Why are there so many prime numbers?

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Outline

Three big theorems about prime numbers
- Euclid’s theorem
- Dirichlet’s theorem
- The prime number theorem

Two proofs of Theorem 1
- God’s proof
- Euclid’s proof

Primes of the form $4m - 1$

Primes of the form $4m + 1$

Other cases of Dirichlet’s theorem

Euler’s proof of Theorem 1

The Riemann hypothesis

Some theorems about primes that every mathematician should know

Theorem 1 (Euclid, 300 BC)

*There are infinitely many prime numbers.*

Euclid’s proof is very elementary, and we will give it shortly.

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Theorem 2 (Dirichlet, 1837, Primes in arithmetic progressions)

Let \( a \) and \( b \) be relatively prime positive integers. Then there are infinitely primes of the form \( am + b \).

Example. For \( a = 10 \), \( b \) could be 1, 3, 7 or 9. The theorem says there are infinitely many primes of the form \( 10m + 1 \), \( 10m + 3 \), \( 10m + 7 \) and \( 10m + 9 \). For other values of \( b \) not prime to 10, there is at most one such prime.

Dirichlet’s proof uses functions of a complex variable.

We will see how some cases of it can be proved with more elementary methods.
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Theorem 3 (Hadamard and de la Vallée Poussin, 1896, Asymptotic distribution of primes)

Let \( \pi(x) \) denote the number of primes less than \( x \). Then

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\lim_{x \to \infty} \frac{\pi(x)}{x/\ln x} = 1.
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In other words, the number of primes less than \( x \) is roughly \( x/\ln x \).

A better approximation is to \( \pi(x) \) is the logarithmic integral

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- Look at the positive integers
  
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- See which of them are primes
  
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Without God’s omniscience, we have to work harder.

Euclid’s proof relies on the *Fundamental Theorem of Arithmetic* (FTA for short), which says that every positive integer can be written as a product of primes in a unique way.

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- Let $S = \{p_1, p_2, \ldots, p_n\}$ be a finite set of primes.
- Let $N = p_1p_2\ldots p_n$, the product of all the primes in $S$.
- The number $N$ is divisible by every prime in $S$.
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- By the FTA, $N + 1$ is a product of one or more primes not in the set $S$.
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We can use Euclid’s method to show there are infinitely many prime of the form $4m − 1$.

- Let $S = \{p_1, \ldots, p_n\}$ be a set of such primes, and let $N$ be the product of all of them.
- The number $4N − 1$ is not divisible by any of the primes in $S$.
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It turns out that the number $4N^2 + 1$ (instead of $4N + 1$) has to be the product of primes of the form $4m + 1$.

Here are some examples.

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To prove this we need some help from Pierre de Fermat, who is best known for his “Last Theorem.”

**Theorem (Fermat’s Little Theorem, 1640)**

If $p$ is a prime, then $x^p - x$ is divisible by $p$ for any integer $x$.

Since $x^p - x = x(x^{p-1} - 1)$, if $x$ is not divisible by $p$, then $x^{p-1} - 1$ is divisible by $p$. In other words, $x^{p-1} \equiv 1$ modulo $p$. 
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Lemma

Any number of the form $4N^2 + 1$ is a product of primes of the form $4m + 1$.

Proof: Let $x = 2N$, so our number is $x^2 + 1$. Suppose it is divisible by a prime of the form $p = 4m + 3$. This means $x^2 \equiv -1$ modulo $p$.

Then $x^{4m} = (x^2)^{2m} \equiv (-1)^{2m} = 1$.

Fermat’s Little Theorem tell us that $x^{p-1} = x^{4m+2} \equiv 1$, but $x^{4m+2} = x^{4m} \cdot x^2 \equiv 1 \cdot -1 = -1$, so we have a contradiction.

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- Let $S = \{p_1, \ldots, p_n\}$ be a set of such primes, and let $N$ be the product of all of them.
- The number $4N^2 + 1$ is not divisible by any of the primes in $S$.
- Therefore $4N^2 + 1$ is the product of some primes not in $S$, all of which must have the form $4m + 1$.
- Therefore $S$ is not the set of all primes of the form $4m + 1$. 
Why are there so many prime numbers?

Outline

Three big theorems about prime numbers
- Euclid’s theorem
- Dirichlet’s theorem
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Some other cases of Dirichlet’s theorem

Similar methods (involving algebra but no analysis) can be used to prove some but not all cases of Dirichlet’s theorem. For example,

- We can show there are infinitely many primes of the forms $3m + 1$ and $3m - 1$.
- We can show there are infinitely many primes of the forms $5m + 1$ and $5m - 1$.
- We can show there are infinitely many primes of the forms $5m + 2$ or $5m + 3$, but not that there are infinitely many of either type alone.
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Euler’s proof that there are infinitely many primes

Euler considered the infinite series

$$\sum_{n \geq 1} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$$

From calculus we know that it converges for $s > 1$ (by the integral test) and diverges for $s = 1$ (by the comparison test), when it is the harmonic series.

Using FTA, Euler rewrote the series as a product

$$\sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left( \sum_{k \geq 0} \frac{1}{p^{ks}} \right)$$

$$= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \ldots\right) \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \ldots\right) \ldots$$
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Euler’s proof (continued)

Each factor in this product is a geometric series. The $p$th factor converges to $1/(1 - p^{-s})$, whenever $s > 0$. Hence

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If there were only finitely many primes, this would give a finite answer for $s = 1$, contradicting the divergence of the harmonic series.

Dirichlet used some clever variations of this method to prove his theorem 100 years later.
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Epilogue: The Riemann zeta function.

In his famous 1859 paper *On the Number of Primes Less Than a Given Magnitude*, Riemann studied Euler’s series

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as a function of a complex variable $s$, which he called $\zeta(s)$.

He showed that the series converges whenever $s$ has real part greater than 1, and that it can be extended as a complex analytic function to all values of $s$ other than 1, where the function has a pole.

He showed that the behavior of this function is intimately connected with the distribution of prime numbers.

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The Riemann hypothesis

When does $\zeta(s)$ vanish?

Riemann showed that $\zeta(s) = 0$ for $s = -2, s = -4, s = -6$ and so on. These are called the trivial zeros.

The Riemann hypothesis is concerned with the non-trivial zeros, and states that:

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\text{The real part of any non-trivial zero of the Riemann zeta function is } 1/2.
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