A stochastic version of the moving boundary problem of Caffarelli and Vazquez

Carl Mueller and Richard Sowers

Abstract

We prove short time existence for a stochastic moving boundary problem arising from flame propagation. Caffarelli and Vazquez studied the deterministic case in a well known paper [CV95].

1 Introduction

Partial differential equations with free boundary conditions have received wide attention in recent years, since such equations arise in a wide variety of physical systems with phase transitions. For example, the Stefan problem describes the temperature of water which is partially frozen. The liquid and solid regions can be described by two different heat equations. But the boundary between the regions can move, and any model must deal with this possibility. Flame propagation is another situation of this kind. In a well known paper [CV95], Caffarelli and Vazquez formulated and studied a parabolic PDE describing the situation.

It is also becoming widely accepted that noise plays an important role in most or all physical systems. Modeling such systems involves stochastic partial differential equations (SPDE), and this field is still rapidly evolving. To our knowledge, the only papers dealing with

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free boundary problems for SPDE are Barbu and Da Prato [BDP02, BDP05], who treat the stochastic Stefan problem.

The purpose of this paper is to study a stochastic version of the moving boundary problem studied by Caffarelli and Vazquez [CV95]. We seek continuous solutions \( u(t, x), t > 0, x \in \mathbb{R} \) to

\[
\begin{align*}
\partial_t u &= \Delta u + h(u) \mathbf{1}(u > K_0) \dot{Z} \\
u(0, x) &= u_0(x)
\end{align*}
\tag{1.1}
\]

Furthermore, we let \( \Omega_t = \{x \in \mathbb{R} : u(t, x) > 0\} \) and let \( \partial \Omega_t \) denote the boundary of \( \Omega_t \). On \( \partial \Omega_t \) we impose the boundary condition

\[|\partial_x u(t, x)| = c\]

for some constant \( c \). Since \( c \) is somewhat arbitrary, we take \( c = 1 \). Here \( \dot{Z} \) is a noise term which we take to be either spacetime white noise \( \dot{W}(t, x) \) or time-dependent white noise \( \dot{B}(t) \). These processes are generalized Gaussian random fields with the following covariances.

\[
E[\dot{W}(t, x) \dot{W}(s, y)] = \delta(t - s) \delta(x - y)
\]
\[
E[\dot{B}(t) \dot{B}(s)] = \delta(t - s)
\]

where \( \delta(x) \) is the Dirac delta function. We believe we could deal with a broader class of noise terms with covariance of the form \( \delta(t - s) h(x - y) \), but for simplicity we restrict ourselves to the above two cases. As for most stochastic PDE, we do not expect the solution \( u(t, x) \) to be differentiable, so we regard (1.1) as a shorthand for an integral equation which we specify later.

Our equation (1.1) is just the Caffarelli-Vazquez equation with an extra noise term. For a much more detailed discussion of the non-stochastic free boundary problem corresponding to (1.1), the reader should consult Caffarelli and Vazquez [CV95]. We will not repeat the extensive information in that paper.

## 2 Weak Form

We regard (1.1) as a shorthand for the following weak form, which is a modification of form given at the bottom of page 412 of [CV95]. First we give the statement, and then we will explain the terminology. We seek a domain \( \Omega \) with Lipschitz continuous lateral boundary \( \Gamma \) and a function \( u \in C(\Omega \cup \Gamma) \) such that

\[\text{2}\]
(i) For every test function $\phi \in C_0^\infty([0,T), \mathbb{R})$
\[
\int_{\Omega} \int u \cdot (\phi_t + \Delta \phi) \, dx \, dt + \int_{\Omega} \phi h(u) 1(u > K_0) \hat{Z} \, dx \, dt + \int_{\Omega_0} u_0 \phi \, dx \\
= \int_{\Gamma} \phi d\Sigma \cos \alpha \tag{2.1}
\]
where the second integral in (2.1) is a stochastic integral in the sense of Walsh [Wal86],
(ii) $u$ vanishes on $\Gamma$, and
(iii) the free boundary $\Gamma$ starts from $\Gamma_0 = \partial \Omega_0$, i.e., the section of $\Gamma_t$ at time $t$ converges to $\Gamma_0$ as $t \to 0$ in some sense.

By “Lipschitz continuous lateral boundary”, we mean that at any given time $t$, the boundary $\Gamma_t$ is either empty or is an interval $[a(t), b(t)]$. Furthermore, we require both functions $a$, $b$ to be Lipschitz in the interior of their domains. Usually solutions of SPDE are very rough, and we would not expect this kind of smoothness on the boundary. But the noise is multiplied by $1(u > K)$, so we expect the noise to vanish in a neighborhood of the boundary.

We now remark on the meaning of the term $d\Sigma \cos \alpha$ in condition (i). We consult [CV95] at the top of page 413. Assuming that $\Omega = [a(t), b(t)]$ is an interval, we find that
\[
d\Sigma \cos \alpha = \frac{|a'(t)|}{\sqrt{1 + a'(t)^2}} + \frac{|b'(t)|}{\sqrt{1 + b'(t)^2}}
\]
In the words of Caffarelli and Vazquez, $d\Sigma \cos \alpha$ is the lateral projection of the area element $d\Sigma$, which in our case is the length elements of the curves $(t, a(t))$ and $(t, b(t))$.

We are now ready to state our main theorem. The main assumption, (4.2), is stated in Section 4.

**Theorem 1** Suppose that $u_0$ satisfies assumption (4.2). Then there exists an almost surely positive random time $T = T(\omega)$ and a solution $u(t, x)$ to (2.1) valid for $0 < t < T, x \in \mathbb{R}$.

### 3 Outline

There are three main points to the argument. We produce a sequence of approximate solutions parameterized by $\varepsilon > 0$ and then show compactness as $\varepsilon \to 0$. In the main part of the compactness proof we
exhibit a barrier function which keeps the support of $u$ compact, at least for a short time. Thirdly, we show that a limit point satisfies our weak form (2.1). Especially important is the behavior of the solution near the boundary. For this part we heavily use the analysis of Caffarelli and Vazquez.

It may be useful to point out that in deterministic moving boundary problems, one usually starts with some boundary regularity, otherwise it is very hard to make progress. We will do the same here.

First, we use linearization as on page 437 of [CV95]. Since $\Gamma$ is assumed to be Lipschitz, there is a dense set of points $(t_0, x_0) \in \Gamma$ at which $\Gamma$ is differentiable. Also, near the boundary $\Gamma$ we have $u < K_0$ and hence the stochastic term is not present. Then we can use the asymptotic results of Caffarelli and Vazquez to draw conclusions about the solution. We need $\nabla u$ to be bounded near the boundary $\Gamma$, and here is where we need the noise coefficient to vanish for $u < K_0$.

Inspired by [CV95] 437, Lemma 10.2, we let

$$u_\lambda(x, t) = \frac{1}{\lambda} u(\lambda(x - x_0), \lambda(t - t_0))$$

where $(t_0, x_0) \in \Gamma$ is a point of the type mentioned above. Suppose that the support test function is supported in a small neighborhood of the boundary $\Gamma$, and that our solution $u < K_0$ on this region. This is a random event, but for $\omega$ in this event, the weak form (2.1) reduces to

$$\int \int_{\Omega} u \cdot (\phi_t + \Delta \phi) dx dt + \int_{\Omega_0} u_0 \phi dx = \int_{\Gamma} \phi d\Sigma \cos \alpha$$

We have (as in Caffarelli and Vazquez),

$$\lambda \frac{\partial u_\lambda}{\partial t} = \Delta u_\lambda$$

and

$$\int \int (\lambda u_\lambda \phi_t + u_\lambda \Delta \phi) dx dt = 0$$

for admissible $\phi$.

Next we let $\lambda \to 0$ along a subsequence $\lambda_n$ and denote the limit as $V(t, x)$. This implies that $\Delta V = 0$; here $\Delta$ equals the second derivative in $x$, and therefore $V$ is linear in $x$ for each value of $t$.

But, in our neighborhood of the boundary we need

1. $|\nabla u_\lambda|$ bounded
2. $\partial_t u_\lambda \leq 0$
If these two conditions hold, we can integrate to get
\[ \partial_t u_\lambda \in L^1_{\text{loc}} \]

4 Compactness of Approximating Solutions

By translating over the $x$-axis if necessary, we may assume that the support of $u_0$ is contained in $[0, L]$ for some $L > 0$. Following Caffarelli and Vazquez, we introduce an approximate solution $u_\varepsilon$ defined as follows. Recall that Caffarelli and Vazquez introduce a nonnegative function $\beta(x)$ on $\mathbb{R}$ with the following properties.

1. $\beta(s)$ is positive on $[0, 1]$ and 0 otherwise.
2. $\beta(s)$ is a $C^\infty$ function on $[0, \infty)$.
3. $\beta(s)$ is increasing on $(0, \frac{1}{2})$ and decreasing on $(\frac{1}{2}, 1)$.
4. $\int_0^1 \beta(s) ds = \frac{1}{2}$.

We also require the condition
\[ \sup_{s \in \mathbb{R}} \beta(s) \leq 1 \]
which is clearly compatible with the other conditions on $\beta$. Then we define
\[ \beta_\varepsilon(s) = \frac{1}{\varepsilon} \beta \left( \frac{s}{\varepsilon} \right). \]

Now we set up the approximate solution $u^{(\varepsilon)}$ which solves the following approximate Caffarelli-Vazquez equation.

\begin{align*}
\partial_t u &= \Delta u - \beta_\varepsilon(u) + h(u)1(u > K_0)\tilde{Z} \\
u(0, x) &= u_0(x)
\end{align*}

The term $-\beta_\varepsilon(u)$ is meant to enforce our moving boundary condition.

We will show that over a short random time interval, the support of the solution can be restricted to $[0, L]$, assuming the initial function satisfies conditions (1) and (2) of the previous section, in a neighborhood of the boundary. For simplicity, we will only give the argument...
near the left endpoint 0. In fact, we will fix \( \varepsilon > 0 \). Consider the barrier function
\[
b(x) = b_\varepsilon(x) = \begin{cases} 
0 & \text{if } x < 0 \\
x^2/2 & \text{if } 0 \leq x \leq \varepsilon \\
x - \varepsilon & \text{if } x > \varepsilon
\end{cases}
\]
and note that \( b(x) \) and \( b'(x) \) are continuous at \( x = \varepsilon \). Assume that
\[
u_0(x) \leq \frac{b(x) \wedge b(L - x)}{2}. \tag{4.2}
\]
Note that (4.2) implies that \( u_0(x) \) is supported on \([0, L]\).

Our next goal is to show the following lemma. Note that our solution \( u(t, x) = u(\varepsilon)(t, x) \) depends implicitly on \( \varepsilon \).

**Lemma 1** For each \( \varepsilon > 0 \) there exists a random time \( T = T_\varepsilon(\omega) \) with \( T > 0 \) a.s. such that the following holds. For \( 0 \leq s < T \) and for all values of \( x \),
\[
u(s, x) \leq b(x) \wedge b(L - x). \tag{4.3}
\]
Furthermore, \( T_\varepsilon \) is bounded away from 0 in probability, uniformly in \( \varepsilon \), in the sense that
\[
\lim_{\delta \downarrow 0} \liminf_{\varepsilon \downarrow 0} P(T_\varepsilon > \delta) = 1
\]
We will compare \( u \) to a function \( w \) which solves the following integral equation.
\[
w(t, x) = \int_0^L H(t, x, y)u_0(y)dy + \int_0^t \int_{L-x}^{L-x-\varepsilon} H(t - s, x, y)w(s, y)\dot{Z}dyds
\]
where we recall that \( \dot{Z} \) could represent either \( \dot{W}(t, x) \) or \( \dot{W}(t) \), and where \( H(t, x, y) \) is the fundamental solution of
\[
\partial_t H = \Delta_x H - \phi_\varepsilon(x)H \\
\phi_\varepsilon(x) = \frac{2}{x^2}1(0 < x < \varepsilon) + \frac{2}{(L-x)^2}1(L - \varepsilon < x < L) \\
H(0, x, y) = \delta(x - y) \tag{4.5}
\]
where $\Delta_x$ denotes the Laplacian with respect to $x$. To explain (4.5), note that as long as $u(t,x) \leq b(x)$ for $0 \leq x \leq \varepsilon$, we have, for all values of $x$,

$$-eta_\varepsilon(x) = -\frac{1}{\varepsilon u} u \leq -\frac{1}{\varepsilon b(x)} u = -\frac{2}{x^2} u$$

and a similar argument shows that

$$-eta_\varepsilon(x) \leq \frac{2}{(L-x)^2} u$$

**Definition 1** Let $\sigma$ be the infimum of times $t$ for which

$$w(t,x) > b(x) \land b(L-x)$$

for some $x \in [0,L]$.

That is, $\sigma$ is the first time $t$ at which the condition $w(t,x) \leq b(x) \land b(L-x)$ fails.

By the usual comparison theorems (see [Shi94] Shiga for example), which roughly state that smaller initial conditions lead to smaller solutions, for $0 \leq t < \sigma$ and for all values of $x$ we have

$$u(t,x) \leq w(t,x) \leq b(x) \land b(L-x) \quad (4.6)$$

Therefore, to prove Lemma 1, it suffices to prove it for $w$ in place of $u$.

### 4.1 Start of the proof of Lemma 1; a bound on $w$

We will use a grid argument, also called a chaining argument. A short calculation, which we leave to the reader, shows that $u_0(x)$ is a supersolution of (4.5), so

$$\int_0^\infty H(s,x,y)u_0(y)dy \leq u_0(x) \leq \frac{b(x) \land b(L-x)}{2}.$$  \quad (4.7)

for all $(t,x)$. Therefore, it suffices to show that if

$$N(t,x) := \int_t^L \int_0^{L-x} H(t-s,x,y)w(s,y) \hat{Z} dy ds$$

then there exists an almost surely positive random time $T = T(\omega)$ such that

$$\sup_{0 \leq t \leq T(\omega)} \sup_{0 \leq x \leq L} N(t,x) \leq \frac{b(x) \land b(L-x)}{2}.$$ \quad (4.8)
The domain of integration is $[\varepsilon, L - \varepsilon]$ for the following reason. As long as (4.8) holds, we have $w(t, x) \leq b(x) \wedge b(L - x)$, and hence the noise coefficient $1(u > K_0)$ vanishes except when $x \in [\varepsilon, L - \varepsilon]$.

We will use the fact that for $t \leq t_0$

$$M_{t_0}(t) = \int_0^t \int_{x}^{L-\varepsilon} H(t_0 - s, x, y) b(y) \frac{\hat{W}dy}{2} ds$$

is a martingale with time variable $t$, with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ where $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\hat{Z}(s): s \leq t$.

### 4.2 Truncation

Let $\tau$ denote the infimum of times $t$ that for some $x \in [0, L]$ we have $w(t, x) > b(x) \wedge b(L - x)$. It is more convenient to work up to a fixed time $T$, so from now on we will let $w(t, x)$ denote the solution of (1.1) with $\hat{Z}(t, x)$ replaced with $\hat{Z}(t, x)1(t \leq \tau)$. Clearly, the two solutions agree for $t \leq \tau$. With this definition of $w(t, x)$, we see that $w(t, x) \leq b(x) \wedge b(L - x)$ holds for all values of $t, x$.

### 4.3 A remark about $H$

We need the following estimates on $H$.

**Lemma 2** For $T, L > 0$ and $\delta \in (0, L/2)$ there exists $C > 0$ such that the following holds. Let $0 < \varepsilon < 4L$, and suppose that $0 \leq t \leq T$ and $x, y \in [\delta, L - \delta]$. Then for $\varepsilon$ small enough,

$$\int_0^t \int_{2\varepsilon}^{L - 2\varepsilon} [H(r, x, z) - H(r, y, z)]^2 dz dr \leq C|x - y|^2 \tag{4.10}$$

$$\int_0^t \int_{2\varepsilon}^{L - 2\varepsilon} [H(r + s, x, z) - H(r, x, z)]^2 dz dr \leq Cs^\frac{1}{2} \tag{4.11}$$

$$\int_0^t \int_{2\varepsilon}^{L - 2\varepsilon} H(r, x, z)^2 dz dr \leq Ct^\frac{1}{2} \tag{4.12}$$

**Proof.** Lemma 2 is well known when $H$ is replaced by the ordinary heat kernel or the Dirichlet heat kernel. In fact, $H$ is just the Dirichlet heat kernel on the interval $[0, L]$ with extra killing on $[0, \varepsilon]$ and $[L - \varepsilon, L]$. In fact, $H(t, x, z) = 0$ for $z$ outside of the interval $[0, L]$, just as for the Dirichlet heat kernel.

To begin with, (4.12) follows because

$$H(t, x, z) \leq G(t, x, z)$$
where $G$ is the heat kernel on $\mathbb{R}$, and so
\[
\int_0^t \int_{2\varepsilon}^{L-2\varepsilon} H(r, x, z)^2 dz dr \leq \int_0^t \int_{-\infty}^\infty G(r, x, z)^2 dz dr \leq Ct^{\frac{3}{2}} \quad (4.13)
\]

The second inequality in (4.13) is well known.

Secondly, let us consider (4.10). Note that for $\varepsilon$ sufficiently small and for $x \in [0, \varepsilon] \cup [L - \varepsilon, L]$, we have that for some constant $C = C(\delta, L)$,
\[
\sup_{t > 0} \sup_{z \in \delta[L-\delta]} \left| \frac{\partial}{\partial x} G(t, x, z) \right| \leq C \quad (4.14)
\]
where $G(t, x, z)$ is the Dirichlet heat kernel on the interval $[0, L]$.

Now we use the interpretation of $H(t, x, z)$ in terms of a Brownian motion $B_{\sqrt{2t}}$ killed upon hitting the boundaries of $[0, L]$. Call this process $X_t$. We can interpret $H(t, x, z)$ as the transition kernel for the process $X_t$ killed according to the function $\phi_\varepsilon$. Call this process $Y_t$, and let $\sigma$ denote the killing time for $Y_s : s \leq t$. Then for $x, z \in \delta[L-\delta]$,
\[
H(t, x, z) dz = P_x(Y_t \in dz) \quad (4.15)
= P_x(X_t \in dz) - E_x P_{X_\sigma}(X_{t-\sigma} \in dz)
= G(t, x, z) - E_x \left[ G(t - \sigma, X_{t-\sigma}, z) \right] dz
\]

Now (4.10) holds with $G$ in place of $H$. This observation and (4.14) now give us (4.10).

Finally, the argument for (4.11) is almost the same as for (4.10). We leave the details to the reader.

4.4 The grid argument

We will work on the time interval $[0, T]$.

\[
\mathcal{G}_n = \mathcal{G}_n^{[\varepsilon]} = \left\{ (t, x) = \left( \frac{k}{2^n}, \frac{\ell}{2^n} \right) \in [0, T] \times [2\varepsilon, L - 2\varepsilon] \right\}
\]

**Lemma 3** For all indices $k, \ell, n$ such that both points $\left( \frac{k}{2^n}, \frac{\ell}{2^n} \right)$ and $\left( \frac{k+1}{2^n}, \frac{\ell+1}{2^n} \right)$ lie in $\mathcal{G}_n$, and such that $2^{-n} \leq \varepsilon$,
\[
P \left\{ N \left( \frac{k}{2^n}, \frac{\ell+1}{2^n} \right) - N \left( \frac{k}{2^n}, \frac{\ell}{2^n} \right) > \lambda 2^{-\frac{n}{2}} \right\} \leq C_0 \exp \left( -C_1 \lambda^2 \frac{1}{2^n} \right)
\]
\[
P \left\{ N \left( \frac{k+1}{2^n}, \frac{\ell}{2^n} \right) - N \left( \frac{k}{2^n}, \frac{\ell}{2^n} \right) > \lambda 2^{-\frac{n}{2}} \right\} \leq C_0 \exp \left( -C_1 \lambda^2 \frac{1}{2^n} \right)
\]
where $C_0, C_1$ can be chosen not to depend on $\varepsilon$. 


Here is an outline of the proof of the first inequality; the second is similar. Let \( t = \frac{k}{2^m}, \ x = \frac{\ell}{2^n} \) and \( y = \frac{\ell+1}{2^n} \). Then, because the coefficient of noise is supported on \([\varepsilon, L - \varepsilon]\), we have

\[
N \left( \frac{k}{2^m}, \frac{\ell + 1}{2^n} \right) - N \left( \frac{k}{2^m}, \frac{\ell}{2^n} \right) = \int_0^t \int_{2\varepsilon}^{L-2\varepsilon} \left[ H(r, x, z) - H(r, y, z) \right] \dot{Z}(r, z) dz dr
\]

(4.16)

Now suppose we are in the white noise case, so \( \dot{Z}(t, x) = \dot{W}(t, x) \). Then the expression in (4.16) is a time-changed Brownian motion \( B_\tau \) with

\[
\tau \leq \tau \leq \int_0^t \int_{2\varepsilon}^{L-2\varepsilon} \left[ H(r, x, z) - H(r, y, z) \right]^2 dz dr
\]

In the case of noise depending only on time, \( \dot{Z}(t, x) = \dot{B}(t) \), we have

\[
\tau \leq \tau \leq \int_0^t \int_{2\varepsilon}^{L-2\varepsilon} \left[ H(r, x, z) - H(r, y, z) \right]^2 dz dr
\]

for some constant \( C = C(L) > 0 \), by Jensen’s inequality. In either case, the reflection principle asserts that

\[
P \left( \sup_{0 \leq s \leq t} B_s > \lambda \right) = 4P(B_t > \lambda)
\]

Now replacing \( t \) with \( \tau \) and using lemma 2, we get the first inequality in Lemma 3. The second inequality follows in a similar way, and we omit the details. This finishes our discussion of Lemma 3.

Next, by a standard chaining argument we have that for each \( \lambda, \delta > 0 \) there exists \( t_0 \) small enough such that

\[
P \left( \sup_{s \in [0,t_0]} N(s, x) > \lambda b(x) \right) \leq \delta
\]

(4.17)

By taking \( \lambda = \frac{1}{2} \) and \( \delta \) small, we can construct our stopping time \( T = T(\omega) \). We claim that (4.17) holds for both time-dependent and spacetime white noise. In other words, we have an upper bound for our approximate solutions.

Note that the first statement in (4.16) follow from the remark in the previous section.
4.5 Compactness

To prove compactness for our approximate solutions, we study their modulus of continuity.

First, we repeat the argument of [CV95] with some modifications. First we consider section 4 of [CV95]. Recall that

\[ U(x, t) = \frac{1}{\varepsilon} u(\varepsilon x, \varepsilon^2 t) \]

and that A is the region where \( 0 < U < 1 \); B is the region where \( 1 \leq U \leq 2 \). We modify C to be the region where \( 2 < U < \frac{k_0}{2} \), so that \( u < \frac{k_0}{2} \) on C. Finally, D is the region where \( U \geq \frac{k_0}{2} \), so that \( u \geq \frac{k_0}{2} \).

Thus the noise term appears only in D. Let \( K \) be the constant from [CV95], Section 4.

Our first goal is to show that there is a constant \( K_1 \) such that on \( A \cup B \cup C \), we have

\[ |\nabla u| \leq K_1 \max\{1, \sup |\nabla u_0|\} \]  \hspace{1cm} (4.18)

We use the same maximum principle argument as in [CV95]. The only difference is on the region C. First, we only deal with the leftmost and rightmost components of C. Consider the leftmost component. We claim that we can choose \( T \) and \( x_0 < x_1 \), all random, such that \( [0, T] \times [x_0, x_1] \subseteq C \). This follows because \( u(t, x) \) is bounded by \( b(x) \), at least up to time \( T \). Furthermore, a lower bound is provided by the deterministic solution with initial condition \( u_0 \wedge K_0 \). Indeed, since the noise vanishes when \( u \leq K_0 \), and since such a deterministic solution never exceeds \( K_0 \), it follows by the comparison principle that such a deterministic solution is also a solution to (1.1) and gives a lower bound for the true solution.

Next we use the parabolic Harnack inequality on C. Let \( x_2 \) be the midpoint of \( [x_0, x_1] \). Since \( u \) is bounded by \( b \), we have \( |\nabla u(t, x_2)| \leq K_1 = K \sup_{x \in [0, L]} b(x) \) for some constant \( K \). Then we can use the maximum principle on the part of C with \( x < x_2 \) or the corresponding region on the right, where \( x_3 \) is the point playing the role of \( x_2 \).

Next we let \( E \) be the remaining region \( E := [0, T] \times [x_2, x_3] \). Note that we can choose \( \delta > 0 \) such that \( u < K_0 \) on \( E' := E \setminus ([x_2 + \delta, x_3 - \delta] \times [0, T]) \). We claim that the usual integral equation holds, giving the solution of \( u \) in terms of a deterministic integral and a integral of the heat kernel against the noise. However, there is a final term taking into account the boundary conditions at \( x = x_2, x_3 \). But this
boundary is a positive distance from $E'$, and the contribution here is smooth on $E'$. Thus, standard heat kernel estimates give the usual modulus of continuity estimate for $u$ in $E$.

This gives us a modulus in $x$. The modulus in $t$ can be handled in a similar way. For the region in which $|\nabla u|$ is bounded, we argue as in [CV95]. In $E$, we argue as in the previous paragraph, and get the same modulus over time as usual.

Here are a few more details about the above argument about $E$ and $E'$, where we set up the equation for $u$ as follows.

$$\frac{\partial u}{\partial t} = \Delta u + u \dot{Z}$$

$$u|_{\partial E_0} u = u_0$$

$$u|_{\partial E_1} u = u_1$$

where $\partial E_0 = E \cap \{t = 0\}$ and $E_1 = E \cap \{x = x_2 \text{ or } x_3\}$. Fix $(t,x) \in E$, let $B_t$ be a Brownian motion started at $x$, and let $\tau$ be the first time $s \leq t$ that $x + B_s \in E_1$. If there is no such time, let $\tau = t$. Let $\nu_0$ be the probability measure corresponding to $(0,x+B_t)$ on the set $\{\tau = t\}$ and for $i = 2, 3$ let $\nu_i$ be the probability measure of $(t - \tau, x + B_\tau)$ on the set $\{\tau < t, B_\tau = x_i\}$. We claim that

$$u(t,x) = \int_{\partial E_0} u_0(y)\nu_0(dy) + \sum_{i=2}^{3} \int_{\partial E_i} u(s,x_i)\nu_i(ds) + \int_0^t \int_{E_1} G(t-s,x-y)W(dyds)$$

$$+ \sum_{i=2}^{3} \int_{\partial E_i} \nu_i(ds) \int_0^s \int_{E_1} G(s-r,x_i-y)W(dyds)$$

Now each of the terms in (4.19) obeys the modulus of continuity in $E'$, so $u$ obeys this same modulus in $E'$.

Here is a short explanation of why (4.19) holds. For $(t,x) \in E$, we can write

$$u(t,x) = v(t,x) + w(t,x)$$

where

$$\frac{\partial v}{\partial t} = \Delta v + \dot{Z}$$

$$v|_{E_i} = 0 \quad \text{for } i = 0, 2, 3$$

$$\frac{\partial w}{\partial t} = \Delta w$$
\[ w|_{E_0} = u_0 \]
\[ w(t, x_i)|_{E_i} = u(t, x_i) \quad \text{for } i = 2, 3 \]

In (4.19), the first 2 terms on the right correspond to \( v \), and the second two terms correspond to \( w \).

5 Proof that the limit satisfies the weak equation

Indeed, checking the argument in the second part of [CV95], we find that the argument just depends on properties of the solution in a neighborhood of the boundary. In other words, we could fix a boundary condition somewhat to the right of the moving boundary, and then one could avoid the noise altogether.

Here are the places where local arguments are used.

1. Page 430, sentence after (7.4): “The general case can be obtained by an easy modification; cf. the local estimates (3.5).” These local estimates involve taking an arbitrarily small neighborhood of the boundary of the support. Thus we can avoid the support of the noise.


3. Proof of Lemma 8.1, (ii.1) This involves a small neighborhood \( S_\varepsilon \), so it’s OK.

4. Proof of Lemma 8.1, (ii.2) This is an argument about an ODE which has no immediate bearing on the solution \( u \).

5. Proof of Lemma 8.1, (ii.3) This is a local modification of \( u_0 \).

6. Proof of Lemma 8.1, (iii) Here it is proved that the support \( \Omega(t) \) is contained in the support at time \( t = 0 \), namely \( \Omega_0 \). We already know this by our barrier function argument.

7. Section 9 is a local argument involving a smooth test function \( \phi \).

8. Section 10 is entirely a local argument.

Proof of Lemma 8.1, point (i): Here is the argument in [CV95].

"Let \( u_\varepsilon \) be the approximate solution obtained by using \( \beta_\varepsilon \). Differentiating in \( t \), we write \( v_\varepsilon = u_{\varepsilon,t} \) and note that

\[ v_{\varepsilon,t} = \Delta v_\varepsilon - \beta'(u_\varepsilon)v_\varepsilon \]

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The Maximum Principle applies to this equation. Therefore, if we approximate \( u_0 \) in such a way that \( \Delta u_{0\varepsilon} \leq \beta_{0\varepsilon}(u_{0\varepsilon}) \) we will obtain \( \partial u_{0\varepsilon}/\partial t \leq 0 \), hence \( u_{\varepsilon}/\partial t \leq 0 \) in \( Q \). In the limit of \( \varepsilon \to 0 \) we will get \( u_t \leq 0 \).

We will make this argument local by looking at a region contained in \( \{ u < K_0 \} \) in which there is no noise. We will give the argument near the left endpoint, since a symmetric argument will deal with the right endpoint. We will take a neighborhood \( (x_2 - \delta, x_2 + \delta) \) and a random time \( T = T(\omega) > 0 \) such that on \( H := [0, T] \times [x_2 - \delta, x_2 + \delta] \) we have \( u < K_0 \), so there is no noise in \( H \). Therefore \( u \) satisfies the heat equation in \( H \). We also assume, as do [CV95], that \( u_{0,xx}(x) < 0 \) on \( H \cap \{ t = 0 \} \). In fact, we assume there is a constant \( \delta_1 > 0 \) such that \( u_{0,xx}(x) < -\delta_1 \) on \( H \cap \{ t = 0 \} \). All this should be uniform over \( \varepsilon > 0 \).

Now we decompose \( u = u_1 + u_2 \), where \( u_i \) satisfy the heat equation in \( H \), but with boundary conditions as follows. Let \( H_1 = H \cap \{ t = 0 \} \) and let \( H_2 \) be the part of the boundary of \( H \) on which \( x = x_2 - \delta \) or \( x = x_2 + \delta \). Let \( u_i = 0 \) on \( H_i \) for \( i = 1, 2 \). By our conditions on \( u_0 \) and by the maximum principle, and since we can assume that \( u_2t \) is continuous, we have \( u_{2t}(t, x_2) < -\delta_1/2 \) for \( 0 < t < T \wedge T_2 \) where \( T_2 \) is a deterministic constant depending on \( \delta, \delta_1 \). Now by the boundary Harnack inequality, there exists \( T_3 > 0 \) such that \( u_{1t}(t, x_2) < \delta_1/4 \) for \( 0 < t < T \wedge T_2 \). But \( u = u_1 + u_2 \) and thus for some random time \( T_4 > 0 \), almost surely, \( u_t(t, x_2) < 0 \) for \( 0 < t < T_4 \). Then we can apply the maximum principle as in [CV95], proof of Lemma 8.1 (quoted above), point (i) to conclude that \( u_t(t, x) \leq 0 \) for \( 0 < t < T_4 \) and \( x < x_2 \).

References


