Preface

The wave map problem is one of the most beautiful and challenging nonlinear hyperbolic problems, which has kept the attention of mathematicians for more than thirty years now. The study of the problem involves diverse issues, e.g., well-posedness, regularity, formation of singularities, and stability of solitons, and combines intricate tools from analysis, geometry, and topology. Moreover, the wave map system has a natural formulation as the Euler-Lagrange system for a map between manifolds, a special case being the nonlinear sigma model, which is one of the fundamental objects in classical field theory. One of the goals of this book is to offer an up-to-date and self-contained overview of the main regularity theory for wave maps. Another goal is to introduce, to a wide mathematical audience, physically motivated generalizations of the wave map system (e.g., the Skyrme model), which are extremely interesting and pose challenging new questions in their own right.

The topic of wave maps has experienced an incredible advancement in the past ten years. This is precisely the time passed from the moment when the last monograph (i.e., Tao [171]) which tried to give a state-of-the-art for this topic appeared. Our book tries to fill this gap by presenting the most recent developments in the field, e.g., the resolution of the large data global regularity theory for wave maps. These results are very technical, being accessible only to experts in their current format. Our goal is to try to explain them to a wider group, which includes advanced graduate students, in the hope of stimulating new research ideas. Moreover, this book is the first one which discusses, from a mathematical point of view, the time evolution for the models which are extensions of the nonlinear sigma model: the Skyrme, Faddeev, and Adkins-Nappi theories.

Our book starts by introducing the reader to the physical motivation and the mathematical formulation of the wave map problem and its generalizations. This is followed by developing the analytic background needed in the investigation of these problems, which include Strichartz estimates and hyperbolic Sobolev spaces. The third chapter is devoted to the study of the local and small data global well-posedness theories for the wave map equation, where one can see the motivation for the emergence of more and more powerful analytic techniques needed in handling the challenging nature of these topics. This chapter also includes detailed expositions for two results, due to Tao [169] and Shatah-Struwe [146], respectively, which make the case for the important role played by the intrinsic geometric aspect of the wave map problem. Next, we discuss the resolution of the large data regularity theory for wave maps in the energy-critical case by Sterbenz-Tataru's program [160, 161]. We focus on the second part (i.e., [161]) of this work, which provides a complete description of the regimes when a large data, finite energy wave map blows up and when it is global-in-time. Our presentation of this topic reworks Sterbenz-Tataru's argument, including a significant number of refinements and extra details. The fifth chapter addresses well-posedness questions for the classical Skyrme model and its extensions. There, we present Wong's result [191] on the Skyrme model using Christodoulou's regular hyperbolicity framework and Lei-Lin-Zhou's global regularity theorem [103] for the $2+1$-dimensional Faddeev problem, which relies on an adaptation of Klainerman's vector field method. Following this, we discuss equivariant results for all the equations considered in this book, which include non-concentration of energy, small data global well-posedness in critical Besov spaces, and global regularity for sufficiently smooth large data. Finally, we turn our attention to the phenomenon of collapse for wave maps and examine Raphaël-Rodnianski's result [136] on the topic. We put forth a novel approach relying on the associated Hodge system, which makes the structure of the problem more transparent. The book also includes an appendix detailing the basic differential geometry concepts needed for following the presentation of the material.

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