Interacting particles, the stochastic Fisher-Kolmogorov-Petrovsky-Piscounov equation, and duality

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Abstract. The stochastic Fisher-Kolmogorov-Petrovsky-Piscounov equation is
\[ \partial_t U(x,t) = D \partial_{xx} U + \gamma U(1 - U) + \epsilon \sqrt{U(1-U)} \eta(x,t) \]
for \(0 \leq U \leq 1\) where \(\eta(x,t)\) is a gaussian white noise process in space and time. Here \(D, \gamma\) and \(\epsilon\) are parameters and the equation is interpreted as the continuum limit of a spatially discretized set of Itô equations. Solutions of this stochastic partial differential equation have an exact connection to the reaction-diffusion system at appropriate values of the rate coefficients and particles’ diffusion constant. This relationship is called “duality” by the probabilists; it is not via some hydrodynamic description of the interacting particle system. In this paper we present a complete derivation of the duality relationship and use it to deduce some properties of solutions to the stochastic Fisher-Kolmogorov-Petrovsky-Piscounov equation.

INTRODUCTION AND MOTIVATION

The Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [1, 2] is one of the most fundamental models in mathematical biology and ecology [3]. It describes a population \(U(x,t)\) that evolves under the combined effects of spatial diffusion and local logistic growth and saturation. In one space dimension the FKPP equation is
\[ \partial_t U = D \partial_{xx} U + \gamma U(1 - U), \quad 0 \leq U \leq 1 \] (1)
with diffusion coefficient \(D\) and low-density growth rate \(\gamma\), and where the population has been normalized so that the stable saturation level is \(U = 1\).

Logistic dynamics—linear growth and quadratic saturation—is the mean-field description of a variety of microscopic processes. Consider, for example, the chemical reaction scheme
\[ A + B \rightarrow A + A \quad \text{at rate} \quad k_1 \]
\[ A + B \rightarrow B + B \quad \text{at rate} \quad k_2 \] (2)
and let \(N_A(t)\) and \(N_B(t)\) be the number of \(A\) and \(B\) particles in a system of volume \(\Omega\). Neglecting discreteness, fluctuation effects and spatial inhomogeneities altogether leads
to the simple mean-field description of the kinetics for the densities $\rho_A(t) = N_A(t)/\Omega$ and $\rho_B(t) = N_B(t)/\Omega$,

$$\frac{d\rho_A(t)}{dt} = (k_1 - k_2)\rho_A\rho_B = -\frac{d\rho_B(t)}{dt}.$$  

(3)

The reaction conserves the total number of particles so $N = N_A + N_B$ is a constant of the dynamics and the fraction of $A$ particles, $U(t) = N_A(t)/N = \rho_A/(\rho_A + \rho_B)$ satisfies the logistic equation

$$\frac{dU}{dt} = \gamma U (1 - U)$$  

(4)

with $\gamma = (k_1 - k_2)\Omega > 0$ (without loss of generality). This level of modeling predicts that any nonzero initial number of $A$ particles will eventually grow to completely dominate the population.

The same homogeneous mean-field description applies as well to the birth-coagulation process

$$A \rightarrow A + A \text{ at rate } \gamma$$

$$A + A \rightarrow A \text{ at rate } \chi.$$  

(5)

In the leading approximation, the total density of $A$-particles $\rho(t) = N(t)/\Omega$ in volume $\Omega$ evolves according to

$$\frac{d\rho(t)}{dt} = \gamma \rho - \frac{1}{2} \rho^2 + \frac{1}{\Omega} = \frac{1}{\Omega}.$$  

(6)

where $\chi/\Omega$ is taken to be the coagulation rate of distinct pairs of particles. Then the instantaneous particle number normalized by the equilibrium particle number $1 + 2\gamma\Omega/\chi$, $U(t) = \rho(t)/(2\gamma/\chi + 1/\Omega)$, is a logistic variable satisfying (4).

If spatial diffusion of particles with diffusion coefficient $D$ is included, the heuristic hydrodynamic description of either of these reaction processes is the FKPP equation (1) where the space-time dependent $U(x,t)$ is proportional to the local density of $A$-type particles. A particularly interesting aspect of this nonlinear partial differential equation is that it gives a simple description of the invasion of the stable saturated state ($U = 1$) into regions of space occupied by the unstable empty, or extinct, state ($U = 0$), a point to which we will return later.

The logistic dynamics in (4) misses an important qualitative feature of the $A + B$ reaction (3). This is that the number of $A$ particles could, due to fluctuations in the discrete reactions, go to zero where it would stay forever thereafter. The variable $U(t)$ should have some probability of vanishing at finite time and subsequently staying fixed at $U = 0$, but the mean-field dynamics completely fails to capture this extinction phenomenon. This is not an issue for the $A \rightarrow A + A$ system; there is never less than one particle in that system so extinction is impossible.

The kinetic descriptions of the reaction processes in (3) and (6) begin to differ from one another when we include discreteness and fluctuation effects as corrections to the mean-field logistic equation. The mean-field dynamics should become exact in the limit of large equilibrium particle numbers. For the $A + B$ reaction this limit means $N \rightarrow \infty$,
and for the $A \rightleftharpoons A + A$ process this means $2\gamma \Omega \chi \to \infty$. Corrections to the mean-field limits may be derived by expanding the master equations for the Markov process describing the $A$-particle numbers in the systems [4].

Consider the $A + B$ process and let $p_n(t) = \text{Prob}(N_A(t) = n), \, n = 0, 1, \ldots, N$. These probabilities evolve according to the master equation

$$\frac{dp_n}{dt} = -\frac{k_1}{\Omega} n(n-1)p_n + \frac{k_1}{\Omega} (n-1)(N-n+1)p_{n-1}$$

$$- \frac{k_2}{\Omega} n(n-1)p_n + \frac{k_2}{\Omega} (n+1)(N-n-1)p_{n+1}$$

where "boundary conditions" $p_{-1} = 0 = p_{N+1}$ are assigned to these undefined quantities in order to make this formula apply uniformly for the extreme values $n = 0$ and $n = N$. Note that $N_A = 0$ and $N_A = N$ are absorbing states for the process.

Let $u = n/N \in (0,1)$ and consider the continuum $N \to \infty$ limit for the state variable. In the diffusion approximation the master equation (8) implies that the probability density $f(u,t)$ satisfies

$$\frac{\partial f(u,t)}{\partial t} = \frac{\partial}{\partial u} \left[ -\frac{(k_1 - k_2)N}{\Omega} u(1-u) + \frac{1}{2} \frac{k_1 + k_2}{\Omega} \frac{\partial}{\partial u} u(1-u) \right] f(u,t)$$

with boundary conditions $f(0,t) = 0 = f(1,t)$. This is the Fokker-Planck equation for the probability density of the solution $U(t)$ to the Itô stochastic differential equation

$$dU = \gamma U(1-U) \, dt + \sigma \sqrt{U(1-U)} \, dW$$

where $\gamma = (k_1 - k_2)N/\Omega$, $\sigma^2 = (k_1 + k_2)/\Omega$ and $W(t)$ is the usual Brownian motion. The boundary conditions on the density $f(u,t)$ correspond to absorbing boundary conditions on $U(t)$ at the 0 and 1 limits of the state space. Note that in the thermodynamic limit $N \to \infty$ and $\Omega \to \infty$ so that the density $N/\Omega = O(1)$, the noise is "weak", i.e., $\sigma \sim \frac{1}{\sqrt{N}}$.

Discreteness and the accompanying noise qualitatively change the logistic dynamics for this system. Because there is no normalizable steady state solution of the Fokker-Planck equation (8), any initial distribution supported in $(0,1)$ will eventually be absorbed into one or the other boundary at $U = 0$ or $U = 1$, contributing terms proportional to $\delta(u)$ and $\delta(1-u)$ to the distribution. That is, starting from an initial distribution $f(u,0) = f_0(u)$,

$$f(u,t) \to p_{\text{ext}} \delta(u) + (1 - p_{\text{ext}}) \delta(1-u) \quad \text{as} \quad t \to \infty$$

where $p_{\text{ext}}\{f_0\}$ is the probability that the process eventually becomes extinct, a functional of the initial distribution. It is straightforward to compute the probability that an initial population of $A$ particles will eventually become extinct—an eventuality that is simply impossible in the absence of noise. If $U(t)$ starts from (nonrandom) $u_0$, this extinction probability is

$$p_{\text{ext}} = \frac{e^{2\gamma(1-u_0)/\sigma^2} - 1}{e^{2\gamma/\sigma^2} - 1}$$

(11)
which is non-vanishing if the initial population contains any $B$ particles at all, i.e., if $0 \leq u_0 < 1$. There is a significant probability of extinction at low $A$-particle population: the probability of extinction is $1 - O(u_0)$ as $u_0 \to 0$.

In view of these considerations, we are naturally led to study the stochastic Fisher-Kolmogorov-Petrovsky-Piscounov equation

$$
\partial_t U = D \partial_{xx} U + \gamma U(1-U) + \epsilon \sqrt{U(1-U)} \eta(x,t)
$$

(12)

where $\eta(x,t)$ is a gaussian white noise process in space and time satisfying $\langle \eta(x,t) \rangle = 0$ and $\langle \eta(x,t) \eta(y,s) \rangle = \delta(x-y)\delta(t-s)$. This stochastic partial differential equation with multiplicative noise is to be interpreted as the continuum limit of a spatially discretized set of Itô equations; more on this below. We will refer to the coefficient $\epsilon$ as the noise strength, which one may think of as being proportional to $\epsilon \sqrt{N}$ where $N$ is the saturation population at a lattice site or in an appropriately defined correlation volume.

It is tempting to identify the stochastic FKPP equation as the fluctuating hydrodynamic description of a spatially extended reaction-diffusion model of (3), but the extra noise introduced by the diffusive hopping process complicates that connection. The stochastic FKPP equation (12) has, however, been rigorously derived as just such a description of an appropriate hydrodynamic limit of the contact process with long-range interactions [5].

It is also tempting to interpret the stochastic FKPP equation as a fluctuating hydrodynamic description of the birth-coagulation process (6), but a moment’s reflection shows that this cannot be. Indeed, fluctuations around the equilibrium level corresponding to $U = 1$ do not vanish for the $A \rightleftharpoons A + A$ process as they do (by definition) in the stochastic FKPP equation. For (6), a diffusion approximation of the master equation leads us to expect instead that the fluctuation strength should be proportional to at least $\sqrt{U + aU^2}$ (for a constant $a > 0$) up to and beyond the $U = 1$ level.

Nevertheless there is an exact rigorous relationship between the process in (6) and the stochastic Fisher-Kolmogorov-Petrovsky-Piscounov equation (12). The connection is by way of the notion of duality, an idea from the branch of probability theory concerned with interacting particle systems [6]. Duality between the growth-coagulation reaction and the stochastic FKPP equation was discovered in 1986 by Shiga and Uchiyama [7]. It is most easily described in the discrete space setting, and in the remainder of this paper we present a complete—and, we hope, accessible—derivation of this duality relation.

In the next section we focus on the simplest setting of the stochastic logistic equation and its dual relationship with the spatially homogeneous single-species birth-coagulation process. We then use this duality to produce a quick derivation of the formula for the extinction probability in (11). In the following section we present duality for the stochastic FKPP equation and the $A \rightleftharpoons A + A$ reaction-diffusion system and use it to compute an exact formula for the extinction probability of a solution of the stochastic partial differential equation. Subsequently, we use duality to explore the issue of wavefront propagation in the stochastic FKPP equation.
DUALITY FOR THE STOCHASTIC LOGISTIC EQUATION

Here we present the details of the duality relation between the spatially homogeneous \( A \leftrightarrow A + A \) reaction process and the stochastic logistic Itô equation

\[
dU = \gamma U (1 - U) dt + \sigma \sqrt{U (1 - U)} \ dW,
\]

where \( W(t) \) is the usual Brownian motion. These processes will be dual when the reaction coefficients in the birth-coagulation process correspond to the coefficients in the stochastic differential equation as follows:

\[
A \rightarrow A + A \quad \text{at rate } \gamma
\]
\[
A + A \rightarrow A \quad \text{at rate } \sigma^2.
\]

Specifically, the pair coagulation probability rate \( \chi/\Omega \) from (6) is taken as the square of the noise amplitude in the stochastic differential equation, \( \sigma^2 \).

Begin with the reaction system and let \( N(t) \) be the Markov process for the number of particles in the system and time \( t \). The master equation for the evolution of the probability \( P_n(t) = \text{Prob}(N(t) = n) \) for \( n = 1, 2, 3, \ldots \), is

\[
\frac{d}{dt} P_n(t) = \gamma (n-1) P_{n-1} - \gamma n P_n - \sigma^2 \frac{n(n-1)}{2} P_n + \frac{\sigma^2}{2} \frac{n+1}{n} P_{n+1}.
\]

More compactly we may write

\[
\frac{d}{dt} P_n(t) = \sum_{m=1}^{\infty} M_{nm} P_m(t)
\]

where the transition matrix \( M_{nm} \) is

\[
M_{nm} = \gamma m \delta_{n,m+1} - \gamma n \delta_{nm} - \frac{\sigma^2}{2} m (m-1) \delta_{nm} + \frac{\sigma^2}{2} m (m-1) \delta_{n,m-1}.
\]

Now consider the stochastic logistic equation (13), but for notational convenience first change variables to \( Z(t) = 1 - U(t) \) satisfying

\[
dZ = -\gamma Z (1 - Z) dt + \sigma \sqrt{Z (1 - Z)} \ dW.
\]

Recall the Itô formula for the increment of a function of a diffusion process,

\[
dF(Z) = F'(Z) dZ + \frac{1}{2} F''(Z) (dZ)^2,
\]

and the (operational) fact that \( (dW)^2 = dt \). Applied to the function \( F(\zeta) = \zeta^m \) for \( m \geq 1 \), we have

\[
\begin{align*}
\frac{dZ^m}{dt} & = m Z^{m-1} dZ + \frac{1}{2} m (m-1) Z^{m-2} (dZ)^2 \\
& = \left( -m \gamma Z^m + m \gamma Z^{m+1} + \frac{\sigma^2}{2} m (m-1) Z^{m-1} - \frac{\sigma^2}{2} m (m-1) Z^m \right) dt \\
& \quad + \sigma m Z^{m-1} \sqrt{Z (Z-1)} \ dW.
\end{align*}
\]
This may be written

\[ dZ^m = \sum_{n=1}^{\infty} Z^n M_{nm} \, dt + \sigma m Z^{m-1} \sqrt{Z(Z-1)} \, dW \]  

(21)

where the matrix \( M_{nm} \) is \textit{exactly} the same as the transition matrix in (17). Because the stochastic differential \( dW \) “sticks out into the future” and is independent of \( Z(t) \), the expectation \( \langle \cdot \rangle \) of the last term in (21) vanishes. Hence the integer moments of \( Z(t) \) evolve according to

\[ \frac{d}{dt} \langle Z(t)^m \rangle = \sum_{n=1}^{\infty} \langle Z(t)^n \rangle M_{nm}. \]  

(22)

In order to articulate the dual relationship, we run the stochastic differential equation forward in time and consider the particle process running independently backwards in time. Choose a time \( T > 0 \) and consider times \( 0 \leq t \leq T \). Define the random variable

\[ \mathcal{M}(t) = \sum_{m=1}^{\infty} Z(t)^m P_m(T-t). \]  

(23)

The fact is that the mean of \( \mathcal{M}(t) \) does not change with time, i.e.,

\[ \langle \mathcal{M}(T) \rangle = \langle \mathcal{M}(t) \rangle = \langle \mathcal{M}(0) \rangle. \]  

(24)

In the language of mathematical probability, \( \mathcal{M}(t) \) is a Martingale. The proof of this is by direct computation:

\[ \frac{d}{dt} \langle \mathcal{M}(t) \rangle = \sum_{m=1}^{\infty} \left( \frac{d}{dt} \langle Z(t)^m \rangle \right) P_m(T-t) + \sum_{m=1}^{\infty} \langle Z(t)^m \rangle \left( \frac{d}{dt} P_m(T-t) \right) \]

(25)

\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle Z(t)^n \rangle M_{nm} P_m(T-t) - \sum_{m=1}^{\infty} \langle Z(t)^m \rangle \sum_{n=1}^{\infty} M_{mn} P_n(T-t) \]

\[ = 0 \text{ (by an appropriate exchange of order of summation).} \]

For any function of \( \mathcal{F}[n] \) of the integers the expectation over the particle process is

\[ \langle \mathcal{F}[N(t)] \rangle = \sum_{n=1}^{\infty} \mathcal{F}[n] P_n(t), \]  

(26)

so the \textit{t-independent} expectation of the Martingale \( \mathcal{M}(t) \) in (24) can be written

\[ \langle Z(T)^{N(0)} \rangle = \langle Z(t)^{N(T-t)} \rangle = \langle Z(0)^{N(T)} \rangle \]  

(27)

where it is understood that the expectation is over the particle process, the Brownian motion driving the stochastic differential equation, and the initial condition \( Z(0) \).

This is the duality relation between the solution of the stochastic logistic equation (13) and the Markov process description of the interaction particle system (15): for any \( t > 0 \),

\[ \langle [1 - U(t)]^{N(0)} \rangle = \langle [1 - U(0)]^{N(t)} \rangle. \]  

(28)
Given freedom of the choices of the initial condition \( U(0) \) and the initial distribution of the particles through \( P_n(0) \), duality allows us to infer properties of one of the processes at time \( t \) in terms of its own initial condition as well as the initial and final state of the other process.

As an example application of the dual relationship between these processes, choose \( N(0) = 1 \) (i.e., \( P_n(0) = \delta_{n,1} \)) and \( U(0) = u_0 \) where \( 0 < u_0 < 1 \) is a nonrandom number. Duality asserts that

\[
\langle U(t) \rangle = 1 - \left\langle (1 - u_0)^N(t) \right\rangle.
\]  

(29)

The process \( U(t) \) will either eventually be absorbed at 0 and become extinct, or eventually be absorbed at 1, i.e., saturate. Let \( p_{\text{ext}} = \text{Prob}[\text{eventual extinction}|U(0) = u_0] \) so that \( 1 - p_{\text{ext}} = \text{Prob}[\text{eventual saturation}|U(0) = u_0] \). Then as \( t \to \infty \),

\[
\langle U(t) \rangle \to 1 - p_{\text{ext}}.
\]  

(30)

On the other hand, we know that the Markov process for the particle process has a unique invariant equilibrium (Poisson) distribution,

\[
P_{n}^{eq} = \frac{(2\gamma / \sigma^2)^n}{n!} \frac{1}{e^{2\gamma / \sigma^2} - 1} \quad \text{for } n = 1, 2, \ldots,
\]  

(31)

to which \( P_n(t) \) converges as \( t \to \infty \). Hence we deduce

\[
p_{\text{ext}} = \lim_{t \to \infty} \left\langle (1 - u_0)^N(t) \right\rangle = \sum_{n=1}^{\infty} (1 - u_0)^n P_{n}^{eq} = \sum_{n=1}^{\infty} (1 - u_0)^n \frac{(2\gamma / \sigma^2)^n}{n!} \frac{1}{e^{2\gamma / \sigma^2} - 1} = \frac{e^{2(1 - u_0)\gamma / \sigma^2} - 1}{e^{2\gamma / \sigma^2} - 1}.
\]  

(32)

Duality thus gives a simple and direct exact calculation of the extinction probability of the stochastic logistic process in terms of the initial condition and the model parameters. As will be shown in the following section, duality also produces an analogous exact result for the extinction probability of the solution of the spatially extended system described by the stochastic Fisher-Kolmogorov-Petrovsky-Piscounov equation.

**DUALITY FOR THE STOCHASTIC FKPP EQUATION**

We now establish the duality between the (spatially discretized) stochastic Fisher-Kolmogorov-Petrovsky-Piscounov partial differential equation and the spatially inhomogeneous birth-coagulation particle process. Discretize the \( x \)-axis with lattice spacing \( h \) and consider the coupled Itô equations

\[
dU_i(t) = \left[ D \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + \gamma U_i(1 - U_i) \right] dt + \sigma \sqrt{U_i(1 - U_i)} \, dW_i
\]  

(33)
where the $W_i(t)$ are independent Brownian motions at each site, i.e.,

$$dW_i(t) dW_j(t) = \delta_{ij} dt.$$  

(34)

With the identification $U(x,t) = U_{x/h}(t)$, the solution of (33) gives meaning to the stochastic FKPP partial differential equation (12) in the continuum limit $h \to 0$ when $\sigma \to \infty$ so that the noise strength

$$e = \sigma \sqrt{h} = \mathcal{O}(1) \quad \text{as} \quad h \to 0. \quad (35)$$

The birth-coagulation reaction-diffusion system

$$A \to A + A \quad \text{at rate} \quad \gamma$$

$$A + A \to A \quad \text{at rate} \quad \sigma^2,$$

(36)

where the particles independently execute nearest-neighbor diffusion with diffusion coefficient $D$, is defined as the Markov process $N(t) = (\ldots, N_{i-1}(t), N_i(t), N_{i+1}(t), \ldots)$ specifying the number of particles at each lattice site at time $t$, defined in detail below. The pair-coagulation probability rate for the $A + A \to A$ process is the square of the noise amplitude $\sigma^2$ in the discrete stochastic FKPP equation (33). Note that in distinction from the spatially homogeneous (single-site) case, the numbers of particles at each site satisfy $0 \leq N_j(t) < \infty$; in the coupled system sites may be empty at any given time and later be populated by diffusion from neighboring sites.

Let the vector $n = (\ldots, n_{i-1}, n_i, n_{i+1}, \ldots)$, where each integer $0 \leq n_j < \infty$, and let the unit vectors $e_i$ be

$$e_i = (\ldots, 0,1,0,\ldots) \quad (37)$$

where the 1 is in the $i^{th}$ slot. Taking into account all the reaction and diffusion processes, the master equation for the evolution of the probability $P_n(t) = \text{Prob}[N(t) = n]$ is

$$\frac{d}{dt} P_n(t) = \sum_{\{m\}} M_{nm} P_m(t) \quad (38)$$

where the sum is over the number of particles at each site, $0 \leq m_i < \infty$, and the transition matrix is

$$M_{nm} = \sum_{i} [\gamma m_i \delta_{n,m+e_i} - \gamma m_i \delta_{nm}$$

$$- \frac{\sigma^2}{2} m_i(m_i - 1) \delta_{nm} + \frac{\sigma^2}{2} m_i(m_i - 1) \delta_{n,m-e_i}$$

$$+ \frac{D}{h^2} m_{i-1} \delta_{n,m+e_i-e_i-1} - \frac{2D}{h^2} m_i \delta_{nm} + \frac{D}{h^2} m_{i+1} \delta_{n,m+e_i-e_{i+1}}]$$

(39)

Returning to the discretized sFKPP equation (33), for convenience let $Z_i(t) = 1 - U_i(t)$, and let $m = (\ldots, m_{i-1}, m_i, m_{i+1}, \ldots)$, where each $0 \leq m_j < \infty$. Then a straightforward application the Itô formula (19) yields

$$d \left( \prod_j Z_j(t)^{m_j} \right) = \sum_{\{n\}} \left( \prod_j Z_j(t)^{n_j} \right) M_{nm} dt \quad (40)$$
where the matrix $M_{nm}$ is exactly the same as the transition matrix in (40). The expectation of the last term above vanishes because the increments of the Brownian motions at time $t$ are independent of $U_i(t)$ and $Z_i(t)$, so the moments evolve according to

$$
\frac{d}{dt} \langle \prod_j Z_j(t)^{m_j} \rangle = \sum_{\{n\}} \left( \prod_i Z_i(t)^{n_i} \right) M_{nm}.
$$

(41)

Now let $0 \leq t \leq T$ and define the random variable

$$\mathcal{M}(t) = \sum_{\{n\}} \left( \prod_i Z_i(t)^{n_i} \right) P_n(T - t).
$$

(42)

Then $\mathcal{M}(t)$ is a Martingale, i.e., $\frac{d}{dt} \langle \mathcal{M}(t) \rangle = 0$. The proof, as before, is by direct computation using the moment evolution equation (41) and the master equation (38).

The expectation over the particle process of any function $\mathcal{F}[n]$ of the vector $n$ is

$$\langle \mathcal{F}[N(t)] \rangle = \sum_{\{n\}} \mathcal{F}[n] P_n(t),
$$

(43)

so duality between the solutions of the discrete stochastic FKPP equation and the birth-coagulation reaction-diffusion process may be expressed as

$$\langle \prod_i (1 - U_i(0))^{N_i(t)} \rangle = \langle \prod_i (1 - U_i(t))^{N_i(0)} \rangle
$$

(44)

for any $t > 0$ where the expectation is over the initial conditions $U_i(0)$, the Brownian motions $W_i(t)$ and the independent particle process $N(t)$.

As an application of duality for this system, we investigate the long-time behavior of the solution of the stochastic FKPP equation. A quantity of interest, for example, is the likelihood that a given nonrandom initial configuration $U_i(0) = u^0_i$ eventually results in the extinction of the entire process. That is, as $t \to \infty$, either all the variables $U_i(t) \to 0$ or they all saturate to 1. The extinction probability $p_{ext}$ is a function of the system parameters and the initial data.

In order to compute $p_{ext}$ for the stochastic FKPP equation, we consider the long-time behavior of the dual particle process $N(t)$. As long as the birth-coagulation process starts with at least one particle present somewhere, as $t \to \infty$ it approaches the invariate state with equilibrium product Poisson probability distribution

$$P_n^{eq} = \prod_i \frac{(2\gamma/\sigma^2)^{n_i}}{n_i!} e^{-2\gamma/\sigma^2}.
$$

(45)

We now use duality with $N(0)$ in the equilibrium state so that $N(t)$ is also in the equilibrium state. On the one hand,

$$\langle \prod_i (1 - U_i(0))^{N_i(t)} \rangle = \sum_{\{n\}} \prod_i (1 - u^0_i)^{n_i} P_n^{eq}
$$

(46)
On the other hand an identical calculation yields

\[
\prod_i (1 - U_i(t))^{N_i(0)} = \exp \left\{ -\sum_i \frac{2\gamma}{\sigma^2} U_i(t) \right\}.
\]  

(47)

The random variable in the expression on the right-hand side above converges, as \( t \to \infty \), to the indicator function that the process becomes extinct, so the expectation converges to \( p_{\text{ext}} \). Hence duality produces the exact formula

\[
p_{\text{ext}} = \exp \left\{ -\sum_i \frac{2\gamma}{\sigma^2} U_i(t) \right\}.
\]  

(48)

Using the fact that \( \epsilon = \sigma \sqrt{n} \), we have thus proven that in the continuum limit \( h \to 0 \) the solution \( U(x, t) \) to the stochastic FKPP equation

\[
\partial_t U(x, t) = D \partial_{xx} U + \gamma U(1 - U) + \epsilon \sqrt{U(1 - U)} \eta(x, t),
\]  

(49)

starting from initial function \( u_0(x) \), will eventually become extinct with probability

\[
p_{\text{ext}} = \exp \left\{ -\frac{2\gamma}{\epsilon^2} \int u_0(x) dx \right\}.
\]  

(50)

This extinction probability looks remarkably like a large deviations result in that \( \epsilon^{-2} \) appears in the exponent, although it is an exact expression and not just asymptotic as \( \epsilon \to 0 \). It is noteworthy that the extinction probability is exponentially small in the total mass of the initial configuration. For example the extinction probability is zero for an initially saturated half-space as would be taken to study the propagation of a wavefront in the stochastic FKPP equation. (This topic is the subject of the next section.) It is also noteworthy that the extinction probability for the solution of the stochastic FKPP equation in (50) does not depend on the diffusion coefficient \( D \). Diffusion is essential for this result, however, as a different formula results from a product of the uncoupled extinction probability in (33) as would be the expression for a collection of independent stochastic logistic equations if \( D = 0 \).

**WAVEFRONTS IN THE STOCHASTIC FKPP EQUATION**

The FKPP equation (1) describes the invasion of the stable saturated state \( U = 1 \) into regions of the deterministically unstable extinct phase \( U = 0 \). This invasion proceeds via propagation of a front at a constant velocity: to find these travelling waves we look for solutions of the form

\[
U(x, t) = w(x - ct),
\]  

(51)
where the speed $c$ is to be determined. Inserting this ansatz into the FKPP equation yields an ordinary differential equation for the front shape $w(z)$:

$$Dw'' + cw' + \gamma w(1 - w) = 0. \quad (52)$$

The boundary conditions for the front shape function are determined by the setup for $U$. If we consider a right-moving wave where $U \to 1$ as $x \to -\infty$ and $U \to 0$ as $x \to +\infty$, then the conditions for $w$ are $w \to 1$ as the “time” $z \to -\infty$ and $w \to 0$ as $z \to +\infty$. In this case we look for positive velocities, $c > 0$, and the front is identified as the heteroclinic orbit connecting the unstable fixed point $(w, w') = (1, 0)$ and the stable equilibrium state $(w, w') = (0, 0)$ of the two-dimensional dynamical system defined by (52). Because $0 \leq U(x, t) \leq 1$ for any initial condition $0 \leq U(x, 0) \leq 1$, we require that $0 \leq w(z) \leq 1$. Then it is easy to see that the determining factor in the analysis is the need to prevent oscillations around 0 near $w = 0$. This means that the “friction coefficient” $c$ in the dynamical system for $w$ must be sufficiently high that the stable fixed point $(w, w') = (0, 0)$ is a node and not a spiral. Hence we find acceptable solutions for any speed $c \geq 2\sqrt{D\gamma}$.

It turns out that for sufficiently sharp initial fronts in $U(x, 0)$, the minimum speed $c_{\text{min}} = 2\sqrt{D\gamma}$ is selected [1, 2], albeit very slowly [8, 9]. That is, for sharp initial fronts any reasonable definition of the instantaneous velocity $v(t)$ satisfies $v(t) = c_{\text{min}} + \mathcal{O}(t^{-1})$. This algebraic relaxation of the speed, combined with the fact that the speed is ultimately selected by the properties of the system far ahead of the bulk of the population front, leads to the terminology that the FKPP equation has a pulled front with weak velocity selection. The situation here is to be contrasted with reaction-diffusion fronts connecting metastable states. The velocity of such a pushed front is determined by the nonlinear dynamics in transition region defining the front and the speed adjusts itself exponentially fast, termed strong velocity selection.

Upon reflection it is clear that the extreme sensitivity of the pulled front to the details of the process near the unstable state cannot be robust or, really, physical. Indeed, any modification of the dynamics at the lowest level where $U \approx 0$—for example due to population discreteness effects—would invalidate the velocity selection mechanism described above. We would expect low population levels, such as those in the leading edge of a traveling wavefront in the FKPP equation, to naturally extinguish themselves. It is thus apparent that no mechanism which relies on the exponentially small population out in front of the wave should be physically relevant in the dynamics.

This lack of “structural stability” of the pulled front has been explored in alot of detail recently. In 1997 and 1998 Brunet and Derrida [10] and Kessler, Ner and Sander [11] considered the influence of discreteness in the population variable $u$ on the front speed by replacing the $u(1 - u)$ growth and saturation dynamics in the FKPP equation with $u(1 - u)\Theta(u - N^{-1})$, where $\Theta(\zeta)$ is the step function vanishing when $\zeta < 0$. Here $N^{-1} \ll 1$ represents the discreteness level. Repeating the dynamical systems argument for the velocity of a traveling wavefront, they concluded that the selected speed is unique, strongly selected, and given for large $N$ by

$$c \sim \sqrt{D\gamma} \left[ 2 - \frac{\pi^2}{(\log N)^2} \right] = c_{\text{min}} - \mathcal{O}((\log N)^{-2}). \quad (53)$$
This is slower than the minimum speed previously available. Moreover, it displays extremely slow convergence to the “continuum” limit for $u$ as $N \to \infty$. Soon thereafter Pechenik and Levine [12] computationally verified this kind of velocity dependence on $N$ in another stochastic model. Most recently Brunet and Derrida conjectured as well that in a discrete stochastic model, the position of the wavefront should diffuse with an $O((\log N)^{-3})$ diffusion coefficient [13].

It is very interesting to explore wavefront propagation from the point of view of the stochastic FKPP equation. The multiplicative noise in the stochastic FKPP has profound qualitative effects on the front. In 1995 Mueller and Sowers [15] proved that the sFKPP equation (12) has the so-called compact support property. This means, for example, that if the initial data $U(x,0)$ for the stochastic partial differential equation satisfies $U(x,0) = 1$ for $x \leq a$ and $U(x,0) = 0$ for $x \geq b$ where $-\infty < a < b < \infty$, then for all subsequent $t > 0$ the solution will satisfy $U(x,t) = 1$ for $x \leq a(t)$ and $U(x,t) = 0$ for $x \geq b(t)$ for some values $a(t)$ and $b(t)$ satisfying $-\infty < a(t) < b(t) < \infty$. That is, a front with sharp leading and trailing edges will always have sharp leading and trailing edges. This is despite the tendency for the diffusion to propagate $U$ into the vacuous regions in front of the wave, and to decrease the population from the saturation level at the back of the front. The noise simply overwhelms the diffusive transport near the boundaries of the state variable at $U = 0$ and $U = 1$. This compact support property can be seen in numerical simulations. In Figure 1 we plot the results of direct numerical simulations of the FKPP equation (1) and stochastic sFKPP equation (12), both starting from the same step function initial condition, explicitly showing the qualitative difference in the

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1 There have been many studies of the effects of spatial inhomogeneities and environmental randomness of the FKPP front dynamics, too [14], but we focus on this statistically spatially homogeneous model.
leading and trailing edges.

Duality allows us to express moments of the solution of the stochastic FKPP equation (33) at time $t$ in terms of its initial data and the solution of the birth-coagulation reaction-diffusion system. In particular, if we choose the initial condition for the particles so that there is just one particle at site $i$ and none anywhere else (i.e., $N(0) = e_i \Leftrightarrow N_j(0) = \delta_{ij}$), then

$$
\langle U_i(t) \rangle = 1 - \left( \prod_j (1 - U_j(0))^{N_j(t)} \right).
$$

Furthermore, if the initial data for $\{U_j\}$ is $U_j(0) = 1$ for $j \leq 0$ and $U_j(0) = 0$ for $j > 0$, the setup for a wavefront propagating to the right, then duality produces the exact representation

$$
\langle U_i(t) \rangle = \text{Prob(any site } j \leq 0 \text{ has a particle at time } t \mid \text{ one particle at site } i \text{ at time 0}).
$$

The growth-coagulation process never becomes extinct; if we start with one or more particles we are ensured that there will always be at least one particle somewhere on the line (or lattice). In fact, starting with only one singly occupied site $i$, the growth process will dominate until the unique equilibrium steady state (45) with average occupation number $2\gamma/\sigma^2$ at all lattice sites is achieved. This connection implies that the asymptotic front speed for the stochastic FPKK equation should be the same as the asymptotic front speed for the $A \leftrightarrow A + A$ reaction-diffusion system when the coagulation rate $\chi$ is properly identified as the square of the noise amplitude $\sigma^2$.

In general we do not know an exact expression for the velocity of a front in this interacting particle model. However, we do know many things about a certain limit of the single-species growth-coagulation reaction-diffusion system, the so-called diffusion-controlled limit.

The diffusion-controlled $A \leftrightarrow A + A$ process is the reaction-diffusion dynamics in (6) in the limit where $\chi \to \infty$ so that $A + A \to A$ instantaneously when two particles land on the same lattice site [16]. The equilibrium density $\rho_{eq}$ of particles on the lattice is then nonzero only if the growth rate $\gamma$ is also scaled up appropriately as $\chi \to \infty$. (By “density” we mean that $\rho_{eq} \rho$ is the equilibrium occupation probability—equivalently the average occupation number—of a site.) Noting the correspondence between $\chi$ and $\sigma^2$ in the duality relation, this means that the strong-noise limit of the sFKPP equation is dual to the diffusion-controlled limit of the interacting particle process. Thus we can make a quantitative connection between the system near the limit and the diffusion-controlled limit itself if we identify the equilibrium occupation probabilities $\rho_{eq} \rho$ and $2\gamma/\sigma^2 = 2\gamma h/\epsilon^2$.

The front propagation problem for the diffusion-controlled single-species growth-coagulation reaction-diffusion system has been solved exactly and in great detail [16, 17, 18]. In particular we know that the speed of the wavefront for the diffusion-limited process is exactly $c = D\rho_{eq}$ [17, 18]. From this and the duality connection, then, we can conjecture the behavior of the velocity of the sFKPP wave in the strong noise or weak growth limit:

$$
c = D\rho_{eq} \sim D \frac{2\gamma}{h\sigma^2} = \frac{2D\gamma}{\epsilon^2}.
$$

(55)
FIGURE 2. Wave speed (normalized by the no-noise speed) vs. dimensionless noise strength from the simulations (data), and the asymptotic conjectures for weak and strong noise (lines).

In Figure 2 we plot the front velocities normalized by the minimum no-noise speed \((2\sqrt{D\gamma})\) as measured in direct numerical simulations of the (discrete) stochastic FKPP equation (33) versus the dimensionless noise strength \(\epsilon/(D\gamma)^{1/4}\). The solid lines are the asymptotic conjectures (53) for weak noise (using \(\tilde{N} = 2\gamma/\sigma^2\)) and (55) for strong noise. The data suggest that both conjectures are quantitatively accurate.

SUMMARY AND CONCLUSIONS

In this paper we have presented a simple and complete derivation of the duality relation, first discovered by Shiga and Uchiyama [7], between solutions of the stochastic FKPP equation and the \(|\_|N\_|N\) birth-coagulation reaction-diffusion interacting particle system. We then used duality to obtain an exact formula for the eventual extinction probability of any solution of the stochastic FKPP equation as a function of the initial configuration and the system parameters. 

The dual particle process becomes the diffusion-controlled \(\mathbb{A} \rightarrow A + A\) reaction in the strong noise (or weak growth) limit of the stochastic FKPP equation. Combined with previously derived exact results for the stochastic wavefront in the diffusion-controlled process, we used this correspondence to conjecture the dependence of the propagation velocity on the system parameters for large values of the relevant dimensionless measure of the noise variance \(\epsilon^2/\sqrt{D\gamma}\). Direct numerical simulations of the stochastic FKPP equation support this conjecture. This result predicts, for example, that in the presence of noise at any level the speed crosses over from being proportional to \(\sqrt{\gamma}\) at large values of the birth rate to being proportional to \(\gamma\) as \(\gamma \rightarrow 0\).

A natural question is whether the weak and strong noise asymptotic expressions
for the front velocity may be proven by analysis of the stochastic partial differential equation (with or without the use of duality). Another open issue is that of the statistics of the leading and trailing points of the front. Mueller and Sowers proved that the fluctuating width of the front attains a stationary distribution as \( t \to \infty \) [15]. Can the stationary distribution of the front width be determined? What is the “typical” shape of the stochastic wavefront? And what about front diffusion? In the other direction, it will be interesting to know if the stochastic partial differential equation can tell us any new facts about the nonequilibrium statistical mechanics of the \( A \rightleftharpoons A + A \) growth-coagulation reaction-diffusion process.

Many of the ideas developed and discussed here can also be generalized to higher spatial dimensions. Front motion in higher spatial dimensions is of great interest for a variety of growth processes. In higher spatial dimensions even an initially smooth or flat front can develop structure in the transverse direction(s) and fluctuations may play an even more dramatic role in the dynamics. Much recent effort has gone into studying particle systems whose mean-field descriptions are the higher dimensional FKPP equation [19, 20, 21, 22]. In these systems the effects of discreteness, fluctuations and noise may be extremely subtle and the quantitative behavior is difficult to discern. It remains to be seen if duality will help to shed some light on these issues.

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