

**An Introduction to Separation of Variables  
with Fourier Series**

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Tim McCrossen

Professor Haessig

**Abstract:** This paper aims to give students who have not yet taken a course in partial differential equations a valuable introduction to the process of separation of variables with an example. After this introduction is given, there will be a brief segue into Fourier series with examples. Then, there will be a more advanced example, incorporating the process of separation of variables and the process of finding a Fourier series solution. There will be three appendices. **Appendix A** will be a glossary of terms, **Appendix B** will contain some useful integrals in Fourier analysis, and **Appendix C** will contain *Mathematica* code and graphics of Fourier series.

## Introduction:

In most every scientific field, one can encounter a particularly challenging class of mathematical problems. These problems, known as differential equations, relate a function to both itself and its derivatives. Differential equations, when occurring in multiple dimensions, are called partial differential equations, and will be the discussion of this paper. Oftentimes, partial differential equations require great skill, powerful computers, or a large amount of luck to solve. However, this paper aims to discuss a certain class of partial differential equations: those that fall under the category of **separable**. These equations can, with a bit of practice, be solved readily by hand, with a process known as **separation of variables**.

## The Form of a Partial Differential Equation:

If you have never seen a partial differential equation before, then the statement “a partial differential equation is a differential equation that occurs in multiple dimensions” may be entirely meaningless. To show what a partial differential equation can look like, the following **linear** differential equation will be presented:

$$(1). \quad \frac{\partial T}{\partial t} = D \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right), 0 < x < a, 0 < y < b, 0 < z < c, t > 0, D \text{ is a constant } > 0$$

In Equation (1), which governs heat flow in two spatial coordinates, it is clear the symbol  $\frac{\partial}{\partial(\ )}$  is used to represent a derivative. This is used, rather than the usual  $\frac{d}{d(\ )}$ , because, in this case, the temperature  $T$  is a function of  $x$ ,  $y$ ,  $z$ , and  $t$ . We now have a partial differential equation for  $T(x, y, z, t)$ , but this does not formulate a fully posed problem.

As you may recall from an early calculus sequence, whenever an indefinite integral is calculated, there will be a constant (or multiple constants, if the function is integrated multiple times) of integration. In partial differential equations, this is no different. If Equation (1) were solved, without knowing anything about the **initial conditions** or **boundary conditions** of the temperature  $T$ , then there would be a number of arbitrary constants. This number of constants, generally speaking, will be equal to the sum of the **order of the derivatives** in each spatial dimension, plus the order of the derivative in the time dimension. So, Equation (1) will yield 7 constants (6 from the sum of the order of the derivative in each spatial dimension, and 1 from the order of the derivative in the time dimension). To solve for these 7 constants, we will then require 6 boundary conditions and 1 initial condition to make Equation (1) a well posed problem:

- (2).  $T(0, y, z, t) = 0$
- (3).  $T(x, 0, z, t) = 0$
- (4).  $T(x, y, 0, t) = 0$
- (5).  $T(a, y, z, t) = 0$
- (6).  $T(x, b, z, t) = 0$
- (7).  $T(x, y, c, t) = 0$
- (8).  $T(x, y, z, 0) = f(x, y, z)$

Equation (1) coupled with Equations (2-8) constitute a full linear, partial differential equation, with homogeneous boundary conditions and an initial temperature distribution of  $f(x, y, z)$ .

### The Separation Process:

The idea of separation of variables is quite simple. When a problem, such as our problem for  $T(x, y, z, t)$  is posed, one can look for product solutions in the form of  $T(x, y, z, t) = E(t) * F(x) * G(y) * H(z)$ . These solutions can be done by inserting  $E(t) * F(x) * G(y) * H(z)$  into the partial differential equation for the variable  $T$ , and then separating the variables so that each side of the equation depends on only one variable. Once the equation has been broken up into separate equations of one variable, the problem can be solved like a normal **ordinary differential equation**.

### A Worked Example:

In order to understand this process, an example concerning the heat conduction in a slab will be worked. This slab heat conduction means that we will be working Equation (1) when the temperature  $T(x, t)$  only depends on the spatial coordinate,  $x$ , and time,  $t$ . Our problem is now as follows:

- (9).  $\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, 0 < x < a$
- (10).  $T(0, t) = 0$
- (11).  $T(a, t) = 0$
- (12).  $T(x, 0) = f(x)$

In Equation (9), it is clear that  $T$  depends on both  $x$  and  $t$ , so we will look for separated solutions of the form  $T(x, t) = F(x) * G(t)$ . Plugging  $F$  and  $G$  into (9) yields:

$$(13). \frac{\partial(FG)}{\partial t} = D \frac{\partial^2(FG)}{\partial x^2}$$

At first, this equation may look complicated. However, one can note that  $F(x)$  depends only on  $x$ , and  $G(t)$  depends only on  $t$ . Thus, we are able to pull  $F(x)$  out of the time derivative and  $G(t)$  out of the  $x$  derivative. Then, dividing by  $D * FG$ , we get:

$$(14). \frac{1}{DG} \frac{dG}{dt} = \frac{1}{F} \frac{d^2F}{dx^2}$$

This is a rather strange equation, though. How can something that depends only on  $t$  (the left hand side of the equation) be equal to something that depends only on  $x$  (the right hand side of the equation)? The answer is simple: this can only occur if each side is equal to a constant, which we will call  $-\lambda$ . Equation (14) then becomes Equation (15), which can be broken up into two separate equations:

$$(15). \frac{1}{DG} \frac{dG}{dt} = \frac{1}{F} \frac{d^2F}{dx^2} = -\lambda, \text{ where } \lambda > 0$$

$$(16). \frac{1}{G} \frac{dG}{dt} = -\lambda D$$

$$(17). \frac{d^2F}{dx^2} = -\lambda F$$

Equations (16, 17), give two ordinary differential equations that can be integrated and solved.

Starting with Equation (16), we note that the equation can be separated into  $x$  and  $t$  equations, and then each side can be integrated. This yields the following, with intermediate steps filled in:

$$(18). \frac{dG}{G} = -\lambda D dt$$

$$(19). \int \frac{dG}{G} = \int -\lambda D dt$$

$$(20). \ln(G) = -\lambda D t + C$$

$$(21). G(t) = A e^{-\lambda D t}$$

So,  $G(t)$  has been solved for, and it depends on a negative exponential in  $t$ .

Next, Equation (17) will be manipulated to solve for  $F(x)$ . Looking at Equation (17), it is clear that the second derivative of  $F(x)$  should yield  $F(x)$  itself multiplied by a negative scalar constant. This can occur with both exponentials and with trigonometric functions. For trigonometric functions the solution is as follows:

$$(22). F(x) = B * \cos(\sqrt{\lambda} x) + C * \sin(\sqrt{\lambda} x)$$

Now that we have a solution for  $F(x)$ , we can begin to use the boundary conditions and the initial condition to solve for the three expected constants, A, B, and C.

Given Equations (10, 11), we can solve for the constants B and C in Equation (22). First, Equation (10) yields the following:

$$(23). T(0, t) = 0 = B$$

Then, Equation (11) yields:

$$(24). T(a, t) = 0 = C * \sin(\sqrt{\lambda} a)$$

In order to satisfy this equality (without having a **trivial** solution), the argument of  $\sin()$ , must be an integer multiple of  $\pi$ , because this is when  $\sin()$  vanishes. This means we have:

$$(25). \sqrt{\lambda} a = n \pi, n \in \mathbb{Z}$$

$$(26). \lambda = \frac{n^2 \pi^2}{a^2}$$

This implies that there exists a countably infinite of set of  $\lambda$  values, and due to the linearity of the problem our product solution  $T(x, t) = F(x) * G(t)$  becomes:

$$(27). T(x, t) = \sum_{n=1}^{\infty} E_n e^{-\frac{n^2 \pi^2 D t}{a^2}} \sin\left(\frac{n \pi x}{a}\right)$$

where the value of  $\lambda$  has been plugged in for both Equation (21) and Equation (22) and  $E_n$  is a constant to be determined from the initial condition.

Now that we have an expression for  $T(x, t)$  that has only one constant, we can solve for this constant by implementing the initial condition. Since  $T(x, 0) = f(x)$ , plugging in  $t = 0$  and setting the equation equal to  $f(x)$ , we get:

$$(28). f(x) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n \pi x}{a}\right)$$

This equation is quite remarkable. We have a non-harmonic function,  $f(x)$ , which is equal to the sum of a series of **harmonic** functions,  $\sin\left(\frac{n \pi x}{a}\right)$ , multiplied by a constant value,  $E_n$ . This is a strange idea which, at first, caught much of the mathematics community by surprise.

### A New Branch of Mathematics:

The idea that a non-harmonic function could be represented by an infinite sum of harmonic functions was, for all intents and purposes, unheard of. That is, until Joseph Fourier introduced a revolutionary paper on the propagation of heat in 1807. In this paper, Fourier not only derived the equation for the transfer of heat, but also claimed that an arbitrary function

could indeed be represented by a sum of harmonic functions (a Fourier series) [Grattan-Guinness, pp. 220].

The proof of the convergence of a Fourier series is out of the scope of this text, however, from this theorem, we can derive two important results [Haberman, pp. 92]:

If  $f(x)$  is **piecewise smooth** on the interval  $-L \leq x \leq L$ , then the Fourier series of  $f(x)$  converges

1. To the **periodic extension** of  $f(x)$ , where the periodic extension is continuous;
2. To the average of the two limits, usually  $\frac{1}{2}[f(x+) + f(x-)]$ , where the periodic extension has a **jump discontinuity**.

where, in these results, the Fourier series of  $f(x)$  is:

$$(29). f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

These are profound results. We now need only a method for calculating  $a_0$ ,  $a_n$  and  $b_n$ , and then we will be able to represent arbitrary functions with a sum of harmonic functions. These coefficients, whose derivations are once again out of the scope of this paper, can be found in the following manner [Haberman, pp. 91]:

$$(30). a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$(31). a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$(32). b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Note: In these integrals, **orthogonality** is a very useful property of functions. To see some common sample integrals that take into account the orthogonality of functions, see **Appendix B**.

### Fourier Series Examples:

To see some of the inner workings of Fourier series calculations, we will go through two examples. First, we will calculate a Fourier series for the function  $f(x) = 3x - x^4$  (from  $-1 < x < 1$  and periodically extended), and we will examine the series' convergence. Then, we will calculate a Fourier series for  $f(x) = x - x^3$  (from  $-1 < x < 1$  and periodically extended), and we will examine the series' convergence.

First, for the function  $f(x) = 3x - x^4$ , we need to calculate the Fourier Coefficients  $a_0$ ,  $a_n$  and  $b_n$ . This is done using Equations (30-32), and the results are tabulated below:

$$(33). a_0 = \frac{1}{2L} \int_{-L}^L 3x - x^4 dx = -\frac{1}{5}$$

$$(34). \quad a_n = \frac{1}{L} \int_{-L}^L 3x - x^4 \cos\left(\frac{n\pi x}{L}\right) dx = -\frac{8(-1)^n(-6+n^2\pi^2)}{n^4\pi^4}$$

$$(35). \quad b_n = \frac{1}{L} \int_{-L}^L 3x - x^4 \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{6(-1)^n}{n\pi}$$

The calculation of these Fourier Coefficients leaves us with a fully defined Fourier series for  $f(x) = 3x - x^4$ . The results of this Fourier series calculation, along with a number of plots for the viewing of convergence, can be seen in **Part A** of **Appendix C**. Note: In these plots, you can see a sort of “ringing” at the edges of the discontinuities. This phenomenon is called **Gibb’s Phenomenon**, and is unavoidable, even at very large term counts.

The next function for which we will calculate a Fourier series is  $f(x) = x - x^3$ . This is once again done by using Equations (30-32), and the results are tabulated below:

$$(36). \quad a_0 = \frac{1}{2L} \int_{-L}^L x - x^3 dx = 0$$

$$(37). \quad a_n = \frac{1}{L} \int_{-L}^L x - x^3 \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$(38). \quad b_n = \frac{1}{L} \int_{-L}^L x - x^3 \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{12(-1)^n}{n^3\pi^3}$$

This calculation of the Fourier Coefficients once again leaves us with a fully defined Fourier series for  $f(x) = x - x^3$ . The results of this Fourier series calculation, along with plots, can be seen in **Part B** of **Appendix C**.

### **A Note on Fourier Series Convergence:**

As you may have noticed, looking at the plots of the convergence of the Fourier series for  $f(x) = 3x - x^4$  and  $f(x) = x - x^3$ , the Fourier series for the latter converges much more rapidly than that of the former. This is because of the following convergence guideline for a Fourier series [Al Clark, Math 281]:

*Suppose  $f_{ext}(x)$ , the periodic extension of  $f(x)$ , is piecewise smooth, and suppose  $f_{ext}$ ,  $f'_{ext}$ ,  $f''_{ext}$ , ...,  $f^{k-1}_{ext}(x)$  are all continuous. Suppose  $f^k_{ext}(x)$  is piecewise continuous but has at least 1 discontinuity in each period. Then, the Fourier coefficients decrease with  $n$  like  $\frac{1}{n^{k+1}}$ . The more rapidly the Fourier coefficients decrease with  $n$ , the more rapidly the series will converge.*

The convergence of the functions  $f(x) = 3x - x^4$  and  $f(x) = x - x^3$  should now make sense. As is evident in the plot of  $f(x) = 3x - x^4$  in **Part A** of **Appendix C**, the function itself has a jump discontinuity in the zeroth derivative. Thus, we would expect the Fourier coefficients

to decrease like  $\frac{1}{n}$ . This is exactly what occurs, as the  $b_n$  coefficients are proportional to  $\frac{1}{n}$ . On the other end of the spectrum, and  $f(x) = x - x^3$  does not have any discontinuities in the periodic extension until the 2<sup>nd</sup> derivative, as can be extrapolated in **Part B of Appendix C**. Thus, we would expect the Fourier coefficients to decrease like  $\frac{1}{n^3}$ . This is exactly what occurs, as the Fourier coefficient  $b_n$  is proportional to  $\frac{1}{n^3}$ .

### Separation of Variables, Revisited:

Now that we have developed the method of finding Fourier series to represent non-harmonic functions in sums of harmonic functions, we can attempt to solve a more advanced partial differential equation. This equation, known as Laplace's Equation in a rectangle, is presented as a fully posed problem below:

$$(39). \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, 0 < x < a, 0 < y < b$$

$$(40). \quad \phi(x, 0) = 0$$

$$(41). \quad \frac{\partial \phi}{\partial y}(x, b) = 1$$

$$(42). \quad \frac{\partial \phi}{\partial x}(0, y) = 0$$

$$(43). \quad \phi(a, y) = 0$$

This equation will be solved much like the original worked example (heat diffusion in a slab), in that we will look for product solutions of the form  $\phi = F(x) * G(y)$ . So, plugging this value in for  $\phi$ , and dividing by FG, we get the following:

$$(44). \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{G} \frac{d^2 G}{dy^2} = -\lambda, \text{ where } \lambda > 0$$

So, we first have the equation for  $F(x)$ , which is as follows:

$$(45). \quad \frac{d^2 F}{dx^2} + \lambda F = 0$$

From this first worked example, we recognize that the solution to this equation is trigonometric, and that  $F(x)$  can be defined as follows:

$$(46). \quad F(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$$

We can solve for the constants A and B by using the two boundary conditions in the x spatial coordinate, Equations (42, 43). This is done below:



$$(47). \frac{dF(x)}{dx} = -A\sqrt{\lambda} \sin(\sqrt{\lambda}x) + B\sqrt{\lambda} \cos(\sqrt{\lambda}x)$$

$$(48). \frac{dF(0)}{dx} = B\sqrt{\lambda} \cos(0) = 0$$

$$(49). B = 0, \text{ because } \cos(0) = 1 \text{ and } \lambda = 0$$

Solving for A:

$$(50). F(x) = A \cos(\sqrt{\lambda}x)$$

$$(51). F(a) = A \cos(\sqrt{\lambda}a) = 0$$

We note that for  $\cos(\cdot)$  to be zero,  $\sqrt{\lambda}a$  must be defined as follows:

$$(52). \sqrt{\lambda}a = \left(n - \frac{1}{2}\right)\pi, n \in \mathbb{Z}$$

$$(53). \lambda = \frac{\left(n - \frac{1}{2}\right)^2 \pi^2}{a^2}, n \in \mathbb{Z}$$

$$(54). F(x) = A \cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{a}x\right)$$

Since we have a value for  $\lambda$  we are done, for now, with the x spatial coordinate.

Next, we will use the homogeneous boundary condition in the y spatial coordinate, Equation (40). First, we recognize the equation for G(y):

$$(55). \frac{d^2G}{dy^2} - \lambda G = 0$$

This equation, which is very similar to the Equation (45), is off by only a negative sign. This leads us to expect hyperbolic solutions in y (because the derivatives of  $\sinh(x)$  and  $\cosh(x)$  are positive, if  $x > 0$ ):

$$(56). G(y) = C \cosh(\sqrt{\lambda}y) + D \sinh(\sqrt{\lambda}y)$$

$$(57). G(0) = C = 0, \text{ so } C = 0$$

$$(58). G(y) = D \sinh\left(\frac{\left(n - \frac{1}{2}\right)\pi}{a}y\right), n \in \mathbb{Z}$$

Now that we have solved for  $F(x)$  and  $G(y)$ , we can formulate our product solution as follows:

$$(59). \phi(x, y) = \sum_{n=1}^{\infty} E_n \sinh\left(\frac{\left(n - \frac{1}{2}\right)\pi}{a}y\right) \cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{a}x\right)$$

In this equation, there is still a constant value,  $E_n$ , that is yet to be determined. This is done by using the last boundary condition, Equation (41):

$$(60). \frac{\partial \phi}{\partial y} = \sum_{n=1}^{\infty} E_n \frac{\left(\frac{n-1}{2}\right)\pi}{a} \cosh\left(\frac{\left(\frac{n-1}{2}\right)\pi y}{a}\right) \cos\left(\frac{\left(\frac{n-1}{2}\right)\pi x}{a}\right)$$

$$(61). \frac{\partial \phi}{\partial y}(x, b) = 1 = \sum_{n=1}^{\infty} E_n \frac{\left(\frac{n-1}{2}\right)\pi}{a} \cosh\left(\frac{\left(\frac{n-1}{2}\right)\pi b}{a}\right) \cos\left(\frac{\left(\frac{n-1}{2}\right)\pi x}{a}\right)$$

This means we must use a Fourier series to solve for  $E_n$ . This is done below (and in **Part C of Appendix C**):

$$(62). E_n \cosh\left(\frac{\left(\frac{n-1}{2}\right)\pi b}{a}\right) \frac{\left(\frac{n-1}{2}\right)\pi}{a} = \frac{1}{a} \int_{-a}^a \cos\left(\frac{\left(\frac{n-1}{2}\right)\pi x}{a}\right) dx$$

$$(63). E_n \cosh\left(\frac{\left(\frac{n-1}{2}\right)\pi b}{a}\right) \frac{\left(\frac{n-1}{2}\right)\pi}{a} = \frac{2(-1)^{n+1}}{\pi\left(\frac{n-1}{2}\right)}$$

$$(64). E_n = \frac{\frac{2(-1)^{n+1}}{\pi\left(\frac{n-1}{2}\right)}}{\cosh\left(\frac{\left(\frac{n-1}{2}\right)\pi b}{a}\right) \frac{\left(\frac{n-1}{2}\right)\pi}{a}}$$

Now, the problem is complete. We have solved for the last unknown,  $E_n$ , and our final solution for  $\phi(x, y)$  is as follows:

$$\phi(x, y) = \sum_{n=1}^{\infty} \frac{2a(-1)^{n+1} \sinh\left(\frac{\left(n - \frac{1}{2}\right)\pi y}{a}\right) \cos\left(\frac{\left(n - \frac{1}{2}\right)\pi x}{a}\right)}{\left(n - \frac{1}{2}\right)^2 \pi^2 \cosh\left(\frac{\left(n - \frac{1}{2}\right)\pi b}{a}\right)}$$

### Conclusion:

At this point, after developing some of the basics of separation of variables and of Fourier series approximations, the reader should be somewhat comfortable with the solution of simple heat diffusion equations and with Laplace's Equation. There is still much to be learned, though, and as a possibility for further learning one can consider taking a course in partial differential equations (such as Math 281 at the University of Rochester), or one can consider independently studying from *Applied Partial Differential Equations with Fourier series and Boundary Value Problems* by Richard Haberman, a book I have found very helpful.

**References:**

*Joseph Fourier 1768-1930*, Ivor Grattan-Guinness, MIT Press, 1972.

*Applied Partial Differential Equations with Fourier series and Boundary Value Problems*,  
Richard Haberman, Fourth Edition, Prentice Hall, 2004.

Math 281 Lectures, Al Clark, University of Rochester, 2009.

## Appendix A: Glossary of Terms

**Boundary Condition:** A condition at the boundary of a problem. Example:  $F(a) = 0, F(0) = 0$ . (F at a is equal to zero, and F at 0 is equal to zero).

**Gibb's Phenomenon:** A characteristic of a Fourier series at a jump discontinuity. At this jump discontinuity, there are large oscillations, which will not die out even with an infinite number of terms.

**Harmonic:** A harmonic function is a function that is twice continuously differentiable, and satisfies Laplace's Equation. Example:  $\sin(x), \cos(x)$ .

**Initial Condition:** The condition that specifies the initial state of a system.

**Jump Discontinuity:** A discontinuity that can occur in the periodic extension of a function. At this point, the function value exists both to the left and the right of the jump discontinuity, but these values are not equal.

**Linear:** Something is said to be linear if it satisfies additivity and homogeneity. That is,  $f(x + y) = f(x) + f(y)$  and  $f(cx) = cf(x)$ .

**Order of Derivative:** The order of the derivative is the number of derivatives that are being taken. For example,  $f''(x)$  is order 2, and  $f'''(x)$  is order 3.

**Ordinary Differential Equation:** An ordinary differential equation is a differential equation that depends only on one independent variable.

**Orthogonality:** Two functions,  $f$  and  $g$  are said to be orthogonal if their inner product,  $\langle f, g \rangle$  is zero.

**Periodic Extension:** A periodic extension of a function takes the function on a given interval and repeats its characteristics on this interval for the entire meaningful domain. See **Appendix C** for a visualization of this.

**Piecewise Smooth:** A function is piecewise smooth if both the function and its derivatives are continuous.

**Separable:** A differential equation is said to be separable if you can find product solutions of the differential equation. Example:  $T(x, y) = F(x) * G(y)$ .

**Separation of Variables:** The process in which solutions are found for separable differential equations.

**Trivial:** The trivial solution occurs when you set all constants equal to zero in the homogeneous case.

## Appendix B: Some Useful Integrals

You may find these integrals useful when presented with a Fourier series calculation that must be done by hand:

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$$

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

**If  $f(x)$  is odd:**

$$\int_{-L}^L f(x) dx = 0$$

**If  $f(x)$  is even:**

$$\int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$$

Note:  $\sin(x)$  is odd, and  $\cos(x)$  is even. An odd function multiplied by an odd function is an even function, an odd function multiplied by an even function is an odd function, and an even function multiplied by an even function is even.

## Appendix C: Mathematica Code

### Part A

This is the coding for the calculation of the Fourier Series of  $f(x) = 3x - x^4$ .

Below, I will define the length scale  $L = 1$ , the function  $f(x) = 3x - x^4$ , and its periodic extension.

```
In[1]:= L = 1;
```

```
In[2]:= f[x_] := 3 x - x^4
```

```
In[3]:= fext[x_] := f[Mod[x, 2, -1]]
```

Below, I will calculate the Fourier Coefficients  $a_0$ ,  $a_n$ , and  $b_n$ .

```
In[4]:= a0 = 1 / (2 L) * Integrate[fext[x], {x, -L, L}]
```

```
Out[4]= - $\frac{1}{5}$ 
```

```
In[5]:= an = Simplify[(1 / L) * Integrate[fext[x] * Cos[(n π x) / L], {x, -L, L}], n ∈ Integers]
```

```
Out[5]= - $\frac{8 (-1)^n (-6 + n^2 \pi^2)}{n^4 \pi^4}$ 
```

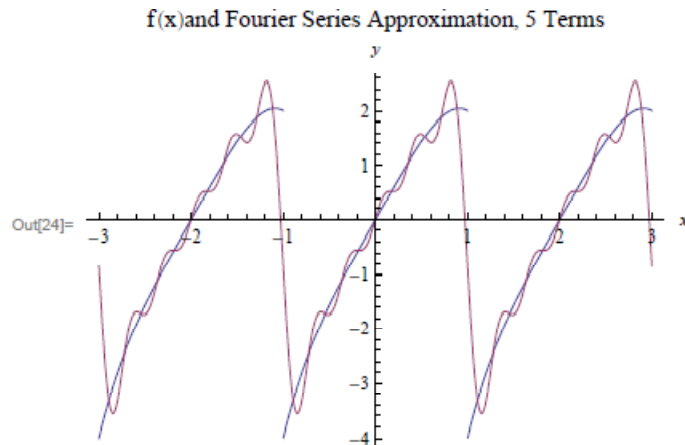
```
In[6]:= bn = Simplify[(1 / L) * Integrate[fext[x] * Sin[(n π x) / L], {x, -L, L}], n ∈ Integers]
```

```
Out[6]= - $\frac{6 (-1)^n}{n \pi}$ 
```

Below, I will define a sum of the Fourier coefficients. Then, I will plot the periodically extended function alongside its Fourier Series approximation, for 5, 10, and 50 terms.

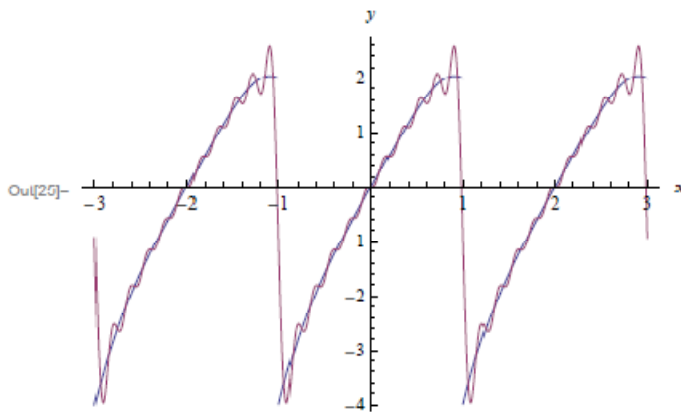
```
In[7]:= sum[x_, nterms_] := a0 +  $\sum_{n=1}^{nterms} (an * Cos[n * \pi * x / L] + bn * Sin[n * \pi * x / L])$ 
```

```
In[24]:= Plot[{fext[x], sum[x, 5]}, {x, -3, 3},  
PlotLabel -> "f(x) and Fourier Series Approximation, 5 Terms", AxesLabel -> {x, y}]
```



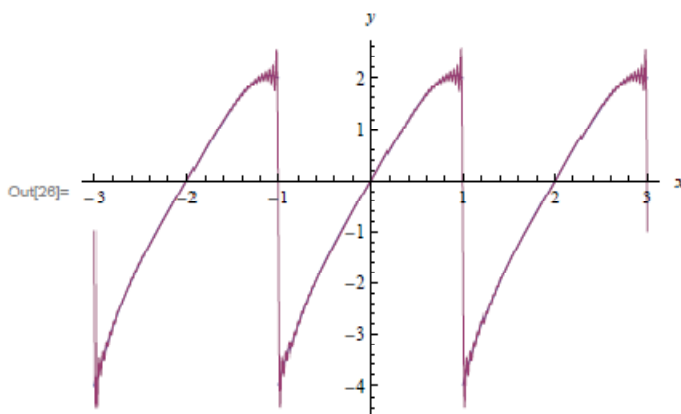
```
In[25]= Plot[{fext[x], sum[x, 10]}, {x, -3, 3},
  PlotLabel -> "f(x)and Fourier Series Approximation, 10 Terms", AxesLabel -> {x, y} ]
```

f(x)and Fourier Series Approximation, 10 Terms



```
In[28]= Plot[{fext[x], sum[x, 50]}, {x, -3, 3},
  PlotLabel -> "f(x)and Fourier Series Approximation, 50 Terms", AxesLabel -> {x, y} ]
```

f(x)and Fourier Series Approximation, 50 Terms



Now, out of curiosity, I will calculate the value of the Fourier Series at  $x = 0$  (at the jump discontinuity). We expect this value to be  $-1$ , as this is the average of the left and right limits of the extended function  $((2 + -4)/2)$ .

```
In[20]= N[sum[1, 1000]]
```

```
Out[20]= -0.99919
```

This is approximately  $-1$ , as expected.

## Appendix C: Mathematica Code

### Part B

This is the coding for the calculation of the Fourier Series of  $f(x) = x - x^3$ .

Below, I will define the length scale  $L = 1$ , the function  $f(x) = x - x^3$ , and its periodic extension.

```
In[1]:= L = 1;
```

```
In[2]:= f[x_] := x - x^3
```

```
In[3]:= fext[x_] := f[Mod[x, 2, -1]]
```

Below, I will calculate the Fourier Coefficients  $a_0$ ,  $a_n$ , and  $b_n$ .

```
In[4]:= a0 = 1 / (2 L) * Integrate[fext[x], {x, -L, L}]
```

```
Out[4]= 0
```

```
In[5]:= an = Simplify[(1 / L) * Integrate[fext[x] * Cos[(n π x) / L], {x, -L, L}], n ∈ Integers]
```

```
Out[5]= 0
```

```
In[6]:= bn = Simplify[(1 / L) * Integrate[fext[x] * Sin[(n π x) / L], {x, -L, L}], n ∈ Integers]
```

```
Out[6]= 
$$-\frac{12 (-1)^n}{n^3 \pi^3}$$

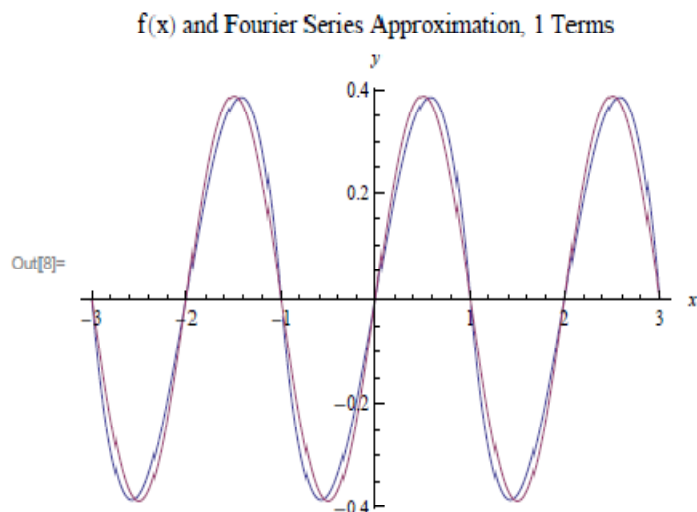
```

Below, I will define a sum of the Fourier coefficients. Then, I will plot the periodically extended function alongside its Fourier Series approximation, for 1, 2, and 10 terms.

```
In[7]:= sum[x_, nterms_] := a0 + 
$$\sum_{n=1}^{nterms} (an * Cos[n * π * x / L] + bn * Sin[n * π * x / L])$$

```

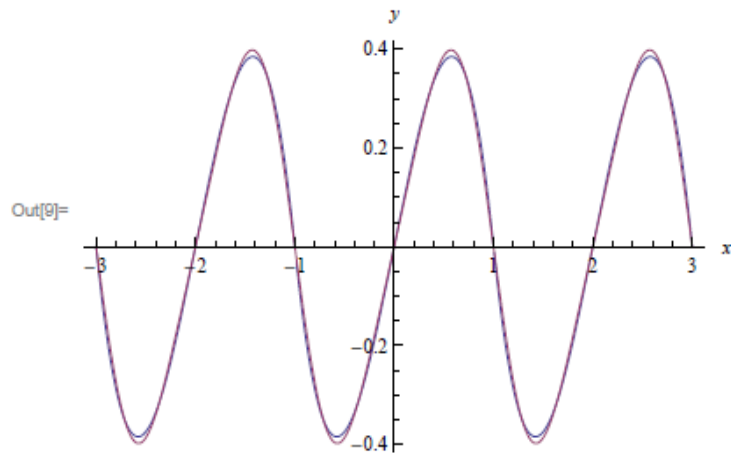
```
In[8]:= Plot[{fext[x], sum[x, 1]}, {x, -3, 3},  
PlotLabel -> "f(x) and Fourier Series Approximation, 1 Terms", AxesLabel -> {x, y}]
```





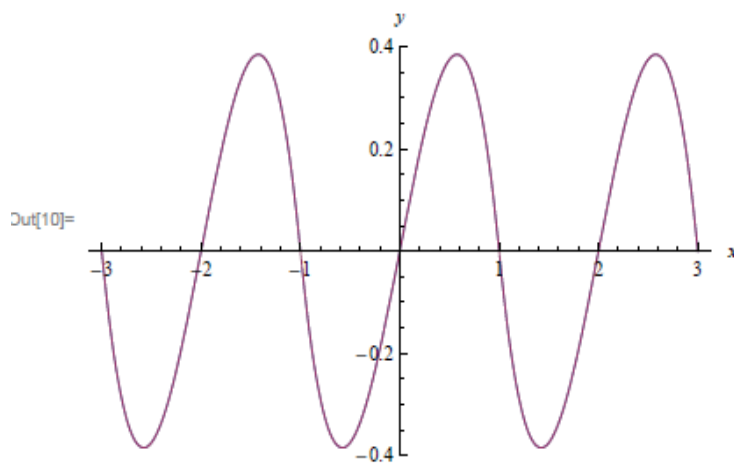
```
In[9]:= Plot[{fext[x], sum[x, 2]}, {x, -3, 3},  
  PlotLabel -> "f(x) and Fourier Series Approximation, 2 Terms", AxesLabel -> {x, y} ]
```

f(x) and Fourier Series Approximation, 2 Terms



```
In[10]:= Plot[{fext[x], sum[x, 10]}, {x, -3, 3},  
  PlotLabel -> "f(x) and Fourier Series Approximation, 10 Terms", AxesLabel -> {x, y} ]
```

f(x) and Fourier Series Approximation, 10 Terms



**Appendix C: *Mathematica Code*****Part C**

This is the coding for the calculation of the coefficient  $E_n$  in the Laplace's Equation example.

```
In[13]= En = Simplify[1 / a * Integrate[Cos[(n - 1 / 2) * π * x / a], {x, -a, a}], n ∈ Integers]
```

$$\text{Out[13]= } \frac{4 (-1)^n}{\pi - 2 n \pi}$$