The Ihara zeta function for graphs and 3-adic convergence of the Sierpiński gasket

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1 Introduction

Imagine you were going for a run downtown. You have a set distance you want to go, but you don't like running the same path two days in a row. You don't like stopping to turn around and take the same road since this breaks your stride, and just repeating some loop multiple times makes you bored. So, how many options do you have? This situation could be modeled with graph theory, where each intersection is a vertex and each road is an edge. The Ihara zeta function counts these special paths, which we call prime paths. Given a finite graph, G, the number of prime paths of a specific length k is finite. Finding the number of prime paths for large k can be immensely difficult. So, with the use of a generating function like the Ihara zeta function, we have the ability to more easily analyze something discrete in a continuous manner, and thus have the ability to use the full power of complex, real, and p-adic analysis to study it.

The Ihara zeta function was first introduced in the 1960's by Yasutaka Ihara. In it's original form, it is an infinite product, given by $\zeta(u, G) = \prod(1 - u^{v(P)})^{-1}$ where the product runs over the set of equivalence classes of prime paths in the graph. Bass proved that this function is in fact a rational polynomial, and, most importantly, given by data obtainable from the graph, including the adjacency matrix, the degree of each vertex, and the size of the edge and vertex sets of the graph. This now gives us a simple way to count the number of prime paths of a large length k without needing to do it by hand. We also have the ability to compare graphs through their respective zeta functions. For example, Storm in [4] looked at comparison of clique numbers, counting Hamiltonian cycles, and determining if a graph is perfect or chordal, although for this he utilized the edge based equivalent of the Ihara zeta function which was introduced by Stark and Terras.

Here, we will look at two aspects of the Ihara zeta function. We will look at the zeta function of a fractal, the Sierpiński gasket. Using Maple 10, we have created a program which will create the m^{th} iteration of the gasket, which has been included in the appendix. Although the Ihara zeta function is only defined for a finite graph, we will approximate it using a limiting approach of graphs and strive to show that it converges 3-adically. This is not the first time a fractal graph has been studied from the point of view of the Ihara zeta function. However, where [2] has normalized the zeta function so that it converges on the complex plane, we will look at the zeta function from the p-adic point of view which will not necessiate changing the function at all. Finally, we will look at the zeta functions of a graph, G, and it's unramified covering, H, and show that the zeta function of the covering is always divisible by that of the graph. Though Terras already proved this in her paper [5], we will expand the proof to fill in the details. We will also offer a specific example, a d-sheeted covering of the graph consisting of the complete graph of 4 vertices and show that divisibility holds.

2 Ihara Zeta Function

We must begin with some preliminary work to be able to define the function. First, we will assume that all graphs we are working with are undirected, leafless, finite (for the time being), and connected. Then, given G, a graph with vertex set $V(G)$ and edge set $E(G)$, a path $P = a_1 a_2 \cdots a_n$ where $a_i \in E(G)$, has a backtrack if $a_{j+1} = a_j^{-1}$ for any $j \in \{1, ..., n-1\}$. That is to say, it crosses the same edge twice in a row. A tail is a backtrack where $a_n = a_1^{-1}$. Really, this is equivalent to saying that the ending edge and beginning edge are the same.

So, we can define the prime path which the Ihara zeta function counts. A prime path $C = a_1 \cdots a_n$ is a closed path in the graph G such that there are no backtracks or tails, and $C \neq D^f$ for some closed path D. We can see by the definition that this still allows for repetitions of edges in G. As such, there are an infinite number of prime paths possible in any graph that has a closed path and which is not a cycle. We are also given an infinite number of options for new prime paths given two or more other prime paths. This is because although D^f cannot be considered a prime path, CD is, as well as CD^2, and so forth.

We can also place an equivalence relation on this set of prime paths in the graph. Two closed paths are equivalent if they are the same path with a different starting point. Note, however, that this means that the path $P = a_1 a_2 \cdots a_{n-1} a_n$ is not equivalent to the same path traversed backwards, $P^{-1} = a_n a_{n-1} \cdots a_2 a_1$. It follows that the equivalence class of C, $[C]$, is the set $[C] = \{a_1 \cdots a_n, a_2 \cdots a_n a_1, \cdots, a_n a_1 \cdots a_{n-1}\}$ which is the prime path started at each point in the path.

The Ihara Zeta Function is

$$\zeta(u, G) = \prod_{[P]} (1 - u^{v(P)})^{-1} \tag{2.1}$$

with $v(P)$ as the number of vertices in $[P]$, and where the product runs over all prime equivalence classes. In this way we have an infinite product to work with. Bass proved that this infinite product is in fact a rational function. Let A be the adjacency matrix of G, where the adjacency matrix is the $|V(G)| \times |V(G)|$ matrix such that for $v_i, v_j \in V(G)$,

$$a_{ij} = \begin{cases} \text{number of undirected edges connecting } v_i \text{ to } v_j, & \text{if } i \neq j \\ 2 \times \text{number of loops at } v_i, & \text{if } i = j \end{cases}$$

The graphs we are working with have no loops or multiple edges, so A will only have entries of 0 (if vertices v_i and v_j are not adjacent) and 1 (if they are adjacent). Let Q be the diagonal matrix such that q_i is the degree of the j^{th} vertex minus 1, and let $r - 1 = |E(G)| - |V(G)|$. Then the Ihara zeta function is equivalent to

$$\zeta(u, G)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2). \tag{2.2}$$

Let us look at the logarithmic derivative of the Ihara zeta function, $u \frac{d}{du} \log[\zeta(u, G)]$. If we begin by taking the log:

$$\log \zeta(u, G) = \log \left(\prod_{[P]} (1 - u^{v(P)})^{-1} \right) = -\sum_{[P]} log(1 - u^{v(P)})$$

and next the derivative:

$$\begin{aligned} \frac{d}{du} \left[-\sum_{[P]} \log(1 - u^{v(P)}) \right] &= -\sum_{[P]} \frac{1}{1 - u^{v(P)}} (-v(P)) u^{v(P)-1} \\ &= \sum_{[P]} \frac{(v(P)) u^{v(P)-1}}{1 - u^{v(P)}} \end{aligned}$$

and finally multiply by u:

$$u \frac{d}{du} \log[\zeta(u, G)] = u \sum_{[P]} \frac{(v(P)) u^{v(P)-1}}{1 - u^{v(P)}} = \sum_{[P]} \frac{(v(P)) u^{v(P)}}{1 - u^{v(P)}}$$

Using the geometric series identity, $\sum_{n=0}^{\infty} u^n = \frac{1}{1-u}$, we have:

$$\begin{aligned} u \frac{d}{du} \log[\zeta(u, G)] &= \sum_{[P]} v(P) u^{v(P)} (1 + u^{v(P)} + u^{2v(P)} + ...) \\ &= \sum_{[P]} v(P) (u^{v(P)} + u^{2v(P)} + u^{3v(P)} + ...) \end{aligned}$$

If we let $M_k = \sum v(P)$ where the sum runs over all $[P]$ such that $v(P)|k$ then we have:

$$u\frac{d}{du}\log\left(\zeta(u,G)\right) = \sum_{k=1}^{\infty} M_k u^k \tag{2.3}$$

What this means is that the coefficient of the term u^k will count the number of prime paths with a number of vertices which divides k. This is extremely useful if we look at a prime k but will be more difficult to interpret for composite k. Let us see an example.

K_4 is the complete graph with 4 vertices. The A and Q matrices are

$$A_{K_4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad Q_{K_4} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$(I - Au + Qu^2) = \begin{pmatrix} 1+2u^2 & -u & -u & -u \\ -u & 1+2u^2 & -u & -u \\ -u & -u & 1+2u^2 & -u \\ -u & -u & -u & 1+2u^2 \end{pmatrix}$$

Thus by (2.2)

$$\begin{aligned}\zeta(u,K_4)^{-1} &= 16u^{12} - 24u^{10} - 16u^9 - 3u^8 + 24u^7 + 16u^6 - 6u^4 - 8u^3 + 1 \\ &= (-1+2u)(u+1)^2(u-1)^3(2u^2+u+1)^3\end{aligned} \tag{2.4}$$

Let us apply (2.3). Let $g(u) = (\zeta(u,G))^{-1}$, thus $g(u)$ for K_4 is equal to equation (2.4) above. Then

$$u\frac{d}{du}\log[\zeta(u,G)] = \frac{-u \cdot g'(u)}{g(u)},$$

so we can use Maple 10 to give us

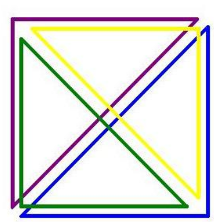
$$u\frac{d}{du}\log[\zeta(u,K_4)] = \frac{-24u^3(2u^2-1)}{(u^2-1)(4u^3+u-1)}.$$

We can also make Maple give us the Taylor series approximation, so that we can see how this equation can take the form of equation (2.3). So, if we set the above equation to $h(u)$, then by typing in "taylor($h(u), u = 0, 120$)," Maple spits out

$$u\frac{d}{du}\log[\zeta(u,K_4)] = 24u^3 + 24u^4 + 96u^6 + 168u^7 + 168u^8 + 528u^9 + 1200u^{10} + 1848u^{11} + O(u^{12}) \tag{2.5}$$

Take for example the coefficient of u^3, which is M_3. Since 3 is prime, we don't have to worry about totaling all possible $v(P)|3$. We need only look at any prime path in K_4 which has 3 vertices contained in it. This means we are looking at any cycle C_3. If we look at K_4, we can see that it contains 4 triangles as in Figure 1. In this

Figure 1: The 4 triangles contained in K_4.



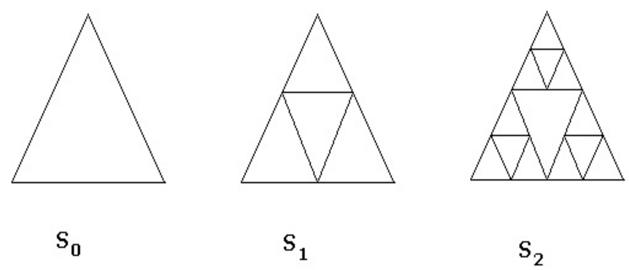
case, $M_3 = \sum v(P)$ over the equivalence classes of the four triangles, $[P_1], [P_2], [P_3], [P_4]$, and their respective backwards paths, $[P_1^{-1}], [P_2^{-1}], [P_3^{-1}], [P_4^{-1}]$. So, $\sum v(P) = 8 \cdot 3 = 24$ since we have 8 equivalence classes, each with $v(P) = 3$. And, as seen in equation (2.5), we have exactly the coefficient of the term u^3.

3 The Sierpiński Gasket

We will prove that a graphic representation of the Sierpiński gasket has an Ihara zeta function which converges 3-adically.

The Sierpiński gasket (Figure 2) is constructed by starting with the triangle C_3, creating three copies of it and attaching them at the vertices. Then, we take this graph, create three copies of it, and attach them at the points of the larger outside triangle. For this case, the points of the graph are the three vertices which have degree 2. Every other vertex in the graph will have degree 4. We use this fact to create a program for Maple 10 which creates each iteration of the gasket. This program has been included in the appendix.

Figure 2: Three iterations of the Sierpinski gasket.



We can also use Maple 10 to calculate some examples of the zeta function. If we look at the Taylor approximation about 0 for the first through the fifth approximations of the gasket, we see that

$$\zeta(u,S_1)^{-1} = 1 + 8\,u^3 + 6\,u^4 + 6\,u^5 + 56\,u^6 + 72\,u^7 + 111\,u^8 + 432\,u^9 + O(u^{10})$$
$$\zeta(u,S_2)^{-1} = 1 + 24\,u^3 + 18\,u^4 + 18\,u^5 + 374\,u^6 + 546\,u^7 + 963\,u^8 + 5690\,u^9 + O(u^{10})$$
$$\zeta(u,S_3)^{-1} = 1 + 72\,u^3 + 54\,u^4 + 54\,u^5 + 2862\,u^6 + 4254\,u^7 + 6609\,u^8 + 87966\,u^9 + O(u^{10})$$
$$\zeta(u,S_4)^{-1} = 1 + 216\,u^3 + 162\,u^4 + 162\,u^5 + 24150\,u^6 + 36114\,u^7 + 52059u^8 + 1894026\,u^9 + O(u^{10})$$
$$\zeta(u,S_5)^{-1} = 1 + 648\,u^3 + 486\,u^4 + 486\,u^5 + 212430\,u^6 + 318318u^7 + 445017\,u^8 + 47223822\,u^9 + O(u^{10})$$

First let us look at the logarithmic derivative of the Ihara zeta function. Recall from section 2 that $u\frac{d}{du}\log\left(\zeta(u,G)\right) = \sum_{k=1}^{\infty} M_k u^k$ where $M_k = \sum v(P)$ and the sum runs over all $[P]$ such that $v(P)|k$. Let $M_k^{(m)}$ be the k^{th} coefficient of $u\frac{d}{du}\zeta(u,S_m)$. The first few functions are

$$u\frac{d}{du}\log\left(\zeta(u,S_1)\right) = 24\,u^3 + 24\,u^4 + 30\,u^5 + 144\,u^6 + 168\,u^7 + 360\,u^8 + 1068\,u^9 + O(u^{10})$$
$$u\frac{d}{du}\log\left(\zeta(u,S_2)\right) = 72\,u^3 + 72\,u^4 + 90\,u^5 + 516\,u^6 + 798\,u^7 + 2952\,u^8 + 8982\,u^9 + O(u^{10})$$
$$u\frac{d}{du}\log\left(\zeta(u,S_3)\right) = 216\,u^3 + 216\,u^4 + 270\,u^5 + 1620\,u^6 + 2562\,u^7 + 10104\,u^8 + 30618\,u^9 + O(u^{10})$$
$$u\frac{d}{du}\log\left(\zeta(u,S_4)\right) = 648\,u^3 + 648\,u^4 + 810\,u^5 + 4932\,u^6 + 7854\,u^7 + 31560\,u^8 + 95526\,u^9 + O(u^{10})$$
$$u\frac{d}{du}\log\left(\zeta(u,S_5)\right) = 1944\,u^3 + 1944\,u^4 + 2430\,u^5 + 14868\,u^6 + 23730\,u^7 + 95928\,u^8 + 290250\,u^9 + O\left(u^{10}\right)$$

First, let us look at $k=3$. For $k=3$, it should be noted that $M_3^{(m)}$ is the same as the number of equivalence classes of triangles in the graph S_m times 3. If we look at S_1, we can see that there are 4 triangles, and so there are 8 equivalence classes, and thus we have $M_3^{(1)} = 8\cdot 3 = 24$. In S_2, we have taken these 8 equivalence classes and repeated them 3 times, since they appear in each copy of S_1. So, $M_3^{(2)} = 3(M_3^{(1)}) = 3\cdot 24 = 72$. Likewise, $M_3^{(3)} = 3(M_3^{(2)}) = 216$. Because of the construction of the Sierpiński gasket, we will always have that $M_3^{(m)} = 3(M_3^{m-1}) = 8\cdot 3^m$. Let us look at the 3-adic numbers to understand this.

***p*-adic Numbers:**

There is nothing interesting about $M_3^{(m)}$ in the real numbers, since this just goes to infinity. So, let's look at it in the 3-adic numbers. What this does is place a new metric on our numbers with the end goal of finding convergence. The idea is that a number can be represented as

$$\sum a_i 3^i$$

with $a_i \in \{0, 1, 2\}$

Then we have a new metric. Fix p prime, then $|n|_p = \frac{1}{p^s}$ where $n = mp^s$. This means numbers highly divisible by p end up very small.

Example 3.1. Let $x = \frac{20}{21} = \frac{2^2 \cdot 5}{3 \cdot 7}$. Then $|x|_3 = 3, |x|_2 = \frac{1}{4}, |x|_5 = \frac{1}{5}$, and $|x|_7 = 7$.

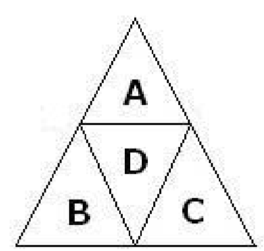
In $\mathbb{R}$, $\frac{1}{1-x} = 1 + x + x^2 + \cdots \Leftrightarrow |x| < 1$. However, with the new metric, these are convergent for numbers highly divisible by p. So in the 3-adic numbers, $1 + 3 + 3^2 + \cdots = \frac{1}{1-3} = -\frac{1}{2}$.

So, we can see that in the 3-adic numbers, M_3 converges to 0 since $|8 \cdot 3^m|_3 = \frac{1}{3^m}$.

More coefficients:

Similarly, $k = 4$ and $k = 5$ are of the form $M_k^{(m)} = 3(M_k^{m-1})$. However, the case where $k = 6$ is far more interesting. Because the minimum size of a prime path is 3, $M_6^{(m)} = 3 \cdot (\text{number of } [P] \text{ of length } 3) + 6 \cdot (\text{number of } [P] \text{ of length } 6)$. So, looking at S_1, we know that there are 8 prime paths of length 3. Now, let us look at prime paths of length 6. These will either be made up of two triangles, or will be C_6, the cycle of length 6. Let the triangles be labeled as in Figure 3, we will let $[A]$ denote triangle A followed clockwise. Then we can see that there are 18 equivalence classes: 12 which consist of pairs of A, B, and C, and 6 which consist of B paired with one of the other 3. These paths are $[AB], [AB^{-1}], [AC], [AC^{-1}], [BC], [BC^{-1}]$ and their respective inverses, and $[DA^{-1}], [DB^{-1}], [DC^{-1}]$ and their respective inverses. Then there are two classes of C_6, consisting of the outer triangle, and the outer triangle followed clockwise and counterclockwise. Thus we have $M_6^{(1)} = 3 \cdot 8 + 6 \cdot (18 + 2) = 144$. So now we look at S_2. As we saw previously, there are $8 \cdot 3 = 24$ prime paths of length 3. As to paths of

Figure 3: Triangles in S_1



length 6, we have 3 copies of each of the 20 we had in S_1, 2 from the cycle C_6 in the large inner triangle, and then 12 completely new ones which were not counted in S_1. These are the prime paths which are comprised of two triangles where each are in a separate copy of S_1 which was used to build S_2 and go through the vertex which connects the two copies in S_2. Then we have

$$M_6^{(2)} = 3(3 \cdot 8) + 6(3 \cdot 20 + 2 + 4 \cdot 3).$$

Next we look at S_3. Again, we have 3 times as many prime paths of length 3 as in S_2. There will also be 3 copies of each prime path of length 6 from S_2, but this time we only have 12 new paths to count from the paths which cross two copies of S_2 since the inner triangle is now long enough that a path of length 6 will not go around it. So we have

$$M_6^{(3)} = 3(3^2 \cdot 8) + 6[3(3 \cdot 20 + 2 + 4 \cdot 3) + 12].$$

Now that we have reached a large enough m, S_m will have 3 times the number of prime paths of length 6 in S_{m-1} plus the 12 new ones created between the copies of S_{m-1}. So doing some simple algebra, and noting that the number of prime paths of length 6 in S_{m-1} is 3 times the number of prime paths of length 6 in S_{m-2} plus 12. So for a large enough m, we know

$$\begin{aligned} M_6^{(m)} &= 3|P_3^{(m-1)}| + 6|P_6^{(m)}| \\ |P_6^{(m)}| &= 3|P_6^{(m-1)}| + 12 \\ |P_6^{(m-1)}| &= \frac{1}{6}\left(M_6^{(m-1)} - 3^m \cdot 8\right) \\ \Rightarrow M_6^{(m)} &= 3M_6^{(m-1)} + 72 \end{aligned}$$

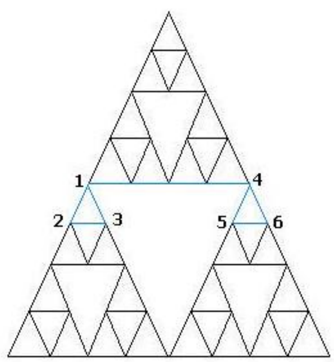
So we can see that as m gets very large, $M_6^{(m)}$ will converge 3-adically to -36.

$$\begin{aligned}
M_6^{(2)} &= 3M_6^{(1)} + 72 \\
M_6^{(3)} &= 3\left(3M_6^{(1)} + 72\right) + 72 \\
M_6^{(m)} &= 3^{m-1}M_6^{(1)} + 3^{m-2}\cdot 72 + 3^{m-3}\cdot 72 + \cdots + 72 \\
&= 3^{m-1}M_6^{(1)} + (3^{m-2} + 3^{m-3} + \cdots + 1)\cdot 72 \\
\lim_{m\to\infty} M_6^{(m)} &= -\frac{1}{2}72 = -36
\end{aligned}$$

Lemma 3.1. *Let L_k be the number of prime paths of length k in S_m which are contained in two copies of S_{m-1} used to create S_m. Then, given k and m such that $k < \min\{2^m+6, 3\cdot 2^{m-1}\}$, L_k is a constant which is independent of m.*

Proof. What needs to be shown is that L_k is not based on S_m for m large enough or, more specifically, that a prime path of length k in S_m cannot be contained in all three copies of S_{m-1}. Suppose we are looking at graph S_m and k such that $k < \min\{2^m + 6, 3\cdot 2^{m-1}\}$. There are two ways in which a prime path could be contained in all three copies. First, it could go around the large, inner triangle. This triangle has length $3\cdot 2^{m-1}$ since each side has length 2^{m-1}. The second is that it could be long enough to cross one side of the large, inner triangle twice, with at least a triangle on each end contained in the other two copies. This can be seen in Figure 4, where the path goes from vertex 1 to 2 to 3, along the edge of the large inner triangle to 4, to 5 to 6 back to 4, and then returning along the edge of the inner triangle back to 1. This path has length $2(2^{m-1}) + 2\cdot 3 = 2^m + 6$. Thus, if a prime path has a length less than both of these, it can be contained in at most two copies. So, for a large enough

Figure 4: The shortest path in a Sierpinski gasket which has edges in all three copies of the previous iteration.



m, each subsequent iteration will have a set number of new paths of length k which were not contained entirely in the previous iteration. Since locally, the area around the vertices where the copies meet will look exactly the same, there will be a set number of prime paths which go through that meeting vertex and are contained in both copies. And so, L_k is a constant which is not based on the graph S_m once m is large enough. □

Lemma 3.2. *Let $|P_k^{(m)}| = \#\{[P] : v(P) = k$ and $[P]$ is in $S_m\}$. If we fix k, $\lim_{m\to\infty}|P_k^{(m)}|$ exists 3-adically and is equal to $-\frac{1}{2}L_k$.*

Proof. The proof follows from Lemma 3.1. Note that once we reach m large enough, $|P_k^{(m)}| = 3|P_k^{(m-1)}| + L_k$ since any prime path in S_m can seen as either a copy of a prime path from S_{m-1} or is contained in two copies of S_{m-1}. Thus

$$|P_k^{(m)}| = 3^{m-1}|P_k^{(1)}| + (3^{m-2} + \cdots + 3 + 1)L_k.$$

So in the 3-adic numbers,

$$\lim_{m\to\infty} = -\frac{L_k}{2}.$$

□

Theorem 3.3. *Let $\{n_1, n_2, \cdots n_t\}$ be the divisors of k, where $n_t = k$, and let $M_k^{(m)}$ be the coefficient of u^k in $u\frac{d}{du}\log(\zeta(u, S_m))$, the logarithmic derivative of the Ihara zeta function of the m^{th} iteration of the Sierpiński* *gasket. Then* $\lim_{m\to\infty} M_k^{(m)}$ *exists 3-adically and is equal to*

$$\sum_{i=1}^{t} n_i L_{n_i}.$$

Proof. From Lemma 3.1, we know that L_k is a constant for a large enough m. From Proposition 3.2 we know that $|P_k^{(m)}| = 3|P_k^{(m-1)}| + L_k$, again, for a large enough m. Recall that $M_k = \sum v(P)$ where the sum runs over all $[P]$ such that $v(P)|k$. We can expand the definition of $M_k^{(m)}$ to see that

$$M_k^{(m)} = \sum_{i=1}^{t} n_i |P_{n_i}^{(m)}|$$

Since we know that $|P_{n_i}^{(m)}| = -\frac{1}{2}L_{n_i}$, we know that 3-adically,

$$\lim_{m\to\infty} M_k^{(m)} = \sum_{i=1}^{t} -\frac{n_i}{2} L_{n_i}$$

□

Now we can look at the Ihara zeta function itself, rather than at the logarithmic derivative. Let F_m be the Ihara zeta function of S_m, the m^{th} iteration of the Sierpiński gasket with the iterations labeled as in Figure 2. Then

$$F_m(u) = \zeta(u, S_m) = \prod_{[P]} \frac{1}{1 - u^{v(P)}} = \prod_{[P]} (1 + u^{v(P)} + \cdots) \tag{3.1}$$

$$= \sum_{D \geq 0} u^{v(D)} = \sum_{k=0}^{\infty} N_k^{(m)} u^k \tag{3.2}$$

where the sum is over all finite formal sums, $D = \sum n_i [P_i]$, with n_i nonnegative integers. Then if we define $v(D) := \sum n_i v(P_i)$, we have $N_k^{(m)} = |\{D \geq 0 | v(D) = k\}|$.

So what the N_k is doing is counting all possible combinations of the form $n_1 v(P_1) + n_2 v(P_2) + \cdots = k$, whether the P_i's are connected or not. This means we are not truly counting prime paths of length k, but how many ways we can combine the lengths of the shorter paths to add up to k. It should also be noted that there are a finite number of prime paths of any length which are not comprised of a combination of shorter prime paths, and paths of length 3, 4, or 5 cannot be created from a combination of shorter prime paths. So, the simplest example to look at in the Sierpiński gaskets are the $N_3^{(m)}$. As shown previously, there are 8 equivalence classes $[P]$ of length 3 in S_1. Since for $k = 3$, we can only directly count equivalence classes, we have $N_3^{(1)} = 8$. Similarly, S_2 has 24 equivalence classes, and thus $N_3^{(2)} = 24$. What we see is that every subsequent S_m will have 3 times as many triangles as the previous, so $N_3^{(m)} = 8 \cdot 3^{m-1}$.

The cases where $k = 4$ or 5 will be similar in that we are counting prime paths of length k, so let us look at the case where $k = 6$ in S_1, which is far more interesting. First, we can look at how many ways we can take these triangles we have been looking at and pair them up. So these are any D of the form $D = 1 \cdot [P_1] + 1 \cdot [P_2]$ where $[P_1]$ and $[P_2]$ are each triangles in S_1. Since we have 8 prime paths in S_1, we have $\binom{8}{2} = 28$ options of D here. Next we have the case where $D = 2 \cdot [P_1]$. So, there are 8 combinations of a path with itself. Finally, we can count the Ds of the form $D = 1 \cdot P_1$, so truly counting the prime paths of length 6. As shown previously, we have 20 prime paths of length 6 in S_1. Thus $N_6^{(1)} = 28 + 8 + 20 = 56$.

Let us look at the next gasket. We have 24 prime paths of length 3 and 74 prime paths of length 6 in S_2. So, we will have $\binom{24}{2} = 276$ options of D of the form $D = 1 \cdot [P_1] + 1 \cdot [P_2]$, 24 options of $D = 2 \cdot [P_1]$, and 74 options of $D = 1 \cdot [P_1]$. Notice that $276 + 24 + 74 = 374 = N_6^{(2)}$.

Theorem 3.4. *Let* $N_k^{(m)} := \#\{D \geq 0 | v(D) = k\}$. *Then* $\lim_{m\to\infty} N_k^{(m)}$ *converges 3-adically.*

Proof. To begin, notice that

$$N_k^{(m)} = |P_k^{(m)}| + \left| \left\{ D \geq 0 |_{\substack{v(D)=k \text{ and } D \text{ consists} \\ \text{of more than one prime path}}} \right\} \right|.$$

From Lemma 3.1 we know that $|P_k^{(m)}|$ converges 3-adically, so we should look at the other portion. We can expand this into

$$\left|\left\{D \geq 0 \Big|^{v(D)=k \text{ and } D \text{ consists}}_{\substack{\text{of more than} \\ \text{one prime path}}}\right\}\right| = \sum_{l=2}^{k} \sum_{\substack{n_i=1 \\ i=1,2,\cdots,l}}^{k} \left\{\text{the number of } D \geq 0 \Big|^{D=n_1[P_1]+\cdots+n_l[P_l]}_{\text{and } v(D)=k}\right\}$$

$$= \sum_{l=2}^{k} \sum_{\substack{n_i=1 \\ i=1,2,\cdots,l}}^{k} \sum_{\substack{a_i>0 \\ \text{such that} \\ n_1a_1+\cdots+n_la_l=k}} \left(|P_{a_1}^{(m)}| + |P_{a_2}^{(m)}| + \cdots + |P_{a_l}^{(m)}|\right)$$

So we can see that $N_k^{(m)}$ is equal to a fixed number of $|P_a^{(m)}|$. Because from proposition 3.2 we know that $\lim_{m\to\infty} |P_{a_i}^{(m)}|$ exists 3-adically, we know that $\lim_{m\to\infty} N_k^{(m)}$ exists 3-adically. □

We still have many questions about the Ihara zeta function of the gasket. Most notably is that we have found an exact way to write out the logarithmic derivative based on discernable data. But just how easy is it to write out?

Question 3.1. Is there a simple way to calculate L_k for any k?

And, even more interesting, is there an easy way to write out the zeta function itself? We do not have a simple number like L_k from the logarithmic derivative to work with. And, once we have this, what else can we say about the function?

Question 3.2. What do the coefficients of the Ihara zeta function of the Sierpiński gasket converge to? What is the radius of convergence of the function? Is it 3-adic meromorphic? Is it a rational function?

Finally, this leads us to wonder to where this idea of p-adic analysis can be extended. Some of the reason that the Sierpiński gasket lends itself so well to a study of prime paths is the method in which the gasket is created. Since there is only a vertex which connects two copies together, prime paths which are contained in two copies are necessarily combinations of shorter prime paths. If we look at an example like the Sierpiński carpet, copies of the previous iteration are attached by far more vertices, and will thus be more complicated to study.

Question 3.3. Are there any other fractals whose Ihara zeta function converges p-adically? If so, is there a way to generalize what p will be necessary?

4 Unramified Coverings

We will now look at unramified coverings of a graph. This concept is quite similar to the idea of a fractal graph, however a covering is not the same as a fractal. The basic idea is that a graph is a cover of another graph if we can create an onto function from the vertex set of the cover to the vertex set of the graph such that neighborhoods are preserved. Although the m^{th} iteration of the Sierpiński gasket is very close to being a covering of the $m-1^{st}$ iteration in that we take reproductions of the graph and connect them, the method of connecting the copies means we do not end up with an integer multiple of the number of vertices from the m^{th} iteration, and so it is not an unramified cover.

A neighborhood of a vertex v in a directed graph G is the set of edges adjacent to v. Then an undirected finite graph H is an unramified covering of an undirected graph G if, after arbitrarily directing the edges of G, there is an assignment of directions to the edges of H and an onto covering map $\pi : H \to G$ sending neighborhoods of H both one to one and onto neighborhoods of G preserving directions. We will say that this cover is d-sheeted if the size of the set $\pi^{-1}(v)$, $v \in V(G)$, which is the set of vertices in H which get mapped to v, is d.

The simplest example is a cycle, C_m. Any graph of the form C_{mn}, $n \in \mathbb{Z}$ is an unramified cover of cycle C_m. Assume we label the vertices in C_{mn} following the cycle. Then the covering map will send vertex $v_i \in V(C_{mn})$ to vertex w_j so long as $i \equiv j \pmod{m}$. Looking at equation (2.1), we can see that cycle C_m has the zeta function $\zeta(u, C_m) = (1-u^m)^{-2}$. This is because there are only two prime path equivalence classes in a cycle, $[C]$ and $[C^{-1}]$. Likewise, cycle C_{mn} has the zeta function $\zeta(u, C_{mn}) = (1-u^{mn})^{-2}$.

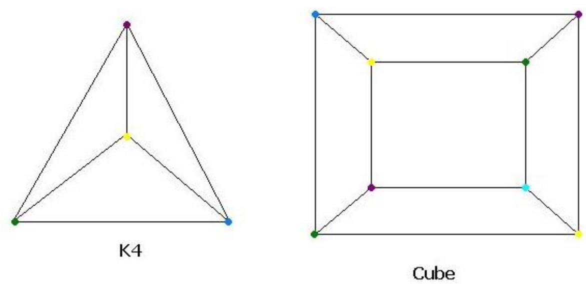
Now, let us look at a slightly more complicated example, K_4. The cube is an unramified covering of K_4, as seen in Figure 5.

We have already shown that

$$\zeta(u, K_4)^{-1} = (-1+2u)(u+1)^2(u-1)^3(2u^2+u+1)^3.$$

Now we can solve for the zeta function of the cube. For terminology ease, let the cube be graph Υ.

Figure 5: The cube as an unramified covering of K_4.



Using Maple, we obtain

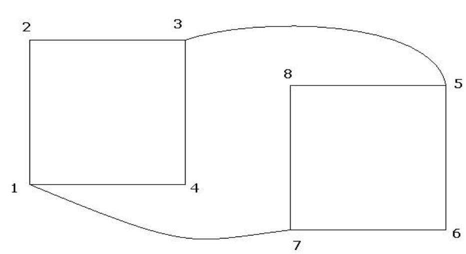
$$\zeta(u,\Upsilon)^{-1} = (2u-1)(2u+1)(2u^2-u+1)^3(2u^2+u+1)^3(u-1)^5(u+1)^5$$

Later, we will show that the zeta function of a graph divides its cover. For the time being, we can see that

$$\frac{\zeta(u,\Upsilon)^{-1}}{\zeta(u,K_4)^{-1}} = (2u+1)(2u^2-u+1)^3(u-1)^2(u+1)^3.$$

Now we can begin to work with a more complicated example. The graph $K_4 - e$ is the complete graph of four vertices with any edge removed. This can be covered by what we will call a necklace. A necklace is a graph N_d which consists of d copies of C_4 connected to each other by a single edge attached to opposite vertices in any copy of the cycle. That is, any vertex of degree 3 is distance 2 from other vertex of degree 3 in the same cycle. Notice that the graph N_d is a d-sheeted covering of $K_4 - e$. So, for starters, we will work with N_2 as in Figure 6. Using equation (2.2), it is easy to compute that

Figure 6: N_2



$$\zeta(K_4-e,\ u)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

Now, if we look at N_2, we have

$$\zeta(N_2,u)^{-1} = (u-1)^3(u+1)^3(2u^2+u+1)(2u^2-u+1)(2u^3-u^2+1)(2u^3+u^2-1)(1+u^2)^2$$

which gives us

$$\frac{\zeta(N_2,u)^{-1}}{\zeta(K_4-e,u)^{-1}} = -(1+u^2)(2u^3-u^2+1)(2u^2-u+1)(u-1)(u+1)^2$$

and again we see that the zeta function of a cover is divisible by the zeta function of the graph.

Now let us look at why these functions are always divisible. Terras proved this in her paper [5] but we will present a more detailed proof.

Theorem 4.1. *Given H, an unramified covering of G, $\zeta(u,G)^{-1}$ divides $\zeta(u,H)^{-1}$.*

Proof. By Bass' theorem, we need to look at

$$\frac{(1-u^2)^{r_H-1}\det(I-A_Hu+Q_Hu^2)}{(1-u^2)^{r_G-1}\det(I-A_Gu+Q_Gu^2)}.$$

Because we form a d-covering graph H by creating multiples of graph G, $|V_H| = d|V_G|$ and $|E_H| = d|E_G|$. Thus $(r_H - 1) = |E_H| - |V_H| = d(|E_G| - |V_G|) = d(r_G - 1)$. Then $r_H - 1 > r_G - 1$, so $(1-u^2)^{r_G-1}$ divides $(1-u^2)^{r_H-1}$

So what remains is to look at

$$\frac{\det(I-A_Hu+Q_Hu^2)}{\det(I-A_Gu+Q_Gu^2)}.$$

First, we need to organize the labeling of vertices of H, so begin by labeling sheets $1\ldots d$ and the vertices $v_1 \ldots v_n$. So if $|V_G| = n$, then we take the n vertices in H corresponding to sheet 1, and label them $\{1\ldots n\}$, then sheet 2 gets $\{(n+1)\ldots 2n\}$, and so on until sheet d gets $\{(d-1)n\ldots dn\}$, making sure that any vertex in H which maps to v_k in G has a label which is equivalent to k mod n.

The adjacency matrix A_H is a mess to define as it is, but since we need only look at the determinant of the end matrix, we can fool with it a bit. Look at just the chunk of the matrix which consists of the first n rows. What we are looking at is all the edges adjacent to sheet 1. In the first n columns of this, we have non-zero entries wherever an edge was directly lifted from G, and is thus adjacent only to vertices in the sheet. If we compare this $n \times n$ matrix with A_G, discrepancies will occur wherever there is an edge in H which goes between sheets of the graph. So, the 1 that occurs as the i, j element in A_G must occur elsewhere in these first n rows of A_H as element i, k, where $k \equiv j$ mod n. Thus, if we were to split this row matrix into $n \times n$ matrices and sum them, we would end up with A_G. Likewise, if we were to look at any other set of n rows or columns pertaining to a sheet, cut it into $n \times n$ matrices and sum them, we will always have A_G returned to us.

Now we can also look at the Q_H and I_H matrices in terms of similar $n \times n$ chunks There are exactly n copies of Q_G in Q_H, as well as n copies of I_G in I_H in the form

$$Q_H = \begin{pmatrix} Q_G & 0 & \cdots & 0 \\ 0 & Q_G & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Q_G \end{pmatrix}, \quad I_H = \begin{pmatrix} I_G & 0 & \cdots & 0 \\ 0 & I_G & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & I_G \end{pmatrix}.$$

So then, if we look at the matrix $X_H = I - A_Hu + Q_Hu^2$, we can take it in d groups of n columns, and add groups 2 through d to group 1. This has no effect on the determinant, and we now have the first block column as

$$\begin{pmatrix} I - A_Gu + Q_Gu^2 \\ \vdots \\ I - A_Gu + Q_Gu^2 \end{pmatrix}$$

So, if we subtract the first block row from each subsequent block row, we will obtain a matrix of the form

$$\begin{pmatrix} (I - A_Gu + Q_Gu^2)_{n\times n} & N_{n\times(dn-n)} \\ 0_{(dn-n)\times n} & M_{(dn-n)\times(dn-n)} \end{pmatrix}$$

where $N_{n\times(dn-n)}$ and $M_{(dn-n)\times(dn-n)}$ are matrices. Then we know that

$$\det(I-A_Hu+Q_Hu^2) = \det(I-A_Gu+Q_Gu^2)\cdot \det M.$$

Thus, we can see that

$$\frac{\zeta(u,H)^{-1}}{\zeta(u,G)^{-1}} = \frac{(1-u^2)^{r_H}\cdot\det(I-A_Hu+Q_Hu^2)}{(1-u^2)^{r_G}\cdot\det(I-A_Gu+Q_Gu^2)} \tag{4.1}$$

$$= (1-u^2)^{r_H-r_G}\det M. \tag{4.2}$$

And so, $\zeta(u,G)^{-1}$ divides $\zeta(u,H)^{-1}$.

□

Now let us see how this works with the example we were working with previously, namely covers of $K_4 - e$ with more than 2 sheets. If we look at N_d, we can label the vertices such that the n^{th} cycle C_4 in the necklace has the numbers $\{4n+1, 4n+2, 4n+3, 4(n+1)\}$, where the vertices are labeled in ascending order around the cycle. Then we connect the cycles by connecting vertex $4n+3$ to vertex $4(n+1)+1$ in the next C_4. This creates a regular labeling which will give us regular matrices. So, if we let $\alpha = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $\gamma = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then we know that for a necklace with d repetitions of C_4, we have the size $4d \times 4d$ matrices

$$A = \begin{pmatrix} \alpha & \beta & 0 & \cdots & 0 & \beta^T \\ \beta^T & \alpha & \beta & 0 & \cdots & 0 \\ 0 & \beta^T & \ddots & \ddots & & \vdots \\ \vdots & 0 & \ddots & \alpha & \beta & 0 \\ 0 & \vdots & & \beta^T & \alpha & \beta \\ \beta & 0 & \cdots & 0 & \beta^T & \alpha \end{pmatrix} \text{ and } Q = \begin{pmatrix} \gamma & 0 & \cdots & 0 \\ 0 & \gamma & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \gamma \end{pmatrix}$$

Now, let X be the matrix $I - Au + Qu^2$. If we look at X in 4×4 chunks, and define it as

$$X = \begin{pmatrix} D & E & 0 & \cdots & 0 & E^T \\ E^T & D & E & & & 0 \\ 0 & E^T & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & E & 0 \\ 0 & & & E^T & D & E \\ E & 0 & \cdots & 0 & E^T & D \end{pmatrix} \tag{4.3}$$

where D and E are 4×4 matrices, then we can see that

$$D = I - \alpha u + \gamma u^2 = \begin{pmatrix} 1+2u^2 & -u & 0 & -u \\ -u & 1+u^2 & -u & 0 \\ 0 & -u & 1+2u^2 & -u \\ -u & 0 & -u & 1+u^2 \end{pmatrix}$$

and

$$E = -\beta u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can also generalize $r-1$. Necklace N_d, then it has $4d$ vertices, and $5d$ edges. Thus, since $r-1 = |E| - |V|$, we have $r - 1 = 5d - 4d = d$. At this point we have

$$\zeta(N_d, u)^{-1} = (1-u^2)^d \det(X)$$

Notice that $\beta + \beta^T + \alpha = A_{K_4 - e}$ and that $\gamma = Q_{K_4 - e}$. Then if we take matrix X and add every column to the first, we have the matrix

$$X = \begin{pmatrix} D+E+E^T & E & 0 & \cdots & 0 & E^T \\ D+E+E^T & D & E & & & 0 \\ D+E+E^T & E^T & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & E & 0 \\ D+E+E^T & & & E^T & D & E \\ D+E+E^T & 0 & \cdots & 0 & E^T & D \end{pmatrix}$$

We can then subtract the first row from every other to obtain

$$X = \begin{pmatrix} D+E+E^T & E & 0 & \cdots & & 0 & E^T \\ 0 & D-E & E & & & & -E^T \\ & E^T-E & D & \ddots & & \vdots & \vdots \\ \vdots & -E & E^T & \ddots & E & 0 & \\ & \vdots & 0 & \ddots & D & E & -E^T \\ & & \vdots & & E^T & D & E-E^T \\ 0 & -E & 0 & \cdots & 0 & E^T & D-E^T \end{pmatrix}$$

Then we can see that $\det(X)$ is equal to $\det(D+E+E^T)$ times a mess of stuff. Using our definitions of D and E from above, we know that $D+E+E^T = (I-\alpha u+\gamma u^2)+(-\beta u)+(-\beta u)^T = I-(\alpha+\beta+\beta^T)u+\gamma u^2$. And, since we know that $\beta+\beta^T+\alpha = A_{K_4-e}$ and that $\gamma = Q_{K_4-e}$, we have that $D+E+E^T = I - A_{K_4-e}u + Q_{K_4-e}u^2$. Thus the determinant of X is divisible by the determinant of $I - A_{K_4-e}u + Q_{K_4-e}u^2$, and so the zeta function of N_d is divisible by the zeta function of K_4-e.

5 Appendix

Sierpiński Gasket Program for Maple

This program uses the property of the Sierpiński gasket that the degree of every vertex is 4 except for the three points. It creates 3 copies of the previous iteration, then connects the copies together by finding the points of each copy. It should be noted that the program requires the package [1] which is far more useful than the networks package which comes with Maple 10. This package is included with Maple 11.

```
Sierp:=proc(b)
local G, G1, G2, H, m, n, f, L, p, q, v, w;
G:=CompleteGraph(3);

for w from 1 to b do

v:=nops(Vertices(G));
G:=RelabelVertices(G, [seq(i, i=1...v)]);
G1:=RelabelVertices(G, [seq(i, i=(v+1)...(2*v))]);
G2:=RelabelVertices(G, [seq(i, i=(2*v+1)...(3*v))]);
H:=DisjointUnion(G,G1,G2):
f:=0;
for n in Vertices(H) do
  for m in Vertices(H) do
      if Degree(H,m)=2 and Degree(H,n)=2
         and Distance(H,m,n)=infinity
        then H:=AddEdge(H, {m,n});
             H:=Contract(H, {m,n});
             if m<n then H:=AddVertex(H, n) else H:=AddVertex(H,m) end if;
       end if:
  od:
od:
for p in Vertices(H) do
  for q in Vertices(H) do
    if Degree(H,p)=2 and Degree(H,q)=2 then
      if Distance(H,p,q)=3*2^(w-1) then
        H:=AddEdge(H,{p,q}) ;
        H:=Contract(H,{p,q});
        if p<q then H:=AddVertex(H, q) else H:=AddVertex(H,p) end if;
      end if;
    end if;
  od:
od:
```

```
for p in Vertices(H) do
  if Degree(H,p)=0 then H:=DeleteVertex(H,p) end if;
od:
G:=H

od:
return(H);
end:
```

Figure 7: S_1 created by Sierp(1)

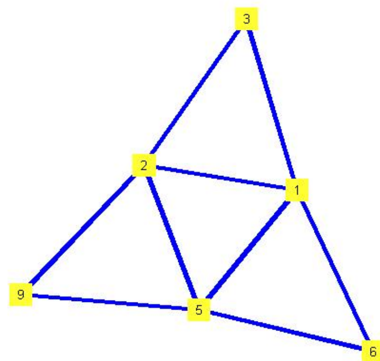
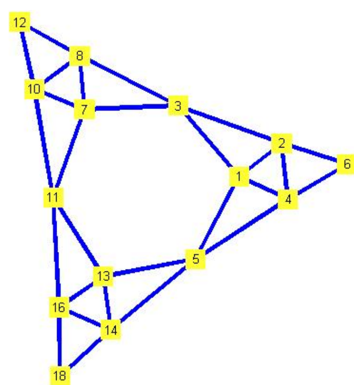


Figure 8: S_2 created by Sierp(2)



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