1st University of Rochester Mathematical Olympiad

Saturday, March 31, 2007

Show all work (each step/computation) to receive full credit. You may use back pages if necessary. No calculators. The test contains 4 problems. Make sure it is complete.

No.	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

REGISTRATION NO. : ____

1. Let n be a positive integer. Compute the following antiderivative:

$$\int \frac{x^{3n-1} - x^{n-1}}{x^{4n} + 1}\, dx$$

2. An equilateral triangle is inscribed on three concentric circles with each vertex being on a different circle. The radii of the circles are 3, 4, 5 respectively. Find the length of the side of the triangle. (Hint: There are two cases depending on whether the common center of the three circles is inside or outside of the triangle.)

3. Let us consider an infinite lattice indexed by the set $\mathbf{Z} \times \mathbf{Z}$ (e.g. the set of points of coordinates (m, n) with $m,\ n \in \mathbf{Z}$). In the vertices of this lattice we place positive integers according to the following rule: the number corresponding to a vertex is the average of the numbers placed in the neighboring vertices (Each vertex has four neighbors corresponding to the four cardinal directions; for example: the neighbors of $(3, 5)$ are $(3, 4)$, $(4, 5)$, $(3, 6)$, and $(2, 5)$. So if the numbers placed at $(3, 4)$, $(4, 5)$, $(3, 6)$, and $(2, 5)$, are 2, 4, 1, respectively 1, then the number placed at $(3, 5)$ is $\frac{2+4+1+1}{4} = 2$). Prove that all the numbers placed on the lattice must be equal.

4. Find with proof all the integer solutions (x, y, z) of the following equation:

$$x^2 + y^2 + z^2 = 2xyz$$