

$$\mathcal{K}(b', b; \rho) = \mathcal{V}(b', b; \rho) + H\mathcal{K}(b', b; \rho + e).$$

Proof. Note, it is sufficient to prove this for $\rho = 0$. Let $u = \sum_{n,s\geq 0} B_{s,n}a^s x^n \in \mathcal{K}(b', b; 0)$. We may write this as

$$u = \sum_{n\geq 0} C_n x^n \quad \text{where} \quad C_n := \sum_{s\geq 0} B_{s,n}a^s.$$

For each $j \geq 0$, define

$$A_j := \sum_{i\geq 0} (-1)^i \left(\frac{a}{d}\right)^i C_{di+d+(j-i)}.$$

With $A_{-1} := 0$, a calculation shows that $u = P + HQ$ where

$$P := \sum_{j=0}^{d-1} (C_j - \frac{a}{d}A_{j-1})x^j \quad \text{and} \quad Q := \frac{1}{d\pi}\sum_{j\geq 0} A_j x^j.$$

It is not hard to show $P(a, x) \in \mathcal{K}(b', b; 0)$ and $Q(a, x) \in \mathcal{K}(b', b; e)$. □

Lemma 2.4. $H\mathcal{K}(b', b) \cap \mathcal{V}(b', b) = \{0\}$.

Proof. Suppose

$$\pi(dx^d + ax)\sum_{j\geq 0} B_j x^j = C_0 + C_1 x + \cdots + C_{d-1}x^{d-1}$$

where $\sum_{j\geq 0} B_j x^j \in \mathcal{K}(b', b)$, $B_j \in L(b')$, and the right-hand side is in $\mathcal{V}(b', b)$. For each $j \geq 0$, the coefficient of x^{d+j} on the left-hand side of the above equals zero:

$$\pi d B_j + \pi a B_{d+j-1} = 0 \quad \Longrightarrow \quad B_j = -\frac{a}{d}B_{d+j-1}.$$

Iterating this i times yields

$$B_j = \left(\frac{-a}{d}\right)^i B_{id-i+j}.$$

Thus, B_j is divisible by a^i for all $i \geq 1$, and so B_j must equal zero. □

Lemma 2.5. *If $f \in \mathcal{K}(b', b)$ and $Hf \in \mathcal{K}(b', b; \rho)$, then $f \in \mathcal{K}(b', b; \rho + e)$.*

Proof. Write $f = \sum_{j\geq 0} A_j x^j$ where $A_j := \sum_{i\geq 0} G_{ij} a^i$. Now,

$$H \sum_{j\geq 0} A_j x^j = \sum_{j\geq 0} C_j x^j \in \mathcal{K}(b', b; \rho)$$

where $C_j := \sum_{i\geq 0} B_{ij} a^i$. In particular, this means

$$\sum_{j\geq d} [dA_{j-d} + aA_{j-1}] x^j = \frac{1}{\pi} \sum_{j\geq d} C_j x^j.$$

Thus,

$$A_j = -\frac{a}{d} A_{j+d-1} + \frac{1}{d\pi} C_{j+d} \quad \text{for every } j \geq 0.$$

Iterating the A's on the right-hand side N times, we obtain the formula:

$$A_j = \frac{(-a)^N}{d^N} A_{j+N(d-1)} + \sum_{i=0}^{N-1} \frac{(-a)^i}{d^i} \frac{1}{d\pi} C_{j+d(i+1)-i}.$$

Specializing a with $ord(a) + b'(1 - \frac{1}{d}) > 0$, then finding $\delta \in \mathbb{R}$ such that $f \in \mathcal{K}(b', b; \delta)$ we have

$$ord(\frac{(-a)^N}{d^N} A_{j+N(d-1)}) \geq N[ord(a) + b(1 - \frac{1}{d})] + bj/d + \delta.$$

Since $b \geq b'$, the coefficient of N is positive. Thus, for a specialized, letting N tend to infinity we have:

$$\begin{aligned} A_j &= \sum_{i\geq 0} \left(\frac{-a}{d} \right)^i \frac{1}{d\pi} C_{j+d(i+1)-i} \\ &= \sum_{r\geq 0} G_{r,j} a^r \end{aligned}$$

where

$$G_{r,j} := \sum_{i=0}^{r} (-1)^i \frac{1}{\pi d^{i+1}} B_{r-i,j+d(i+1)-i}.$$

It follows that $f(a, x) = \sum_{j,r\geq 0} G_{r,j} a^r x^j \in \mathcal{K}(b', b; \rho + e)$. $\square$

Lemma 2.6. *For every $\rho \in \mathbb{R}$, $\mathcal{K}(b', b; \rho) = \mathcal{V}(b', b; \rho) + D_a \mathcal{K}(b', b; \rho + e)$.*

Proof. It is sufficient to prove this for $\rho = 0$. Let $f \in \mathcal{K}(b', b; 0)$. Set $f^{(0)} := f$. Then there exists a unique $\eta^{(0)} \in \mathcal{V}(b', b; 0)$ and $\xi^{(0)} \in \mathcal{K}(b', b; e)$ such that

$$f^{(0)} = \eta^{(0)} + H\xi^{(0)}.$$

Define

$$\begin{aligned} f^{(1)} :&= f^{(0)} - \eta^{(0)} - D_a \xi^{(0)} \\ &= -x \frac{\partial}{\partial x} \xi^{(0)} \in \mathcal{K}(b', b; e). \end{aligned}$$

Then, there exists a unique $\eta^{(1)} \in \mathcal{V}(b', b; e)$ and $\xi^{(1)} \in \mathcal{K}(b', b; 2e)$ such that

$$f^{(1)} = \eta^{(1)} + H\xi^{(1)}.$$

Define

$$f^{(2)} := f^{(1)} - \eta^{(1)} - D_a \xi^{(1)}.$$

Continuing h times we get:

$$f^{(h)} := f^{(h-1)} - \eta^{(h-1)} - D_a \xi^{(h-1)}.$$

Adding all these together, we obtain

$$f^{(h)} = f^{(0)} - \sum_{i=0}^{h-1} \eta^{(i)} - D_a \sum_{i=0}^{h-1} \xi^{(i)} \in \mathcal{K}(b', b; he).$$

Thus, as $h \to \infty$, $f^{(h)} \to 0$ in $\mathcal{H}(b', b)$, leaving

$$f = \sum_{i \geq 0} \eta^{(i)} + D_a \sum_{i \geq 0} \xi^{(i)} \in \mathcal{V}(b', b) + D_a \mathcal{K}(b', b).$$

□

Lemma 2.7. *If $f \in \mathcal{K}(b', b)$ and $D_a f \in \mathcal{K}(b', b; \rho)$, then $f \in \mathcal{K}(b', b; \rho + e)$.*

Proof. If $f \neq 0$, then we may choose $c \in \mathbb{R}$ such that $f \in \mathcal{K}(b', b; c)$ but $f \notin \mathcal{K}(b', b; c + e)$. By hypothesis, $D_a f \in \mathcal{K}(b', b; \rho)$, thus

$$Hf = D_a f - x \frac{\partial}{\partial x} f \in \mathcal{K}(b', b; \rho) + \mathcal{K}(b', b; c) = \mathcal{K}(b', b; l)$$

where $l := \min\{\rho, c\}$. Thus, $f \in \mathcal{K}(b', b; l + e)$ which means $l \neq c$. Hence $l = \rho$ as desired. □

Corollary 2.8. *$ker(D_a | \mathcal{K}(b', b)) = 0$.*

Proof. Suppose $D_a(f) = 0$. Since $0 \in \mathcal{K}(b', b; \rho)$ for all $\rho \in \mathbb{R}$, by the previous lemma, $f \in \mathcal{K}(b', b; \rho + e)$ for all $\rho \in \mathbb{R}$. However, the only element with this property is 0. □

Lemma 2.9. *$D_a \mathcal{K}(b', b) \cap \mathcal{V}(b', b) = \{0\}$.*

Proof. Let $f \in D_a \mathcal{K}(b', b) \cap \mathcal{V}(b', b)$ and suppose $f \neq 0$. Choose $\rho \in \mathbb{R}$ such that $f \in \mathcal{K}(b', b; \rho)$ but $f \notin \mathcal{K}(b', b; \rho + e)$. Also, let $\eta \in \mathcal{K}(b', b)$ such that $D_a \eta = f$. By Lemma 2.7, $\eta \in \mathcal{K}(b', b; \rho + e)$ and so $f - H\eta = x \frac{\partial}{\partial x} \eta \in \mathcal{K}(b', b; \rho + e)$. Thus, there exists $\zeta \in \mathcal{V}(b', b; \rho + e)$ and $w \in \mathcal{K}(b', b; \rho + 2e)$ such that

$$x \frac{\partial}{\partial x} \eta = \zeta + Hw.$$

This means $f = \zeta + H(\eta + w)$. Since $f \in \mathcal{V}(b', b)$ we must have $f = \zeta \in \mathcal{V}(b', b; \rho + e)$. But this contradicts our choice of ρ. □

Remark 2.10. Notice that in the proof of Lemma 2.3 that if u is divisible by x, then P is also divisible by x but Q need not be. From this and the proof of Lemma 2.6, if $f \in \mathcal{K}(b', b; \rho)$ is divisible by x then when we write $f = \sum_{\geq 0} \eta^{(i)} + D_a \sum_{i \geq 0} \xi^{(i)}$ in the decomposition of $\mathcal{K}(b', b; \rho) = \mathcal{V}(b', b; \rho) \oplus D_a \mathcal{K}(b', b; \rho + e)$, then $\sum \eta^{(i)}$ is divisible by x but $\sum \xi^{(i)}$ need not be. This will be important when we define the relative primitive cohomology in §2.4.

2.2 Relative Dual Cohomology

In the last section, we saw that $\{x^i\}_{i=0}^{d-1}$ is a basis of the $L(b')$-module $\mathcal{H}_a(b', b) := \mathcal{K}(b', b) / D_a \mathcal{K}(b', b)$. In this section, we will describe a dual space of $\mathcal{H}_a(b', b)$ and compute its dual basis via a nondegenerate pairing.

Let b^* be a positive real number with $b > b^*$. Define the $L(b')$-module

$$\mathcal{K}^*(b', b^*) := \left\{ \sum_{i,j=0}^{\infty} B_{ij} a^i x^{-j} \,|\, \inf_{i,j} (\text{ord}(B_{ij}) - (b'(1 - \frac{1}{d}) i - b^* j / d)) > -\infty \right\}.$$

Define the pairing $\langle \cdot, \cdot \rangle : \mathcal{K}(b', b) \times \mathcal{K}^*(b', b^*) \to L(b')$ as follows. With $g^* \in \mathcal{K}^*(b', b^*)$ and $h \in \mathcal{K}(b', b)$, thinking of the variable a as a constant, define

$$\langle h, g^* \rangle := \text{constant term w.r.t. } x \text{ in the product } hg^*.$$

For clarity, we will write this out explicitly: with $g^*(a, x) := \sum_{j \geq 0} A_j(a) x^{-j} \in \mathcal{H}^*(b', b^*)$ and $h(a, x) := \sum_{j \geq 0} B_j(a) x^j \in \mathcal{K}(b', b)$,

$$\langle h, g^* \rangle := \sum_{s=0}^{\infty} G_s a^s \quad \text{where} \quad G_s := \sum_{r=0}^{\infty} \sum_{i+j=s} A_{ir} B_{jr}.$$

This defines a *perfect* pairing (or nondegenerate pairing) in the sense that no nonzero element $g^* \in \mathcal{K}^*(b', b^*)$ exists such that $\langle h, g^* \rangle = 0$ for every $h \in \mathcal{K}(b', b)$ and *vice versa.*

Define the truncation operator $Trunc_x : \mathbb{C}_p[[a, x^{\pm 1}]] \to \mathbb{C}_p[[a, x^{-1}]]$ by linearly extending

$$Trunc_x(x^n) := \begin{cases} x^n & \text{if } n \leq 0 \\ 0 & \text{if } n > 0 \end{cases}.$$

Define the $\mathcal{K}^*(b', b^*)$-endomorphism

$$\begin{aligned} D_a^* : &= -x\frac{\partial}{\partial x} + Trunc_x[\pi(dx^d + ax)] \\ &= Trunc_x[e^{\pi(x^d+ax)} \circ -x\frac{\partial}{\partial x} \circ e^{-\pi(x^d+ax)}]. \end{aligned} \tag{2.2}$$

The operators D_a and D_a^* are dual to one another with respect of the pairing. That is, for every $g^* \in \mathcal{K}^*(b', b^*)$ and $h \in \mathcal{K}(b', b)$, we have

$$\langle h, D_a^* g^* \rangle = \langle D_a h, g^* \rangle.$$

To see this, notice that the operator $Trunc_x$ does not affect constant terms (with respect to x), and so, $Trunc_x[\pi(dx^d + ax)]$ is dual to $\pi(dx^d + ax)$. Next, that $-x\frac{\partial}{\partial x}$ is dual to $x\frac{\partial}{\partial x}$ follows from the Leibniz identity

$$x\frac{d}{dx}(hg^*) = x\frac{d}{dx}(h)g^* + hx\frac{d}{dx}(g^*)$$

since the left-hand side has no constant term.

Define the $L(b')$-module

$$\mathcal{R}_a(b', b^*) := ker(D_a^* | \mathcal{K}^*(b', b^*)).$$

We wish to show that $\mathcal{R}_a(b', b^*)$ is the dual space of $\mathcal{H}_a(b', b)$.

Theorem 2.11. *Let $b > \frac{1}{p-1} > b' > 0$, and $b \geq b^* > 0$. Then $\mathcal{R}_a(b', b^*)$ is the algebraic dual of $\mathcal{H}_a(b', b)$. Furthermore, dual to the basis $\{x^i\}_{i=0}^{d-1}$ of $\mathcal{H}_a(b', b)$ is the basis $\{g_i^*(a)\}_{i=0}^{d-1} \subset \mathcal{R}_a(b', b^*)$ which takes the explicit form: $g_0^* = 1$, and for $i = 1, 2, \ldots, d-1$*

$$g_i^*(a) = \sum_{r=i}^{i+d-2} \sum_{l=0}^{\infty} \sum_{j=0}^{\left\lfloor \frac{l-(r-i)}{d} \right\rfloor} \frac{\mathbb{Z}}{(d\pi)^{(d-1)j+(r-i)}} \left(\frac{-a}{d}\right)^{l-(r-i)-dj} x^{-((d-1)l+r)}.$$

where "$\mathbb{Z}$" indicates some determinable nonzero integer.

Proof. It is clear that $g_0^* = 1$ is the dual vector of $1 \in \mathcal{H}_a(b', b)$. For $i = 1, \ldots, d-1$, let $g_i^*(a) := x^{-i} + \sum_{j \geq d} B_j^{(i)} x^{-j}$. Let us determine these $B_j^{(i)}$.

From $D_a^*(g_i^*(a)) = 0$, the coefficients $B_j^{(i)}$ must satisfy the recurrence relation

$$B_{d+j}^{(i)} = \frac{-j}{d\pi} B_j^{(i)} - \frac{a}{d} B_{j+1}^{(i)}.$$

Thus, from the initial conditions given by requiring $\langle x^j, g_i^*(a) \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol, each $B_j^{(i)}$ may be uniquely solved by a series of the form indicated above.

Next, notice that the p-adic order of the coefficient of $a^{l-(r-i)-dj} x^{-((d-1)l+r)}$ in $g_i^*(a)$ is bounded below by $-\frac{(d-1)j+(r-i)}{p-1}$ since d is a p-adic unit. Since $b \geq \frac{1}{p-1} \geq b' > 0$, we have

$$-\frac{(d-1)j + (r-i)}{p-1} \geq b' \left(1 - \frac{1}{d}\right)(l - (r-i) - dj) - \frac{b}{d}((d-1)l + r).$$

This means $g_i^*(a) \in \mathcal{K}^*(b', b^*; 0)$.

To show $\{g_i^*(a)\}_{i=0}^{d-1}$ is a basis of $\mathcal{R}_a(b', b^*)$, we need only show that they span $\mathcal{R}_a(b', b^*)$ since it is clear that they are linearly independent. This is demonstrated as follows: Let $h^* \in \mathcal{R}_a(b', b^*)$ and define $h_i := \langle x^i, h^* \rangle$ for each $i = 0, 1, \ldots, d-1$. Next, define $\tilde{h}^* := h^* - \sum_{i=0}^{d-1} h_i g_i^*(a)$. We wish to show $\tilde{h}^* = 0$. Now, $\tilde{h}^* = \sum_{j \geq 0} \tilde{h}_j^*(a) x^{-j}$, and by construction, $\tilde{h}_i^* = 0$ for $i = 0, 1, \ldots, d-1$. Using the reduction formula $d\pi x^{n+d} = D_a(x^n) - nx^n - \pi a x^{n+1}$ and $D_a^*(\tilde{h}^*) = 0$, it is easy to see that

$$\tilde{h}_{n+d}^* = \langle x^{d+n}, \tilde{h}^* \rangle = \sum_{i=0}^{d-1} A_i \tilde{h}_i^* = 0$$

where $A_i \in L(b')$. Hence, $\tilde{h}^* = 0$ as desired. ∎

2.3 Relative Dwork Operators

In this section, we define the Dwork operator $\bar{\alpha}(a)$ on the relative cohomology $\mathcal{H}_a(b', b)$, and its dual $\bar{\alpha}^*(a)$ on $\mathcal{R}_a(b', b^*)$. Throughout this section we will fix real numbers b and b' such that $\frac{p-1}{p} \geq b > \frac{1}{p-1}$ and $\frac{b}{p} \geq b' > 0$. Fix $\pi \in \mathbb{C}_p$ such that $\pi^{p-1} = -p$.

Dwork's first splitting function on $\mathbb{F}_p$ is $\theta(t) := e^{\pi(t-t^p)} = \sum_{i=0}^{\infty} \theta_i t^i$. It is well-known [12] that $ord(\theta_i) \geq \frac{p-1}{p^2} i$ for every i. Also, $\theta_i = \frac{\pi^i}{i!}$ for each $i = 0, 1, 2, \ldots, p-1$. Next, define

$$F(a, x) := \frac{\exp \pi(x^d + ax)}{\exp \pi(x^{dp} + a^p x^p)} = \theta(x^d)\theta(ax).$$

Writing $F(a, x) = \sum_{r \geq 0} H_r(a) x^r$ with $H_r(a) := \sum_{i=0}^{\lfloor r/d \rfloor} \theta_i \theta_{r-di} a^{r-di}$, then the coefficient of $a^{r-di} x^r$ satisfies $ord(\theta_i \theta_{r-di}) \geq b'(1 - \frac{1}{d})(r - di) + \frac{(b/p)}{d} r$. Consequently, $F(a, x) \in \mathcal{K}(b', b/p)$. It follows that multiplication by $F(a, x)$ is an endomorphism of $\mathcal{K}(b', b/p)$.

From the Cartier operator defined as

$$\psi_x : \mathcal{K}(b', b/p) \to \mathcal{K}(b', b) \quad \text{takes} \quad \sum_{i=0}^{\infty} B_i(a) x^i \longmapsto \sum_{i=0}^{\infty} B_{pi}(a) x^i,$$

we may define the Dwork operator

$$\alpha(a) := \psi_x \circ F(a, x) : \mathcal{K}(b', b) \to \mathcal{K}(b', b).$$

It is useful to keep in mind the formal identity:

$$\alpha(a) = e^{-\pi(x^d + a^p x)} \circ \psi_x \circ e^{\pi(x^d + ax)}. \tag{2.3}$$

Since $px\frac{\partial}{\partial x} \circ \psi_x = \psi_x \circ x\frac{\partial}{\partial x}$, and for any $u \in \mathbb{C}_p[[x]]$, $u(x) \circ \psi_x = \psi_x \circ u(x^p)$, equations (2.1) and (2.3) demonstrate that $\alpha(a) \circ D_a = pD_{a^p} \circ \alpha(a)$. Consequently, $\alpha(a)$ induces two $L(b')$-linear maps on the relative cohomology: $\bar{\alpha}(a) : \mathcal{H}_a(b', b) \to \mathcal{H}_{a^p}(b', b)$ and $\bar{\alpha}(a) : ker(D_a | \mathcal{K}(b', b)) \to ker(D_{a^p} | \mathcal{K}(b', b))$.

Remark 2.12. Let us show $\bar{\alpha}(a)$ is an isomorphism by creating a right-inverse on the co-chain level which will reduce to an isomorphism on cohomology. Denote by $F_{-\pi}(a, x)$ the replacement of π by its conjugate $-\pi$ in $F(a, x)$. Notice that $F_{-\pi}(a, x) = F(a, x)^{-1} \in \mathcal{K}(b', b/p)$. Define the Frobenius map

$$\Phi_x : \mathbb{C}_p[[a, x^{\pm 1}]] \to \mathbb{C}_p[[a, x^{\pm p}]] \quad \text{by linearly extending} \quad x \mapsto x^p.$$

Next, define the endomorphism $\alpha(a)^{-1} := F_{-\pi}(a, x) \circ \Phi_x : \mathcal{K}(b', b) \to \mathcal{K}(b', b)$ and note

$$\alpha(a)^{-1} = e^{-\pi(x^d + ax)} \circ \Phi_x \circ e^{\pi(x^d + a^p x)}.$$

From this and (2.3), $\alpha(a) \circ \alpha(a)^{-1} = id$. Warning, $\alpha(a)^{-1}$ is *not* a left inverse of $\alpha(a)$.

Next, from (2.1) and the relation $p\Phi_x \circ x\frac{\partial}{\partial x} = x\frac{\partial}{\partial x} \circ \Phi_x$, we see that

$$p\alpha(a)^{-1} \circ D_{a^p} = D_a \circ \alpha(a)^{-1}.$$

Hence, $\alpha(a)^{-1}$ induces a mapping $\bar{\alpha}(a)^{-1} : \mathcal{H}_{a^p}(b', b) \to \mathcal{H}_a(b', b)$ which satisfies $\bar{\alpha}(a)\bar{\alpha}(a)^{-1} = id$. Since these maps are acting on free $L(b')$-modules of finite rank, $\bar{\alpha}(a)$ must be an isomorphism.

Dual Dwork Operator. Define the operator $\alpha^*(a) : \mathcal{K}^*(b', b^*) \to \mathcal{K}^*(b', b^*)$ by the following:

$$\begin{aligned} \alpha^*(a) : &= Trunc_x[F(a, x) \circ \Phi_x] \\ &= Trunc_x[e^{\pi(x^d + ax)} \circ \Phi_x \circ e^{-\pi(x^d + a^p x)}]. \end{aligned} \tag{2.4}$$

Notice that ψ_x and Φ_x are dual operators in the sense that for every $g^* \in \mathcal{K}^*(b', b^*)$ and $h \in \mathcal{K}(b', b)$, we have $\langle \psi_x h, g^* \rangle = \langle h, \Phi_x g^* \rangle$. Next, since the operator $Trunc_x$ does not kill any constant terms (with respect to x), $Trunc_x[F(a, x)]$ is dual to $F(a, x)$. Putting these together yields the duality between $\alpha^*(a)$ and $\alpha(a)$. By taking the dual of $\alpha(a) \circ D_a = pD_{a^p} \circ \alpha(a)$, we have $p\alpha^*(a) \circ D^*_{a^p} = D^*_a \circ \alpha^*(a)$. Therefore, $\alpha^*(a)$ induces a mapping $\alpha^*(a) : \mathcal{R}_{a^p}(b', b^*) \to \mathcal{R}_a(b', b^*)$. Moreover, $\bar{\alpha}(a) : \mathcal{H}_a(b', b) \to \mathcal{H}_{a^p}(b', b)$ and $\alpha^*(a) : \mathcal{R}_{a^p}(b', b^*) \to \mathcal{R}_a(b', b^*)$ are dual to one another with respect to the pairing $\langle \cdot, \cdot \rangle$.

2.4 Relative Primitive Cohomology and its Dual

Define the $L(b')$-module

$$x\mathcal{K}(b',b) := \{g \in \mathcal{K}(b',b) | g \text{ is divisible by } x\}.$$

Notice that $\alpha(a)$ and D_a are well-defined endomorphism of $x\mathcal{K}(b',b)$. Define $\mathcal{M}_a(b',b) := x\mathcal{K}(b',b)/D_a\mathcal{K}(b',b)$. From Theorem 2.1 and Remark 2.10, $\mathcal{M}_a(b',b)$ is a free $L(b')$-module with basis $\{x^i\}_{i=1}^{d-1}$.

Since we still have $pD_{a^p} \circ \alpha(a) = \alpha(a) \circ D_a$, $\alpha(a)$ induces a mapping

$$\bar{\alpha}(a) : \mathcal{M}_a(b',b) \to \mathcal{M}_{a^p}(b',b).$$

Using a pairing identical to that between $\mathcal{H}_a(b',b)$ and $\mathcal{R}_a(b',b^*)$, we see that the subspace generated by 1 in $\mathcal{R}_a(b',b^*)$ is the annihilator of $\mathcal{M}_a(b',b)$. Hence, the dual of $\mathcal{M}_a(b',b)$ is the free $L(b')$-module $\mathcal{R}'_a(b',b^*) := \mathcal{R}_a(b',b^*)/\langle 1 \rangle$ with basis $\{g_i^*(a)\}_{i=1}^{d-1}$. Dual to $\bar{\alpha}(a)$ is $\bar{\alpha}^*(a)$, induced by $\alpha^*(a)$, and this map takes $\mathcal{R}'_{a^p}(b',b^*)$ bijectively onto $\mathcal{R}'_a(b',b^*)$. Notice that $\bar{\alpha}^*(a)(1) = 1$.

2.5 Relative Functional Equation

So far, we have fixed a solution π of the equation $x^{p-1} = -p$. Since we are assuming p is odd, $-\pi$ is also a solution. Let us see how we would have proceeded had we used $-\pi$. But before we do this, we need to modify our current notation to keep things clear. Let us denote the operator D_a by $D_{\pi,a}$, the space $\mathcal{M}_a(b',b)$ by $\mathcal{M}_{\pi,a}(b',b)$, and the Dwork operator $\bar{\alpha}(a)$ by $\bar{\alpha}_\pi(a)$. Also, denote by $\mathcal{R}_{\pi,a}(b',b^*)$ and $\alpha^*_\pi(a)$ the dual space $\mathcal{R}_a(b',b^*)$ and dual Dwork operator $\alpha^*(a)$.

Now, had we used $D_{-\pi,a} := x\frac{\partial}{\partial x} - \pi(dx^d + ax)$, then $\mathcal{H}_{-\pi,a}(b',b) := \mathcal{K}(b',b)/D_{-\pi,a}\mathcal{K}(b',b)$ would still be a free $L(b')$-module with basis $\{x^i\}_{i=0}^{d-1}$. Let $\mathcal{M}_{-\pi,a}(b',b) := x\mathcal{K}(b',b)/D_{-\pi,a}\mathcal{K}(b',b)$.

Using the same pairing as that between $\mathcal{H}_{\pi,a}(b',b)$ and $\mathcal{R}_\pi(b',b^*)$, the dual of $\mathcal{H}_{-\pi,a}(b',b)$ is the space $\mathcal{R}_{-\pi,a}(b',b^*) := ker(D^*_{-\pi,a}|\mathcal{K}^*(b',b^*))$. Clearly, a basis for the latter space is $\{g^*_{-\pi,i}\}_{i=0}^{d-1}$, where we have replaced π for $-\pi$ in the formulas given in Theorem 2.11. Next, dual to $\alpha_{-\pi}(a) := \psi_x \circ F_{-\pi}(a,x)$ is $\alpha^*_{-\pi}(a) := Trunc_x \circ F_{-\pi}(a,x) \circ \Phi_x$. Notice that $\alpha^*_{-\pi}(a) : \mathcal{R}_{-\pi,a^p}(b',b^*) \to \mathcal{R}_{-\pi,a}(b',b^*)$.

Define $\Theta_{-\pi,a} := -x\frac{\partial}{\partial x} - \pi(dx^d + ax)$; it is useful to view this map as both $D^*_{-\pi,a}$ without the truncation operator, and as $-D_{\pi,a}$. By definition, $\mathcal{R}_{-\pi,a}(b',b^*)$ is the kernel of $D^*_{-\pi,a} = Trunc_x \circ \Theta_{-\pi,a}$. Thus, if $\xi^* \in \mathcal{R}_{-\pi,a}(b',b^*)$ then $\Theta_{-\pi,a}\xi^*$ will consist of only a finite number of positive powers of x. That is, $\Theta_{-\pi,a}$ defines an $L(b')$-linear map

$$\Theta_{-\pi,a} : \mathcal{R}_{-\pi,a}(b',b^*) \to xL(b')[x].$$

We view the right-hand side as a subset of $\mathcal{K}(b,b')$. Therefore, by reduction, $\Theta_{-\pi,a}$ induces the mapping

$$\overline{\Theta}_{-\pi,a} : \mathcal{R}_{-\pi,a}(b',b^*) \to \mathcal{H}_{\pi,a}(b',b).$$

Notice that $\overline{\Theta}_{-\pi,a}(1) = -D_{\pi,a}(1) = 0$ in $\mathcal{H}_{\pi,a}(b',b)$. Also, writing $g^*_{-\pi,i}(a,x) = x^i + \sum_{j \geq d} B_j^{(i)}(a)x^{-j}$ as in the proof of Theorem 2.11, we see that $\overline{\Theta}_{-\pi,a}(g^*_{-\pi,i}(a,x)) = -d\pi x^{d-i}$ for each $i = 1, 2, \ldots, d-1$ since $g^*_{-\pi,i}$ is in the kernel of $D^*_{-\pi,a}$ and so all negative powers of x vanish, including the constant term.

It follows that $\overline{\Theta}_{-\pi,a}$ induces an $L(b')$-module isomorphism

$$\overline{\Theta}_{-\pi,a} : \mathcal{R}'_{-\pi,a}(b',b^*) \to \mathcal{M}_{\pi,a}(b',b).$$

With $\xi^* \in \mathcal{R}'_{-\pi,a}(b',b)$, we have

$$e^{-\pi(x^d+ax)} \circ \Phi_x \circ e^{\pi(x^d+a^px)}(\xi^*) = \bar{\alpha}^*_{-\pi}(a)(\xi^*) + \eta$$

where $\eta \in L(b')[x]$. Applying $\Theta_{-\pi,a}$ to both sides we have

$$pe^{-\pi(x^d+ax)} \circ \Phi_x \circ e^{\pi(x^d+a^px)}\Theta_{-\pi,a^p}(\xi^*) = \Theta_{-\pi,a}\bar{\alpha}^*_{-\pi}(a)(\xi^*) + \Theta_{-\pi,a}(\eta)$$

which we may rewrite as

$$\Theta_{-\pi,a^p}(\xi^*) = p^{-1}\alpha_\pi(a) \circ \Theta_{-\pi,a} \circ \bar{\alpha}^*_{-\pi}(a)(\xi^*) - D_{\pi,a^p} \circ \alpha_\pi(a)(\eta).$$

It follows that

$$\overline{\Theta}_{-\pi,a^p} = p^{-1}\bar{\alpha}_\pi(a) \circ \overline{\Theta}_{-\pi,a} \circ \bar{\alpha}^*_{-\pi}(a). \tag{2.5}$$

This is the functional equation. Notice that it relates the conjugate dual Dwork operator $\bar{\alpha}^*_{-\pi}(a)$ to the inverse Dwork operator $\bar{\alpha}_\pi(a)^{-1}$.

2.6 Frobenius Estimates

Let $\mathfrak{A}(a) = (\mathfrak{A}_{ij})$ be the matrix of $\bar{\alpha}(a)$ with respect to the basis $\{x^i\}_{i=1}^{d-1}$. In this section, we will determine the p-adic order of $\mathfrak{A}_{ij}$ as a function of a. Fix $b := (p-1)/p$ and $b' := b/p$. Recall from §2.3 that $F(a,x) = \sum_{r\geq 0} H_r(a)x^r \in \mathcal{K}(b', b'; 0)$. Thus, $ord(H_r(a)) \geq b'r/d$ for all $ord(a) > -\left(\frac{d-1}{d}\right)b'$. For $i = 1, 2, \ldots, d-1$, since $x^i \in \mathcal{K}(b', b'; -ib'/d)$, we see that $F(a,x)x^i \in \mathcal{K}(b', b'; -ib'/d)$, and so $\alpha(a)x^i \in \mathcal{K}(b', pb'; -ib'/d)$. From Lemma 2.6,

$$\alpha(a)x^i \subset \sum_{j=1}^{d-1} \mathfrak{A}_{ij}(a)x^j + D_a\mathcal{K}(b', b)$$

for some $\mathfrak{A}_{ij} \in L(b'; \frac{b'}{d}(pj-i))$. This means

$$ord(\mathfrak{A}_{ij}(a)) \geq \frac{b'(pj-i)}{d} \quad \text{for all} \quad ord(a) > -\left(\frac{d-1}{d}\right)b'.$$

A better estimate than this is often needed for applications. For this, we offer the following exact order for $\mathfrak{A}_{ij}$.

Notation. Let $f(a)$ be an analytic function over $\mathbb{C}_p$ convergent on $ord(a) + \rho > 0$. We will write $f(a) = h(a) + o_\rho(>)$ if there is an analytic function $h(a)$ such that $ord(f(a)) = ord(h(a))$ for all $ord(a) + \rho > 0$.

Theorem 2.13. *Suppose $d \geq 2$ and $p \geq d+6$. Then there exists $\epsilon > 0$ which depends on p and d such that*

$$\mathfrak{A}_{ij}(a) = \frac{\pi^{pj-i-(d-1)r_{ij}}}{r_{ij}!(pj-i-dr_{ij})!}a^{pj-i-dr_{ij}} + o_\epsilon(>)$$

where $r_{ij} := \left\lfloor \frac{pj-i}{d} \right\rfloor$. (Note: $\epsilon \to 0^+$ as p tends to infinity.)

Remark 2.14. As a consequence, the p-adic absolute values of the entries of the Frobenius $\mathfrak{A}(a)$ are constant on the unit circle $|a| = 1$.

Proof. Recall from §2.3, we may write $F(a,x) = \sum_{r\geq 0} H_r(a)x^r$ with $H_r(a) := \sum_{i=0}^{\lfloor r/d \rfloor} \theta_i\theta_{r-di}a^{r-di} \in L(b'; b'r/d)$. In $x\mathcal{K}(b', b)$, we have

$$\alpha(a)x^i = \psi_x \circ F(a,x)x^i = \sum_{j\geq 1} H_{pj-i}x^j.$$

From Lemma 2.3, we have

$$\alpha(a)x^i = \mu_{i,1}x + \mu_{i,2}x^2 + \cdots \mu_{i,d-1}x^{d-1} + \pi(x^d + ax)Q,$$

for some Q, where

$$\mu_{i,j} = H_{pj-i} - \frac{a}{d}\sum_{r\geq 0}(-1)^r \left(\frac{a}{d}\right)^r H_{p[dr+d+j-1-r]-i}. \tag{2.6}$$

Step 1: $H_{pj-i} = \frac{\pi^{pj-i-(d-1)r_{ij}}}{r_{ij}!(pj-i-dr_{ij})!}a^{pj-i-dr_{ij}} + o_\epsilon(>)$ where $r_{ij} := \left\lfloor \frac{pj-i}{d} \right\rfloor$.

Recall that $\theta_r = \pi^r/r!$ for $r = 0, 1, \ldots, p-1$. Consequently, notice that for $\left\lceil \frac{p(j-1)-(i-1)}{d} \right\rceil \leq r \leq \left\lfloor \frac{pj-i}{d} \right\rfloor$, since $1 \leq i, j \leq d-1$ and $p > d$, we have both

$$\theta_r = \frac{\pi^r}{r!} \qquad \text{and} \qquad \theta_{pj-i-dr} = \frac{\pi^{pj-i-dr}}{(pj-i-dr)!}.$$

Motivated by this, let us split H_{pj-i} into two sums:

$$H_{pj-i}(a) = \sum_{r=0}^{\left\lceil \frac{p(j-1)-(i-1)}{d} \right\rceil - 1} \theta_r\theta_{pj-i-dr}a^{pj-i-dr} + \sum_{r=\left\lceil \frac{p(j-1)-(i-1)}{d} \right\rceil}^{\left\lfloor \frac{pj-i}{d} \right\rfloor} \frac{\pi^{pj-i-(d-1)r}}{r!(pj-i-dr)!}a^{pj-i-dr}.$$

Step 1 follows after some elementary tedious calculations.

Step 2: $\mu_{ij} = H_{pj-i} + o_\epsilon(>)$.

Now,

$$\left(\frac{a}{d}\right)^{r+1} H_{p[dr+d+j-1-r]-i} \in L(b'; (b'/d)[p(dr+d+j-1-r)-i] - b'(r+1)),$$

and so, the infinite series in (2.6) is an element in $L(b';(b'/d)(pd-pj-p-i)-b')$. Comparing this with H_{pj-i} from Step 1, after tedious calculations, Step 2 follows.

Step 3: $\mathfrak{A}_{ij}=\mu_{ij}+o_\epsilon(>)$.

In the notation of Lemma 2.6, $\mu_{ij}=\eta^{(0)}$, and so, continuing this notation

$$\mathfrak{A}_{ij}-\mu_{ij}=\sum_{r\geq 1}\eta^{(r)}$$

with $\eta^{(r)}\in L(b;(b'/d)(pj-i)+re)$ where $e:=b-\frac{1}{p-1}>0$. Thus, the series on the right-hand side is an element of $L(b';(b'/d)(pj-i)+e)$. Step 3 follows. □

3 Fibres

3.1 L-function of the Fibres: $L(x^d+\bar{z}x,T)$

By elementary Dwork theory [12], Dwork's first splitting function $\theta(t):=\exp\pi(t-t^p)$ converges on the closed unit disk $D^+(0,1)$ and $\theta(1)$ is a primitive p-th root of unity in $\mathbb{C}_p$. Let $\bar{z}$ be an element of a fixed algebraic closure of $\mathbb{F}_p$ and let $s:=[\mathbb{F}_p(\bar{z}):\mathbb{F}_p]$. Denote by z the Teichmüller representative in $\mathbb{C}_p$ of $\bar{z}$; notice that $z^{p^s-1}=1$.

Letting $f_{\bar{z}}(x):=x^d+\bar{z}x$, we may define for each $n\in\mathbb{Z}_{>0}$ the exponential sum

$$S_n^*(f_{\bar{z}}):=\sum_{\bar{x}\in\mathbb{F}^*_{p^{sn}}}\theta(1)^{Tr_{\mathbb{F}_{p^{sn}}/\mathbb{F}_p}(\bar{x}^d+\bar{z}\bar{x})}.$$

The associated L-function is

$$L^*(f_{\bar{z}},T):=\exp(\sum_{n\geq 1}S_n^*(f_{\bar{z}})\frac{T^n}{n}).$$

Dwork's splitting function $\theta(t)$ defines a p-adic analytic representation of the additive character $\theta(1)^{Tr(\cdot)}$. With

$$F(z,x):=\theta(x^d)\theta(zx),$$

by standard Dwork theory, we have (with x the Teichmüller representative of $\bar{x}$ in $\mathbb{C}_p^*$)

$$\theta(1)^{Tr_{\mathbb{F}_{p^{sn}}/\mathbb{F}_p}(\bar{x}^d+\bar{z}\bar{x})}=F(z,x)F(z^p,x^p)\cdots F(z^{p^{ns-1}},x^{p^{ns-1}}).$$

Therefore,

$$S_n^*(f_{\bar{z}})=\sum_{x\in\mathbb{C}_p,x^{p^{ns}-1}=1}F(z,x)F(z^p,x^p)\cdots F(z^{p^{ns-1}},x^{p^{ns-1}}).$$

We now return to our current situation. Let $\bar{z},z$, and s be as above. Let b and b' be real numbers such that $\frac{p-1}{p}\geq b>\frac{1}{p-1}$ and $b/p>b'>0$. Define the spaces

$$K(b)_z:=(\mathcal{K}(b',b)\text{ with the variable }a\text{ specialized at }z)$$
$$xK(b)_z:=\{h\in K(b)_z|h\text{ is divisible by }x\}.$$

From Theorem 2.1 and §2.4,

$$H_z:=K(b)_z/D_zK(b)_z\quad\text{and}\quad M_z:=xK(b)_z/D_zK(b)_z$$

are $\mathbb{C}_p$-vector space with bases $\{x^i\}_{i=0}^{d-1}$ and $\{x^i\}_{i=1}^{d-1}$, respectively. Dual to these are the spaces

$$R_z:=(\mathcal{R}_a(b',b^*)\text{ with }a\text{ specialized at }z)$$
$$R'_z:=(\mathcal{R}'_a(b',b^*)\text{ with }a\text{ specialized at }z).$$

Since, with respect to the basis $\{x^i\}_{i=0}^{d-1}$ of both $\mathcal{H}_a(b',b)$ and $\mathcal{H}_{a^p}(b',b)$, the matrix of $\bar{\alpha}(a)$ has coefficients in $L(b')$, we may specialize $\bar{\alpha}(a)$ at z. Define $\bar{\alpha}_z:=\bar{\alpha}(a)|_{a=z}$. Notice, $\bar{\alpha}_z:H_z\to H_{z^p}$ is an isomorphism from Remark 2.12.

It is well-known that α_z is a nuclear operator on the space $K(b)_z$. Thus, since $z^{p^s-1}=1$, the trace formula for nuclear operators [12, Thm 6.11] tells us that

$$(p^s-1)Tr_{nuc}(\alpha_{z,s}|K(b)_z)=p^sTr(\bar{\alpha}_{z,s}|ker(D_z|K(b)_z))-Tr(\bar{\alpha}_{z,s}|H_z)$$

where

$$\alpha_{z,s} := \alpha_{z^{p^{s-1}}} \circ \cdots \circ \alpha_{z^p} \circ \alpha_z.$$

On the other hand, Dwork's trace formula tells us that

$$(p^s-1)Tr_{nuc}(\alpha_{z,s}|K(b)_z) = \sum_{x^{p^s-1}=1} F(z^{p^{s-1}}, x^{p^{s-1}}) \cdots F(z^p, x^p)F(z,x).$$

Observe that the right-hand side is just Dwork's p-adic analytic representation of the character sum $S_1^*(f_{\bar{z}})$. That is,

$$(p^s-1)Tr_{nuc}(\alpha_{z,s}|K(b)_z) = S_1^*(f_{\bar{z}}).$$

Since $\alpha_{z,sn} = (\alpha_{z,s})^n$ we may use the trace formula to generalize this to

$$S_n^*(f_{\bar{z}}) = p^{sn}Tr(\bar{\alpha}_{z,s}^n|ker(D_z|K(b)_z)) - Tr(\bar{\alpha}_{z,s}^n|H_z).$$

Using the well-known identity $\exp(-\sum_{n=0}^{\infty} Tr(A^n)\frac{T^n}{n}) = det(1-AT)$ for finite square matrices A, we obtain

$$\begin{aligned} L^*(f_{\bar{z}}, T) &= \exp(\sum_{n\geq 1} S_n^*(f_{\bar{z}})\frac{T^n}{n}) \\ &= \frac{det(I - \bar{\alpha}_{z,s}T|H_z)}{det(I - p^s\bar{\alpha}_{z,s}T|ker(D_z|K(b)_z)}. \end{aligned}$$

From §2.4, we know that $\alpha_{z,s}^*$ has 1 as an eigenvector with eigenvalue 1. Thus,

$$det(1-\alpha_{z,s}^*T|R_z) = (1-T)det(1-\bar{\alpha}_{z,s}^*T|R_z').$$

Also, $ker(D_z|K(b)_z) = 0$ by Theorem 2.1. Therefore, since $det(1-\bar{\alpha}_{z,s}^*T|R_z') = det(1-\bar{\alpha}_{z,s}T|M_z)$,

$$\begin{aligned} L^*(f_{\bar{z}}, T) &= (1-T)det(1-\bar{\alpha}_{z,s}T|M_z) \\ &= (1-T)\prod_{i=1}^{d-1}(1-\pi_i(\bar{z})T) \in \mathbb{Z}[\zeta_p][T]. \end{aligned}$$

Equivalently, for every $n \geq 1$,

$$-S_n^*(f_{\bar{z}}) = 1 + \pi_1(\bar{z})^n + \cdots + \pi_{d-1}(\bar{z})^n.$$

Now, had we used $\bar{x} = 0$ in the definition of the character sum $S_n^*(f_{\bar{z}})$, that is,

$$S_n(f_{\bar{z}}) := \sum_{\bar{x}\in\mathbb{F}_{p^{sn}}} \theta(1)^{Tr_{\mathbb{F}_{p^{sn}}/\mathbb{F}_p}(\bar{x}^d + \bar{z}\bar{x})},$$

then since $S_n(f_{\bar{z}}) = 1 + S_n^*(f_{\bar{z}})$, we have

$$L(f_{\bar{z}}, T) := \exp(\sum_{n\geq 1} S_n(f_{\bar{z}})\frac{T^n}{n}) = det(I - \bar{\alpha}_{z,s}T|M_z) = \prod_{i=1}^{d-1}(1-\pi_i(\bar{z})T).$$

Notice that, from the relative functional equation (2.5), there is a nonzero constant c, dependent on p, d, and $deg(\bar{z})$, such that

$$cT^{d-1}\overline{L}(f_{\bar{z}}, p^{-s}T^{-1}) = L(f_{\bar{z}}, T),$$

where the bar on the left means complex conjugation. Note, when d is odd, since $-(x^d+ax) = (-x)^d + a(-x)$, the L-function has real coefficients since the exponential sums are in this case real.

Newton Polygon. Let us determine the Newton polygon of the L-function $L(f_{\bar{z}}, T)$. For simplicity, we will assume $\bar{z} \in \mathbb{F}_p^*$. Set $b := \frac{p-1}{p}$ and $b' := b/p$. Let $\xi \in \mathbb{C}_p$ such that $ord_p(\xi) = b'/d$, and let $\tilde{\xi} := \xi^{p-1}$. Consider the basis $\{\xi^i x^i\}_{i=1}^{d-1}$ for the space $\mathcal{M}_a(b', b)$. From the beginning of §2.6, we have

$$\bar{\alpha}(a)\xi^i x^i = \sum_{j=1}^{d-1}(\mathfrak{A}_{i,j}\xi^{i-j})\xi^j x^j \in \mathcal{M}_{a^p}(b', pb'; 0)$$

Thus, with respect to this basis, the matrix $\mathfrak{A}(a) = (\mathfrak{A}_{ij})$ of $\bar{\alpha}(a)$, acting on the right, may be written as

$$\mathfrak{A}(a) = \begin{pmatrix} A_{0,0} & \tilde{\xi}A_{0,1} & \cdots & \xi^{d-1}A_{0,d-1} \\ A_{1,0} & \tilde{\xi}A_{1,0} & \cdots & \xi^{d-1}A_{1,d-1} \\ \vdots & \vdots & \vdots & \vdots \\ A_{d-1,0} & \tilde{\xi}A_{d-1,1} & \cdots & \tilde{\xi}^{d-1}A_{d-1,d-1} \end{pmatrix}$$

where $\tilde{\xi}^j A_{i,j} = \mathfrak{A}_{ij}\xi^{i-j}$ and $ord(\mathfrak{A}_{i,j}) \geq 0$. Now, $det(1 - \mathfrak{A}(a)T) = \sum_{m=0}^{d-1} c_m T^m$ where

$$c_m := (-1)^m \sum_{1 \leq u_1 < u_2 < \cdots < u_m \leq d-1} \sum_{\sigma} sgn(\sigma) \left(\tilde{\xi}^{u_1} A_{\sigma(u_1),u_1}\right) \left(\tilde{\xi}^{u_2} A_{\sigma(u_2),u_2}\right) \cdots \left(\tilde{\xi}^{u_m} A_{\sigma(u_m),u_m}\right)$$

where the second sum runs over all permutations σ of the $u_1, u_2, \ldots, u_m$ and $sgn(\sigma)$ is the signature of the permutation. Our goal is to determine the precise p-adic order of these c_m when $ord_p(a) = 0$. We do this as follows. Notice that

$$\begin{aligned} c_m &\equiv (-1)^m \sum_{\sigma \in S_m} sgn(\sigma) \left(\tilde{\xi} A_{\sigma(1),1}\right) \left(\tilde{\xi}^2 A_{\sigma(2),2}\right) \cdots \left(\tilde{\xi}^m A_{\sigma(m),m}\right) \quad mod(\tilde{\xi}^{m+1}) \\ &= (-1)^m \sum_{\sigma \in S_m} sgn(\sigma) \mathfrak{A}_{\sigma(1),1} \mathfrak{A}_{\sigma(2),2} \cdots \mathfrak{A}_{\sigma(m),m} \end{aligned}$$

where S_m is the symmetric group on $\{1, 2, \ldots, m\}$. From Theorem 2.13, we have

$$\mathfrak{A}_{\sigma(1),1}\mathfrak{A}_{\sigma(2),2} \cdots \mathfrak{A}_{\sigma(m),m} = \frac{\pi^{(p-1)m(m+1)/2 - (d-1)\sum_{j=1}^m r_{\sigma(j),j}}}{\left(\prod_{j=1}^m r_{\sigma(j),j}!\right)\left(\prod_{j=1}^m [pj - \sigma(j) - dr_{\sigma(j),j}]!\right)} a^{(p-1)\frac{m(m+1)}{2} - d\sum_{j=1}^m r_{\sigma(j),j}} + o_\epsilon(>)$$

Let τ_p be the permutation on $\{1, 2, \ldots, d-1\}$ defined by multiplication by p modulo d. Define $\tilde{r}_j := (pj - \tau_p(j))/d$. Since

$$\sum_{j=1}^m r_{\sigma(j),j} = \sum_{j=1}^m \left\lfloor \frac{\tau_p(j) - \sigma(j)}{d} + \tilde{r}_j \right\rfloor,$$

as σ runs over the permutations in S_m, all of these sums with $\sigma(j) \leq \tau_p(j)$ for $j = 1, \ldots, m$ will be equal and the largest; all other σ will give a strictly smaller sum. Thus,

$$c_m = u_m \pi^{(p-1)m(m+1)/2 - (d-1)(\tilde{r}_1 + \tilde{r}_2 + \cdots + \tilde{r}_m)} a^{(p-1)m(m+1)/2 - d(\tilde{r}_1 + \cdots + \tilde{r}_m)} + o_\epsilon(>)$$

where

$$\begin{aligned} u_m &:= (-1)^m \frac{1!2! \cdots m!}{\tilde{r}_1! \cdots \tilde{r}_m! \tau_p(1)! \tau_p(2)! \cdots \tau_p(m)!} \sum_{\sigma \in S_m, \sigma \leq \tau_p} sgn(\sigma) \binom{\tau_p(1)}{\sigma(1)} \cdots \binom{\tau_p(m)}{\sigma(m)} \\ &= (-1)^{m+1} \frac{1}{\tilde{r}_1! \cdots \tilde{r}_m!} \prod_{1 \leq r < s \leq m} (\tau_p(s) - \tau_p(r)), \end{aligned}$$

where the second equality comes from the following identity:

Lemma 3.1 (J. Zhu [19]). *Fix a positive integer d. For every $\tau \in S_d$ and $1 \leq m \leq d$,*

$$\sum_{\sigma \in S_m, \sigma \leq \tau} sgn(\sigma) \binom{\tau(1)}{\sigma(1)} \cdots \binom{\tau(m)}{\sigma(m)} = -\frac{1}{1!2! \cdots m!} \left(\prod_{j=1}^m \tau(j)\right) \left(\prod_{1 \leq r < s \leq m} (\tau(s) - \tau(r))\right). \tag{3.1}$$

Proof. First, rewrite (3.1) as

$$\begin{aligned} &\frac{1}{1!2! \cdots m!} \sum_{\sigma \in S_m, \sigma \leq \tau} sgn(\sigma) \prod_{i=1}^m \tau(i)(\tau(i) - 1) \cdots (\tau(i) - \sigma(i) + 1) \\ &= \frac{1}{1!2! \cdots m!} \sum_{\sigma \in S_m} sgn(\sigma) \prod_{i=1}^m \tau(i)(\tau(i) - 1) \cdots (\tau(i) - \sigma(i) + 1) \\ &= \frac{1}{1!2! \cdots m!} \det M \end{aligned}$$

where M is the $m \times m$ matrix

$$M := \begin{pmatrix} \tau(1) & \tau(1)(\tau(1)-1) & \cdots \\ \tau(2) & \tau(2)(\tau(2)-1) & \cdots \\ & \vdots & \\ \tau(m) & \tau(m)(\tau(m)-1) & \cdots \end{pmatrix}.$$

Now,

$$det(M) = (-1)^{m+1} det \begin{pmatrix} 1 & \tau(1) & \tau(1)(\tau(1)-1) & \cdots \\ 1 & \tau(2) & \tau(2)(\tau(2)-1) & \cdots \\ \vdots & & \vdots & \\ 1 & \tau(m) & \tau(m)(\tau(m)-1) & \cdots \\ 1 & 0 & 0 & \cdots \end{pmatrix}.$$

This matrix may be transformed into a Vandermonde matrix by column operations:

$$\begin{aligned} det(M) &= (-1)^{m+1} det \begin{pmatrix} 1 & \tau(1) & \tau(1)^2 & \cdots \\ 1 & \tau(2) & \tau(2)^2 & \\ & & \vdots & \cdots \\ 1 & \tau(m) & \tau(m)^2 & \cdots \\ 1 & 0 & 0 & \cdots \end{pmatrix} \\ &= -\left(\prod_{j=1}^{m} \tau(j)\right)\left(\prod_{1 \le r < s \le m} (\tau(s) - \tau(r))\right) \end{aligned}$$

□

Since u_m is a p-adic unit, we have

$$ord_p(c_m) = \frac{m(m+1)}{2} - \left(\frac{d-1}{p-1}\right)(\tilde{r}_1 + \tilde{r}_2 + \cdots + \tilde{r}_m).$$

This proves the following theorem. (Note, the following theorem was first proven by Zhu [19] for the family $x^d + ax$ where $a \in \mathbb{Q}$ and assuming p is sufficiently large.)

Theorem 3.2. *If $p \ge d+6$, and $\bar{z} \in \mathbb{F}_p^*$ then writing*

$$L(x^d + \bar{z}x, T) = (1 - \pi_1(\bar{z})T) \cdots (1 - \pi_{d-1}(\bar{z})T)$$

we have

$$ord_p(\pi_j(\bar{z})) = j - \left(\frac{d-1}{p-1}\right)\left(\frac{pj - \tau_p(j)}{d}\right)$$

where τ_p is the permutation on $\{1, 2, \ldots, d-1\}$ defined by multiplication by p modulo d.

When $d = 3$, Theorem 3.2 says if $p \equiv 1$ mod 3 then the reciprocal roots $\pi_1(\lambda)$ and $\pi_2(\lambda)$ may be ordered such that $ord_p\pi_1(\lambda) = 1/3$ and $ord_p\pi_2(\lambda) = 2/3$ for all λ. If $p \equiv -1$ mod 3, then $ord_p\pi_1(\lambda) = \frac{p+1}{3(p-1)}$ and $ord_p\pi_2(\lambda) = \frac{2(p-2)}{3(p-1)}$. See Figure 1. This was first proven by Sperber [15].

When $p \equiv -1$ mod(d) and p is greater than approximately $2^{(d-1)/2} \cdot d$, Yang [18] has proven Theorem 3.2 by a different method. His proof is interesting in that he computes the Frobenius over the chain complex rather than passing to cohomology. The advantage of this is that the entries of the Frobenius matrix are given explicitly; the disadvantage is that the Frobenius matrix has infinitely many rows and columns. Yang's result follows from a careful diagonalization procedure of this infinite matrix.

Remark 3.3. Blache and Férard [2] have given a lower bound for the generic Newton polygon of the L-function of polynomials of degree d with $p > 3d$. More precisely, in the space of polynomials of degree d, there is a Zariski open set (the complement of an associated Hasse polynomial) of which any polynomial lying in this open set will have the Newton polygon of the associated L-function coinciding with this lower bound. For one-variable polynomials, see also [20], [3]. For higher dimensions, see [17].

Remark 3.4. For $d = 3$, when $p \equiv 1$ mod 3 one may show the existence of a Tate-Deligne mapping (excellent lifting) of the root $\pi_1(\lambda)$. We suspect no such lifting exists when $p \equiv -1$ mod 3.

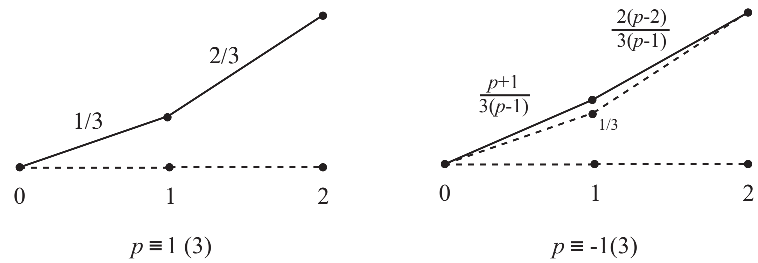

Figure 1: Newton polygon of $L(x^3 + \lambda x/\mathbb{F}_p, T)$.

3.2 (Appendix) Elementary Entire

This section is independent of the rest of the paper. Throughout this section, we will assume $\bar{z}^p = \bar{z}$; the general case is handled similarly. Notice that

$$det(1 - \alpha_z T|K(b)_z) = L^*(f_{\bar{z}}, T)det(1 - p\alpha_z T|K(b)_z).$$

Using this equation recursively gives

$$det(1 - \alpha T|K(b)_z) = \prod_{i=0}^{\infty} L^*(f_{\bar{z}}, p^i T).$$

That is, $det(1 - \alpha T|K(b)_z)$ is the infinite product of the same polynomial, each scaled by a factor of increasing p. We call any entire function with such a factorization *elementary*. We wish to show that this infinite product comes from a vector space decomposition of $K(b)_z$ into an infinite number of finite dimensional subspaces, each of the same dimension and each related to one another by the operator D_z.

From Theorem 2.1, we may write

$$K(b)_z = V_z \oplus D_z K(b)_z \tag{3.2}$$

where $V_z := \mathcal{V}(b', b)|_{a=\bar{z}}$. Notice that D_z is an endomorphism of $K(b)_z/D_z K(b)_z$, and so, denote its image in the quotient by $D_z V_z$. Since $ker(D_z|K(b)_z) = 0$, equation (3.2) becomes a recursive equation, which means we may write

$$K(b)_z = V_z \oplus D_z V_z \oplus D_z^2 K(b)_z.$$

Next, since

$$\begin{array}{ccccccc}
ker(D_z^2|K(b)_z) & \xrightarrow{\text{inj}} & K(b)_z & \xrightarrow{D_z^2} & K(b)_z & \xrightarrow{\text{surj}} & \frac{K(b)_z}{D_z^2 K(b)_z} \\
\downarrow{\scriptstyle p^2\bar{\alpha}_z} & & \downarrow{\scriptstyle p^2\alpha_z} & & \downarrow{\scriptstyle \alpha_z} & & \downarrow{\scriptstyle \bar{\alpha}_z} \\
ker(D_z^2|K(b)_z) & \xrightarrow{\text{inj}} & K(b)_z & \xrightarrow{D_z^2} & K(b)_z & \xrightarrow{\text{surj}} & \frac{K(b)_z}{D_z^2 K(b)_z}
\end{array}$$

where "inj" and "surj" indicate that the maps are either injective or surjective, we get

$$det(1 - \alpha_z T|K(b)_z) = det(1 - \bar{\alpha}_z T|K(b)_z/D_z^2 K(b)_z)det(1 - p^2\alpha_z T|K(b)_z).$$

Since $K(b)_z/D_z^2 K(b)_z$ is isomorphic to $V_z \oplus D_z V_z$,

$$det(1 - \bar{\alpha}_z T|K(b)_z/D_z^2 K(b)_z) = det(1 - \bar{\alpha}_z T|V_z)det(1 - \bar{\alpha}_z T|D_z V_z).$$

It follows that

$$det(1 - \bar{\alpha}_z T|D_z V_z) = det(1 - p\bar{\alpha}_z T|V_z) = L^*(f_{\bar{z}}, pT).$$

Of course, this generalizes to

$$det(1 - \bar{\alpha}_z T|D_z^s V_z) = det(1 - p^s\bar{\alpha}_z T|V_z) = L^*(f_{\bar{z}}, p^s T).$$

That is, the Fredholm determinant of the operator α_z on the decomposition

$$K(b)_z = V_z \oplus D_z V_z \oplus D_z^2 V_z \oplus \cdots$$

is

$$det(1 - \alpha_z T|K(b)_z) = \prod_{i=0}^{\infty} det(1 - \bar{\alpha}_z T|D_z^i V_z) = \prod_{i=0}^{\infty} L^*(f_{\bar{z}}, p^i T).$$

Notice that we have also proven

$$det(1-\alpha_z T|K(b)_z) = \lim_{s\to\infty} det(1-\bar{\alpha}_z T|K(b)_z/D_z^s K(b)_z).$$

A dual statement of this takes the form

$$det(1-\alpha_z^* T|K(b)_z^*) = \lim_{s\to\infty} det(1-\bar{\alpha}_z^* T|ker((D_z^*)^s|K(b)_z^*)).$$

4 Variation of Cohomology (Deformation Theory)

From now on we will fix $d=3$. With $\bar{z}\in\overline{\mathbb{F}}_p$ and z its Teichmüller representative in $\mathbb{C}_p^*$ define the spaces

$$\begin{aligned} K(b)_z^* &:= (\mathcal{K}^*(b',b) \text{ with } a \text{ specialized at } z)\\ R_z &:= ker(D_z^*|K(b)_z^*)\\ R_z' &:= R_z/\langle 1\rangle. \end{aligned}$$

Notice that R_z and R_z' are just $\mathcal{R}_a(b',b^*)$ and $\mathcal{R}_a'(b',b^*)$ with a specialized at z, respectively. Through use of the pairing defined in §2.2 these three spaces are algebraically dual to $K(b)_z$, H_z, and M_z, respectively.

We wish to study how the space R_z' varies as z moves around in $\mathbb{C}_p$. To do this we will define an isomorphism $T_{z,a}$ from R_z' to R_a' as follows: first, for each $a,z\in\mathbb{C}_p$ with $|a-z|<p^{-b'/3}$ define the isomorphism $T_{z,a}: K(b)_z^*\to K(b)_a^*$ by

$$T_{z,a} := Trunc_x \frac{\exp \pi(x^3+ax)}{\exp\pi(x^3+zx)} = Trunc_x \circ e^{\pi(a-z)x}.$$

Next, using (2.2),

$$D_a^*\circ T_{z,a} = T_{z,a}\circ D_z^*.$$

Also, notice that $T_{z,a}(1)=1$. Consequently, for any $a\in\mathbb{C}_p$ close enough to z the mapping $T_{z,a}$ induces an isomorphism $T_{z,a}: R_z'\to R_a'$. It is important to notice that $T_{z,z}=I$.

Deformation Theory. Using notation from Theorem 2.11, fix the bases $\{g_1^*(z), g_2^*(z)\}$ and $\{g_1^*(a), g_2^*(a)\}$ of R_z' and R_a', respectively. Observe that each $g_j^*(z)$ is simply a power series in the variable x with coefficients in $\mathbb{C}_p$. Let $C(z,a)=(c_{i,j})$ be the matrix representation of $T_{z,a}$ with respect to these bases. Note, we are thinking of R_z' as a column space, and so, $C(z,a)$ acts on the left.

The pairing $\langle\cdot,\cdot\rangle: M_z\times R_z'\to\mathbb{C}_p$ allows us to focus on individual entries of $C(z,a)$:

$$c_{ij} = \langle x^i, T_{z,a}(g_j^*(z))\rangle = \text{constant term of } x \text{ in } x^i e^{\pi(a-z)x} g_j^*(z). \tag{4.1}$$

We wish to demonstrate that $C(z,a)$ satisfies a differential equation. Differentiating (4.1) with respect to the variable a, we obtain

$$\frac{dc_{i,j}}{da} = \text{constant term of } x \text{ in } \pi x^{i+1} e^{\pi(a-z)x} g_j^*(z) = \langle \pi x^{i+1}, T_{z,a}(g_j^*(z))\rangle.$$

The pairing in this last line is between M_a and R_a'. Since M_a is a $\mathbb{C}_p$-vector space spanned by the vectors x, x^2 we see that πx^{i+1} is just a scalar multiple of a basis vector if $i=1$. If $i=2$ then we need to rewrite πx^3 in terms of the basis $\{x,x^2\}$. This is done as follows: using $D_a := x\frac{\partial}{\partial x}+\pi(3x^3+ax)$, we have $D_a(1)=3\pi x^3+\pi a x$. Hence, $\pi x^3 \equiv \frac{-\pi a}{3}x$ in M_a.

Putting this together, we have

$$\frac{d}{da}C(z,a) = B(a)C(z,a) \quad \text{and} \quad C(z,z)=I \tag{4.2}$$

where B is the 2×2 matrix

$$B(a) := \begin{pmatrix} 0 & \pi \\ \frac{-\pi a}{3} & 0 \end{pmatrix}$$

In particular, if $(C_1,C_2)^t$ is a solution of this differential equation, then C_1 satisfies the Airy equation $y''+\frac{\pi^2 a}{3}y=0$ and $C_2=\pi^{-1}C_1'$.

Connection with Frobenius. From (2.4),

$$T_{z,a}\circ\bar{\alpha}_z^* = \bar{\alpha}_a^*\circ T_{z^p,a^p}. \tag{4.3}$$

Let us use this and the deformation equation (4.2) to compute $det(\bar{\alpha}_a^*)$.

Fix $|z| < p^{2b'/3}$ and let $a \in \mathbb{C}_p$ be such that $|a - z| < p^{-b'/3}$. Define $w_z(a) := det(T_{z,a})$. Notice that $w_z(a)$ is nonzero on $D^-(z, p^{-b'/3})$ since $T_{z,a}$ is invertible. From (4.3) we may write

$$det(\bar{\alpha}_a^*) = w_z(a) det(\bar{\alpha}_z^*) w_{z^p}(a^p)^{-1}. \tag{4.4}$$

Now, a standard fact from differential equations states that

$$\frac{dw_z(a)}{da} = Tr(B(a)) w_z(a).$$

Since $Tr(B(a)) = 0$, $w_z(a)$ is a locally constant function centered at z.

In particular, if we let $z = 0$ then $w_0(a)$ is a constant function on $|a| < p^{-b'/3}$. Since $w_0(0) = 1$, we must have $w_0(a) = 1$ for all $|a| < p^{-b'/3}$. This means $det(\bar{\alpha}_a^*) = det(\bar{\alpha}_0^*)$ for all $|a| < p^{-b'/3}$. However, from §2.3, $\bar{\alpha}(a)^*$ is an $L(b')$-module isomorphism, and so, $det(\bar{\alpha}(a)) \in L(b')^*$. Thus, the domain of the equality of (4.4) may be extended to that of $L(b')$:

$$det(\bar{\alpha}_a^*) = det(\bar{\alpha}_0^*) \qquad \text{for all } |a| < p^{2b'/3}.$$

In other words, $det(\bar{\alpha}(a)^*)$ is constant for all a in this domain.

Using either the classical theory of Gauss sums or Dwork theory [7], it is not hard to calculate the value of $det(\bar{\alpha}_0^*)$. However, to describe it, we must first recall the definition of Gauss sum. Define Dwork's first splitting function $\theta_s(t) := \exp \pi(t - t^{p^s})$ on the field $\mathbb{F}_{p^s}$. Next, for each $j \in \mathbb{Z}$ and $s \in \mathbb{Z}_{>0}$ define the Gauss sum

$$g_s(j) := - \sum_{t \in \mathbb{C}_p, t^{p^s - 1} = 1} t^{-j} \theta_s(t)$$

Let $q := p^s$. Then

$$det(\bar{\alpha}_{a,s}^*) = \begin{cases} q & \text{if } q \equiv 1 \bmod(3) \\ -g_2(\frac{q^2-1}{3}) & \text{if } q \equiv -1 \bmod(3) \end{cases}.$$

Frobenius Action on Solutions of the Differential Equation. Let $\mathfrak{M}_z$ denote the field of meromorphic functions (in the variable a) near z. Define $\partial_a^* := Trunc_a(-a\frac{d}{da} + \pi a x)$. Notice that we have the commutative diagram

$$\begin{array}{ccc} R'_z \otimes \mathfrak{M}_z & \xrightarrow{T_{z,a}} & R'_a \otimes \mathfrak{M}_z \\ {\scriptstyle -a\frac{d}{da}}\downarrow & & \downarrow{\scriptstyle \partial_a^*} \\ R'_z \otimes \mathfrak{M}_z & \xrightarrow{T_{z,a}} & R'_a \otimes \mathfrak{M}_z. \end{array}$$

As a consequence we have the following important observation: $g^*(a) := C_1(a)g_1^*(a) + C_2(a)g_2^*(a) \in R'_a \otimes \mathfrak{M}_z$ is the image of an element in $R'_z \otimes \mathfrak{M}_z$ by $T_{z,a}$ which is independent of the variable a if and only if $\partial_a^*(g^*) = 0$. Notice that this last equation means the vector $\binom{C_1}{C_2}$ satisfies (4.2).

We wish to define a Frobenius-type endomorphism on the vector space of local solution at z as follows. Let $\binom{C_1}{C_2}$ be a local analytic solution of (4.2) about z. Thus, C_1 and C_2 are power series in $(a - z)$. From the previous paragraph, there exists $\xi_z^* \in R'_z \otimes \mathfrak{M}_z$, independent of a, such that $T_{z,a}(\xi_z^*) = C_1(a)g_1^*(a) + C_2(a)g_2^*(a)$. Now, for some constants b_1 and b_2, $\xi_z^* = b_1 g_1^*(z) + b_2 g_2^*(z)$. It follows that, if we let ϕ denote the operation of replacing $(a - z)$ by $(a^p - z^p)$, then we have

$$T_{z^p,a^p}(\xi_{z^p}^*) = C_1^\phi g_1^*(a^p) + C_2^\phi g_2^*(a^p)$$

with $\xi_{z^p}^* = b_1 g_1^*(z^p) + b_2 g_2^*(z^p)$. Next, apply $\bar{\alpha}_a^*$ to both sides:

$$\bar{\alpha}_a^*(C_1^\phi g_1^*(a^p) + C_2^\phi g_2^*(a^p)) = \bar{\alpha}_a^* \circ T_{z^p,a^p}(\xi_{z^p}^*) = T_{z,a} \circ \bar{\alpha}_z^*(\xi_{z^p}^*).$$

This equation demonstrates that the left-hand side comes from an element of $R'_z \otimes \mathfrak{M}_z$ independent of a, and so, it satisfies the differential equation. In particular, if $\mathfrak{Y}$ is a (local) fundamental solution matrix of (4.2) near z, then there is a constant matrix M such that

$$\mathfrak{A}_a \mathfrak{Y}^\phi = \mathfrak{Y} M$$

where $\mathfrak{A}_a$ is the matrix of $\bar{\alpha}_a^*$. Note, the matrix M depends on z.

The determination of M is an interesting topic. For $z = 0$, $M = \mathfrak{A}_0$ which is easily calculated via Gauss sums. In the next section, we will determine M when $1 < |z| < p^{2b'/3}$.

4.1 Behavior near Infinity

In this section, we wish to determine the matrix M coming from the Frobenius action on the set of local solutions of (4.2) at z near infinity. Once this is done, we will study the solutions of the k-th symmetric power of the scalar equation of (4.2).

With $\pi \in \mathbb{C}_p$ such that $\pi^{p-1} = -p$, define the p-adic Airy equation

$$y'' + \frac{\pi^2 a}{3} y = 0. \tag{4.5}$$

Observe that this equation is regular everywhere except at infinity where it has an irregular singular point. Near infinity, the asymptotic expansions of (4.5) will be power series in the ramified variable $\sqrt{a}$, so, we are naturally led to a change of variables $a \mapsto a^2$. This changes (4.5) into

$$a\tilde{y}'' - \tilde{y}' + \frac{4\pi a^5}{3}\tilde{y} = 0. \tag{4.6}$$

For convenience, let us move infinity to zero by the change of variable $a \mapsto 1/a$ so that (4.6) becomes

$$a^8 y'' + 3a^7 y' + \frac{4\pi^2}{3} y = 0.$$

To remove the irregular singular point at the origin, we consider solutions of this differential equation of the form

$$y(a) = a^{1/2} e^{\kappa\pi a^{-3}} v(a) \quad \text{where} \quad \kappa := \frac{2i}{3\sqrt{3}}.$$

and i is a fixed square root of -1. This means $v(a)$ must satisfy the differential equation

$$v'' + (4a^{-1} - 6\kappa\pi a^{-4})v' + (5/4)a^{-2}v = 0.$$

Since this equation has only a regular singular point at the origin, we may explicitly solve it using the method of Frobenius. It follows that a local solution of (4.6) about infinity is of the form

$$y_1(a) := a^{-1/2}\exp(\kappa\pi a^3)v(1/a)$$

where

$$v(a) := \sum_{n=0}^{\infty} \frac{\left(\frac{7}{6}\right)_n \left(\frac{17}{12}\right)_n}{2^n \kappa^n \pi^n (n+1)!} a^{3n}.$$

Note, $(c)_n := c(c+1)(c+2)\cdots(c+n-1)$. Replacing i with $-i$ obtains another local solution linearly independent over $\mathbb{C}_p$ from $y_1(a)$:

$$y_2(a) := a^{-1/2}\exp(-\kappa\pi a^3)\bar{v}(1/a)$$

where $\bar{v}(a)$ is the series defined by replacing i with $-i$ in the coefficients of $v(a)$. Using [9, p.243], it is not hard to show that $v(a)$ and $\bar{v}(a)$ converge on the open unit disk $D^-(0,1)$.

Remark 4.1. Notice that we can center the solutions about any $|z| > 1$ by choosing a branch of $a^{-1/2}$ and by using $\exp(\kappa\pi(a-z)^3$. In this case, v and $\bar{v}$ do not change.

With $a \mapsto a^2$, the deformation equation (4.2) becomes

$$\frac{d}{da}C(a) = \begin{pmatrix} 0 & 2\pi a \\ -\frac{2\pi a^3}{3} & 0 \end{pmatrix} C(a).$$

Thus, if the vector $\binom{C_1}{C_2}$ satisfies this, then C_1 satisfies (4.6).

For $|z| > 1$, the (local) fundamental solution matrix takes the form

$$\mathfrak{Y}(a) = a^{-1/2}V(a)S(a)$$

where $v_1(a) := v(1/a)$ and $v_2(a) := \bar{v}(1/a)$, and

$$V(a) := \begin{pmatrix} v_1(a) & v_2(a) \\ (-\frac{1}{2}a^{-2} + 3a^2\kappa\pi)v_1 + a^{-1}v_1'(a) & (-\frac{1}{2}a^{-2} - 3a^2\kappa\pi)v_2 + a^{-1}v_2'(a) \end{pmatrix}$$

and

$$S(a) := \begin{pmatrix} e^{\kappa\pi a^3} & 0 \\ 0 & e^{-\kappa\pi a^3} \end{pmatrix}.$$

From the previous section, there is a constant matrix $M := \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ such that $\mathfrak{A}_{a^2}\mathfrak{Y}^\phi = \mathfrak{Y}M$. That is,

$$\mathfrak{A}_{a^2} a^{-p/2} V(a^p)S(a^p) = a^{-1/2}V(a)S(a)M. \tag{4.7}$$

Let us view this equation over the ring $\widehat{R}$ of analytic functions on $1 < |a| < e$, $e > 1$ unspecified, adjoined the algebraic (over $\widehat{R}$) functions $\sqrt{a}$, $e^{\kappa\pi a^3}$, $e^{-\kappa\pi a^3}$, $e^{\kappa\pi a^{3p}}$, $e^{-\kappa\pi a^{3p}}$ satisfying the compatibility relations: $e^{\kappa\pi a^3}e^{-\kappa\pi a^3} = 1$, $e^{\kappa\pi a^{3p}}e^{-\kappa\pi a^{3p}} = 1$, $e^{\kappa\pi a^3}e^{-\kappa\pi a^{3p}} = \theta(\kappa a^3)e^{\pi(\kappa^p-\kappa)a^{3p}}$, and $e^{\kappa\pi a^3}e^{\kappa\pi a^{3p}} = \theta(\kappa a^3)e^{\pi(\kappa^p+\kappa)a^{3p}}$, where θ is Dwork's first splitting function on $\mathbb{F}_p$.

Using the compatibility relations, we may write

$$V(a)^{-1}\mathfrak{A}^*_{a^2}V(a^p)a^{-(p-1)/2} = \begin{pmatrix} m_1\theta(\kappa a^3)e^{\pi(\kappa^p-\kappa)a^{3p}} & m_2\theta(\kappa a^3)e^{\pi(\kappa^p+\kappa)a^{3p}} \\ m_2\theta(-\kappa a^3)e^{-\pi(\kappa^p+\kappa)a^{3p}} & m_4\theta(-\kappa a^3)e^{-\pi(\kappa^p-\kappa)a^{3p}} \end{pmatrix}.$$

Notice that the left-hand side belongs to $\widehat{R}$, thus the ride-hand side must be as well. It follows from Lemma 4.3 below that:

Theorem 4.2. *If $p \equiv 1, 7\ mod(12)$, then*

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_4 \end{pmatrix},$$

else if $p \equiv 5, 11\ mod(12)$, then

$$M = \begin{pmatrix} 0 & m_2 \\ m_3 & 0 \end{pmatrix}.$$

Lemma 4.3. *For $p \equiv 1, 7\ mod(12)$, we have $ord_p(\kappa^p - \kappa) > 0$ and $ord_p(\kappa^p + \kappa) = 0$. Otherwise, for $p \equiv 5, 11\ mod(12)$, we have $ord_p(\kappa^p - \kappa) = 0$ and $ord_p(\kappa^p + \kappa) > 0$*

Proof. Recall, $\kappa := \frac{2i}{3\sqrt{3}}$. Thus,

$$\kappa^p - \kappa = \begin{cases} i\left[\left(\frac{2}{3\sqrt{3}}\right)^p - \left(\frac{2}{3\sqrt{3}}\right)\right] & \text{if } p \equiv 1 \text{ mod}(4) \\ -i\left[\left(\frac{2}{3\sqrt{3}}\right)^p + \left(\frac{2}{3\sqrt{3}}\right)\right] & \text{if } p \equiv 3 \text{ mod}(4). \end{cases}$$

Let us concentrate on $p \equiv 1$ mod(4). By Fermat's little theorem, $2^p = 2 + p\mathcal{O}_{\mathbb{C}_p}$ and $3^p = 3 + p\mathcal{O}_{\mathbb{C}_p}$, where $\mathcal{O}_{\mathbb{C}_p} = \{c \in \mathbb{C}_p : |c| \leq 1\}$. Consequently, we have

$$\kappa^p - \kappa = \frac{6\sqrt{3}i}{(3\sqrt{3})^p}(1 - 3^{(p-1)/2}) + p\mathcal{O}_{\mathbb{C}_p}.$$

Thus, $\kappa^p - \kappa$ has positive p-adic order if and only if $3^{(p-1)/2} \equiv 1$ mod(p). By Euler's criterion, $3^{(p-1)/2} \equiv \left(\frac{3}{p}\right)$ mod(p), where the right-hand side is the Legendre symbol. Since $\left(\frac{p}{3}\right) = 1$ if and only if $p \equiv 1$ mod(3), the cases $p \equiv 1, 5$ mod(12) for $\kappa^p - \kappa$ follow from quadratic reciprocity. The other cases are similar. ☐

Notice that $v_1(-a) = v_2(a)$ and so $v_1(-a)' = -v_2(a)$. Thus, if $T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = T^{-1}$, then $V(-a)T = V(a)$. Next, since the change $a \mapsto -a$ does not affect $\mathfrak{A}_{a^2}$, changing $a \mapsto -a$ in (4.7) gives

$$V(a)^{-1}\mathfrak{A}_{a^2}V(a^p)(-1)^{(p-1)/2}a^{-(p-1)/2} = S(a)TMTS(a^p).$$

Thus,

$$(-1)^{(p-1)/2}TM = MT$$

which demonstrates that if $p \equiv 1$ mod(12), then $m_1 = m_4$, else if $p \equiv 7$ mod(12), then $-m_1 = m_4$. Similarly, if $p \equiv 5$ mod(12), then $m_2 = m_3$, else if $p \equiv 11$ mod(12) then $-m_2 = m_3$.

Next, since $det(\mathfrak{Y})$ is locally constant, from $\mathfrak{A}_{a^2}\mathfrak{Y}^\phi = \mathfrak{Y}M$ we see that $det(M) = det(\mathfrak{A}_{a^2})$, and this equals p if $p \equiv 1$ mod(3) and $g := -g_2((p^2-1)/3)$ if $p \equiv -1$ mod(3). Putting everything together, we have:

$$M = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \pm\sqrt{p} \end{pmatrix} \quad \text{if } p \equiv 1, 7 \text{ mod}(12)$$

(of course, $\sqrt{p}$ may not be the positive square root) and

$$M = \begin{pmatrix} 0 & \sqrt{-g} \\ \pm\sqrt{-g} & 0 \end{pmatrix} \quad \text{if } p \equiv 5, 11 \bmod(12)$$

where "$\pm$" means positive if $p \equiv 1,5 \mod(12)$ and negative if $p \equiv 7,11 \mod(12)$.

Symmetric powers at infinity. As demonstrated above, $\{y_1, y_2\}$ forms a formal basis for the solution space (around a point near infinity) of the differential operator $\tilde{l} := \frac{d^2}{da^2} - (1/a)\frac{d}{da} + 4\pi a^4/3$. We define the k-th symmetric power of $\tilde{l}$, denoted by $\tilde{l}_k$, as the unique monic differential equation of order $k+1$ that annihilates every

$$z_j := y_1^{k-j} y_2^j = a^{-k/2} e^{(k-2j)\kappa\pi a^3} v_1(a)^{k-j} v_2(a)^j$$

for $j = 0, 1, \ldots, k$.

Define

$$\widehat{R} := \text{ analytic functions on } 1 < |a| < e, \; e \text{ unspecified.}$$

We are interested in studying the solutions of $\tilde{l}_k$ over the ring $\widehat{R}$. Let $Y(a) \in \widehat{R}$ be a solution of $\tilde{l}_k$ which converges on $1 < |a| < e$. Fix $1 < |z| < e$. Since $\{z_j\}_{j=0}^k$ is a formal basis of $\tilde{l}_k$, if Y is a solution of $\tilde{l}_k$ over $\widehat{R}$, then locally (in $D^-(z,1)$) using Remark 4.1 we have the relation

$$Y(a) = \sum_{j=0}^{k} c_j a^{-k/2} v^j(1/a) \bar{v}^{k-j}(1/a) [e^{\kappa\pi(a-z)^3}]^{2j-k}$$

for some constants $c_j \in \mathbb{C}_p$. Let us rewrite this as

$$a^{k/2} Y(a) = \sum_{r=0}^{p-1} S_r(a) [e^{\kappa\pi(a-z)^3}]^r \tag{4.8}$$

where for each $r = 0, 1, \ldots, p-1$,

$$S_r(a) := \sum_{j \in E_r} c_j v^j(1/a) \bar{v}^{k-j}(1/a) [e^{p\kappa\pi(a-z)^3}]^{\frac{(2j-k)-r}{p}}$$

with

$$E_r := \{j \in \mathbb{Z} \cap [0,k] : (2j-k) \equiv r \bmod(p)\}.$$

Since $v(1/a)$ and $\bar{v}(1/a)$ converge on $D^+(z,1) \subset D^-(\infty,1)$, and $e^{\pm p\kappa\pi(a-z)^3}$ and $Y(a)$ converge on $D^+(z,1)$, equation (4.8) shows us that the functions $\exp(r\kappa\pi(a-z)^3)$, for $r = 0, 1, \ldots, p-1$, are linearly dependent over $\mathfrak{M}^+(z,1)[\sqrt{a}]$, where $\mathfrak{M}^+(z,1)$ is the field of meromorphic functions on the closed unit disc around z. However, Lemma 4.4 below shows us that these functions are actually linearly independent over $\mathfrak{M}^+(z,1)[\sqrt{a}]$. Consequently, we must have $S_r = 0$ for each $r \geq 1$, and so,

$$a^{k/2} Y(a) = S_0(a) = \sum_{(2j-k) \equiv 0 \bmod(p)} c_j v^j(1/a) \bar{v}^{k-j}(1/a) \exp[(2j-k)\kappa\pi(a-z)^3].$$

Notice that the right-hand side now converges on $D^+(z,1)$. Thus, if k is odd, since $Y(a)$ converges on $D^+(z,1)$, we see that $a^{k/2} \in \mathfrak{M}^+(z,1)$. However, by Lemma 4.5 below, this cannot be. Therefore, for k odd we must have $Y = 0$.

Thus, for k even $\{z_j : 0 \leq j \leq k, p|(k-2j)\}$ is a basis for $ker(\tilde{l}_k|\widehat{R})$, else if k is odd then $ker(\tilde{l}_k|\widehat{R}) = 0$.

Symmetric powers of the Airy operator. (cf. [13, §7]) Define the Airy differential operator $l := \frac{d^2}{da^2} + \frac{\pi^2 a}{3}$ and denote by l_k its k-th symmetric power. We wish to study $ker(l_k|\widehat{R})$.

For k odd, $ker(l_k|\widehat{R}) = 0$ since any solution $Y \in \widehat{R}$ of l_k produces a solution $Y(a^2)$ to $\tilde{l}_k$. Assume $2|k$. Notice that $v_1(-a) = v_2(a)$. Consequently, for $0 \leq j \leq k/2$ and $p|(k-2j)$,

$$\widetilde{w}_j^+(a) := e^{(k-2j)\kappa\pi a^3} v_1(a)^{k-j} v_2(a)^j + e^{-(k-2j)\kappa\pi a^3} v_1(a)^j v_2(a)^{k-j}$$

is an even function while

$$\widetilde{w}_j^-(a) := e^{(k-2j)\kappa\pi a^3} v_1(a)^{k-j} v_2(a)^j - e^{-(k-2j)\kappa\pi a^3} v_1(a)^j v_2(a)^{k-j}$$

is an odd function. Clearly, $\{a^{-k/2}\widetilde{w}_j^+, a^{-k/2}\widetilde{w}_j^-|0 \leq j \leq k/2, p|(k-2j)\}$ is a basis for $ker(\tilde{l}_k|\widehat{R})$.

From this, if $4|k$ then $\{a^{-k/2}\widetilde{w}_j^+(a)|0 \leq j \leq k/2, p|(k-2j)\}$ is a set of even functions whereas $\{a^{-k/2}\widetilde{w}_j^-(a)|0 \leq j \leq k/2, p|(k-2j)\}$ are odd. Thus $ker(l_k|\widehat{R}) = span\{w_j^+(a)|0 \leq j \leq k/2, p|(k-2j)\}$ where w_j^+ is defined by $w_j^+(a^2) = a^{-k/2}\widetilde{w}_j^+(a)$. Similarly, if $2|k$ but $4 \nmid k$, then $ker(l_k|\widehat{R}) = span\{w_j^-(a)|0 \leq j < k/2, p|(k-2j)\}$, where w_j^- is defined by $w_j^-(a^2) = a^{-k/2}\widetilde{w}_j^-(a)$. Thus,

$$dim_{\mathbb{C}_p} ker(l_k|\widehat{R}) = \begin{cases} 1 + \left\lfloor \frac{k}{2p} \right\rfloor & \text{if } 4|k \\ \left\lfloor \frac{k}{2p} \right\rfloor & \text{if } 2|k \text{ but } 4 \nmid k \\ 0 & \text{if } k \text{ odd.} \end{cases}$$

Appendix.

Lemma 4.4. *Let $\mathfrak{M}^+(0,1)$ denote the field of meromorphic functions on $D^+(0,1)$ in the variable a. Then the polynomial $X^p - e^{p\pi a} \in \mathfrak{M}^+(0,1)[X]$ is irreducible over $\mathfrak{M}^+(0,1)[\sqrt{a}]$.*

Proof. It is clear that $X^p - e^{p\pi a}$ is either irreducible or completely reducible over $\mathfrak{M}^+(0,1)[\sqrt{a}]$. If the latter, then we would have

$$e^{\pi a} = H_1(a) + \sqrt{a} H_2(a)$$

for some $H_1, H_2 \in \mathfrak{M}^+(0,1)$. This implies that $e^{\pi a}$ satisfies a quadratic polynomial whose coefficients lie in $\mathcal{M}^+(0,1)$. Thus, $X^p - e^{p\pi a}$ is completely reducible over $\mathfrak{M}^+(0,1)$. In particular, $e^{\pi a} \in \mathfrak{M}^+(0,1)$. However, this is not true by the following lemma. ☐

Lemma 4.5.

(a) $\exp(\pi a) \notin \mathfrak{M}^+(0,1)$.

(b) Let $a^{1/2}$ be a branch of square root centered around $z \in \mathbb{C}_p^*$. Then $a^{1/2} \notin \mathfrak{M}^+(z,1)$.

Proof. We know that $\exp(\pi a)$ converges on $D^-(0,1)$ and is divergent on the boundary $|a| = 1$. Suppose there exists analytic functions $f(a)$ and $g(a)$ on $D^+(0,1)$ such that we have the equality $e^{\pi a} = f(a)/g(a)$ whenever $|a| < 1$. Raising this to the p-th power gives the equality $e^{p\pi a} = f(a)^p/g(a)^p$ for all $|a| \leq 1$. Since $\exp(p\pi a)$ has no poles, $g(a)$ must be invertible on $D^+(0,1)$. Let $h(a) := 1/g(a)$ and note this converges on $D^+(0,1)$. Then $\exp(\pi a) = f(a)h(a)$ which shows we may extend $\exp(\pi a)$ to an analytic function on $D^+(0,1)$. Contradiction. This proves (a). A similar argument proves (b). ☐

5 Cohomological Formula for $M_k(T)$

Define

$$M_k^*(T) := \prod_{\lambda \in |\mathbb{G}_m|} \prod_{i=0}^{k} (1 - \pi_1(\lambda)^i \pi_2(\lambda)^{k-i} T^{deg(\lambda)})^{-1}$$

where $|\mathbb{G}_m|$ denotes the set of Zariski closed points of $\mathbb{G}_m := \mathbb{P}^1_{\mathbb{F}_p} \setminus \{0, \infty\}$. In this section, we will define a Frobenius operator $\bar{\beta}_k$ on the (symmetric power) cohomology spaces H_k^0 and H_k^1 such that

$$M_k^*(T) = \frac{det(1 - \bar{\beta}_k T | H_k^1)}{det(1 - p\bar{\beta}_k T | H_k^0)}.$$

From this, we will deduce that $M_k(T)$ (and $M_k^*(T)$) is a rational function. We will also find its field of definition.

The relationship between $M_k(T)$ and $M_k^*(T)$ is as follows. If we write $M_k(T) = \exp \sum_{s \geq 1} N_s \frac{T^s}{s}$, then taking the logarithmic derivative of the definition of $M_k(T)$ shows

$$N_s = \sum_{\bar{a} \in \mathbb{F}_{p^s}} \sum_{j=0}^{k} deg(\bar{a}) [\pi_1(\bar{a})^{k-j} \pi_2(\bar{a})^j]^{\frac{s}{deg(\bar{a})}}.$$

Using the same procedure, we may calculate N_s^*, where $M_k^*(T) = \exp \sum_{s \geq 1} N_s^* \frac{T^s}{s}$. It follows that

$$N_s = \sum_{j=0}^{k} (\pi_1(0)^{k-j} \pi_2(0)^j)^s + N_s^*. \tag{5.1}$$

Differential Operator. Define the operator $\partial_a := a\frac{d}{da} + \pi a x$ on $\mathcal{K}(b', b)$, and notice that, formally

$$\partial_a = e^{-\pi(x^3+ax)} \circ a\frac{d}{da} \circ e^{\pi(x^3+ax)}.$$

It follows that ∂_a and D_a commute, and so ∂_a defines an operator on both $\mathcal{H}_a(b', b)$ and $\mathcal{M}_a(b', b)$.

Denote by $\mathcal{M}_a^{(k)}(b', b)$ the k-th symmetric power of $\mathcal{M}_a(b', b)$ over $L(b')$. It is easy to see that ∂_a induces an endomorphism on $\mathcal{M}_a^{(k)}(b', b)$ by extending linearly

$$\partial_a(u_1 \cdots u_k) := \sum_{i=1}^{k} u_1 \cdots \hat{u}_i \cdots u_k \partial_a(u_i)$$

where $\hat{u}_i$ means we are leaving it out of the product.

Matrix representation of ∂_a. With the $L(b')$-basis $\{v, w\}$ of $\mathcal{M}_a(b', b)$, where $v := x$ and $w := x^2$, fix the ordered basis $(v^k, v^{k-1}w, \ldots, w^k)$ of $\mathcal{M}_a^{(k)}(b', b)$. Since $D_a(1) = 3\pi x^3 + \pi a x$, we see that $x^3 \equiv -\frac{ax}{3}$ in $\mathcal{M}_a(b', b)$. Therefore, $\partial_a(v) = \pi a w$ and $\partial_a(w) = \frac{-\pi a^2}{3} v$. For each $j = 0, 1, \ldots, k$, we have

$$\begin{aligned}\partial_a(v^j w^{k-j}) &= j v^{j-1} \partial_a(v) w^{k-j} + (k-j) v^j w^{k-j-1} \partial_a(w) \\ &= j\pi a v^{j-1} w^{k-j+1} + (k-j)\frac{-\pi a^2}{3} v^{j+1} w^{k-j-1}.\end{aligned}$$

Thus, ∂_a, acting on *row vectors*, has the matrix representation $a\frac{d}{da} - G_k$ where

$$G_k := \begin{pmatrix} 0 & k\pi a & & & & \\ \frac{-\pi a^2}{3} & 0 & (k-1)\pi a & & & \\ & \frac{-2\pi a^2}{3} & & & \ddots & \\ & & & \ddots & & \pi a \\ & & & & \frac{-k\pi a^2}{3} & 0 \end{pmatrix}.$$

Dwork Operator. For $\frac{p-1}{p} \geq b > \frac{1}{p-1}$ and $b/p \geq b' > 0$, $\bar{\alpha}(a)$ is an isomorphism of $\mathcal{M}_a(b', b)$ onto $\mathcal{M}_{a^p}(b', b)$ satisfying $\partial_{a^p}\bar{\alpha}(a) = \bar{\alpha}(a)\partial_a$ (since $a\frac{d}{da}$ and ψ_x commute with no p-factor). Thus $\bar{\alpha}(a)$ induces an isomorphism $\bar{\alpha}_k(a) : \mathcal{M}_a^{(k)}(b', b) \to \mathcal{M}_{a^p}^{(k)}(b', b)$ defined by acting on the image of $(u_1, \ldots, u_k) \in \mathcal{M}_a(b', b)^k$ in $\mathcal{M}_a^{(k)}(b', b)$ by

$$\bar{\alpha}_k(a)(u_1 \cdots u_k) := \bar{\alpha}(a)(u_1) \cdots \bar{\alpha}(a)(u_k).$$

Note, $\partial_{a^p}\bar{\alpha}_k(a) = \bar{\alpha}_k(a)\partial_a$.

Define the operator $\psi_a : L(b') \to L(pb')$ by

$$\psi_a : \sum_{i=0}^{\infty} B_i a^i \longmapsto \sum_{i=0}^{\infty} B_{pi} a^i.$$

We may extend this map to $\mathcal{M}_a^{(k)}(b', b)$ as follows. Fix the basis $\{v^i w^{k-i}\}_{i=0}^k$ of $\mathcal{M}_a^{(k)}(b', b)$. Define the map $\psi_a : \mathcal{M}_{a^p}^{(k)}(b', b) \to \mathcal{M}_a^{(k)}(pb', b)$ by

$$\sum_{i=0}^{k} h_i v^i w^{k-i} \longmapsto \sum_{i=0}^{k} \psi_a(h_i) v^i w^{k-i}.$$

It is not hard to show that $\psi_a \partial_{a^p} = p\partial_a \psi_a$.

Denote by $\mathcal{M}_a^{(k)}(b)$ the space $\mathcal{M}_a^{(k)}(b, b)$. We define the Dwork operator $\beta_k := \psi_a \circ \bar{\alpha}_k(a)$ on $\mathcal{M}_a^{(k)}(b)$ by noting

$$\mathcal{M}_a^{(k)}(b) \xrightarrow{\bar{\alpha}_k(a)} \mathcal{M}_a^{(k)}(b/p, b) \xrightarrow{\psi_a} \mathcal{M}_a^{(k)}(b).$$

This is a nuclear operator on $\mathcal{M}_a^{(k)}(b)$ in the sense of [14].

Cohomology. Define the cohomology spaces

$$\begin{aligned} H_k^0 &:= ker(\partial_a | \mathcal{M}_a^{(k)}(b)) \\ H_k^1 &:= \mathcal{M}_a^{(k)}(b) / \partial_a \mathcal{M}_a^{(k)}(b). \end{aligned}$$

Since

$$\beta_k \circ \partial_a = \psi_a \circ \bar{\alpha}_k(a) \circ \partial_a = \psi_a \circ \partial_{a^p} \circ \bar{\alpha}_k(a) = p\partial_a \circ \psi_a \circ \bar{\alpha}_k(a) = \partial_a \circ p\beta_k,$$

β_k induces endomorphisms

$$\bar{\beta}_k : H^0_k \to H^0_k \quad \text{and} \quad \bar{\beta}_k : H^1_k \to H^1_k.$$

These maps are in fact isomorphisms; this follows by defining a right-inverse to β_k as in Remark 2.12.

Dwork Trace Formula. From the following exact sequence and chain map:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0_k & \longrightarrow & \mathcal{M}^{(k)}_a(b) & \xrightarrow{\partial_a} & \mathcal{M}^{(k)}_a(b) & \longrightarrow & H^1_k & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle p^s\bar{\beta}^s_k} & & \downarrow{\scriptstyle p^s\beta^s_k} & & \downarrow{\scriptstyle \beta^s_k} & & \downarrow{\scriptstyle \bar{\beta}^s_k} & & \downarrow \\
0 & \longrightarrow & H^0_k & \longrightarrow & \mathcal{M}^{(k)}_a(b) & \xrightarrow{\partial_a} & \mathcal{M}^{(k)}_a(b) & \longrightarrow & H^1_k & \longrightarrow & 0
\end{array}$$

it follows from [14, Proposition 8.3.10] that the alternating sum of the traces is zero:

$$Tr(p^s\bar{\beta}^s_k|H^0_k) - Tr_{nuc}(p^s\beta^s_k|\mathcal{M}^{(k)}_a(b)) + Tr_{nuc}(\beta^s_k|\mathcal{M}^{(k)}_a(b)) - Tr(\bar{\beta}^s_k|H^1_k) = 0.$$

Hence,

$$(p^s - 1)Tr_{nuc}(\beta^s_k|\mathcal{M}^{(k)}_a(b)) = p^s Tr(\bar{\beta}^s_k|H^0_k) - Tr(\bar{\beta}^s_k|H^1_k). \tag{5.2}$$

Next, we have Dwork's trace formula

$$(p^s - 1)Tr_{nuc}(\beta^s_k|\mathcal{M}^{(k)}_a(b)) = \sum_{a\in\mathbb{C}^*_p, a^{p^s-1}=1} Tr_{\mathcal{M}^{(k)}_a(b)}\bar{\alpha}_k(a;s) \tag{5.3}$$

where

$$\bar{\alpha}_k(a;s) := \bar{\alpha}_k(a^{p^{s-1}}) \circ \bar{\alpha}_k(a^{p^{s-2}}) \circ \cdots \circ \bar{\alpha}_k(a)$$

Cohomological Formula for $M_k(T)$. Recall from §3.1, if $\bar{z} \in \overline{\mathbb{F}}^*_p$ and z is its Teichmüller representative in $\mathbb{C}_p$, then $\bar{\alpha}_z$ was defined to be the operator $\bar{\alpha}(a)$ specialized at $a = z$. Using this, define

$$\bar{\alpha}^{(k)}_{z,s} := \bar{\alpha}_k(a^{p^{s-1}}) \circ \bar{\alpha}_k(a^{p^{s-2}}) \circ \cdots \circ \bar{\alpha}_k(a)|_{a=z}.$$

Using (5.2) and (5.3) for the first two equalities (resp.) of the following, we have

$$\begin{aligned}
\frac{det(1-\bar{\beta}_kT|H^1_k)}{det(1-p\bar{\beta}_kT|H^0_k)} &= \exp\sum_{s\geq 1}(p^s-1)Tr_{nuc}(\beta^s_k|\mathcal{M}^{(k)}_a(b))\frac{T^s}{s} \\
&= \exp\sum_{s\geq 1}\sum_{a\in\mathbb{C}_p, a^{p^s-1}=1} Tr_{\mathcal{M}^{(k)}_a(b)}(\bar{\alpha}_k(a;s))\frac{T^s}{s} \\
&= \exp\sum_{r\geq 1}\sum_{\substack{\bar{a}\in\overline{\mathbb{F}}^*_p, deg(\bar{a})=r \\ a=Teich(\bar{a})}}\sum_{m\geq 1} Tr_{\mathcal{M}^{(k)}_a(b)}(\bar{\alpha}_k(a;rm))\frac{T^{rm}}{rm} \\
&= \exp\sum_{r\geq 1}\sum_{\substack{\bar{a}\in\overline{\mathbb{F}}^*_p, deg(\bar{a})=r \\ a=Teich(\bar{a})}}\sum_{m\geq 1} Tr_{\mathbb{C}_p}(\bar{\alpha}^{(k)}_{a,rm})\frac{T^{rm}}{rm} \\
&= \exp\sum_{r\geq 1}\sum_{\substack{\bar{a}\in\overline{\mathbb{F}}^*_p, deg(\bar{a})=r \\ a=Teich(\bar{a})}}\sum_{m\geq 1} Tr_{\mathbb{C}_p}((\bar{\alpha}^{(k)}_{a,r})^m)\frac{T^{rm}}{rm} \\
&= \prod_{r\geq 1}\prod_{\substack{\bar{a}\in\overline{\mathbb{F}}^*_p, deg(\bar{a})=r \\ a=Teich(\bar{a})}}[det_{\mathbb{C}_p}(1-\bar{\alpha}^{(k)}_{a,r}T^r)]^{-1/r}.
\end{aligned} \tag{5.4}$$

From §3.1 we know that $\bar{\alpha}_{a,r}$ has two eigenvalues $\pi_1(a)$ and $\pi_2(a)$ on M_a. Since $\bar{\alpha}^{(k)}_{a,r}$ is the k-th symmetric power of $\bar{\alpha}_{a,r}$, its eigenvalues are $\pi_1(a)^{k-j}\pi_2(a)^j$ for $j = 0, 1, \ldots, k$. Thus,

$$det(1-\bar{\alpha}^{(k)}_{a,r}T) = \prod_{j=0}^{k}(1-\pi_1(a)^{k-j}\pi_2(a)^jT).$$

This means (5.4) equals

$$\prod_{r\geq 1}\prod_{\substack{\bar{a}\in\overline{\mathbb{F}}_p^*, deg(\bar{a})=r\\ a=Teich(\bar{a})}}\prod_{j=0}^{k}(1-\pi_1(a)^{k-j}\pi_2(a)^jT^r)^{-1/r}$$

which is precisely $M_k^*(T)$.

Theorem 5.1. *$M_k(T)$ is a rational function with coefficients in $\mathbb{Z}[\zeta_p]$. If $p\equiv 1\ mod(3)$, then the coefficients of $M_k(T)$ lie in the ring of integers of the unique subfield L of $\mathbb{Q}(\zeta_p)$ with $[L:\mathbb{Q}]=3$. However, if $p\equiv -1\ mod(3)$ then $M_k(T)$ has integer coefficients.*

Proof. We will first prove rationality by using a result of Borel-Dwork. From the cohomological description, $M_k(T)$ is p-adic meromorphic. Next, write $M_k(T)=\exp\sum_{s\geq 1}N_s\frac{T^s}{s}$ as in the introduction, with

$$N_s=\sum_{\bar{a}\in\mathbb{F}_{p^s}}\sum_{j=0}^{k}deg(\bar{a})[\pi_1(\bar{a})^{k-j}\pi_2(\bar{a})^j]^{\frac{s}{deg(\bar{a})}}.$$

Since $|\pi_j(\bar{a})|_{\mathbb{C}}\leq p^{deg(\bar{a})/2}$ by the Riemann Hypothesis for curves, $|N_s|_{\mathbb{C}}\leq csp^{(k+1)s/2}$ for a positive integer c independent of s. Thus $\sum_{s\geq 1}\frac{N_s}{s}T^s$ has at least a $p^{-(k+1)/2}$ radius of convergence on $\mathbb{C}$. Consequently, $M_k(T)$ has a positive $\mathbb{C}$-radius of convergence.

Let ζ_p is a primitive p-th root of unity. Since $L(x^3+\bar{a}x,T)$ is a polynomial with entries in $\mathbb{Z}[\zeta_p]$, we see that the roots live in $\mathbb{Z}[\zeta_p]$-rational orbits under the action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_p))$. Hence, the symmetric polynomial $\prod_{j=0}^{k}(1-\pi_1(\bar{a})^{k-j}\pi_2(\bar{a})^jT)$ must also have coefficients in $\mathbb{Z}[\zeta_p]$. It follows that $M_k(T)$ has coefficients in $\mathbb{Z}[\zeta_p]$. Since $M_k(T)$ is p-adic meromorphic and converges on a disc of positive radius in $\mathbb{C}$, and the coefficients lie in a fixed number field, by the Borel-Dwork theorem [4, Theorem 3] $M_k(T)$ is a rational function with coefficients in $\mathbb{Z}[\zeta_p]$.

Next, with each $\lambda\in\mathbb{F}_p^*$ we may define $\sigma_\lambda\in Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, where , by $\sigma_\lambda(\zeta_p):=\zeta_p^\lambda$; note, every element of $Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ may be represented uniquely by such a σ_λ. Now,

$$\begin{aligned}\sum_{\bar{x}\in\mathbb{F}_{p^k}}\zeta_p^{Tr_{\mathbb{F}_{p^k}/\mathbb{F}_p}(\bar{x}^3+\lambda^2\bar{a}\bar{x})} &= \sum_{\bar{x}\in\mathbb{F}_{p^k}}\zeta_p^{Tr_{\mathbb{F}_{p^k}/\mathbb{F}_p}[\lambda^3(\bar{x}^3+\bar{a}\bar{x})]}\\ &=\sigma_{\lambda^3}\left(\sum_{\bar{x}\in\mathbb{F}_{p^k}}\zeta_p^{Tr_{\mathbb{F}_{p^k}/\mathbb{F}_p}(\bar{x}^3+\bar{a}\bar{x})}\right).\end{aligned}$$

This means $\sigma_{\lambda^3}(L(x^3+\bar{a}x,T))=L(x^3+\lambda^2\bar{a}x,T)$. Consequently,

$$\sigma_{\lambda^3}(M_k(T))=M_k(T),$$

and so $M_k(T)$ is fixed by the subgroup $\{\lambda^3|\lambda\in\mathbb{F}_p^*\}$. Since this subgroup has index $r:=gcd(3,p-1)$ in the Galois group, the coefficients of $M_k(T)$ must lie in the ring of integers of the unique subfield of $\mathbb{Q}(\zeta_p)$ of degree r over $\mathbb{Q}$. □

6 Cohomology of Symmetric Powers

6.1 H_k^0 for all odd k, or for all $k<2p$ even

Recall, with respect to the basis $\{x,x^2\}$ of $\mathcal{M}_a(b',b)$, ∂_a has the matrix representation (acting on *row* vectors):

$$\partial_a=a\frac{d}{da}-\begin{pmatrix}0 & \pi a\\ \frac{-\pi a^2}{3} & 0\end{pmatrix}.$$

Thus, if (C_1,C_2) is a solution of this differential equation then C_2 must satisfy the Airy differential equation: $y''+\frac{\pi^2 a}{3}y=0$.

Theorem 6.1. *Let k be a positive integer. If k is odd, or k is even with $k<2p$, then $H_k^0:=ker(\partial_a|\mathcal{M}_a^{(k)}(b',b))=0$.*

Proof. Every element of $\mathcal{M}_a^{(k)}(b',b)$ which is annihilated by ∂_a corresponds to a unique solution in $L(b')$, and hence $\widehat{R}$, of the scalar equation l_k, where l_k was defined in §4.1. By the methods of §4.1, if k is odd, then no such solution exists in $\widehat{R}$. If $2|k$ but $4\nmid k$, and $k<2p$, then again, no such solution exists. If $4|k$, then one solution of l_k exists in $\widehat{R}$, however, this corresponds to $v_1(a)^{k/2}v_2(a)^{k/2}$ which is clearly not an element of $L(b')$. □

Conjecture 6.2. *$H_k^0=0$ for all positive integers k.*

6.2 H_k^1 for odd $k < p$

We will assume k is odd and $k < p$ throughout this section. The two vectors $v := x$ and $w := x^2$ form an $L(b)$-basis of $\mathcal{M}_a(b)$. Thus, the set $\{v^j w^{k-j} | 0 \le j \le k\}$ is an $L(b)$-basis of $\mathcal{M}_a^{(k)}(b)$. With $\delta_j := \frac{bj}{3}$, define

$$\begin{aligned}\mathcal{M}_a^{(k)}(b; \boldsymbol{\delta} + \rho) &:= \bigoplus_{j=0}^{k} L(b; \delta_{j+1} + \rho) v^{k-j} w^j \\ &= L(b; \delta_1 + \rho) \oplus L(b; \delta_2 + \rho) \oplus \cdots \oplus L(b; \delta_{k+1} + \rho)\end{aligned}$$

and

$$\mathcal{M}_a^{(k)}(b) := \bigcup_{\rho \in \mathbb{R}} \mathcal{M}_a^{(k)}(b; \boldsymbol{\delta} + \rho).$$

We will write $\mathcal{M}_a^{(k)}(b; \rho)$ with the $\boldsymbol{\delta}$ implied. Define

$$\begin{aligned}V_k &:= \mathbb{C}_p^{k+1} \oplus \bigoplus_{j=0}^{(k-1)/2} (\mathbb{C}_p a) v^{k-2j} w^{2j} \qquad (6.1)\\ V_k(b; \rho) &:= V_k \cap \mathcal{M}_a^{(k)}(b; \rho) \\ V_k(b) &:= \bigcup_{\rho \in \mathbb{R}} V_k(b; \rho).\end{aligned}$$

Theorem 6.3. *Let $p \ge 5$ be a prime. Suppose k is odd and $k < p$. Let b be a real number such that $e := b - \frac{1}{p-1} > 0$. Then we have the following decomposition:*

$$\mathcal{M}_a^{(k)}(b) = V_k(b) \oplus \partial_a \mathcal{M}_a^{(k)}(b).$$

Furthermore, ∂_a is injective on $\mathcal{M}_a^{(k)}(b)$.

It will be useful to recall, if we order the basis of $\mathcal{M}_a^{(k)}(b)$ as $(v^k, v^{k-1}w, \ldots, w^k)$, then ∂_a has the matrix representation $\partial_a = a\frac{d}{da} - G_k$, acting on *row vectors*, where

$$G_k := \begin{pmatrix} 0 & k\pi a & & & \\ \frac{-\pi a^2}{3} & 0 & (k-1)\pi a & & \\ & \frac{-2\pi a^2}{3} & & \ddots & \\ & & \ddots & & \pi a \\ & & & \frac{-k\pi a^2}{3} & 0 \end{pmatrix}$$

That is, if we write the row vector $\xi = (\xi_1, \ldots, \xi_{k+1})$ for $\sum_{j=1}^{k=1} \xi_j v^{k-j} w^j$ then

$$\partial_a \xi \quad \text{means} \quad (\xi_1, \ldots, \xi_{k+1})(a\frac{d}{da} - G_k).$$

Lemma 6.4. *$\mathcal{M}_a^{(k)}(b; 0) \subset V_k(b; 0) + \mathcal{M}_a^{(k)}(b; e) G_k$.*

Proof. **Odd Column Case.** For $n \ge 2$, consider the vector $(0, \ldots, B_n a^n, \ldots, 0)$ where a^n is in the $(2j+1)$-entry with all others zero, and $B_n a^n \in L(b; \delta_{2j+1})$.

Suppose $n \le (k+1)/2 - j$. Then

$$(0, \ldots, \overbrace{B_n a^n}^{2j+1 \text{ entry}}, \ldots, 0) = (0, 0, \ldots, 0, C_{2(j+1)}, 0, C_{2(j+2)}, \cdots) G_k + (0, \ldots, \overbrace{\eta_{j,n}}^{2(n-1+j)+1 \text{ entry}}, \ldots, 0)$$

where

$$\eta_{j,n} := \prod_{l=0}^{n-2} \frac{3[k - (2(j+l)+1)]}{[2(j+l)+1]} B_n a \in L(b; \delta_{2(n-1+j)+1})$$

and for $i = 1, 2, \ldots, n-1$, we have

$$C_{2(j+i)} = \frac{-3^i}{\pi} \left(\frac{\prod_{l=0}^{i-2} [k - (2(j+l)+1)]}{\prod_{l=0}^{i-1} [2(j+l)+1]} \right) B_n a^{n-1-i} \in L(b; \delta_{2(j+i)} + e)$$

Next, suppose $n \geq (k+1)/2 - j + 1$. Then

$$(0, \dots, \overbrace{B_n a^n}^{2j+1 \text{ entry}}, \dots, 0) = (0, 0, \dots, 0, C_{2(j+1)}, 0, C_{2(j+2)}, \cdots)G_k$$

where, for $m = 1, 2 \dots, (k+1)/2 - j$,

$$C_{2(j+m)} := \frac{(-3)^m (j - \frac{k-1}{2})_{m-1}}{2\pi(j + \frac{1}{2})_m} B_n a^{n-m-1}.$$

Notice that, since $k < p$, the constants in front of the series are p-adic units.

Even Column Case. Fix $j \in \{1, \dots, \frac{k+1}{2}\}$. Let $g(a) := \sum_{i=0}^{\infty} B_i a^i \in L(b; \delta_{2j})$. Then we may write

$$(0, \dots, 0, \overbrace{g(a)}^{\text{entry } 2j}, 0, \dots, 0) = \xi G_k + (0, \dots, 0, \overbrace{B_0}^{\text{entry } 2j}, 0, \dots, 0)$$

where

$$\xi := (\xi_1, 0, \xi_3, 0, \dots, \xi_{2j-1}, 0, 0, \dots, 0)$$

where, for $m = 0, 1, \dots, j-1$,

$$\xi_{2j-1-2m} := \frac{(1-j)_m}{2(-3)^m \pi (\frac{k}{2} - j + 1)_{m+1}} \sum_{i=0}^{\infty} B_{i+1} a^{i+m} \in L(b; \delta_{2j-1-2m} + e).$$

□

Lemma 6.5. $V_k(b) \cap \left(\mathcal{M}_a^{(k)}(b) G_k\right) = \{0\}$.

Proof. We will proceed by induction. Let $\eta = (\eta_1, \dots, \eta_{k+1}) \in V_k \cap \left(\mathcal{M}_a^{(k)}(b) G_k\right)$. Then there is a $\xi \in \mathcal{M}_a^{(k)}(b)$ such that $\xi G_k = \eta$. In particular, $\eta_1 = \frac{-\pi a^2}{3} \xi_2$. Hence, $\xi_2 = 0$. Suppose $\xi_{2(j-1)} = 0$, then for some p-adic units c_1 and c_2 we may write

$$\eta_{2j-1} = c_1 a \xi_{2(j-1)} + c_2 a^2 \xi_{2j}$$

which shows $\xi_{2j} = 0$. Thus, $\xi_{2j} = 0$ for $j = 1, 2, \dots, \frac{k+1}{2}$. Next, since $\eta_{k+1} = \pi a \xi_k$ we see that $\xi_k = 0$. Suppose $\xi_{2j+1} = 0$, then since there are units c_1 and c_2 such that

$$\eta_{2j} = c_1 a \xi_{2j-1} + c_2 a^2 \xi_{2j+1}$$

we see that $\xi_{2j-1} = 0$. □

Lemma 6.6. *Let $u \in \mathcal{M}_a^{(k)}(b)$ such that $uG_k \in \mathcal{M}_a^{(k)}(b; \rho)$. Then $u \in \mathcal{M}_a^{(k)}(b; \rho + e)$. Consequently, G_k is an injective operator on $\mathcal{M}_a^{(k)}(b)$.*

Proof. We will proceed by induction. Let $\xi := uG_k$. Then $\xi_1 = \frac{-\pi}{3} a^2 u_2 \in L(b; \delta_1)$. Thus, $u_2 \in L(b; \delta_2 + e)$ as desired. Next, suppose $u_{2j} \in L(b; \delta_{2j} + e)$. Then $\pi a u_{2j} \in L(b; \delta_{2j+1})$. From $\xi = uG_k$, there are p-adic units c_1 and c_2 such that

$$\xi_{2j+1} = c_1 \pi a u_{2j} - \frac{c_2 \pi a^2}{3} u_{2j+2} \in L(b; \delta_{2j+1}).$$

Hence, $\frac{c_2 \pi a^2}{3} u_{2j+2} \in L(b; \delta_{2j+1})$, and this implies $u_{2j+2} \in L(b; \delta_{2j+2} + e)$.

Next, by hypothesis, $\xi_{k+1} = \pi a u_k \in L(b; \delta_{k+1})$ and so, $u_k \in L(b; \delta_k + e)$. Suppose $u_{2j+1} \in L(b; \delta_{2j+1} + e)$. Then $\pi a^2 u_{2j+1} \in L(b; \delta_{2j})$. As before, from $\xi = uG_k$, there are p-adic units c_1 and c_2 such that

$$\xi_{2j} = c_1 \pi a u_{2j-1} - c_2 \pi a^2 u_{2j+1} \in L(b; \delta_{2j}).$$

As before, it follows that $u_{2j-1} \in L(b; \delta_{2j-1} + e)$. □

Lemma 6.7. $\mathcal{M}_a^{(k)}(b; 0) \subset V_k(b; 0) + \partial_a \mathcal{M}_a^{(k)}(b; e)$.

Proof. Let $u \in \mathcal{M}_a^{(k)}(b;0)$. Set $u^{(0)} := u$. We know there exists unique $\eta^{(0)} \in V_k(b;0)$ and $\xi^{(0)} \in \mathcal{M}_a^{(k)}(b;e)$ such that

$$u^{(0)} = \eta^{(0)} + \xi^{(0)} G_k.$$

Set $E := a\frac{d}{da}$. Then

$$\begin{aligned} u^{(1)} &:= u^{(0)} - \eta^{(0)} + \partial_a \xi^{(0)} \\ &= E\xi^{(0)} \in \mathcal{M}_a^{(k)}(b;e). \end{aligned}$$

Again, there exists unique $\eta^{(1)} \in V_k(b;e)$ and $\xi^{(1)} \in \mathcal{M}_a^{(k)}(b;2e)$ such that

$$u^{(1)} = \eta^{(1)} + \xi^{(1)} G_k.$$

Set $u^{(2)} := u^{(1)} - \eta^{(1)} + \partial_a \xi^{(1)}$. Continue this procedure h-times:

$$u^{(h)} := u^{(h-1)} - \eta^{(h-1)} + \partial_a \xi^{(h-1)} \in \mathcal{M}_a^{(k)}(b;he).$$

Adding these together produces

$$u^{(h)} = u^{(0)} - \sum_{i=0}^{h-1} \eta^{(i)} + \partial_a \sum_{i=0}^{h-1} \xi^{(i)}.$$

Letting $h \to \infty$ we see that $u^{(h)} \to 0$ in $\mathcal{M}_a^{(k)}(b)$. Thus,

$$u^{(0)} = \sum_{i=0}^{\infty} \eta^{(i)} - \partial_a \sum_{i=0}^{\infty} \xi^{(i)} \in V_k(b) + \partial_a \mathcal{M}_a^{(k)}(b).$$

$\square$

Lemma 6.8. *Let $u \in \mathcal{M}_a^{(k)}(b)$ be such that $\partial_a u \in \mathcal{M}_a^{(k)}(b;\rho)$. Then $u \in \mathcal{M}_a^{(k)}(b;\rho+e)$.*

Proof. Suppose $u \neq 0$. Choose $c \in \mathbb{R}$ such that $u \in \mathcal{M}_a^{(k)}(b;c)$ but $u \notin \mathcal{M}_a^{(k)}(b;c+e)$. By hypothesis, $\partial_a u \in \mathcal{M}_a^{(k)}(b;\rho)$. Thus

$$-uG_k = \partial_a u - Eu \in \mathcal{M}_a^{(k)}(b;\rho) + \mathcal{M}_a^{(k)}(b;c) = \mathcal{M}_a^{(k)}(b;l).$$

where $l := \min\{\rho, c\}$. Hence, $u \in \mathcal{M}_a^{(k)}(b;l+e)$ which means $l = \rho$. $\square$

Corollary 6.9. *∂_a is an injective operator on $\mathcal{M}_a^{(k)}(b)$.*

Lemma 6.10. *$V_k(b) \cap \partial_a \mathcal{M}_a^{(k)}(b) = \{0\}$.*

Proof. Let $u \in V_k(b) \cap \partial_a \mathcal{M}_a^{(k)}(b)$. Assume $u \neq 0$. Then, we may find $\rho \in \mathbb{R}$ such that $u \in \mathcal{M}_a^{(k)}(b;\rho)$ but $u \notin \mathcal{M}_a^{(k)}(b;\rho+e)$. By hypothesis, there exists $\eta \in \mathcal{M}_a^{(k)}(b)$ such that $\partial_a \eta = u$. Thus, $\eta \in \mathcal{M}_a^{(k)}(b;\rho+e)$. Now,

$$u + \eta G_k = E\eta \in \mathcal{M}_a^{(k)}(b;\rho+e)$$

which means that we may find $\zeta \in V_k(b;\rho+e)$ and $w \in \mathcal{M}_a^{(k)}(b;\rho+2e)$ such that

$$E\eta = \zeta + wG_k.$$

Therefore, $u = \zeta + (w - \eta)G_k$. Since $u \in V_k(b)$, by uniqueness, $u = \zeta$ (and $\eta = w$). However, this means $u \in V_k(b;\rho+e) \subset \mathcal{M}_a^{(k)}(b;\rho+e)$ which is a contradiction. $\square$

This concludes the proof of Theorem 6.3.

6.3 H_k^1 for even $k < p$

In this section, while we attempt to proceed in a similar manner to the last, we are only able to prove a partial decomposition theorem. The crucial difference is that when k is even, G_k has a kernel.

Assume k is even and $k < p$. With $\delta_j := \frac{bj}{3}$, define

$$\mathcal{M}_a^{(k)}(b; \boldsymbol{\delta} + \rho) := \bigoplus_{j=0}^{k} L(b; \rho + \delta_{j+1}) v^{k-j} w^j$$

As we did in the previous section, we will write $\mathcal{M}_a^{(k)}(b; \rho)$ with the $\boldsymbol{\delta}$ implied. Ordering the basis as $(v^k, v^{k-1}w, \dots, w^k)$, define the subspace $\widetilde{\mathcal{M}}_a^{(k)}(b; \rho) \subset \mathcal{M}_a^{(k)}(b; \rho)$ as

$$\widetilde{\mathcal{M}}_a^{(k)}(b; 0) := L(b; \delta_1) \oplus L(b; \delta_2) \oplus \cdots \oplus L(b; \delta_k) \oplus \{0\}$$

Next, define

$$V_k := \mathbb{C}_p^{k+1} \oplus \bigoplus_{j=0}^{k/2-1} \mathbb{C}_p a v^{k-2j} w^{2j},$$

and the vector

$$\mathbf{k} := (K_1, 0, K_3, 0, \dots, 0, K_{k+1})$$

where for $j = 0, 1, \dots, k/2$

$$K_{2j+1} := \frac{\prod_{i=1}^{j} [\frac{k}{2} - (i-1)]}{j!\, 3^{k/2-j}} a^{k/2-j}.$$

Notice that the kernel of G_k is $L(b)\mathbf{k}$. Define

$$\mathcal{I}_{ker}(b; \rho) = aL(b)\mathbf{k} \cap \mathcal{M}_a^{(k)}(b; \rho) \quad \text{and} \quad \mathcal{I}_{ker}(b) := aL(b)\mathbf{k}.$$

Theorem 6.11. *Let $p \geq 5$. Suppose k is even and $k < p$. Let b be a real number such that $b - \frac{1}{p-1} > 0$. Then*

$$\mathcal{M}_a^{(k)}(b) = V_k(b) \oplus \partial_a \widetilde{\mathcal{M}}_a^{(k)}(b) \oplus \mathcal{I}_{ker}(b).$$

The proof will consist of several lemmas.

Lemma 6.12. *Set $e := b - \frac{1}{p-1} > 0$. Then*

$$\mathcal{M}_a^{(k)}(b; 0) \subset V_k(b; 0) + \widetilde{\mathcal{M}}_a^{(k)}(b; e) G_k + \mathcal{I}_{ker}(b; 0).$$

The proof of this is a bit involved.

Even Entry Case. Consider $(0, \dots, g(a), \dots, 0)$ where $g(a) = \sum_{i \geq 0} B_i a^i \in L(b; \delta_{2j})$ is in the $2j$-entry, and all other entries are zero. Then

$$(0, \dots, \overbrace{g(a)}^{2j \text{ entry}}, \dots, 0) = (C_1, 0, C_3, 0, \dots, C_{2j-1}, 0, 0, \dots, 0) G_k + (0, \dots, \overbrace{B_0}^{2j \text{ entry}}, \dots, 0)$$

where

$$C_{2j-2l-1} = \frac{\prod_{m=0}^{l-1}(j - l + m)}{2\pi 3^l \prod_{m=1}^{l+1}(\frac{k}{2} - j + m)} \sum_{i \geq 1} B_i a^{i-1+l} \in L(b; \delta_{2j-2l-1} + e)$$

for $l = 0, 1, \dots, j-1$ and all other coordinates are zero. Note, C_{k+1} equals zero.

Odd Entry Case. This is a bit more technical. For $n \geq 2$, consider the vector $(0, \dots, B_n a^n, \dots, 0)$ where a^n is in the $(2j+1)$-entry with all others zero, and $B_n a^n \in L(b; \delta_{2j+1})$.

First, suppose $n \leq k/2 - j$. Then

$$(0, \dots, \overbrace{B_n a^n}^{2j+1 \text{ entry}}, \dots, 0) = (0, 0, \dots, 0, C_{2(j+1)}, 0, C_{2(j+2)}, \cdots) G_k + (0, \dots, \overbrace{\eta_{j,n}}^{2(n-1+j)+1 \text{ entry}}, \dots, 0)$$

where

$$\eta_{j,n} := \prod_{l=0}^{n-2} \frac{3[k - (2(j+l)+1)]}{[2(j+l)+1]} B_n a \in L(b; \delta_{2(n-1+j)+1})$$

and for $i = 1, 2, \ldots, n-1$, we have

$$C_{2(j+i)} = \frac{-3^i}{\pi}\left(\frac{\prod_{l=0}^{i-2}[k-(2(j+l)+1)]}{\prod_{l=0}^{i-1}[2(j+l)+1]}\right) B_n a^{n-1-i} \in L(b; \delta_{2(j+i)} + e)$$

Next, suppose $n \geq \frac{k}{2} - j + 1$. We will demonstrate that

$$(0, \ldots, \overbrace{B_n a^n}^{2j+1 \text{ entry}}, \ldots, 0) = (0, C_2, 0, C_4, 0, \ldots, C_k, 0)G_k + \mathbf{k}h \tag{6.2}$$

where h is given by (6.5) below and, for $l = 1, 2, \ldots, k/2$,

$$C_{2l} = \frac{\mathbb{Z}}{2^k (k/2)!\pi} B_n a^{n+j-l-1} \in L(b; \delta_{2j} + e)$$

where "$\mathbb{Z}$" indicates some determinable integer.

In (6.2), since the even numbered entries are all zero, let us *delete them* from this equation to obtain

$$(0, \ldots, \overbrace{B_n a^n}^{j^{\text{th}} \text{ entry}}, \ldots, 0) = (C_2, C_4, \ldots, C_k)\widetilde{G}_k + \mathbf{k}_{odd} h \tag{6.3}$$

where the vector on the left has $(k/2)+1$ entries and $B_n a^n$ is in the j-th entry, $\mathbf{k}_{odd}$ is the vector $\mathbf{k}$ with the even entries deleted, and $\widetilde{G}_k$ is the $(k/2) \times (k/2+1)$-matrix

$$\widetilde{G}_k = \begin{pmatrix} -\frac{\pi a^2}{3} & (k-1)\pi a & & & \\ & -\frac{3\pi a^2}{3} & (k-3)\pi a & & \\ & & \ddots & \ddots & \\ & & & -\frac{(k-1)\pi a^2}{3} & \pi a \end{pmatrix}.$$

Thus, we may rewrite this as

$$(0, \ldots, \overbrace{B_n a^n}^{j+1 \text{ entry}}, \ldots, 0) = (h, C_2, C_4, \ldots, C_k)N_k \quad \text{where} \quad N_k := \begin{pmatrix} \mathbf{k}_{odd} \\ \widetilde{G}_k \end{pmatrix}.$$

Note, N_k is a square matrix. Furthermore, $detN_k = \frac{2^k (k/2)!}{3^{k/2}} \pi^{k/2} a^k$ by the following lemma.

Lemma 6.13. $detN_k = \frac{2^k (k/2)!}{3^{k/2}} \pi^{k/2} a^k$

Proof. Expanding the determinant always using the left column, we get

$$detN_k = \frac{\pi^{k/2} a^k}{3^{k/2}} \sum_{j=0}^{k/2} \frac{k!}{2^{k/2} \cdot (k/2)!} \binom{k/2}{j}^2 \binom{k}{2j}^{-1}.$$

Thus, the lemma follows from the following general formula:

$$\sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n}{2j}^{-1} = \frac{(2^n \cdot n!)^2}{(2n)!}. \tag{6.4}$$

To prove this last formula (of which I thank Zhi-Wei Sun for the following), notice that the above may be rewritten as

$$\sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n}{2j}^{-1} = 4^n \binom{2n}{n}^{-1}.$$

Therefore, since

$$\binom{2n}{n}\binom{n}{j}^2 \binom{2n}{2j}^{-1} = \binom{2j}{j}\binom{2n-2j}{n-j}$$

we have (6.4) equal to

$$\sum_{j=0}^{n} \binom{2j}{j}\binom{2n-2j}{n-j} = 4^n$$

which is a well-known combinatorial formula. ∎

By Cramer's rule, with $N_k(1)$ denoting the matrix N_k with the first row replaced by $(0,\ldots,B_na^n,\ldots,0)$, where B_na^n is in the $(j+1)$-entry, we have

$$h=\frac{detN_k(1)}{detN_k}=\frac{3^{\frac{k}{2}-j}}{2^k(k/2)!}\left(\prod_{i=1}^{j}(2i-1)\right)\left(\prod_{i=j}^{\frac{k}{2}-1}[k-(2i+1)]\right)B_na^{n-\frac{k}{2}+j}. \tag{6.5}$$

Notice that h is divisible by a by our hypothesis that $n\geq\frac{k}{2}-j+1$. Using this and (6.3), it follows that $\mathbf{k}h\in\mathcal{I}_{ker}(b;0)$ and

$$C_{2l}=\frac{\mathbb{Z}}{2^k(k/2)!\pi}a^{n+j-l-1}\qquad\text{for } l=0,1,2,\ldots,k/2.$$

This finishes the proof of Lemma 6.12.

Lemma 6.14. *We have*

1. $V_k\cap\left(\mathcal{M}_a^{(k)}(b)G_k\right)=\{0\}$
2. $\mathcal{I}_{ker}(b)\cap V_k=\{0\}$
3. $\mathcal{I}_{ker}(b)\cap\left(\widetilde{\mathcal{M}}_a^{(k)}(b)G_k\right)=\{0\}$

Proof. To prove the first statement, consider $\eta\in V_k\cap\left(\mathcal{M}_a^{(k)}(b)G_k\right)$. Then $\eta_1=\frac{-\pi a^2}{3}C_2$. Thus, η_1 is divisible by a^2 which means $\eta_1=C_2=0$. Next, $\eta_3=(k-1)\pi aC_2-\pi a^2C_4$. Thus, $C_4=0$. Continuing, we see that $C_{2j}=0$ for each j. Consequently, all the odd coordinates of η are zero. Next, since all the even coordinates of η are constants and the image of $\mathcal{M}_a^{(k)}(b)$ by G_k is divisible by a, η must be zero.

The second statement is clear.

Lastly, let us prove the third. Suppose there exists $h\in L(b)$ such that $(C_1,C_2,\ldots,C_k,0)G_k=h\mathbf{k}$. Looking at the last entry in this, we must have $C_k=\frac{1}{\pi a}(\mathbb{Q}_{>0})h$ where "$\mathbb{Q}_{>0}$" is some positive rational number. Working backwards from this using entries $k-1,k-3,\ldots,3$, we see $C_{k-2l}=\frac{1}{\pi a}(\mathbb{Q}_{>0})ha^l$ for $l=0,1,\ldots,k/2-1$. However, the first entry says $\frac{-\pi a^2}{3}C_2=h\frac{1}{3^{k/2}}a^{k/2}$ which means $C_2=-\frac{1}{\pi a}(\mathbb{Q}_{>0})ha^{k/2}$. This contradicts the positivity of the rational constant. ∎

Corollary 6.15. $\mathcal{M}_a^{(k)}(b)=V_k\oplus\widetilde{\mathcal{M}}_a^{(k)}(b)G_k\oplus\mathcal{I}_{ker}(b)$.

Lemma 6.16. *If $u\in\widetilde{\mathcal{M}}_a^{(k)}(b)$ and $uG_k\in\mathcal{M}_a^{(k)}(b;\rho)$, then $u\in\widetilde{\mathcal{M}}_a^{(k)}(b;\rho+e)$.*

Proof. We will proceed by induction. Let $\xi=uG_k$. Then $\xi_1=-\frac{\pi a^2}{3}u_2\in L(b;\delta_1)$. Thus, $u_2\in L(b;\delta_2+e)$. Next, suppose $u_{2j}\in L(b;\delta_{2j}+e)$. Then there are p-adic units c_1 and c_2 such that

$$\xi_{2j+1}=c_1\pi au_{2j}+c_2\pi a^2u_{2j+2}\in L(b;\delta_{2j+1}).$$

Since $c_1\pi au_{2j}\in L(b;\delta_{2j+1})$, $u_{2j+2}\in L(b;\delta_{2j+2}+e)$ as desired. This finishes the *odd* coordinates of ξ. We now move on to the even.

By the form of G_k, we can find p-adic units c_1 and c_2 such that

$$\xi_k=c_1\pi au_{k-1}+c_2\pi a^2u_{k+1}\in L(b;\delta_k).$$

However, since $u\in\widetilde{\mathcal{M}}_a^{(k)}(b)$, we have $u_{k+1}=0$. Hence, $u_{k-1}\in L(b;\delta_{k-1}+e)$. Next, suppose $u_{2j+1}\in L(b;\delta_{2j+1}+e)$. Again, we may find p-adic units c_1 and c_2 such that

$$\xi_{2j}=c_1\pi au_{2j-1}+c_2\pi a^2u_{2j+1}\in L(b;\delta_{2j}).$$

It follows that $u_{2j-1}\in L(b;\delta_{2j-1}+e)$. ∎

Proceeding as in the previous section, Theorem 6.11 follows.

7 Dual Cohomology of Symmetric Powers

In this section, we will define a dual space H_k^{1*} to the cohomology space H_k^1 and a dual operator $\bar{\beta}_k^*$ to the Dwork operator $\bar{\beta}_k$.

Dagger Spaces. Define

$$\begin{aligned}
\widehat{R} &:= \text{ analytic functions on } 1 < |a| < e, \ e \text{ unspecified} \\
L^{\dagger} &:= \text{ analytic functions on } |a| < e, \text{ where } e > 1 \text{ unspecified} \\
L^{\dagger *} &:= \text{ analytic functions on } 1 < |a| \leq \infty.
\end{aligned}$$

Define a perfect pairing $\langle \cdot, \cdot \rangle : \widehat{R} \times \widehat{R} \to \mathbb{C}_p$ by $\langle u, v \rangle :=$ the constant term of the product uv. By [14], this places $\widehat{R}$ into (topological) duality with itself. Further, by the Mittag-Leffler theorem, $\widehat{R} = aL^{\dagger} \oplus L^{\dagger *}$. Thus, since the annihilator of $L^{\dagger}$ in $\widehat{R}$ is $aL^{\dagger}$, the pairing places $L^{\dagger}$ into duality with $\widehat{R}/aL^{\dagger} = L^{\dagger *}$.

With any $b', b > 0$, define

$$\begin{aligned}
\mathcal{M}_a &:= \mathcal{M}_a(b', b) \otimes_{L(b')} L^{\dagger} \\
\mathcal{M}_a^* &:= \mathcal{M}_a(b', b) \otimes_{L(b')} \widehat{R}/aL^{\dagger}.
\end{aligned}$$

Fixing $\{x, x^2\}$ as a basis of both $\mathcal{M}_a$ and $\mathcal{M}_a^*$, we may place them into (topological) duality by defining the pairing $\langle \cdot, \cdot, \rangle : \mathcal{M}_a \times \mathcal{M}_a^* \to \mathbb{C}_p$ by taking $h = h_1(a)x + h_2(a)x^2 \in \mathcal{M}_a$ and $g = g_1(a)x + g_2(a)x^2 \in \mathcal{M}_a^*$ and defining

$$\langle h, g \rangle := \langle h_1, g_1 \rangle + \langle h_2, g_2 \rangle.$$

We may extend this duality, and the pairing, to the symmetric powers of these spaces as follows. Let $\mathcal{M}_a^{(k)}$ (resp. $\mathcal{M}_a^{(k)*}$) be the k-th symmetric tensor power of $\mathcal{M}_a$ (resp. $\mathcal{M}_a^*$) over $L^{\dagger}$ (resp. $L^{\dagger *}$). For both $\mathcal{M}_a^{(k)}$ and $\mathcal{M}_a^{(k)*}$, fix the basis $\mathcal{B} := \{v^j w^{k-j}\}_{j=0}^k$ where $v := x$ and $w := x^2$. With $(u_1, \ldots, u_k) \in (\mathcal{M}_a)^k$ and $(v_1, \ldots, v_k) \in (\mathcal{M}_a^*)^k$, define a perfect pairing between $\mathcal{M}_a^{(k)}$ and $\mathcal{M}_a^{(k)*}$ by taking their images $u_1 \cdots u_k \in \mathcal{M}_a^{(k)}$ and $v_1 \cdots v_k \in \mathcal{M}_a^{(k)*}$ and defining

$$(u_1 \cdots u_k, v_1 \cdots v_k) := \sum_{\sigma \in \mathcal{S}_k} \prod_{i=1}^{k} \langle u_i, v_{\sigma(i)} \rangle$$

where $\mathcal{S}_k$ denotes the full symmetric group on $\{1, 2, \ldots, k\}$.

Differential Operator. With respect to the basis $\{x, x^2\}$ of $\mathcal{M}_a$, we may write the matrix representation of ∂_a as $a\frac{d}{da} - G$ with

$$G := \begin{pmatrix} 0 & \pi a \\ \frac{-\pi a^2}{3} & 0 \end{pmatrix}.$$

Note, we are thinking of $\mathcal{M}_a$ as a *row space*, and so, G acts on the right. Define the endomorphism of $\mathcal{M}_a^*$

$$\partial_a^* := -a\frac{d}{da} - Trunc_a \circ G.$$

The map ∂_a^* acts on the *column space* $\mathcal{M}_a^*$. Similar to that of D_a and D_a^*, ∂_a and ∂_a^* are dual to one another with respect to the pairing $\langle \cdot, \cdot \rangle$.

As we did in §5, we extend ∂_a^* to an operator on $\mathcal{M}_a^{(k)*}$ by extending linearly the action

$$\partial_a^*(u_1 \cdots u_k) := \sum_{i=1}^{k} u_1 \cdots \hat{u}_i \cdots u_k \partial_a^*(u_i)$$

where $\hat{u}_i$ means we are leaving it out of the product. It is not hard to verify that ∂_a^* and ∂_a are dual to one another via the pairing $(\cdot, \cdot)$.

We may now define

$$H_k^1 := \mathcal{M}_a^{(k)}/\partial_a \mathcal{M}_a^{(k)} \quad \text{and} \quad H_k^{1*} := ker(\partial_a^* | \mathcal{M}_a^{(k)*}).$$

While we have already defined H_k^1 as the quotient space of $\mathcal{M}_a^{(k)}(b', b)$ by $\partial_a \mathcal{M}_a^{(k)}(b', b)$, the two definitions are the same. Since H_k^1 is of finite dimension, from [14, Proposition 7.2.2], $\partial_a \mathcal{M}_a^{(k)}$ is a closed subspace of $\mathcal{M}_a^{(k)}$. It follows that H_k^{1*} is dual to H_k^1, and hence, they have the same dimension.

Dwork Operators. Similar to that of §5, define the Dwork operator

$$\beta_k := \psi_a \circ \bar{\alpha}_k(a) : \mathcal{M}_a^{(k)} \to \mathcal{M}_a^{(k)}.$$

Observe $\beta_k \circ \partial_a = p\partial_a \circ \beta_k$, and so β_k induces a map $\bar{\beta}_k : H_k^1 \to H_k^1$.

With respect to the ordered basis $\mathcal{B} := (v^k, v^{k-1}w, \ldots, w^k)$, we may denote by $Sym^k(\mathfrak{A}(a))$ the matrix of $\bar{\alpha}_k(a)$. This makes the Dwork operator representable as $\beta_k = \psi_a \circ Sym^k(\mathfrak{A}(a))$. From this perspective, it is natural to define the dual of β_k as

$$\beta_k^* := Trunc_a \circ Sym^k(\mathfrak{A}(a)) \circ \Phi_a : \mathcal{M}_a^{(k)*} \to \mathcal{M}_a^{(k)*}.$$

Notice that since the matrix entries of $\mathfrak{A}(a)$ lie in $L^\dagger$, β_k^* is well-defined.

Dual to $p\partial_k \circ \beta_k = \beta_k \circ \partial_k$ is $p\beta_k^* \circ \partial_k^* = \partial_k^* \circ \beta_k^*$. Hence, β_k^* induces a map $\bar{\beta}_k^* : H_k^{1*} \to H_k^{1*}$. It follows that

$$det(1 - \bar{\beta}_k T | H_k^1) = det(1 - \bar{\beta}_k^* T | H_k^{1*}).$$

7.1 Primitive Cohomology and its Dual

In this section, we will decompose the (conjugate) dual cohomology into three subspaces: a constant subspace, a trivial subspace, and a primitive part. The eigenvectors and associated eigenvalues of the dual Frobenius on the first two spaces are explicitly given.

As we did in §2.5, we need to distinguish between π and its conjugate $-\pi$. Using notation from §2.5, for any $b', b > 0$, define the spaces

$$\mathcal{M}_{-\pi,a} := \mathcal{M}_{-\pi,a}(b',b) \otimes_{L(b')} L^\dagger$$
$$\mathcal{M}_{-\pi,a}^* := \mathcal{M}_{-\pi,a}(b',b) \otimes_{L(b')} \widehat{R}/aL^\dagger.$$

Denote by $\mathcal{M}_{-\pi,a}^{(k)}$ and $\mathcal{M}_{-\pi,a}^{(k)*}$ the k-th symmetric tensor powers of these spaces over $L^\dagger$ and $L^{\dagger *}$, respectively. The differential operator on $\mathcal{M}_{-\pi,a}^{(k)}$ is $\partial_{-\pi,a} := a\frac{d}{da} + G_k$, and the differential operator on $\mathcal{M}_{-\pi,a}^{(k)*}$ is $\partial_{-\pi,a}^* := -a\frac{d}{da} + Trunc_a \circ G_k$. From this, we may define $H_{-\pi,k}^1 := \mathcal{M}_{-\pi,a}^{(k)}/\partial_{-\pi,a}\mathcal{M}_{-\pi,a}^{(k)}$ and $H_{-\pi,k}^{1*} := ker(\partial_{-\pi,a}^* | \mathcal{M}_{-\pi,a}^{(k)*})$.

Constant Subspace $\mathbb{C}_p^{k+1}$. Using the basis $\mathcal{B}$ defined above, define $\mathbb{C}_p^{k+1} := \bigoplus_{j=0}^k \mathbb{C}_p v^j w^{k-j}$. Notice that $\mathbb{C}_p^{k+1} \subset H_{-\pi,k}^{1*}$.

Subspace $\mathfrak{T}_k$ coming from infinity. (cf. §4.1) If k is odd, define $\mathfrak{T}_k := 0$. Assume k is even. Define $\kappa := \frac{2i}{3\sqrt{3}}$ where i is a square root of -1 in $\mathbb{C}_p$, and define

$$v_1(a) := \sum_{n=0}^\infty \frac{\left(\frac{7}{6}\right)_n \left(\frac{17}{12}\right)_n}{2^n \kappa^n \pi^n (n+1)!} a^{-3n},$$

and $v_2(a)$ by replacing i with $-i$ in v_1. By reduction, define the vectors in $\mathcal{M}_a^*$

$$u_1 := v_1 v + \left(-\frac{1}{2}a^{-2}v_1 + 3a^2\kappa\pi v_1 + a^{-1}v_1'\right) w$$
$$u_2 := v_2 v + \left(-\frac{1}{2}a^{-2}v_2 - 3a^2\kappa\pi v_2 + a^{-1}v_2'\right) w.$$

Next, for each $0 \le j \le k/2$ and $p|(k-2j)$, define

$$\eta_j^+ := a^{-k/2}\left(e^{(k-2j)\kappa\pi a^3} u_1^{k-j} u_2^j + e^{-(k-2j)\kappa\pi a^3} u_1^j u_2^{k-j}\right)$$
$$\eta_j^- := a^{-k/2}\left(e^{(k-2j)\kappa\pi a^3} u_1^{k-j} u_2^j - e^{-(k-2j)\kappa\pi a^3} u_1^j u_2^{k-j}\right).$$

If $4|k$, then for each $0 \le j \le k/2$ and $p|(k-2j)$, define $\omega_j(a)$ by $\omega_j(a^2) = \eta_j^+(a)$. Else, if $2|k$ but $4 \nmid k$, then for each $0 \le j < k/2$ and $p|(k-2j)$, define $\omega_j(a^2) = \eta_j^-(a)$.

It follows from §4.1 that $\mathfrak{T}_k := span_{\mathbb{C}_p}\{Trunc_a(\omega_j(a))\}$ is a subspace of $H_{-\pi,k}^{1*}$ of dimension

$$dim_{\mathbb{C}_p}\mathfrak{T}_k = \begin{cases} 1 + \left\lfloor \frac{k}{2p} \right\rfloor & \text{if } 4|k \\ \left\lfloor \frac{k}{2p} \right\rfloor & \text{if } 2|k \text{ but } 4 \nmid k \\ 0 & \text{if } k \text{ odd.} \end{cases}$$

To see this, from §4.1 $\omega_j(a)$ is a solution of $a\frac{d}{da} - G_k$ whose coordinates live in $\widehat{R}$. By Mittag-Leffler, $Trunc_a(\omega_j(a))$ is analytic on $D^-(\infty,1)$, and while $\left(a\frac{d}{da} - G_k\right) Trunc_a(\omega_j(a))$ is not zero, it only consists of positive powers of the variable a. Hence, it is zero when we apply $\partial^*_{-\pi,a}$.

Primitive Cohomology. Define the space

$$PH^{1*}_{-\pi,k} := H^{1*}_{-\pi,k}/(\mathbb{C}_p^{k+1} \oplus \mathfrak{T}_k). \tag{7.1}$$

Using this, we define the subspace $PH^1_{-\pi,k} \subset H^1_{-\pi,k}$ as the annihilator of $\mathbb{C}_p^{k+1} \oplus \mathfrak{T}_k$ in $H^1_{-\pi,k}$ via the pairing. Thus, $PH^{1*}_{-\pi,k}$ and $H^1_{-\pi,k}$ are dual.

Frobenius on the Constant Subspace. Let us show

$$\bar{\beta}^*_{-\pi,k}|_{\mathbb{C}_p^{k+1}} = Sym^k(\mathfrak{A}(0)). \tag{7.2}$$

To prove this, recall $\bar{\beta}^*_{-\pi,k} := Trunc_a \circ Sym^k\mathfrak{A}_{-\pi}(a) \circ \Phi_a$. The operator Φ_a is the identity map on $\mathbb{C}_p^{k+1}$ and $Sym^k\mathfrak{A}_{-\pi}(a)$ is a matrix with entries in $L^\dagger$; in particular, they are power series in the variable a. Therefore, only the constant terms of these power series will have an effect on $\mathbb{C}_p^{k+1}$, everything else would be killed by the truncation operator. That is,

$$\begin{aligned} Trunc_a \circ Sym^k\mathfrak{A}_{-\pi}(a) \circ \Phi_a(\mathbb{C}_p^{k+1}) &= Trunc_a \circ Sym^k\mathfrak{A}_{-\pi}(a)(\mathbb{C}_p^{k+1}) \\ &= Trunc_a \circ Sym^k\mathfrak{A}(0)(\mathbb{C}_p^{k+1}) \end{aligned}$$

which proves (7.2). Further, since $\mathfrak{A}(0)$ is invertible, we have $\bar{\beta}^*_{-\pi,k}(\mathbb{C}_p^{k+1}) = \mathbb{C}_p^{k+1}$.

Frobenius on the Trivial Subspace $\mathfrak{T}_k$. Recall from §4.1 that

$$\mathfrak{A}^*_{-\pi}(a^2)a^{-p/2}V(a^p)S(a^p) = a^{-1/2}V(a)S(a)M$$

where $V(a)$ is the matrix $(u_1 \ \ u_2)$, $S(a) := \begin{pmatrix} e^{\kappa\pi a^3} & 0 \\ 0 & e^{-\kappa\pi a^3} \end{pmatrix}$, and M is a constant matrix given by

$$M = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \pm\sqrt{p} \end{pmatrix} \quad \text{if } p \equiv 1, 7 \bmod(12)$$

(of course, $\sqrt{p}$ may not be the positive square root) and

$$M = \begin{pmatrix} 0 & \sqrt{-g} \\ \pm\sqrt{-g} & 0 \end{pmatrix} \quad \text{if } p \equiv 5, 11 \bmod(12)$$

where "$\pm$" means positive if $p \equiv 1, 5 \bmod(12)$ and negative if $p \equiv 7, 11 \bmod(12)$, and $g := g_2((p^2-1)/3)$.

From this, we may calculate that for $4|k$

$$\bar{\beta}^*_{-\pi,k}(Trunc_a(\omega_j)) = \begin{cases} p^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 1 \text{mod}(12) \\ (-1)^j p^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 7 \text{mod}(12) \\ g^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 5 \bmod(12) \\ (-1)^j g^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 11 \bmod(12). \end{cases}$$

and for $2|k$ but $4 \nmid k$,

$$\bar{\beta}^*_{-\pi,k}(Trunc_a(\omega_j)) = \begin{cases} p^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 1 \text{mod}(12) \\ (-1)^j p^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 7 \text{mod}(12) \\ (-g)^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 5 \bmod(12) \\ (-1)^{j+1}(-g)^{k/2}Trunc_a(\omega_j) & \text{if } p \equiv 11 \bmod(12). \end{cases}$$

This demonstrates that not only is $\bar{\beta}^*_{-\pi,k}$ stable on $\mathfrak{T}_k$, but the basis $\{Trunc_a(\omega_j)\}$ consists of eigenvectors.

Using either Gauss sums or Dwork theory, one may show

$$P_{-\pi,k}(T) := det(1 - \bar{\beta}^*_{-\pi,k}T|\mathbb{C}_p^{k+1}) = \prod_{j=0}^{k}(1 - \pi_1(0)^j\pi_2(0)^{k-j}T)$$

where $L(x^3/\mathbb{F}_p, T) = (1-\pi_1(0)T)(1-\pi_2(0)T)$; note, we are using the (conjugate) splitting function $\exp(-\pi(t-t^p))$ to compute this L-function. Letting $N_{-\pi,k}(T) := det(1 - \bar{\beta}^*_{-\pi,k}T|\mathfrak{T}_k)$, we see that

$$\overline{M}^*_k(T) = \frac{det(1 - \bar{\beta}^*_{-\pi,k}T|H^{1*}_{-\pi,k})}{det(1 - p\bar{\beta}_{-\pi,k}T|H^0_{-\pi,k})} = \frac{P_{-\pi,k}(T)N_{-\pi,k}(T)det(1 - \bar{\beta}_{-\pi,k}T|PH^1_{-\pi,k})}{det(1 - p\bar{\beta}_{-\pi,k}T|H^0_{-\pi,k})}$$

where we have used duality for the second equality and the bar on the left means complex conjugation. Note, conjugation acts as the identity on $M^*_k(T)$ since it has real coefficients. It follows from (5.1) that

$$M_k(T) = \frac{N_{\pi,k}(T)det(1 - \bar{\beta}_{\pi,k}T|PH^1_{\pi,k})}{det(1 - p\bar{\beta}_{\pi,k}T|H^0_{\pi,k})}$$

7.2 Functional Equation

In this section, we will define an isomorphism $\Theta_k : PH^{1*}_{-\pi,k} \to PH^1_{\pi,k}$ which relates the Frobenius $\bar{\beta}_{\pi,k}$ with its conjugate dual $\bar{z}^*_{-\pi,k}$; see equation (7.7).

For convenience, denote by $\mathfrak{A}_{-\pi,k,a}$ the matrix $Sym^k(\mathfrak{A}_{-\pi}(a))$. Dual to $\beta_{-\pi,k} := \psi_a \circ \mathfrak{A}_{-\pi,k,a} : \mathcal{M}^{(k)}_{-\pi,a} \to \mathcal{M}^{(k)}_{-\pi,a}$ is the isomorphism $z^*_{-\pi,k} := Trunc_a \circ \mathfrak{A}_{-\pi,k,a} \circ \Phi_a : \mathcal{M}^{(k)*}_{-\pi,a} \to \mathcal{M}^{(k)*}_{-\pi,a}$. For $\xi^* \in ker(\partial^*_{-\pi,a}|\mathcal{M}^{(k)*}_{-\pi,a})$, we have (in $\mathbb{C}_p[[a^{\pm 1}]]^{k+1}$)

$$\mathfrak{A}_{-\pi,k,a} \circ \Phi_a(\xi^*) = z^*_{-\pi,k}(\xi^*) + \eta \tag{7.3}$$

for some $\eta \in \mathbb{C}_p[a]^{k+1}$.

Recall the space $\mathcal{R}'_{-\pi,a}(b', b)$ from §2.5. Define $\mathcal{R}'_{-\pi,a} := \mathcal{R}'_{-\pi,a}(b', b) \otimes_{L(b')} L^\dagger$. Define

$$\Theta'_{k,a} := -a\frac{d}{da} + G_k : ker(\partial^*_{-\pi,a}|\mathcal{M}^{(k)*}_{-\pi,a}) \to Sym^k(\mathcal{R}'_{-\pi,a}).$$

It will be useful to note that $\Theta'_{k,a}$ is both $\partial^*_{-\pi,a}$ without the truncation map (and hence it p-commutes with $\mathfrak{A}_{-\pi,k,a} \circ \Phi_a$) and $-\partial_{\pi,a}$. Applying $\Theta'_{k,a}$ to (7.3), we have (in $Sym^k(R'_{-\pi,a})$)

$$p\,\mathfrak{A}_{-\pi,k,a} \circ \Phi_a \circ \Theta'_{k,a}(\xi^*) = \Theta'_{k,a} \circ z^*_{-\pi,k}(\xi^*) + \Theta'_{k,a}(\eta). \tag{7.4}$$

Recall from §2.5 the isomorphism $\overline{\Theta}_{-\pi,a} : \mathcal{R}'_{-\pi,a}(b', b^*) \to \mathcal{M}_{\pi,a}(b', b)$ which satisfies $\overline{\Theta}_{-\pi,a^p} = p^{-1}\bar{\alpha}_\pi(a) \circ \overline{\Theta}_{-\pi,a} \circ \bar{\alpha}^*_{-\pi}(a)$. Denoting by $\overline{\Theta}_{-\pi,k,a}$ the k-th symmetric power of $\overline{\Theta}_{-\pi,a}$, and using that the matrix of $\bar{\alpha}^*_{-\pi}(a)$ equals $\mathfrak{A}_{-\pi}(a)$, we have

$$p\mathfrak{A}^{-1}_{\pi,k,a} \circ \overline{\Theta}_{-\pi,k,a^p} = \overline{\Theta}_{-\pi,k,a} \circ \mathfrak{A}_{-\pi,k,a} \tag{7.5}$$

Thus, applying $\overline{\Theta}_{-\pi,k,a}$ to both sides of (7.4), we have

$$p^{k+1}\mathfrak{A}^{-1}_{\pi,k,a} \circ \overline{\Theta}_{-\pi,k,a^p} \circ \Phi_a \circ \Theta'_{k,a}(\xi^*) = \overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}z^*_{-\pi,k}(\xi^*) + \overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}(\eta)$$

which equals

$$p^{k+1}\mathfrak{A}^{-1}_{\pi,k,a} \circ \Phi_a \circ \overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}(\xi^*) = \overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}z^*_{-\pi,k}(\xi^*) + \overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}(\eta).$$

Thus,

$$\overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}(\xi^*) = p^{-(k+1)}\beta_{\pi,k}\overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}z^*_{-\pi,k}(\xi^*) + \beta_{\pi,k}\overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}(\eta). \tag{7.6}$$

Now,

$$\begin{aligned}
\beta_{\pi,k}\overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}(\eta) &= \psi_a \circ p\overline{\Theta}_{-\pi,k,a^p}(\bar{\alpha}^*_{-\pi,k,a})^{-1}\Theta'_{k,a}(\eta) \\
&= p\overline{\Theta}_{-\pi,k,a} \circ \psi_a \circ \Theta'_{k,a^p}(\bar{\alpha}^*_{-\pi,k,a})^{-1}(\eta) \\
&= p^2\Theta'_{k,a}\overline{\Theta}_{-\pi,k,a}\psi_a \circ (\bar{\alpha}^*_{-\pi,k,a})^{-1}(\eta) \\
&\equiv 0 \bmod(\partial_{\pi,a}\mathcal{M}^{(k)}_{\pi,a})
\end{aligned}$$

where the first equality comes from (7.5). Finally, define $\Theta_k : ker(\partial^*_{-\pi,k}|\mathcal{M}^{(k)*}_{-\pi,a}) \to H^1_{\pi,k}$ by

$$\Theta_k(\xi^*) := \overline{\Theta}_{-\pi,k,a}\Theta'_{k,a}(\xi^*) \quad \bmod(\partial_{\pi,a}\mathcal{M}^{(k)}_{\pi,a}).$$

Then (7.6) shows us that

$$p^{k+1}\bar{\beta}^{-1}_{\pi,k}\Theta_k = \Theta_k\bar{z}^*_{-\pi,k}.$$

However, this is not quite the functional equation since the map Θ_k has a kernel.

Lemma 7.1. *$ker\ \Theta_k = \mathbb{C}_p^{k+1} \oplus \mathfrak{T}_k$.*

Proof. Since $z^*_{-\pi,k}(\mathbb{C}_p^{k+1}) = 0$, we see by (7.6) that $\Theta_k(\mathbb{C}_p^{k+1}) = 0$. Next, since $\Theta'_{k,a}$ is simply $\partial^*_{-\pi,k}$ without the truncation map, we have $\Theta'_{k,a}(\mathfrak{T}_k) = 0$, and so, $\Theta_k(\mathfrak{T}_k) = 0$.

Suppose $\xi^* \in \mathcal{M}^{(k)*}_{-\pi,a}$ is in the kernel of Θ_k and is not constant. Then there exists $h \in \mathcal{M}^{(k)}_{\pi,a}$ such that $\Theta_k(\xi^*) = \partial_{\pi,a}(h)$. Now, $\overline{\Theta}_{-\pi,k,a}\Theta'_{k,a} = \Theta'_{k,a}\overline{\Theta}_{-\pi,k,a}$, and so

$$\left(a\frac{d}{da} - G_k\right)(-\xi^* + \tilde{h}) = 0$$

where $\tilde{h} := \overline{\Theta}^{-1}_{-\pi,k,a}(h)$. Thus, $-\xi^* + \tilde{h}$ corresponds to a solution in $ker(l_k|\widehat{R})$. But these solutions are in one-to-one correspondence with $\mathfrak{T}_k$. ☐

Thus, if we restrict to $\bar{z}^*_{-\pi,k}$ on $PH^{1*}_{-\pi,k}$ and $\bar{\beta}_{\pi,k}$ on $PH^1_{\pi,k}$, then $\Theta_k : PH^{*1}_{-\pi,k} \to PH^1_{\pi,k}$ is an isomorphism which satisfies

$$p^{k+1}\bar{\beta}^{-1}_{\pi,k}\Theta_k = \Theta_k \bar{z}^*_{-\pi,k}. \tag{7.7}$$

Corollary 7.2. *$dim_{\mathbb{C}_p} PH^1_k = dim_{\mathbb{C}_p} PH^{1*}_{-\pi,k} \le k$.*

Proof. From the explicit description of the matrix G_k, the image of the *injective* function $\Theta'_k : PH^{1*}_{-\pi,k} \to \mathbb{C}_p[a]^{k+1}$ is contained within the subspace $\{0\} \times a\mathbb{C}_p^k$ which has dimension k. ☐

8 Newton Polygon of $M_k(T)$

For k odd and $k < p$, from Theorem 6.3 and Section 7.1, we have

$$M_k(T) = det(1 - \bar{\beta}_k T | PH^1_k)$$

where

$$PH^1_k = \bigoplus_{j=0}^{(k-1)/2} (\mathbb{C}_p a) v^{k-2j} w^{2j}.$$

Using this explicit description of the cohomology, and the effective decomposition theorem (Theorem 6.3), we may prove the following.

Theorem 8.1. *Suppose k is odd and $k < p$. Writing*

$$M_k(T) = 1 + c_1 T + c_2 T^2 + \cdots + c_{(k+1)/2} T^{(k+1)/2}$$

we have

$$ord(c_m) \ge \frac{(p-1)^2}{3p^2}(m^2 + m + mk).$$

for every $m = 0, 1, \ldots, (k+1)/2$.

Proof. Let $b := (p-1)/p$ and $b' := b/p$. From §2.6, we know $\alpha(a)x^i \equiv \mathfrak{A}_{i1}(a)x + \mathfrak{A}_{i2}(a)x^2 \mod(D_a\mathcal{K}(b', b))$ with $\mathfrak{A}_{ij} \in L(b'; \frac{b'}{3}(pj - i))$. From §6.2, we know $\{av^{k-2i}w^{2i}\}_{i=0}^{(k-1)/2}$ is a basis of PH^1_k. Since $v := x$ and $w := x^2$,

$$\bar{\alpha}_k(a)(av^{k-2i}w^{2i}) = a(\bar{\alpha}(a)v)^{k-2i}(\bar{\alpha}(a)w)^{2i} = \sum_{r=0}^{k} aB_r(a)v^{k-r}w^r$$

where

$$B_r(a) := \sum_{\substack{n+m=k-r \\ n=0,1,\ldots,k-2i \\ m=0,1,\ldots,2i}} \binom{k-2i}{n}\binom{2i}{m}(\mathfrak{A}_{11})^n(\mathfrak{A}_{21})^{k-2i-n}(\mathfrak{A}_{12})^m(\mathfrak{A}_{22})^{2i-m}.$$

After some calculation, we see that

$$aB_{2r} \in L(b'; \frac{b'}{3}[(p-1)k + pr - 2i] - 2b'/3) = L(b'; \frac{b'}{3}[(p-1)k - 2i] + \delta_{r+1} - \frac{b}{3} - \frac{2b'}{3}).$$

where $\delta_{r+1} := \frac{b(r+1)}{3}$. Using notation from §6.2, this means

$$\bar{\alpha}_k(a)(av^{k-2i}w^{2i}) \in \mathcal{M}_{a^p}^{(k)}(b'; \boldsymbol{\delta} + \frac{b'}{3}[(p-1)k-2i] - \frac{b}{3} - \frac{2b'}{3}).$$

Consequently,

$$\beta_k(av^{k-2i}w^{2i}) \in \mathcal{M}_a^{(k)}(b; \boldsymbol{\delta} + \frac{b'}{3}[(p-1)k-2i] - \frac{b}{3} - \frac{2b'}{3}).$$

It follows from Lemma 6.7 that for some constants A_{ij} we may write

$$\beta_k(av^{k-2i}w^{2i}) \subset \sum_{j=0}^{(k-1)/2} A_{ij} av^{k-2j}w^{2j} + \partial_a \mathcal{M}_a^{(k)}(b)$$

with the sum being an element of $V_k(b; \boldsymbol{\delta} + \frac{b'}{3}[(p-1)k-2i] - \frac{b}{3} - \frac{2b'}{3})$. Therefore,

$$A_{ij}a \in L(b; \delta_{2j+1} + \frac{b'}{3}[(p-1)k-2i] - \frac{b}{3} - \frac{2b'}{3}),$$

and so,

$$ord(A_{ij}) \geq \frac{2b'}{3}(pj-i) + \frac{b'}{3}(p-1)k + \frac{2}{3}(b-b').$$

Let us rewrite this. Let $\xi \in \mathbb{C}_p$ such that $ord_p(\xi) = 2b'/3$. Fix $\tilde{\xi} := \xi^{p-1}$. Then, if we had used the basis $\{\xi^i av^{k-2i}w^{2i}\}_{i=0}^{(k-1)/2}$, then

$$\bar{\beta}_k(\xi^i av^{k-2i}w^{2i}) = \sum_{j=0}^{(k-1)/2} \left(\xi^{i-j}A_{ij}\right)\xi^j av^{k-2j}w^{2j}.$$

Hence, the matrix of $\bar{\beta}_k$ on PH_k^1, with respect to this new basis, takes the form

$$\text{matrix of } \bar{\beta}_k = p^{\frac{b'}{3}(p-1)k+\frac{2}{3}(b-b')} \begin{pmatrix} B_{00} & \tilde{\xi}B_{01} & \cdots & \tilde{\xi}^{(k-1)/2}B_{0,(k-1)/2} \\ B_{10} & \tilde{\xi}B_{11} & \cdots & \tilde{\xi}^{(k-1)/2}B_{1,(k-1)/2} \\ \vdots & \vdots & \vdots & \vdots \\ B_{(k-1)/2,0} & \tilde{\xi}B_{(k-1)/2,1} & \cdots & \tilde{\xi}^{(k-1)/2}B_{(k-1)/2,(k-1)/2} \end{pmatrix}, \tag{8.1}$$

where $\tilde{\xi}^j B_{ij} = \xi^{i-j}A_{ij}$, and so, $ord_p(B_{ij}) \geq 0$. It follows that, if we write

$$det(1 - \bar{\beta}_k T | PH_k^1) = 1 + c_1 T + \cdots + c_{(k+1)/2}T^{(k+1)/2}$$

then since $b := (p-1)/p$ and $b' := b/p$,

$$ord(c_m) \geq \frac{(p-1)^2}{3p^2}(m^2 + m + mk).$$

□

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