A REVIEW OF PRIME PATTERNS

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1 Introduction

The first records of studies of prime numbers come from the Ancient Greeks. Euclid proved that there are an infinite number of primes as well as the Fundamental Theorem of Arithmetic which states that every natural number greater than 1 can be written as a unique product of prime numbers. Since the Greeks, prime numbers have been found to apply to more than just pure mathematics, but have applications in cryptography and even animation. As of now, there is no one pattern that can find all prime numbers but there are many other patterns that can find finite sequences of prime and satisfy other conditions. Ben Green of Cambridge University in England and Terry Tao of UCLA in the US published a paper [4] in 2005 which impacted the prime world significantly. They showed that for any integer k there are infinitely many k-term arithmetic progressions of primes, that is, there exist infinitely many distinct pairs of nonzero integers a, d such that $a, a+d, \ldots, a+(k-1)d$ are all primes. A related paper [2] by Antal Balog deals with the prime k-tuplets conjecture on average and Balog squares. Andrew Granville discusses and expands on these two papers in [3].

2 Prime number patterns: Results and examples

Where Green and Tao [4] proved the existence of numerous patterns, Granville [3] sought to find examples of each of these patterns, find the smallest examples, and attempt to predict how large the smallest sample is with some generality. He shows how the results of Green and Tao generate all sorts of mathematically and aesthetically desirable patterns.

2.1 Arithmetic progressions of primes

Before we begin to analyze the various patterns of primes, we shall define the following:
Definition An arithmetic progression of primes is a set of primes of the form $p_1 + kd$ for fixed p_1 and d and consecutive k, i.e., $\{p_1, p_1 + d, p_1 + 2d, ...\}$.

One example of the smallest arithmetic progression of length 5 is given by 5, 11, 17, 23, 29. When we say "smallest" we mean the example in which the largest prime in the set is smallest. If there is a tie, the set in which the second largest prime is smallest.

Length k	Arithmetic Progression ($0 \leq n \leq k-1$)	Last Term
3	3 + 2n	7
4	5 + 6n	23
5	5 + 6n	29
6	7 + 30n	157
7	7 + 150n	907
8	199 + 210n	1669
9	199 + 210n	1879
10	199 + 210n	2089
11	110437 + 13860n	249037
12	110437 + 13860n	262897
13	4943 + 60060n	725663
14	31385539 + 420420n	36850999
15	115453391 + 4144140n	173471351
16	53297929 + 9699690n	198793279
17	3430751869 + 87297210n	4827507229
18	4808316343 + 717777060n	17010526363
19	8297644387 + 4180566390n	83547839407
20	214861583621 + 18846497670n	572945039351
21	5749146449311 + 26004868890n	6269243827111

The k-term arithmetic progression of primes with smallest last term.

Granville asks if it is it is possible to predict the size of the last term or the smallest k-term arithmetic progression of primes. He expects that the smallest k-term arithmetic progression of primes has largest prime around

$$\left(\frac{e^{1-\gamma}k}{2}\right)^{k/2}$$

where γ is the Euler Mascheroni constant defined by $\gamma = \lim_{N\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\ldots+\frac{1}{N}\right) - logN$

2.2 Generalized arithmetic progressions of primes:

Definition Generalized arithmetic progressions of primes (**GAPs**) are sets of integers of the form $a + n_1b_1 + n_2b_2 + \ldots + n_db_d$,

The **GAP** above has dimension d and volume $N_1, \ldots N_d$. While the integers in a **GAP** are not necessarily distinct, there is no linear dependence among the b_js which implies that they must be distinct. It is possible to show that there is a **GAP** that generates distinct integers for any given dimension and volume. All other cases follow from the proof of the dimension 1 case, which is the arithmetic progression we previously discussed.

Proof:

Let $N = max_{1\leq j\leq d}N_j$ and k=N . Suppose that we have a k-term arithmetic progression of primes, $a+jq, 0 \leq j \leq k-1$. Let $b_i = (N_i-1)q$ for each i, so that $a+n_1b_1+n_2b_2+\ldots+n_db_d = a+jq$

where we write j in base N as $j = n_1 + n_2N + n_3N^2 + \ldots + n_dN^d - 1$. Therefore the **GAP** is a subset of our k-term arithmetic progression. Since each j has a unique expansion in base N, no two elements of the **GAP** are equal. Hence the **GAP** is made up entirely of distinct primes, as desired. A few other examples of smallest **GAP**s are:

5	17	29
47	59	71
89	101	113

29	41	53
59	71	83
89	101	113

The 3-by-3 **GAP**s 5 + 12i + 42j and 29 + 12i + 30j .

11	47	83
101	137	173
191	227	263
281	317	353

503	1721	2939	4157
863	2081	3299	4517
1223	2441	3659	4877
1583	2801	4019	5237

The 4-by-3 **GAP** 11 + 90i + 36j , and the 4-by-4 **GAP** 503 + 360i + 1218j .

Let's go through the first 3-by-3 **GAP** given by 5 + 12i + 42j to clarify any doubts. The first element in (i,j)=(0,0) is 5 which is 5 + 12(0) + 42(0). Letting (i,j)=(1,0) gives us 5 + 12(1) + 42(0) which is 17. Lets look at one more, (i,j)=(1,2), 5 + 12(1) + 42(2) =101.

2.3 Balog cubes

Balog cubes are similar to n-by-n **GAP**s. Balog proved that there are infinitely many 3-by-3 squares of distinct primes where each row and each column forms an arithmetic progression. He also proved that there are infinitely many 3-by-3-by-3 cubes of distinct primes where each row and each column and each vertical line forms an arithmetic progression. Balog's concept has been expanded by Green and Tao to include an N-by-N-by- ...-by-N Balog cube of primes. This is due to the nature of any **GAP** of distinct primes with dimension d and $N_1 = N_2 = \ldots = N_d = N$

11	17	23
59	53	47
107	89	71

The smallest 3-by-3 Balog cube of primes.

We can see here that every row and column for an arithmetic progression. The first row is given by 11 + 6n, the second is 59 – 6n, and the third is 107 – 18n. Similarly the columns are given by 11 + 48n, 17 + 36n, and 23 + 24n respectively. This also clearly shows that this cube is not a 3-by-3 **GAP**. Now adding another dimension gives us the following:

47	383	719
179	431	683
311	479	647

149	401	653
173	347	521
197	293	389

251	419	587
167	63	359
83	107	131

A 3-by-3-by-3 Balog cube of primes.

Now we have a 3-by-3-by-3 cube so let us check a few of the vertical lines. The first one is 47, 149, 251 which comes from 47 + 102n. Another one is 479, 293, 107 which is given by 479 – 186n.

2.4 Sets of primes, averaging in pairs:

Another of Balog's results is that statement that there exist arbitrarily large sets $\mathcal{A}$ of distinct primes such that for any a, b$\in \mathcal{A}$ the average $\frac{a+b}{2}$ is also prime (and all of these averages are distinct). This follows from the result of Green and Tao [4]. Suppose that we want $\mathcal{A}$ to have n elements. If we did not mind whether the averages were all distinct then we could take any k-term arithmetic progression of primes, where $k = 2n,\ a + jd,\ 0 \leq j \leq k-1$, and let $\mathcal{A} = \{a + 2jd : 0 \leq j \leq n-1\}$. In this case $\frac{1}{2}((a + 2id) + (a + 2jd)) = a + (i + j)d$ is prime, since whenever $0 \leq i, j \leq n-1$ we have $0 \leq i + j < k - 1$. However, we do want all the averages to be distinct. To do this, we must introduce Sidon sequences.

Definition A Sidon sequences is a sequence of integers $b_1 < b_2 < \ldots < b_n$ in which all of the sums $b_i + b_j, i < j$, are distinct.

n	Set of primes
2	3, 7
3	3, 7, 19
4	3, 11, 23, 71
5	3, 11, 23, 71, 191
6	3, 11, 23, 71, 191, 443
7	5, 17, 41, 101, 257, 521, 881
8	257, 269, 509, 857, 1697, 2309, 2477, 2609
9	257, 269, 509, 857, 1697, 2309, 2477, 2609, 5417
10	11, 83, 251, 263, 1511, 2351, 2963, 7583, 8663, 10691
11	757, 1009, 1117, 2437, 2749, 4597, 6529, 10357, 11149, 15349, 21757
12	71, 1163, 1283, 2663, 4523, 5651, 9311, 13883, 13931, 14423, 25943, 27611

Sets of n primes whose pairwise averages are all distinct primes

Going through n=3, we see that $\frac{3+7}{2} = 5,\quad \frac{7+19}{2} = 13$, and $\quad \frac{3+19}{2} = 11$.

2.5 Sets of primes, averaging all subsets

Now what if we wanted a set of integers $\mathcal{A}$ where each nontrivial subset $\mathcal{S}$ of $\mathcal{A}$ is also a prime. For a set $\mathcal{A}$ of integers and nontrivial subset $\mathcal{S}$ of $\mathcal{A}$, let $\mu_{\mathcal{S}}$ be the average of the values in $\mathcal{S}$. If we did not mind whether the $\mu_{\mathcal{S}}$ were all distinct or not, then we could take any $k(= n(n!))$-term arithmetic progression of primes $a+jd, 0 \leq j \leq k-1$, and let $\mathcal{A} = \{a+j(n!)d : 0 \leq j \leq n-1\}$. Then for any nonempty subset $\mathcal{J}$ of $\{1, 2, \ldots, n\}$, and corresponding $\mathcal{S} = \mathcal{S}\mathcal{J} = \{a + j(n!)d : j \in J\}$, we have $\mu_{\mathcal{S}} = \frac{1}{|J|}\sum_{j\in\mathcal{J}}(a + j(n!)d)$. We can pull out the d and a since neither depend on j to get $\mu_{\mathcal{S}} = a + d\left(\sum_{j\in\mathcal{J}}(j(n!)\right)$. We can rewrite this as $\mu_{\mathcal{S}} = a + d\left(\sum_{j\in\mathcal{J}}(j)\right)\frac{n!}{|\mathcal{J}|}$. This shows that $\mu_{\mathcal{S}}$ is an element of our progression and is also prime since $\frac{n!}{|\mathcal{J}|}$ is an integer and $0 \leq n! \sum_{j \in \mathcal{J}} \left(\frac{j}{|\mathcal{J}|} \right) < n(n!) = k$.

What if we wanted the averages to be distinct? Consider any set $\mathcal{B} = b_1 < b_2 < \ldots < b_n$ of integers which average to $\mu_{\mathcal{S}}$ are all distinct where we let $\mathcal{S} \subset \mathcal{B}, \mathcal{S} \neq \emptyset$. Now, letting $k = (b_n - b_1)n!$, take any k-term arithmetic progression of primes $a+jd$, $0 \leq j \leq k-1$, and then let $\mathcal{A} = \{a + (b_j - b_1)(n!)d : 1 \leq j \leq n\}$. By a similar argument as above, we can show that the averages of any nontrivial subset $\mathcal{S}$ of $\mathcal{B}$ is a prime and that each average is distinct. Here are some examples:

n	Minimal set of primes
2	3, 7
3	7, 19, 67
4	5, 17, 89, 1277
5	209173, 322573, 536773, 1217893, 2484733

2.6 Monochromatic arithmetic progressions of primes

Green and Tao [4] proved the following theorem:

Theorem: Fix any $\delta > 0$ and any integer $k \geq 3$. If x is sufficiently large and if $\mathcal{P}$ is a subset of the primes up to x containing at least $\delta\pi(x)$ elements then $\mathcal{P}$ contains a k-term arithmetic progression of primes.

In this theorem, $\pi(x)$ denotes the number of primes $\leq x$. If we want an arithmetic progression of length k then let $\delta = \frac{1}{r}$ in the result above with x sufficiently large. A nice visualization is if you color the primes with r colors, then there will be a monochromatic arithmetic progressions of primes. If we let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be the partition of the primes up to x into their assigned colors, then at least one of the $\mathcal{P}_j$has at least $\delta\pi(x)$ elements. Therefore it contains a k-term arithmetic progression of primes of color j by the Green-Tao theorem from [4].

2.7 Magic squares of primes

Magic squares are fun little puzzles to solve where there is an n-by-n array of distinct integers so that the sum of any given row, column, or diagonal is the same. Here is one example of a 3-by-3 magic square:

4	9	8
11	7	3
6	5	10

Note that each row, column and diagonal sums to 21. Here is another example :

17	89	71
113	59	5
47	29	101

The sum for this magic square is 177. In this magic square, each entry is a distinct prime and there is a relation between this 3-by-3 magic square and 3-by-3 **GAP**s. Here, the magic square is a rearrangement of the 3-by-3 **GAP** given by 5 + 12i + 42j. Granville [4] claims that there is a 1-to-1 correspondence between 3-by-3 magic squares and 3-by-3 **GAP**s.

Magic squares can be made more complicated with the idea of bi-magic squares. Bi-magic squares are magic squares that when the entries are squared, it also forms a magic square. Here is a 6-by-6 bi-magic square:

17	36	55	124	62	114
58	40	129	50	111	20
108	135	34	64	38	49
87	98	92	102	1	28
116	25	86	7	96	78
22	74	12	81	100	119

Since $\sum (a + m_{i,j}b)^2 = a^2 \sum 1 + 2ab \sum m_{i,j} + b^2 \sum m_{i,j}^2$ we see that if there is at least one n-by-n bi-magic square, then there are infinitely many n-by-n bi-magic squares of primes. The following 4-by-4 magic squares have a very interesting property. The one on the left contains every prime between 31 and 101 while the one on the right contains all primes between 37 and 103.

37	83	97	41
53	61	71	73
89	67	59	43
79	47	31	101

41	71	103	61
97	79	47	53
37	67	83	89
101	59	43	73

3 Concluding remarks

This article surveyed recent developments in the mathematics of prime numbers, specifically results following the Green and Tao paper [4]. By showing there are infinitely many k-tem arithemtic progressions of primes, it was possible to expand this result into the various topics I discussed. Granville did a wonderful job at summarizing and generalizing consequences of [4].

4 References

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