9th University of Rochester Math Olympiad

Saturday, February 7, 2015

9:30 AM - 12:30 PM

Show all work (each step and computation) to receive full credit. You may use back pages if necessary. Calculators are not permitted. The olympiad consists of four problems. Please submit your work for all the problems, even if some pages are blank.

No.	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

REGISTRATION NO. : ______________________________

1. Show that, for any triplet (a, b, ϕ) of real numbers, the following inequality holds:

$$1 + \left(\frac{a+b}{2} \right)^2 \geq (\sin \phi + a \cos \phi) (\sin \phi + b \cos \phi) .$$

2. Let $p \geq 3$ be a prime number. Prove that

$$\lfloor (2+\sqrt{5})^p \rfloor \; - \; 2^{p+1}$$

is divisible by p, where $\lfloor x \rfloor$ represents the floor function (e.g., $\lfloor \pi \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$).

3. Prove that if $f : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying

$$2015 \int_0^x f^2(t)\, dt \le \left(\int_0^x f(t)\, dt \right)^2, \qquad \forall\, x \ge 0,$$

then $f(x) = 0$ for all $x \ge 0$.

4. Let $n \geq 1$ be an integer and consider a set A of $2n+3$ points in a plane such that no three of them lie on the same line and no four of them lie on the same circle. Show that there exists a circle in the plane satisfying simultaneously the following conditions:

- it passes through three points from A;
- it contains n points from A in its interior;
- the remaining n points from A lie in its exterior.