ON THE NON-EXISTENCE OF ELEMENTS OF KERVAIRE INVARIANT ONE

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Abstract. We show that the Kervaire invariant one elements $\theta_j \in \pi_{2^{j+1}-2}S^0$ exist only for $j \leq 6$. By Browder's Theorem, this means that smooth framed manifolds of Kervaire invariant one exist only in dimensions 2, 6, 14, 30, 62, and possibly 126. Except for dimension 126 this resolves a longstanding problem in algebraic topology.

Contents

1. Introduction

The existence of smooth framed manifolds of Kervaire invariant one is one of the oldest unresolved issues in differential and algebraic topology. The question originated in the work of Pontryagin in the 1930's. It took a definitive form in the paper [28] of Kervaire in which he constructed a combinatorial 10-manifold with no smooth structure, and in the work of Kervaire-Milnor [29] on h-cobordism classes of manifolds homeomorphic to a sphere. The question was connected to homotopy theory by Browder in his fundamental paper [7] where he showed that smooth framed manifolds of Kervaire invariant one exist only in dimensions of the form $(2^{j+1}-2)$, and that a manifold exists in that dimension if and only if the class

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

in the E_2-term of the classical Adams spectral represents an element

$$\theta_j \in \pi_{2^{j+1}-2} S^0$$

in the stable homotopy groups of spheres. The classes θ_j for $j \leq 5$ were shown to exist by Barratt-Mahowald, and by Barratt-Jones-Mahowald (see [5]).

The purpose of this paper is to prove the following theorem

Theorem 1.1. *For $j \geq 7$ the class $h_j^2 \in \operatorname{Ext}_A^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ does not represent an element of the stable homotopy groups of spheres. In other words, the Kervaire invariant elements θ_j do not exist for $j \geq 7$.*

Smooth framed manifolds of Kervaire invariant one therefore exist only in dimensions 2, 6, 14, 30, 62, and possibly 126. At the time of writing, our methods still leave open the existence of θ_6.

Many open issues in algebraic and differential topology depend on knowing whether or not the Kervaire invariant one elements θ_j exist for $j \geq 6$. The following results represent some of the issues now settled by Theorem 1.1. In the statements, the phrase "exceptional dimensions" refers to the dimensions 2, 6, 14, 30, 62, and 126. In all cases the situation in the dimension 126 is unresolved. By Browder's work [7] the results listed below were known when the dimension in question was not 2 less than a power of 2. Modulo Browder's result [7] the reduction of the statements to Theorem 1.1 can be found in the references cited.

Theorem 1.2 ([29, 31]). *Except in the six exceptional dimensions, every stably framed smooth manifold is framed cobordant to a homotopy sphere.* □

In the first five of the exceptional dimensions it is known that not every stably framed manifold is framed cobordant to a homotopy sphere. The situation is unresolved in dimension 126.

Theorem 1.3 ([29]). *Let M^m be the manifold with boundary constructed by plumbing together two copies of the unit tangent bundle to S^{2k+1} (so $m = 4k+2$), and set $\Sigma^{m-1} = \partial M^m$. Unless m is one of the six exceptional dimensions, the space M^m/Σ^{m-1} is a triangulable manifold which does not admit any smooth structure, and the manifold Σ^{m-1} (the Kervaire sphere) is homoemorphic but not diffeomorphic to S^{m-1}.* □

In the first five of the exceptional cases, the Kervaire sphere is known to be diffeomorphic to the ordinary sphere, and the Kervaire manifold can be smoothed.

Theorem 1.4 ([29, 31]). *Let Θ_n be the Kervaire-Milnor group of h-cobordism classes of homotopy n-spheres. Unless $(4k+2)$ is one of the six exceptional dimensions,*

$$\Theta_{4k+2} \approx \pi_{4k+2} S^0$$

and

$$|\Theta_{4k+1}| = a_k \left|\pi_{4k+1} S^0\right|,$$

where a_k is 1 if k is even, and 2 if k is odd.

Theorem 1.5 ([4]). *Unless n is 1, or one of the six exceptional dimensions, the Whitehead square $[\iota_{n+1}, \iota_{n+1}] \in \pi_{2n+1} S^{n+1}$ is not divisible by 2.* □

1.1. **Outline of the argument.** Our proof builds on the strategy used by the third author in [43] and on the homotopy theoretic refinement developed by the second author and Haynes Miller (see [47]).

We construct a multiplicative cohomology theory Ω and establish the following results:

Theorem 1.6 (The Detection Theorem). *If $\theta_j \in \pi_{2^{j+1}-2}S^0$ is an element of Kervaire invariant 1, then the image of θ_j in $\pi_{2^{j+1}-2}\Omega$ is non-zero.*

Theorem 1.7 (The Periodicity Theorem). *The cohomology theory Ω is 256-fold periodic: $\pi_*\Omega = \pi_{*+256}\Omega$.*

Theorem 1.8 (The Gap Theorem). *The groups $\pi_i\Omega$ are zero for $-4 < i < 0$.*

These three results easily imply Theorem 1.1. The Periodicity Theorem and the Gap Theorem imply that the groups $\pi_i\Omega$ are zero for $i \equiv -2 \mod 256$. By the Detection Theorem, if θ_j exists it has a non-zero image in $\pi_{2^{j+1}-2}\Omega$. But this latter group is zero if $j \geq 7$.

1.2. **The cohomology theory Ω.** Write C_n for the cyclic group of order n. Our cohomology theory Ω is part of a pair $(\Omega, \Omega_{\mathbb{O}})$ analogous to the orthogonal and unitary K-theory spectra KO and KU. The role of complex conjugation on KU is played by an action of C_8 on $\Omega_{\mathbb{O}}$, and Ω arises as its fixed points. It is better to think of $\Omega_{\mathbb{O}}$ as generalizing Atiyah's C_2-equivariant $K_{\mathbb{R}}$-theory [3], and in fact $\Omega_{\mathbb{O}}$ is constructed from the corresponding real bordism spectrum, as we now describe.

Let $MU_{\mathbb{R}}$ be the C_2-equivariant *real bordism* spectrum of Landweber [30] and Fujii [16]. Roughly speaking one can think of $MU_{\mathbb{R}}$ as describing the cobordism theory of *real manifolds*, which are stably almost complex manifolds equipped with a conjugate linear action of C_2, such as the space of complex points of a smooth variety defined over $\mathbb{R}$. A real manifold of real dimension $2n$ determines a homotopy class of maps

$$S^{n\rho_2} \to MU_{\mathbb{R}}$$

where $n\rho_2$ is the direct sum of n copies of the real regular representation of C_2, and $S^{n\rho_2}$ is its one point compactification.

Write

$$MU^{(C_8)} = MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$

for the C_8-equivariant spectrum gotten by smashing 4 copies of $MU_{\mathbb{R}}$ together and letting C_8 act by

$$(a, b, c, d) \mapsto (\bar{d}, a, b, c).$$

Very roughly speaking, $MU^{(C_8)}$ can be thought of as the cobordism theory of stably almost manifolds equipped with a C_8-action, with the property that the restriction of the action to $C_2 \subset C_8$ determines a real structure. If M is a real manifold then $M \times M \times M \times M$ with the C_8-action

$$(a, b, c, d) \mapsto (\bar{d}, a, b, c)$$

is an example. A suitable C_8-manifold M of real dimension $8n$ determines a homotopy class of maps

$$S^{n\rho_8} \to MU^{(C_8)},$$

where $n\rho_8$ is the direct sum of n copies of the real regular representation of C_8, and $S^{n\rho_8}$ is its one point compactification.

To define Ω we invert an equivariant analogue

$$D : S^{\ell\rho_8} \to MU^{(C_8)}$$

of the Bott periodicity class and form C_8-equivariant spectrum $\Omega_{\mathbb{O}} = D^{-1}MU^{(C_8)}$ (in fact ℓ works out to be 19). The cohomology theory Ω is defined to be the homotopy fixed point spectrum of the C_8-action on $\Omega_{\mathbb{O}}$.

There is some flexibility in the choice of D, but it needs to be chosen in order that the Periodicity Theorem holds, and in order that the map from the fixed point spectrum of $\Omega_{\mathbb{O}}$ to the homotopy fixed point spectrum be a weak equivalence. It also needs to be chosen in such a way that the Detection Theorem is preserved. That such an D can be chosen with these properties is a relatively easy fact, albeit mildly technical. It is specified in Corollary 9.16. It can be described in the form $M \times M \times M \times M$ for a suitable real manifold M, though we do not do so.

1.3. **The detection theorem.** Since the non-equivariant spectrum $\Omega_{\mathbb{O}}$ underlying $\Omega_{\mathbb{O}}$ is complex orientable, there is a map

$$\begin{array}{ccc}
\mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*\Omega) & \Longrightarrow & \pi_{t-s}\Omega \\
\downarrow & & \| \\
H^s(C_8; \pi_t\Omega_{\mathbb{O}}) & \Longrightarrow & \pi_{t-s}\Omega
\end{array}$$

from the Adams-Novikov spectral sequence for Ω to the C_8 homotopy fixed point spectral sequence. This map is actually an isomorphism of spectral sequences, though we do not need this fact. The inclusion of the unit $S^0 \to \Omega$ gives a map from the Adams-Novikov spectral sequence for the sphere, to the Adams-Novikov spectral sequence for Ω. Composing with the map described above gives the horizontal arrow in the diagram of spectral sequences below

$$\begin{array}{ccc}
\begin{array}{c}\text{Adams-Novikov}\\ \text{spectral sequence}\end{array} & \longrightarrow & \begin{array}{c}C_8 \text{ homotopy}\\ \text{fixed point}\\ \text{spectral sequence}\end{array} \\
\downarrow & & \\
\begin{array}{c}\text{Classical Adams}\\ \text{spectral sequence}\end{array} & &
\end{array}$$

The Detection Theorem is proved by investigating this diagram, and follows from the purely algebraic result below.

Theorem 1.9 (Algebraic Detection Theorem). *If*

$$x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$$

is any element mapping to h_j^2 in the E_2-term of the classical Adams spectral sequence, then the image of x in $H^2(C_8; \pi_{2^{j+1}}\Omega_{\mathbb{O}})$ is nonzero.

To deduce the Detection Theorem from the Algebraic Detection Theorem suppose that $\theta_j : S^{2^{j+1}-2} \to S^0$ is a map represented by h_j^2 in the classical Adams spectral sequence. Then θ_j has Adams filtration 0, 1 or 2 in the Adams-Novikov spectral sequence, since the Adams filtration can only increase under a map. Since both

$$\mathrm{Ext}^{0,2^{j+1}-2}_{MU_*MU}(MU_*, MU_*) \quad \text{and} \quad \mathrm{Ext}^{1,2^{j+1}-1}_{MU_*MU}(MU_*, MU_*)$$

are zero, the class θ_j must be represented in Adams filtration 2 by some element x which is a permanent cycle. By the Algebraic Detection Theorem, the element x has a non-trivial image $b_j \in H^2(C_8; \pi_{2^{j+1}}\Omega_{\mathbb{O}})$, representing the image of θ_j in $\pi_{2^{j+1}-2}\Omega$. If this image is zero then the class b_j must be in the image of the differential

$$d_2 : H^0(C_8; \pi_{2^{j+1}-1}\Omega_{\mathbb{O}}) \to H^2(C_8; \pi_{2^{j+1}}\Omega_{\mathbb{O}}).$$

But $\pi_{\text{odd}}\Omega_{\mathbb{O}} = 0$, so this cannot happen.

The proof of the Algebraic Detection Theorem is given in §11. It consists of surveying the known list of elements in the Adams-Novikov E_2-term mapping to h_j^2 and checking that they all have non-zero image in $H^2(C_8; \pi_{2^j}\Omega_{\mathbb{O}})$. The method is very similar to that used in [43], where an analogous result is established at primes greater than 3.

1.4. **The slice filtration and the Gap Theorem.** While the Detection Theorem and the Periodicity Theorem involve the homotopy fixed point spectral sequence for Ω, the Gap Theorem results from studying $\Omega_{\mathbb{O}}$ as an honest equivariant spectrum. What permits the mixing of the two approaches is the following result, which is part of Theorem 10.8.

Theorem 1.10 (Homotopy Fixed Point Theorem)**.** *The map from the fixed point spectrum of $\Omega_{\mathbb{O}}$ to the homotopy fixed point spectrum of $\Omega_{\mathbb{O}}$ is a weak equivalence.*

In particular, for all n, the map

$$\pi_n^{C_8}\Omega_{\mathbb{O}} \to \pi_n \Omega_{\mathbb{O}}^{hC_8} = \pi_n\Omega$$

is an isomorphism, where the symbol $\pi_n^{C_8}\Omega_{\mathbb{O}}$ denotes the group of equivariant homotopy classes of maps from S^n (with the trivial action) to $\Omega_{\mathbb{O}}$.

We will study the equivariant homotopy type of $\Omega_{\mathbb{O}}$ using an analogue of the Postnikov tower. We call this tower the *slice tower.* Versions of it have appeared in work of Dan Dugger [13], Hopkins-Morel (unpublished), Voevodsky[53, 54, 55], and Hu-Kriz [26].

The slice tower is defined for any finite group G. For a subgroup $K \subseteq G$, let ρ_K denote its regular representation and write

$$\widehat{S}(m, K) = G_+ \underset{K}{\wedge} S^{m\rho_K} \qquad m \in \mathbb{Z}.$$

Definition 1.11. The set of *slice cells* (for G) is

$$\mathcal{A} = \{\widehat{S}(m, K), \Sigma^{-1}\widehat{S}(m, K) \mid m \in \mathbb{Z}, K \subseteq G\}.$$

Definition 1.12. A slice cell $\widehat{S}$ is *free* if it is of the form $G_+ \wedge S^m$ for some m. An *isotropic* slice cell is one which is not free.

We define the *dimension* of a slice cell $\widehat{S} \in \mathcal{A}$ by

$$\dim \widehat{S}(m, K) = m|K|$$
$$\dim \Sigma^{-1}\widehat{S}(m, K) = m|K| - 1.$$

Finally the *slice section* P^nX is constructed by attaching cones on slice cells $\widehat{S}$ with $\dim \widehat{S} > n$ to kill all maps $\widehat{S} \to X$ with $\dim \widehat{S} > n$. There is a natural map

$$P^nX \to P^{n-1}X$$

The *n-slice of X* is defined to be its homotopy fiber P_n^nX.

In this way a tower $\{P^n X\}$, $n \in Z$ is associated to each equivariant spectrum X. The homotopy colimit $\operatorname{holim}_{\overrightarrow{n}} P^n X$ is contractible, and $\operatorname{holim}_{\overleftarrow{n}} P^n X$ is just X. The *slice spectral sequence* for X is the spectral sequence of the slice tower, relating $\pi_* P^n_n X$ to $\pi_* X$.

The key technical result of the whole paper is the following.

Theorem 1.13 (The Slice Theorem)**.** *The C_8-spectrum $P^n_n MU^{(C_8)}$ is contractible if n is odd. If n is even then $P^n_n MU^{(C_8)}$ is weakly equivalent to $H\underline{\mathbb{Z}} \wedge W$, where $H\underline{\mathbb{Z}}$ is the Eilenberg-Mac Lane spectrum associated to the constant Mackey functor $\underline{\mathbb{Z}}$, and W is a wedge of isotropic slice cells of dimension n.*

The Slice Theorem actually holds more generally for the spectra $MU^{(C_{2^k})}$ formed like $MU^{(C_8)}$, using the smash product of 2^k copies of $MU_{\mathbb{R}}$. The more general statement is Theorem 6.1

Just to be clear, the expression *weak equivalence* refers to the notion of weak equivalence in the usual model category structure for equivariant stable homotopy theory (and not, for example to an equivariant map which is a weak homotopy equivalence of underlying, non-equivariant spectra). Between fibrant-cofibrant objects it corresponds to the notion of an equivariant homotopy equivalence. Similarly, the term *contractible* means weakly equivalent to the terminal object.

The Gap Theorem depends on the following result.

Lemma 1.14 (The Cell Lemma)**.** *Let $G = C_{2^n}$ for some $n \neq 0$. If $\widehat{S}$ is an isotropic slice cell of even dimension, then the groups $\pi^G_k H\underline{\mathbb{Z}} \wedge \widehat{S}$ are zero for $-4 < k < 0$.*

This is an easy explicit computation, and reduces to the fact that the orbit space $S^{m\rho_G}/G$ is simply connected, being the suspension of a connected space.

Since the restriction of ρ_G to a subgroup $K \subseteq G$ is isomorphic to $(|G/K|)\,\rho_K$ there is an equivalence

$$S^{m\rho_G} \wedge (G_+ \underset{K}{\wedge} S^{n\rho_K}) \approx G_+ \underset{K}{\wedge} S^{(n+m|G/K|)\rho_K}.$$

It follows that if $\widehat{S}$ is a slice cell of dimension d, then for any m, $S^{m\rho_G} \wedge \widehat{S}$ is a slice cell of dimension $d + m|G|$. Moreover, if $\widehat{S}$ is isotropic, then so is $S^{m\rho_G} \wedge \widehat{S}$. The Cell Lemma and the Slice Theorem then imply that for any m, the group

$$\pi^{C_8}_i S^{m\rho_{C_8}} \wedge MU^{(C_8)}$$

is zero for $-4 < i < 0$. Since

$$\pi^{C_8}_i \Omega_{\mathbb{O}} = \varinjlim \pi_i S^{-m\ell\rho_{C_8}} MU^{(C_8)}$$

this implies that

$$\pi^{C_8}_i \Omega_{\mathbb{O}} = \pi_i \Omega = 0$$

for $-4 < i < 0$, which is the Gap Theorem.

The Periodicity Theorem is proved with a small amount of computation in the $RO(C_8)$-graded slice spectral sequence for $\Omega_{\mathbb{O}}$. It makes use of the fact that $\Omega_{\mathbb{O}}$ is an equivariant commutative ring spectrum. Using the nilpotence machinery of [10, 24] instead of explicit computation, it can be shown that the groups $\pi_*\Omega$ are periodic with *some* period which a power of 2. This would be enough to show that only finitely many of the θ_j can exist. Some computation is necessary to get the actual period stated in the Periodicity Theorem.

All of the results are fairly easy consequences of the Slice Theorem, which in turn reduces to a single computational fact: that the quotient of $MU^{(C_8)}$ by the analogue of the "Lazard ring" is the Eilenberg-Mac Lane spectrum $H\underline{\mathbb{Z}}$ associated to the constant Mackey functor $\underline{\mathbb{Z}}$. We call this the *Reduction Theorem* and its generalization to C_{2^n} appears as Theorem 6.5. It is proved for $G = C_2$ in Hu-Kriz [26], and the analogue in motivic homotopy theory is the main result of the (unpublished) work of the second author and Morel mentioned earlier, where it is used to identify the Voevodsky slices of MGL. It would be very interesting to find a proof of Theorem 6.5 along the lines of Quillen's argument in [42].

1.5. **Summary of the contents.** We now turn to a more detailed summary of the contents of this paper. In §2 we recall the basics of equivariant stable homotopy theory, establish many conventions and explain some simple computations. One of our main new constructions, introduced in §2.3.2 is the multiplicative *norm functor*. We merely state our main results about the norm, deferring the details of the proofs to the appendices. Another innovation, the *method of polynomial algebras*, is described in §2.4. It is used in constructing convenient filtrations of rings, and in forming the quotient of an equivariant commutative ring spectrum by a regular sequence, in the situation in which the group is acting non-trivially on the sequence.

Section 4 introduces the slice filtration and establishes many of its basic properties, including the strong convergence of the slice spectral sequence (Theorem 4.38), and an important result concerning the distribution of groups in the E_2-term (Corollary 4.39). The notions of *pure spectra*, *isotropic spectra*, and *spectra with cellular slices* are introduced in §4.6.2. These are very convenient classes of equivariant spectra, and the point of the Slice Theorem is to show that $MU^{(G)}$ belongs to this class. Most of the material of these first sections makes no restriction on the group G. The exception is in §4.6.3, where special methods for determining the slices of C_{2^n}-spectra are described. Strictly speaking, these methods are not necessary for the proof of Theorem 1.1, but they are very useful for other problems, and so have been included.

From §5 forward we restrict attention to the case in which G is cyclic of order a power of 2, and we localize all spectra at the prime 2. The spectra $MU^{(G)}$ are introduced and some of the basic properties are established. The groundwork is laid for the proof of the Slice Theorem. The Reduction Theorem (Theorem 6.5) is stated in §6. The Reduction Theorem is the backbone of the Slice Theorem, and is the only part that is not "formal" in the sense that it depends on the outcome of certain computations.

The Slice Theorem is proved in §6, assuming that the Reduction Theorem holds. The proof of the Reduction Theorem is in §7. The Gap Theorem in proved in §8, the Periodicity theorem in §9. The Homotopy Fixed Point Theorem is proved in §10, and the Detection Theorem in §11.

The paper concludes with two long appendices devoted to foundational aspects of equivariant stable homotopy theory. Two factors contribute to the length of this material. One is simply the wish to make this paper as self-contained as possible and to collect material central to the focus of our investigation in one place. But there is a mathematical reason as well. We have chosen to do equivariant stable homotopy theory in the category of equivariant orthogonal spectra [36] since it possesses the convenience of a symmetric monoidal smash product. This enables us to streamline our theory of the norm. But it comes at a cost. Basing the norm on the symmetric monoidal product forces us to work with commutative ring spectra (unital commutative monoids under the smash product) rather than E_∞-ring spectra. Unfortunately, the spectra underlying commutative rings are never cofibrant. This means that care must be taken to ensure that a construction like the norm realizes the correct homotopy type when applied to cofibrant commutative rings. One of the main technical results of Appendix B (Proposition B.63) shows that this is the case. Establishing this property of commutative rings involves details of the foundations of equivariant orthogonal spectra, and cannot be done at the level of user interface. Because of this, a relatively complete survey of equivariant orthogonal spectra is required.

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Finally, and most importantly, the authors would like to thank Mark Mahowald for a lifetime of mathematical ideas and inspiration, and for many helpful discussions in the early stages of this project.

2. Equivariant stable homotopy theory

We will work in the category of equivariant orthogonal spectra [37, 36]. In this section we survey some of the main properties of the theory and establish some notation. The definitions, proofs, constructions, and other details are explained in Appendices A and B. The reader is also referred to the survey of Greenlees and May [18] for an overview of equivariant stable homotopy theory, and for further references.

2.1. **G-spaces.** Let G be a finite group, and $\mathcal{T}^G$ the topological category of pointed left G-spaces and spaces of equivariant maps. Let $\mathcal{T}_G$ be the G-equivariant topological category of G-spaces and G-spaces of continuous maps (on which G acts by conjugation). Thus

$$\mathcal{T}^G(X,Y) = \mathcal{T}_G(X,Y)^G.$$

The category $\mathcal{T}^G$ is a closed symmetric monoidal category under the smash product operation. The tensor unit is the 0-sphere S^0 equipped with the trivial G-action. Since it is closed, $\mathcal{T}^G$ may be regarded as enriched over itself. This enriched category is, of course, just $\mathcal{T}_G$.

The homotopy set (group, for $n > 0$) $\pi_n^H(X)$ of a pointed G-space is defined for $H \subseteq G$ and $n \geq 0$ to be the set of H-equivariant homotopy classes of pointed maps

$$S^n \to X.$$

This is the same as the ordinary homotopy set (group) $\pi_n(X^H)$ of the space of H fixed-points in X.

A *weak equivalence* in $\mathcal{T}^G$ is a map inducing an isomorphism on equivariant homotopy groups π_n^H for all $H \subset G$ and all $n \geq 0$. A *fibration* is a map $X \to Y$ which for every $H \subset G$ is a Serre fibration on fixed points $X^H \to Y^H$. These two classes of maps make $\mathcal{T}^G$ into a topological model category. The smash product of G-spaces makes $\mathcal{T}^G$ into a symmetric monoidal category in the sense of Schwede-Shipley [49, Definition 3.1], and $\mathcal{T}_G$ into an *enriched model category*.

Every pointed G-space is weakly equivalent to a G-CW complex constructed inductively from the basepoint by attaching equivariant cells of the form

$$G/H \times S^{n-1} \to G/H \times D^n.$$

As is customary we'll write both

$$\operatorname{ho}\mathcal{T}^G(X, Y) \text{ and } [X, Y]^G$$

for the set of maps from X to Y in the homotopy category of $\mathcal{T}^G$. When X is cofibrant and Y is fibrant this can be calculated as the set of homotopy classes of maps

$$[X, Y]^G = \pi_0 \, \mathcal{T}^G(X, Y) = \pi_0^G \, \mathcal{T}_G(X, Y).$$

An important role is played by the equivariant spheres S^V arising as the one point compactification of real orthogonal representations V of G. When V is the trivial representation of dimension n, S^V is just the n-sphere S^n with the trivial G-action. We combine these two notations and write

$$S^{n+V} = S^{\mathbb{R}^n \oplus V}.$$

Associated to S^V is the equivariant homotopy group

$$\pi_V^G X$$

defined to the be the set $[S^V, X]^G$ of homotopy classes of G-equivariant maps from S^V to X.

2.2. **Equivariant stable homotopy theory.**

2.2.1. *G-spectra.* The category of G-spectra is obtained from the category of G-spaces by formally inverting the smash product operations $S^V \wedge (-)$ with V a finite dimensional representation of G. We will use the category $\mathcal{S}^G$ of equivariant orthogonal spectra [36], in its *positive stable model structure* for our model of equivariant stable homotopy theory, and refer to it simply as the *category of G-spectra*. When G is the trivial group we will drop the superscript and use $\mathcal{S}$. We summarize here the main properties of this category, referring the reader to Appendices A and B and to [36] for an expanded discussion,

An *orthogonal G-spectrum* consists of a collection of pointed G-spaces X_V indexed by the finite dimensional orthogonal representations V of G, an action of the orthogonal group $O(V)$ (of non-equivariant maps) on X_V, and for each (not necessarily G-equivariant) orthogonal inclusion $t : V \subset W$ a map $S^{W-t(V)} \wedge X_V \to X_W$, in which $W - t(V)$ denotes the orthogonal complement of the image of V in W.

These maps are required to be compatible with the actions of G and $O(V)$. For a more detailed description see §A.2.1.

Depending on the context, we will refer to orthogonal G-spectra as "equivariant orthogonal spectra," "orthogonal spectra," "G-spectra," and sometimes just as "spectra".

As with G-spaces, there are two useful ways of making the collection of G-spectra into a category. There is the topological category $\mathcal{S}^G$ just described, and the G-equivariant topological category $\mathcal{S}_G$ of equivariant orthogonal spectra and G-spaces of non-equivariant maps. Thus for equivariant orthogonal spectra X and Y there is an identification

$$\mathcal{S}^G(X,Y) = \mathcal{S}_G(X,Y)^G.$$

If V and W are two orthogonal representations of G the same dimension, and $O(V,W)$ is the G-space of (not necessarily equivariant) orthogonal maps, then

$$O(V,W)_+ \underset{O(V)}{\wedge} X_V \to X_W$$

is a G-equivariant homeomorphism. In particular an orthogonal G-spectrum X is determined by the X_V with V a trivial G-representation. This means that the category $\mathcal{S}^G$ is equivalent to the category of objects in $\mathcal{S}$ equipped with a G-action (Proposition A.11). This is a very useful fact when studying the category theoretic properties of $\mathcal{S}^G$, though not so convenient from the point of view of the model category structures.

As with G-spaces, we will write both

$$\operatorname{ho}\mathcal{S}^G(X,Y) \text{ and } [X,Y]^G$$

for the set of maps from X to Y in the homotopy category of $\mathcal{S}^G$. When X is cofibrant and Y is fibrant this can be calculated as the set of homotopy classes of maps

$$[X,Y]^G = \pi_0\mathcal{S}^G(X,Y) = \pi_0^G\mathcal{S}_G(X,Y).$$

The category of G-spectra is related to the category of G-spaces by a pair of adjoint functors, the "suspension spectrum" and "0-space" functors

$$\Sigma^\infty : \mathcal{T}^G \leftrightarrows \mathcal{S}^G : \Omega^\infty. \tag{2.1}$$

The functor Σ^∞ is symmetric monoidal, but it does not, in general preserve cofibrations.[1] Because of this the functors (2.1) do not form a Quillen pair. We will often not distinguish in notation between the suspension spectrum of a G-space and the G-space itself.

2.2.2. *Smash product and stable spheres.* The category $\mathcal{S}_G$ is a closed symmetric monoidal category under the smash product operation. The tensor unit is the sphere spectrum S^0. For each finite dimensional G-representation V there is a G-spectrum S^{-V} equipped with a weak equivalence

$$S^{-V} \wedge S^V \to S^0,$$

and more generally

$$S^{-V\oplus W} \wedge S^W \to S^{-V}. \tag{2.2}$$

[1]The suspension spectrum of the 0-sphere, for example, is not cofibrant. However, if X is a cofibrant G-space then $S^{-1} \wedge S^1 \wedge X$ is a cofibrant replacement of $\Sigma^\infty X$.

There are canonical identifications

$$S^{-V} \wedge S^{-W} \approx S^{-V \oplus W}$$

and in fact the association

$$V \mapsto S^{-V}$$

is a symmetric monoidal functor from the category of finite dimensional representations of G (and isomorphisms) to $\mathcal{S}_G$.

Using the spectra S^{-V_1} and the spaces S^{V_0} one can associate a stable "sphere" to each virtual representation V of G. To do so, one must first represent V as difference $[V_0] - [V_1]$ of representations, and then set

$$S^V = S^{-V_1} \wedge S^{V_0}.$$

If (V_0, V_1) and (W_0, W_1) are two pairs of orthogonal representations representing the same virtual representation

$$V = [V_0] - [V_1] = [W_0] - [W_1] \in RO(G),$$

then there is a representation U, and an isomorphism

$$W_1 \oplus V_0 \oplus U \approx V_1 \oplus W_0 \oplus U.$$

A choice of such data gives weak equivalences

$$\begin{aligned} S^{-W_1} \wedge S^{W_0} \leftarrow S^{-W_1 \oplus V_0 \oplus U} \wedge S^{W_0 \oplus V_0 \oplus U} & \\ \approx S^{-V_1 \oplus W_0 \oplus U} \wedge S^{W_0 \oplus V_0 \oplus U} & \to S^{-V_1} \wedge S^{V_0} \end{aligned}$$

Thus, up to weak equivalence

$$S^V = S^{-V_1} \wedge S^{V_0}$$

depends only on V. However, the weak equivalence between the spheres arising from different choices depends on data not specified in the notation. This leads to some subtleties in grading equivariant stable homotopy groups over the real representation ring $RO(G)$. See [38, Chapter XIII] and [18, §3].

In the positive stable model structure, the spectrum just written will be cofibrant if and only if the dimension of the fixed point space V_1^G is positive.

Definition 2.3. Suppose that V is a virtual representation of G. A *positive representative* of V consists of a pair of representations (V_0, V_1) with $\dim V_1^G > 0$ and for which

$$V = [V_0] - [V_1] \in RO(G).$$

The *sphere* associated to a positive representative (V_0, V_1) of V is

$$S^V = S^{-V_1} \wedge S^{V_0}.$$

We will often abuse the terminology just refer to S^V as the *sphere associated to* V.

Remark 2.4. There is some potential for confusion with this notation. For example if V is an actual representation the symbol S^V has been defined to mean the suspension spectrum of the one point compactification of V. On the other hand the same symbol also refers to the sphere associated to a positive representation of the virtual representation determined by V, which is not actually defined until we specify a pair (V_0, V_1) with $\dim V_1^G > 0$. In the main body of the paper (Sections 4 through 11) our emphasis is on homotopy theory and the homotopy category, and there the symbol S^V will mean the sphere associated to a positive representative of V. In most cases the statements will not depend in a significant way on the choice of positive representative, or there will be an evident way of manufacturing a positive representative out of previous choices. In this section and the appendices, which are more foundational in nature, we will be more careful about the language, and will specify which meaning S^V is to have.

The smash product makes $\mathcal{S}_G$ into a *symmetric monoidal model category* in the sense of Hovey [25, Definition 4.2.6] and Schwede-Shipley [49, Definition 3.1]. This means that the analogue of Quillen's axiom SM7 holds (called the *pushout-product* axiom [49]), and for any cofibrant X, the map

$$\tilde{S}^0 \wedge X \to X$$

is a weak equivalence, where $\tilde{S}^0 \to S^0$ is a cofibrant approximation (called the *unit* axiom [49]). For more details see §B.1.3.

2.2.3. *The canonical homotopy presentation.* Up to weak equivalence, every G-spectrum X can be functorially presented as the homotopy colimit of a sequence

$$\cdots \to S^{-V_n} \wedge X_{V_n} \to S^{-V_{n+1}} \wedge X_{V_{n+1}} \to \cdots, \tag{2.5}$$

in which $\{V_n\}$ is a fixed increasing sequence of representations eventually containing every finite dimensional representation of G, and each X_{V_n} is a G-space. The G-space X_{V_n} is given by

$$X_{V_n} = \mathcal{S}_G(S^{-V_n}, X).$$

and the transition maps are constructed from the diagram

$$S^{-V_n} \wedge X_{V_n} \xleftarrow{\sim} S^{-V_{n+1}} \wedge S^{V_{n+1}-V_n} \wedge X_{V_n} \to S^{-V_{n+1}} \wedge X_{V_{n+1}}$$

in which the leftmost arrow is a weak equivalence. We will often just abbreviate this important *canonical homotopy presentation* as

$$X = \underset{\overrightarrow{V}}{\operatorname{holim}}\, S^{-V} \wedge X_V. \tag{2.6}$$

For the actual construction §B.1.5.

It will be convenient to have a name for the key property of the sequence $\{V_n\}$.

Definition 2.7. An increasing sequence $V_n \subset V_{n+1} \subset \cdots$ of finite dimensional representations of G is *exhausting* if any finite dimensional representation V of G admits an equivariant embedding in some V_n.

Clearly any two exhausting sequences are cofinal in each other, and so any two presentations (2.6) can be nested into each other.

2.2.4. *Change of group.* For a subgroup $H \subset G$, there is a symmetric monoidal restriction functor

$$i^* = i_H^* : \mathcal{S}_G \to \mathcal{S}_H.$$

It is derived from an analogous functor $i_H^* : \mathcal{T}^G \to \mathcal{T}^H$ formed by simply restricting the G action to H. Under the equivalence of $\mathcal{S}^G$ with the category of objects in $\mathcal{S}$ equipped with a G-action (Proposition A.18), the restriction functor is formed as with spaces, by restriction the G-action to H.

In case H is the trivial group, we'll write the restriction functor as

$$i_0^* : \mathcal{S}_G \to \mathcal{S}.$$

The functor i_H^* has both a left and right adjoint

$$G_+ \underset{H}{\wedge} (\,-\,) \text{ and } F_H(G_+, -\,),$$

and the functors

$$G_+ \underset{H}{\wedge} (\,-\,) : \mathcal{S}_H \leftrightarrows \mathcal{S}_G : i_H^*$$

form a Quillen pair.

The Wirthmüller isomorphism ([56], [18, Theorem 4.10]) gives an equivariant weak equivalence

$$G_+ \underset{H}{\wedge} (\,-\,) \to F_H(G_+, -\,).$$

Because of this we will make little mention of the functor $F_H(G_+, X)$.

2.2.5. *Equivariant stable homotopy groups.* Let X be a G-spectrum. For an integer $k \in \mathbb{Z}$ and a subgroup $H \subset G$ set

$$\pi_k^H(X) = [S^k, X]^H.$$

As H varies, the abelian groups $\pi_k^H X$ fit together to define a *Mackey functor* (see §2.6.1 below) denoted $\underline{\pi}_k X$. We will use the shorthand $\pi_k^u X$ for the homotopy group $\pi_k^H(X)$, when $H \subset G$ is the trivial group. The group $\pi_k^u X$ is the k^{th} homotopy group of the non-equivariant spectrum underlying X, and the superscript u is intended to indicate *underlying*.

The collection of Mackey functors $\underline{\pi}_k X$, $k \in \mathbb{Z}$, will be called the *stable homotopy groups* of X. The *weak equivalences* in the topological model category $\mathcal{S}_G$ are the maps inducing isomorphisms of stable homotopy groups.

In terms of the canonical homotopy presentation

$$X = \operatorname{ho}\varinjlim S^{-V_n} \wedge X_{V_n},$$

the group π_k^H can be calculated as

$$\pi_k^H(X) = \varinjlim \pi_{k+V_n}^H X_{V_n},$$

with the transition maps in the system formed using the maps

$$S^{V_{n+1}-V_n} \wedge X_{V_n} \to X_{V_{n+1}}.$$

Generalizing this, one associates to a virtual representation V of G the group

$$\pi_V^H(X) = [S^V, X]^H.$$

These groups fit together to form the *$RO(G)$-graded Mackey functor* $\underline{\pi}_\star(X)$. These are discussed a little more fully in §2.6.1 below.

2.2.6. *Equivariant generalized cohomology theories.* We will also want to use orthogonal G-spectra to represent generalized equivariant homology theories. When doing so we will use a symbol like E for the spectrum representing the theory, and write $E_* X$ and $E^* X$ for the equivariant homology and cohomology groups of another G-spectrum X, as well as $E^\star X$ and $E_\star X$ for the $RO(G)$-graded analogues. Collecting our various notations, we have

$$\begin{aligned} E_V X &= \pi_V^G(E \wedge X) = [S^V, E \wedge X]^G \\ E^V X &= [X, S^V \wedge E]. \end{aligned}$$

2.3. **Multiplicative properties.**

2.3.1. *Commutative and associative algebras.*

Definition 2.8. A *commutative algebra* is a unital commutative monoid in $\mathcal{S}^G$ with respect to the smash product operation. An *associative algebra* is a unital associative monoid with respect to the smash product.

There is a weaker "up to homotopy notion" that sometimes comes up.

Definition 2.9. A *homotopy associative algebra* is an associative algebra in $\operatorname{ho}\mathcal{S}_G$. A *homotopy commutative algebra* is a commutative algebra in $\operatorname{ho}\mathcal{S}_G$.

The G-equivariant topological category of commutative algebras in $\mathcal{S}_G$ will be denoted $\mathbf{Comm}_G$. It is tensored and cotensored over $\mathcal{T}_G$ and is an enriched model category. The tensor product of an equivariant commutative algebra R and a G-space T will be denoted

$$R \otimes T$$

to distinguish it from the smash product. By definition

$$\mathbf{Comm}_G(R \otimes T, E) = \mathcal{T}_G(T, \mathbf{Comm}_G(R, E)).$$

The forgetful functor and its left adjoint, the free commutative algebra functor form a Quillen morphism

$$\mathrm{Sym} : \mathcal{S}_G \leftrightarrows \mathbf{Comm}_G.$$

Modules over an equivariant commutative ring are defined in the evident way using the smash product. The category of left modules over R will be denoted $\mathcal{M}_R$. It is an enriched symmetric monoidal model under the operation

$$M \underset{R}{\wedge} N$$

where M is regarded as a right R-module via

$$M \wedge R \xrightarrow{\text{flip}} R \wedge M \to M,$$

and $M \underset{R}{\wedge} N$ is defined by the coequalizer diagram

$$M \wedge R \wedge N \rightrightarrows M \wedge N \to M \underset{R}{\wedge} N.$$

The "free module" and "forgetful" functors

$$X \mapsto R \wedge X : \mathcal{S}_G \leftrightarrows \mathcal{M}_R : M \mapsto M$$

are adjoint and form a Quillen morphism. A map of R-modules is a weak equivalence if and only if the underlying map of spectra is a weak equivalence.

Remark 2.10. There are also the related notions of E_∞ and A_∞ algebras. It can be shown that the categories of E_∞ and commutative algebras are Quillen equivalent, as are those of A_∞ and associative algebras.

2.3.2. *Indexed monoidal products and norm induction.* We now turn to a multiplicative analogue of the functor

$$G_+ \underset{H}{\wedge} (\,-\,).$$

This notion first appears in group cohomology (Evens [15]), and is often referred to as the "Evens transfer" or the "norm transfer." The analogue in stable homotopy theory appears in Greenlees-May [19].

The fact that the category $\mathcal{S}_G$ is equivalent to the category of objects in $\mathcal{S}$ equipped with a G-action has an important and useful consequence. It means that if a construction involving spectra happens to produce something with a G-action, it defines a functor with values in G-spectra. For example, the group C_2 acts on both the iterated wedge $X \vee X$ and the iterated smash product $X \wedge X$. These constructions therefore defines functors from $\mathcal{S}$ to $\mathcal{S}_{C_2}$. The first is just the left adjoint to the restriction functor. The second is the "norm."

Suppose that G is a finite group, and J is a finite set on which G acts. Write $\mathcal{B}_J G$ for the category with object set J, in which a map from j to j' is an element $g \in G$ with $g \cdot j = j'$. We abbreviate this to $\mathcal{B}G$ in case $J = \text{pt}$. Given a functor

$$X : \mathcal{B}_J G \to \mathcal{S}$$

define the *indexed wedge* and *indexed smash product* of X to be

$$\bigvee_{j \in J} Xj \quad \text{and} \quad \bigwedge_{j \in J} Xj$$

respectively. The group G acts on the indexed wedge and indexed smash product and so they define functors from the category of $\mathcal{B}_J G$-diagrams of spectra to $\mathcal{S}_G$. For more details, see §A.3.2.

Suppose that H is a subgroup of G and $J = G/H$. In this case $\mathcal{B}_J G$ is equivalent to $\mathcal{B}H$, so the category of $\mathcal{B}_J G$-diagrams of spectra is equivalent to $\mathcal{S}_H$. Under this equivalence, the indexed wedge works out to be the functor

$$G_+ \underset{H}{\wedge} (-).$$

The indexed smash product is the *norm functor*

$$N_H^G : \mathcal{S}_H \to \mathcal{S}_G.$$

The norm functor can be described as sending an H-spectrum X to the G-spectrum

$$\bigwedge_{j \in G/H} X_j$$

where

$$X_j = (H_j)_+ \underset{H}{\wedge} X,$$

and H_j is the right coset indexed by j.

Remark 2.11. When the context is clear, we will sometimes abbreviate the N_H^G simply to N in order to avoid clustering of symbols.

The norm distributes over wedges in much the same way as the iterated smash product. A precise statement of the general "distributive law" appears in §A.3.3.

The functor N_H^G is symmetric monoidal, commutes with filtered colimits (Propositions A.46), and takes weak equivalences between cofibrant objects to weak equivalences (Proposition B.42). As described in §A.4, the fact that $V \mapsto S^{-V}$ is symmetric monoidal implies that

$$N_H^G S^{-V} = S^{-\operatorname{ind}_H^G V}, \tag{2.12}$$

where $\operatorname{ind}_H^G V$ is the induced representation. From the definition, one also concludes that for a pointed G-space T,

$$N_H^G S^{-V} \wedge T = S^{-\operatorname{ind}_H^G V} \wedge N_H^G T,$$

where $N_H^G T$ is the analogous norm functor on spaces. Combining these one finds a useful description of $N_H^G X$ in terms of the canonical homotopy presentation

$$N_H^G X = \operatorname{holim}_{\overrightarrow{V_n}} S^{-\operatorname{ind}_H^G V_n} \wedge N_H^G X_{V_n}.$$

Because it is symmetric monoidal, the functor N take commutative algebras to commutative algebras, and so induces a functor

$$N = N_H^G : \mathbf{Comm}_H \to \mathbf{Comm}_G.$$

The following result is proved in the Appendices, as Corollaries A.49 and B.30.

Proposition 2.13. *The functor*

$$N : \mathbf{Comm}_H \to \mathbf{Comm}_G.$$

is left adjoint to the restriction functor i^*. Together they form a Quillen morphism of model categories. □

Corollary 2.14. *There is a natural isomorphism*

$$N_H^G(i_H^* R) \to R \otimes (G/H),$$

under which the counit of the adjunction is identified with the map

$$R \otimes (G/H) \to R \otimes (pt)$$

given by the unique G-map $G \to pt$. □

A useful consequence Corollary 2.14 is that the group $N(H)/H$ of G-automorphisms of G/H acts naturally on $N_H^G(i_H^*)R$. The result below is used in the main computational assertion of Proposition 5.49.

Corollary 2.15. *For $\gamma \in N(H)/H$ the following diagram commutes:*

$$\begin{tikzcd} N_H^G(i_H^* R) \arrow[rr, "\gamma"] \arrow[dr] & & N_H^G(i_H^* R) \arrow[dl] \\ & R & \end{tikzcd}$$

Proof: Immediate from Corollary 2.14. □

At this point a serious technical issue arises. The spectra underlying commutative rings are almost never cofibrant. This means that there is no guarantee that the norm of a commutative ring has the correct homotopy type. The fact that it does is one of the main results of Appendix B. The following is a consequence of Proposition B.63.

Proposition 2.16. *Suppose that R is a cofibrant commutative H-algebra, and $\tilde{R} \to R$ is a cofibrant approximation of the underlying H-spectrum. The map*

$$N_H^G(\tilde{R}) \to N_H^G(R)$$

is a weak equivalence. Furthermore, if $Z \to Z'$ is a weak equivalence of G-spectra, then the map

$$Z \wedge N_H^G(R) \to Z' \wedge N_H^G(R)$$

is a weak equivalence. □

We refer to the properties exhibited in Proposition 2.16 by saying that cofibrant commutative rings are *very flat*.

2.3.3. *Other uses of the norm.* There are several important constructions derived from the norm functor which also go by the name of "the norm."

Suppose that R is a G-equivariant commutative ring spectrum, and X is an H-spectrum for a subgroup $H \subset G$. Write

$$R^0_H(X) = [X, i^*_H R]^H.$$

There is a norm map

$$N^G_H : R^0_H(X) \to R^0_G(NX)$$

defined to be the composite

$$NX \to N(i^*_H R) \to R$$

in which the second map is the counit of the restriction-norm adjunction. This is the *norm map on equivariant spectrum cohomology*, and is the form in which the norm is described in Greenlees-May [19].

When V is a representation of H and $X = S^V$ the above gives a map

$$N = N^G_H : \pi^H_V R \to \pi^G_{\operatorname{ind} V} R$$

in which $\operatorname{ind} V$ is the induced representation.

Now suppose that X is a pointed G-space. There is a norm map

$$N^G_H : R^0_H(X) \to R^0_G(X)$$

sending

$$x \in R^0_H(X) = [S^0 \wedge X, i^*_H R]^H$$

to the composite

$$S^0 \wedge X \to S^0 \wedge N(X) \approx N(S^0 \wedge X) \to N(i^*_H R) \to R,$$

in which the equivariant map of pointed G-spaces

$$X \to N^G_H(X)$$

is the "diagonal"

$$X \to \prod_{j \in G/H} X_j \to \bigwedge_{j \in G/H} X_j$$

whose j^{th} component is the inverse to the isomorphism

$$X_j = (Hj)_+ \underset{H}{\wedge} X \to X$$

given by action map. That this is actually equivariant is probably most easily seen by making the identification

$$X_j \approx \hom_H(Hj^{-1}, X)$$

in which Hj^{-1} denotes the *left* H-coset consisting of the inverses of the elements of Hj, and then writing

$$\prod_{j \in G/H} X_j \approx \hom_H(G, X).$$

Under this identification, the "diagonal" map is the map

$$X \to \hom_H(G, X)$$

adjoint to the action map

$$G \underset{H}{\times} X \to X,$$

which is clearly equivariant.

One can combine these construction to define the *norm on $RO(G)$-graded cohomology* of a G-space X

$$N_H^G : R_H^V(X) \to R_G^{\operatorname{ind} V}(X)$$

sending

$$S^0 \wedge X \xrightarrow{a} S^V \wedge i_H^* R$$

to the composite

$$S^0 \wedge X \to S^0 \wedge NX \xrightarrow{Na} S^{\operatorname{ind} V} \wedge N i_H^* R \to S^{\operatorname{ind} V} \wedge R.$$

2.4. **The method of polynomial algebras.**

2.4.1. *Equivariant polynomial algebras.* In this section we construct a class of associative algebras which are in some sense equivariant polynomial extensions of other rings. A word of warning, though. These ring spectra are not necessarily commutative algebras, and are not free commutative algebras.

We start with a subgroup H of G, and a positive representative (V_0, V_1) of a virtual representation V of H. Let

$$S^0[S^V] = \bigvee_{k \geq 0} (S^V)^{\wedge k}$$

be the free H-equivariant associative algebra generated by $S^V = S^{-V_1} \wedge S^{V_0}$, and

$$\bar{x} \in \pi_V^H S^0[S^V]$$

the homotopy class of the generating inclusion. The spectrum $S^0[S^V]$ is not a commutative algebra, though the $RO(H)$-equivariant homotopy groups (§2.2.5) make it appear so, since $\pi_\star^H S^0[S^V]$ is a free module over $\pi_\star^H S^0$ with basis $\{1, \bar{x}, \bar{x}^2, \dots\}$. It will be convenient to use the notation

$$S^0[\bar{x}] = S^0[S^V].$$

Using the norm functor we can then form the G-equivariant "polynomial" algebra

$$N_H^G(S^0[S^V]) = S^0[G_+ \underset{H}{\wedge} S^V].$$

Things will look cleaner, and better resemble the algebra we are modeling if we use the alternate notation

$$S^0[G \cdot S^V] \text{ and } S^0[G \cdot \bar{x}]$$

to refer to $N_H^G(S^0[S^V])$. Though the symbol H is omitted in this notation, it is still referenced. The representation V is representation of H, and $\bar{x}$ is an H-equivariant map with domain S^V.

By smashing examples like these together we can make associative algebras that "look like" equivariant polynomial algebras over S^0, in which the group G is allowed to act on the polynomial generators. More explicitly, suppose we are given a sequence (possibly infinite) of subgroups $H_i \subset G$ and for each i a positive representative $((V_i)_0, (V_i)_1)$ of a virtual representation V_i of H_i. For each i form

$$S^0[G \cdot \bar{x}_i]$$

as described above, smash the first m together to make

$$S^0[G \cdot \bar{x}_1, \dots, G \cdot \bar{x}_m],$$

and then pass to the colimit to construct the G-equivariant associative algebra

$$T = S^0[G \cdot \bar{x}_1, G \cdot \bar{x}_2, \dots],$$

which we think of as an *equivariant polynomial algebra* over S^0.

All of this can be done relative to an equivariant commutative algebra R by defining

$$R[G\cdot\bar{x}_1, G\cdot\bar{x}_2,\dots]$$

to be

$$R\wedge S^0[G\cdot\bar{x}_1, G\cdot\bar{x}_2,\dots].$$

Because they can fail to be commutative, these polynomial algebras do not have all of the algebraic properties one might hope for. But it is possible to naturally construct all of the equivariant *monomial ideals*. Here is how.

Let J be the left G-set defined by

$$J=\coprod_i G/H_i.$$

By the distributivity of the norm over wedges (§A.3.3), the ring spectrum T is the indexed wedge

$$T=\bigvee_{f\in\mathbb{N}_0^J} S^{V_f}$$

in which f is running through the set of functions

$$J\to\mathbb{N}_0=\{0,1,2,\dots\}$$

taking non-zero values on only finitely many elements (*finitely supported* functions). The group G acts on the set $\mathbb{N}_0^J$ through its action on J, and V_f is the virtual representation

$$V_f=\sum_{j\in J} f(j)\cdot V_j$$

of the stabilizer H_f of f, with the evident positive representation

$$\left(\bigoplus_{j\in J}(V_j)_0^{f(j)}, \bigoplus_{j\in J}(V_j)_1^{f(j)}\right).$$

The G-set $\mathbb{N}_0^J$ is a commutative monoid under addition of functions, and the ring structure on T is the indexed sum of the obvious isomorphisms

$$S^{V_f}\wedge S^{V_g}\approx S^{V_f\oplus V_g}\approx S^{V_{f+g}}.$$

Recall (for example from [9]) that an ideal in a commutative monoid L is a subset $I\subset L$ with the property that $L+I\subset I$. Given a G-stable ideal $I\subset\mathbb{N}_0^J$ form the G-spectrum

$$T_I=\bigvee_{f\in I} S^{V_f}.$$

The formula for the multiplication in T implies that T_I is an equivariant sub bi-module of T, and that the association $I\mapsto T_I$ is an inclusion preserving function from the set of ideals in $\mathbb{N}_0^J$ to the set of sub-bimodules of T. For a more general and systematic discussion see §A.3.6.

Example 2.17. The monomial ideal corresponding to the set I of all non-zero elements of $\mathbb{N}_0^J$ is the augmentation ideal–the fiber of the map $T\to S^0$. It is convenient to denote this T bi-module as $(G\cdot\bar{x}_1, G\cdot\bar{x}_2,\dots)$. More generally, for an integer $n>0$ the set $nI=I+\cdots+I$ of n-fold sums of elements of I is a monoid ideal.

It corresponds to the monomial ideal given by the n^{th} power of the augmentation ideal.

Example 2.18. Let $\dim : \mathbb{N}_0^J \to \mathbb{N}_0$ be the function given by

$$\dim f = \dim V_f = \sum_{j\in J} f(j) \dim V_j.$$

If for all i, $\dim V_i > 0$, then the set $\{f \mid \dim f \geq d\}$ is a monoid ideal in $\mathbb{N}_0^J$ and corresponds to the monomial ideal $M \subset T$ consisting the wedge of spheres of dimension greater than d. The quotient bimodule M_d/M_{d-1} can be identified with the indexed coproduct

$$\bigvee_{\dim f = d} S^{V_f}$$

on which T is acting through the augmentation $T \to S^0$. These monomial ideals play an important role in the proof of the Slice Theorem in §6.

2.4.2. *The method of polynomial algebras.*

Definition 2.19. Suppose that

$$f_i : B_i \to A, \quad i = 1, \dots m$$

are algebra maps from an associative algebra to a commutative algebra R. The *smash product* of the f_i is the algebra map

$$\bigwedge^m f_i : \bigwedge^m B_i \to \bigwedge^m R \to R,$$

in which the right most map is the iterated multiplication. If B is an H-equivariant associative algebra, and $f : B \to i_H^* R$ is an algebra map, the *norm of f* is the G-equivariant algebra map

$$N_H^G B \to R$$

given by

$$N_H^G B \to N_H^G (i_H^* R) \to R.$$

These constructions make it easy to map a polynomial algebra to a commutative algebra. Suppose that R is a fibrant G-equivariant commutative algebra, and we're given a sequence

$$\bar{x}_i \in \pi_{V_i}^{H_i} R, \qquad i = 1, 2, \dots.$$

A choice of cofibrant representative $((V_0)_i, (V_1)_i)$ of V_i and a map

$$S^{V_i} \to R$$

representing $\bar{x}_i$ determines an associative algebra map

$$S^0[\bar{x}_i] \to R.$$

Applying the norm gives a G-equivariant associative algebra map

$$S^0[G \cdot \bar{x}_i] \to R.$$

By smashing these together we can make a sequence of equivariant algebra maps

$$S^0[G \cdot \bar{x}_1, \dots, G \cdot \bar{x}_m] \to R.$$

Passing to the colimit gives an equivariant algebra map

$$S^0[G \cdot \bar{x}_1, G \cdot \bar{x}_2, \dots] \to R \tag{2.20}$$

representing the sequence $\bar{x}_i$. We will refer to this process by saying that the map (2.20) is constructed by the *method of polynomial algebras*. The whole construction can also be made relative to a commutative algebra S, leading to an S-algebra map

$$S[G\cdot\bar{x}_1, G\cdot\bar{x}_2,\dots]\to R \tag{2.21}$$

when R is a commutative S-algebra.

2.4.3. *Quotient modules.* One important construction in ordinary stable homotopy theory is the formation of the quotient of a module M over a commutative algebra R by the ideal generated by a sequence $\{x_1, x_2, \dots\}\subset\pi_*R$. This is done by inductively forming the cofibration sequence of R-modules

$$\Sigma^{|x_n|}M/(x_1,\dots,x_{n-1})\to M/(x_1,\dots,x_{n-1})\to M/(x_1,\dots,x_n) \tag{2.22}$$

and passing to the colimit in the end. There is an evident equivalence

$$M/(x_1,\dots)\approx M\underset{R}{\wedge}R/(x_1,\dots)$$

in case M is a cofibrant R-module. The situation is slightly trickier in equivariant stable homotopy theory, where the group G might be permuting the elements x_i, and preventing the inductive approach described above. The method of polynomial algebras (§2.4.1) can be used to get around this difficulty.

Suppose that R is a fibrant equivariant commutative algebra, and that

$$\bar{x}_i\in\pi^{H_i}_{V_i}(R)\qquad i=1,2,\dots$$

is a sequence of equivariant homotopy classes. Using the method of polynomial algebras, construct an associative R-algebra map

$$T=R[G\cdot\bar{x}_1, G\cdot\bar{x}_2,\dots]\to R, \tag{2.23}$$

and a second associative algebra map $T\to R$ sending each $\bar{x}_i$ to 0. Using the map (2.23), we may regard an equivariant R-module M as a T-module.

Definition 2.24. The *quotient module* $M/(G\cdot\bar{x}_1,\dots)$ is the R-module

$$\tilde{M}\underset{T}{\wedge}R$$

in which $\tilde{M}$ is a cofibrant approximation of M as a T-module.

Since cofibrant T-modules restrict to cofibrant R-modules, once one has formed

$$R/(\bar{x}_1,\dots),$$

the associativity of the smash product implies that the R-module $M/(\bar{x}_1,\dots)$ is given by

$$M/(\bar{x}_1,\dots)\approx M\underset{R}{\wedge}R/(\bar{x}_1,\dots).$$

Let us check that this construction reduces to the usual one when the group acting is the trivial group and M is a cofibrant R-module. For ease of notation, write

$$\begin{aligned}T&=R[x_1,\dots]\\T_n&=R[x_1,\dots,x_n].\end{aligned}$$

Since

$$R[x_1,\dots]=R[x_1,\dots,x_n]\underset{R}{\wedge}R[x_{n+1},\dots]$$

one can construct an associative algebra map

$$T \to R[x_{n+1}, \dots]$$

by smashing the map

$$R[x_1, \dots, x_n] \to R$$

sending each x_i to 0, with the identity map of $R[x_{n+1}, \dots]$. By construction, the evident map of T-algebras

$$\varinjlim R[x_{n+1}, \dots] \to R$$

is an isomorphism. The associativity of the smash product gives an equivalence

$$\tilde{M} \underset{T}{\wedge} R[x_{n+1}, \dots] \approx \tilde{M} \underset{T_n}{\wedge} R = M/(x_1, \dots, x_n).$$

It follows that

$$M/(x_1, \dots) \approx \varinjlim M \underset{T}{\wedge} R[x_{n+1}, \dots] \approx \operatorname{ho}\varinjlim M/(x_1, \dots, x_n).$$

To compare $M/(x_1, \dots, x_{n-1})$ with $M/(x_1, \dots, x_n)$ use a similar construction to write

$$M/(x_1, \dots, x_{n-1}) \approx \tilde{M} \underset{T_n}{\wedge} R[x_n],$$

and conclude that

$$M/(x_1, \dots, x_n) \approx M/(x_1, \dots, x_{n-1}) \underset{R[x_n]}{\wedge} R.$$

The cofibration sequence (2.22) is derived from this expression by applying the functor

$$M/(x_1, \dots, x_{n-1}) \underset{R[x_n]}{\wedge} (-). \tag{2.25}$$

to the pushout diagram of $R[x_n]$ bimodules

$$\begin{CD} (x_n) @>>> R[x_n] \\ @VVV @VVV \\ * @>>> R. \end{CD} \tag{2.26}$$

One needs to make use of the fact that (2.26) is actually homotopy cocartesian since $(x_n) \to R$ is the inclusion of a wedge summand, and hence an h-cofibration (§A.5), and that by [36, Proposition III.7.7] the diagram remains so after applying (2.25), since $M/(x_1, \dots, x_{n-1})$ is a cofibrant $R[x_n]$-module. The bimodule in the upper left corner is the monomial ideal consisting of the non-zero powers of x_n (Example 2.17).

A similar discussion applies to the equivariant situation, giving

$$M/(G \cdot \bar{x}_1, \dots) \approx \varinjlim M/(G \cdot \bar{x}_1, \dots, G \cdot \bar{x}_n),$$

a relation

$$M/(G \cdot \bar{x}_1, \dots, G \cdot \bar{x}_n) \approx M/(G \cdot \bar{x}_1, \dots, G \cdot \bar{x}_{n-1}) \underset{R[G \cdot \bar{x}_n]}{\wedge} R,$$

and a cofibration sequence

$$(G{\cdot}\bar{x}_n){\cdot}M/(G{\cdot}\bar{x}_1, \dots, G{\cdot}\bar{x}_{n-1}) \to M/(G{\cdot}\bar{x}_1, \dots, G{\cdot}\bar{x}_{n-1}) \to M/(G{\cdot}\bar{x}_1, \dots, G{\cdot}\bar{x}_n),$$

derived by applying the functor

$$M/(G \cdot \bar{x}_1, \dots, G \cdot \bar{x}_{n-1}) \underset{R[G \cdot \bar{x}_n]}{\wedge} (-)$$

to

$$(G\cdot \bar{x}_n)\to R[G\cdot \bar{x}_n]\to R.$$

One can also easily deduce the equivalences

$$R/(G\cdot \bar{x}_1,\dots,G\cdot \bar{x}_n)\approx R/(G\cdot \bar{x}_1)\underset{R}{\wedge}\cdots\underset{R}{\wedge} R/(G\cdot \bar{x}_1) \tag{2.27}$$

and

$$R/(G\cdot \bar{x}_1,\dots)\approx \varinjlim R/(G\cdot \bar{x}_1)\underset{R}{\wedge}\cdots\underset{R}{\wedge} R/(G\cdot \bar{x}_n). \tag{2.28}$$

These expressions play an important role in the proof Lemma 7.7, which is a key step in the proof the Reduction Theorem.

2.5. **Fixed points, isotropy separation and geometric fixed points.**

2.5.1. *Fixed point spectra.* The *fixed point spectrum* of a G-spectrum X is defined to be the spectrum of G fixed points in the underlying, non-equivariant spectrum i_0^*X. In other words it is given by

$$X\mapsto i_0^*X^G.$$

The notation $i_0^*X^G$ can get clumsy and we will usually abbreviate it to X^G.

The functor of fixed points is right adjoint to the functor sending $S^{-V}\wedge X_V\in \mathcal{S}$ to $S^{-V}\wedge X_V\in \mathcal{S}_G$, where in the latter expression, V is regarded as a representation of G with trivial G-action and X_V is regarded as a space with trivial G-action. The fixed point functor on spectra doesn't always have the properties one might be led to expect by analogy with spaces. For example it does not generally commute with smash products, or with the formation of suspension spectra.

2.5.2. *Isotropy separation and geometric fixed points.* A standard approach to getting at the equivariant homotopy type of a G-spectrum X is to nest X between two pieces, one an aggregate of information about the spectra i_H^*X for all proper subgroups $H\subset G$, and the other a localization of X at a "purely G" part. This is the *isotropy separation sequence* of X.

More formally, let $\mathcal{P}$ denote the family of proper subgroups of G, and $E\mathcal{P}$ the "classifying space" for $\mathcal{P}$, characterized by the property that the space of fixed points $E\mathcal{P}^G$ is empty, while for any proper $H\subset G$, $E\mathcal{P}^H$ is contractible. The space $E\mathcal{P}$ can be constructed as the join of infinitely many copies of G/H with H ranging through the proper subgroups of G. It can also be constructed as the unit sphere in the sum of infinitely many copies of the reduced regular representation of G. It admits an equivariant cell decomposition into cells of the form $(G/H)_+\wedge D^n$ with H a proper subgroup of G. Let $\tilde{E}\mathcal{P}$ be the mapping cone of $E\mathcal{P}\to \mathrm{pt}$, with the cone point taken as base point. Smashing with a G spectrum X gives the *isotropy separation sequence*

$$E\mathcal{P}_+\wedge X\to X\to \tilde{E}\mathcal{P}\wedge X. \tag{2.29}$$

The term on the left can be computed in terms of the action of proper subgroups $H\subset G$ on X. The homotopy type of the term on the right is determined by its fixed point spectrum

$$\Phi^G(X)=(\tilde{E}\mathcal{P}\wedge X)^G_f,$$

in which the subscript f indicates a functorial fibrant replacement.

The functor $\Phi^G(X)$ is the *geometric fixed point functor* and has many remarkable properties.

Proposition 2.30. i) *The functor Φ^G commutes with directed colimits.*

ii) *The functor Φ^G sends weak equivalences to weak equivalences.*

iii) *For a G-spectrum X and a G-space T the spectra*

$$\Phi^G(X \wedge T) \quad \text{and} \quad \Phi^G(X) \wedge T^G$$

are related by a natural chain of weak equivalences.

iv) *For G-spectra X and Y the spectra*

$$\Phi^G(X \wedge Y) \quad \text{and} \quad \Phi^G(X) \wedge \Phi^G(Y).$$

are related by a natural chain of weak equivalences.

v) $\Phi^G(S^{-V}) = S^{-V_0}$ *where $V_0 \subset V$ is the subspace of G-invariant vectors.*

□

These properties imply that, in terms of the canonical homotopy presentation

$$X \approx \operatorname*{holim}_{\overrightarrow{V}} S^{-V} \wedge X_V$$

one has

$$\Phi^G X \approx \operatorname*{holim}_{\overrightarrow{V}} S^{-V_0} X_V^G. \tag{2.31}$$

Remark 2.32. When $G = C_{2^n}$, the space $E\mathcal{P}$ is the space EC_2 with G acting through the epimorphism $G \to C_2$. Taking S^∞ with the antipodal action as a model of EC_2, this leads to an identification

$$\tilde{E}\mathcal{P} \approx \lim_{n\to\infty} S^{n\sigma},$$

in which $S^{n\sigma}$ denotes the one point compactification of the direct sum of n copies of the real sign representation of G.

Remark 2.33. The isotropy separation sequence often leads to the situation of needing to show that a map $X \to Y$ of cofibrant G-spectra induces a weak equivalence

$$\tilde{E}\mathcal{P} \wedge X \to \tilde{E}\mathcal{P} \wedge Y.$$

Since for every proper $H \subset G$, $\pi_*^H \tilde{E}\mathcal{F} \wedge X = \pi_*^H \tilde{E}\mathcal{P} \wedge Y = 0$, this is equivalent to showing that the map of geometric fixed point spectra $\Phi^G X \to \Phi^G Y$ is a weak equivalence.

Remark 2.34. Since for every proper $H \subset G$, $\pi_*^H \tilde{E}\mathcal{P} \wedge X = 0$, it is also true that

$$[T, \tilde{E}\mathcal{P} \wedge X]_*^G = 0$$

for every G-CW spectrum T built entirely from G-cells of the form $G_+ \underset{H}{\wedge} D^n$ with H a proper subgroup of G. Similarly, if T is gotten from T_0 by attaching G-cells induced from proper subgroups, then the restriction map

$$[T, \tilde{E}\mathcal{P} \wedge X]_*^G \to [T_0, \tilde{E}\mathcal{P} \wedge X]_*^G$$

is an isomorphism. This holds, for example, if T is the suspension spectrum of a G-CW complex, and $T_0 \subset T$ is the subcomplex of G-fixed points.

Remark 2.35. For a subgroup $H \subset G$ and a G-spectrum X it will be convenient to use the abbreviation

$$\Phi^H X$$

for the more correct $\Phi^H i_H^* X$. This situation comes up in our proof of the "homotopy fixed point" property of Theorem 10.8, where the more compound notation becomes a little unwieldy.

We end this section with a simple result whose proof illustrates a typical use of the geometric fixed point spectra.

Proposition 2.36. *Suppose that X is a G-spectrum with the property that for all $H \subseteq G$, the geometric fixed point spectrum $\Phi^H X$ is contractible. Then X is contractible as a G-spectrum.*

Proof: By induction on $|G|$ we may assume that for proper $H \subset G$, the spectrum $i_H^* X$ is contractible. It then follows that $T \wedge X$ is contractible for any G CW-complex built entirely from cells of the form $G_+ \underset{H}{\wedge} D^n$ with $H \subset G$ proper. This applies in particular to $T = E\mathcal{P}_+$. The isotropy separation sequence then shows that

$$X \to \tilde{E}\mathcal{P} \wedge X$$

is a weak equivalence. But Remark 2.33 and our assumption that $\Phi^G X$ is contractible imply that $\tilde{E}\mathcal{P} \wedge X$ is contractible. □

2.5.3. *The monoidal geometric fixed point functor.* For some purposes it is useful to have a version of the geometric fixed point functor which is lax symmetric monoidal. For example, such a functor automatically takes (commutative) algebras to (commutative) algebras. The geometric fixed point functor defined in [36, §V.4] has this property. We denote it Φ_M^G and refer to it as the *monoidal geometric fixed point functor* in order to distinguish it from Φ^G. The construction is described in §B.5.

Proposition 2.37. *The monoidal geometric fixed point functor has the following properties:*

i) *It preserves acyclic cofibrations.*

ii) *It is lax symmetric monoidal.*

iii) *If X and Y are cofibrant, the map*

$$\Phi_M^G(X) \wedge \Phi_M^G(Y) \to \Phi_M^G(X \wedge Y)$$

is an isomorphism.

iv) *It commutes with cobase change along a closed inclusion.*

v) *It commutes with directed colimits.*

Property iii) implies that Φ_M^G is weakly symmetric monoidal in the sense of the definition below.

Definition 2.38 (Schwede-Shipley [49])**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between (symmetric) monoidal model categories is *weakly (symmetric) monoidal* if it is lax (symmetric) monoidal, and the map

$$F(X) \wedge F(Y) \to F(X \wedge Y)$$

is a weak equivalence when X an Y are cofibrant.

The next result is [36, Proposition V.4.17], and is discussed in more detail as Proposition B.88.

Proposition 2.39. *There are natural transformations*

$$\Phi^G(X) \to \tilde{\Phi}^G_M(X) \xleftarrow{\sim} \Phi^G_M(X)$$

in which the rightmost arrow is a always weak equivalence and the leftmost arrow is a weak equivalence when X is cofibrant. □

Because Φ^G is lax monoidal, it determines functors

$$\Phi^G_M : \mathbf{Alg}_G \to \mathbf{Alg}$$
$$\Phi^G_M : \mathbf{Comm}_G \to \mathbf{Comm},$$

and for each associative algebra R a functor

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}.$$

In addition, if R is an associative algebra, M a right R-module and N a left R-module there is a natural map

$$\Phi^G_M(M \underset{R}{\wedge} N) \to \Phi^G_M M \underset{\Phi^G_M R}{\wedge} \Phi^G_M N. \tag{2.40}$$

The argument for [36, Proposition V.4.7] shows that (2.40) is a weak equivalence (in fact an isomorphism) if M and N are cofibrant (see Proposition B.90).

While these properties if Φ^G_M are very convenient, they must be used with caution. The value $\Phi^G_M(X)$ is only guaranteed to have the "correct" homotopy type on cofibrant objects. The spectrum underlying a commutative algebra is rarely known to be cofibrant, making the monoidal geometric fixed point functor difficult to use in that context. The situation is a little better with associative algebras. The weak equivalence (2.40) leads to an expression for the geometric fixed point spectrum of a quotient module which we will use in §7.3. In order to do so, we will need criteria guaranteeing that the monoidal geometric fixed point functor realizes the correct homotopy type. Such criteria are described in §B.5.3.

2.5.4. *Geometric fixed points and the norm.* The geometric fixed point construction interacts well with the norm. Suppose $H \subset G$ is a subgroup, and that X is an H-spectrum. The following result is proved as Proposition B.96. It follows easily from the canonical homotopy presentation.

Proposition 2.41. *There is a natural map*

$$\Phi^H_M X \to \Phi^G_M(N^G_H X)$$

which is a weak equivalence on cofibrant objects.

Because of Proposition 2.39 and the fact that the norm preserves cofibrant objects (Proposition B.36), the above result gives a natural zig zag of weak equivalences relating $\Phi^H(X)$ and $\Phi^G(N_H^G X)$ when X is cofibrant. In fact the situation is better with the actual geometric fixed point functor, and there is a natural zig zag of maps

$$\Phi_M^H X \leftrightarrow \Phi_M^G(N_H^G X)$$

which is a weak equivalence not only for cofibrant X, but for suspension spectra of cofibrant G-spaces and for the spectra underlying cofibrant commutative rings. The actual statement is somewhat technical, and is one of the main results of Appendix B. The condition is described in the statement of Proposition B.99. See also Remarks B.100 and B.101.

Proposition 2.42. *For the spectra satisfying the condition of Proposition B.99, the composite functor*

$$\Phi^G \circ N_H^G : \mathcal{S}_H \to \mathcal{S}$$

preserves wedges, directed colimits and cofiber sequences. □

There is another useful result describing the interaction of the geometric fixed point functor with the norm map in $RO(G)$-graded cohomology described in §2.3.3. Suppose that R is a G-equivariant commutative algebra, X is a G-space, and $V \in RO(H)$ a virtual real representation of a subgroup $H \subset G$. In this situation one can compose the norm

$$N : R_H^V(X) \to R_G^{\operatorname{ind} V}(X)$$

with the geometric fixed point map

$$\Phi^G : R_G^{\operatorname{ind} V}(X) \to (\Phi^G R)^{V_0}(X^G),$$

where $V_0 \subset V$ is the subspace of H-fixed vectors, and X^G is the space of G-fixed points in X.

Proposition 2.43. *The composite*

$$\Phi^G \circ N : R_H^V(X) \to (\Phi^G R)^{V_0}(X^G)$$

is a ring homomorphism.

Proof: Multiplicativity is a consequence of the fact that both the norm and the geometric fixed point functors are weakly monoidal. Additivity follows from the fact that the composition $\Phi^G \circ N$ preserves wedges (Proposition 2.42). □

2.6. **Mackey functors.**

2.6.1. *Mackey functors and cohomology.* In equivariant homotopy theory, the role of “abelian group” is played by the notion of a *Mackey functor*. The following formulation is taken from Greenlees-May [18].

Definition 2.44 (Dress [11])**.** A *Mackey functor* consists of a pair $M = (M_*, M^*)$ of functors on the category of finite G-sets. The two functors have the same object function (denote M) and take disjoint unions to direct sums. The functor M_* is covariant, while M^* is contravariant, and together they take a pullback diagram of finite G-sets

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

to a commutative square

$$\begin{tikzcd} M(P) \arrow[r, "\delta_*"] & M(X) \\ M(Y) \arrow[u, "\gamma^*"] \arrow[r, "\beta_*"'] & M(Z) \arrow[u, "\alpha^*"'] \end{tikzcd}$$

where $\alpha^* = M^*(\alpha)$, $\beta_* = M_*(\beta)$, etc.

The contravariant maps $M^*(\alpha)$ are called the *restriction* maps, and the covariant maps $M_*(\beta)$, the *transfer* maps.

A Mackey functor can also be defined as a contravariant additive functor from the full subcategory of $\mathcal{S}_G$ consisting of the suspension spectra $\Sigma^\infty B_+$ of finite G-sets B. It is a theorem of tom Dieck that these definitions are equivalent. See [18, §5].

The equivariant homotopy groups of a G-spectrum X are naturally part of the Mackey functor $\underline{\pi}_n X$ defined by

$$\underline{\pi}_n X(B) = [S^n \wedge B_+, X]^G.$$

For $B = G/H$ one has

$$\underline{\pi}_n X(B) = \pi_n^H X.$$

Just as every abelian group can occur as a stable homotopy group, every Mackey functor M can occur as an equivariant stable homotopy group. In fact associated to each Mackey functor M is an equivariant Eilenberg-Mac Lane spectrum HM, characterized by the property

$$\underline{\pi}_n HM = \begin{cases} M & n = 0 \\ 0 & n \neq 0. \end{cases}$$

The homology and cohomology groups of a G-spectrum X with coefficients in M are defined by

$$H_k^G(X; M) = \pi_k^G HM \wedge X$$
$$H_G^k(X; M) = [X, \Sigma^k HM]^G.$$

For a pointed G-space Y one defines

$$H_n^G(Y; M) = H_n^G(\Sigma^\infty Y; M)$$
$$H_G^n(Y; M) = H_G^n(\Sigma^\infty Y; M).$$

We emphasize that the equivariant cohomology groups of pointed G-spaces Y we consider will always be *reduced* cohomology groups.

The Mackey functor homology and cohomology groups of a G-CW spectrum Y can be computed from a chain complex analogous to the complex of cellular chains. Write $Y^{(n)}$ for the n-skeleton of Y so that

$$Y^{(n)}/Y^{(n-1)} \approx B_+ \wedge S^n$$

with B a discrete G-set. Set

$$\begin{aligned} C_n^{\text{cell}}(Y;M) &= \underline{\pi}_n^G HM \wedge Y^{(n)}/Y^{(n-1)} = \underline{\pi}_0^G HM \wedge B_+ \\ C^n_{\text{cell}}(Y;M) &= [Y^{(n)}/Y^{(n-1)}, \Sigma^n HM]^G = [\Sigma^\infty B_+, HM]^G. \end{aligned}$$

The map

$$Y^{(n)}/Y^{(n-1)} \to \Sigma Y^{(n-1)}/Y^{(n-2)}$$

defines boundary and coboundary maps

$$\begin{aligned} C_n^{\text{cell}}(Y;M) &\to C_{n-1}^{\text{cell}}(Y;M) \\ C^{n-1}_{\text{cell}}(Y;M) &\to C^n_{\text{cell}}(Y;M). \end{aligned}$$

The equivariant homology and cohomology groups of Y with coefficients in M are the homology and cohomology groups of these complexes. By writing the G-set B as a coproduct of finite G-sets B_α one can express $C_n^{\text{cell}}(Y;M)$ and $C^n_{\text{cell}}(Y;M)$ in terms of the values of the Mackey functor M on the B_α.

2.6.2. *Examples of Mackey functors.* The free Mackey functor on one generator is the *Burnside ring* Mackey functor $\underline{A}$. For a G-set S, the value $\underline{A}(S)$ is the group completion of the monoid of finite G-sets $T \to S$ over S under disjoint union. The restriction maps are given by pullback, and the transfer maps by composition. The group $\underline{A}(G/H)$ is the abelian group underlying the Burnside ring of finite H-sets.

Another important example is the "constant" Mackey functor $\underline{\mathbb{Z}}$ represented on the category of G-sets by the abelian group $\mathbb{Z}$ with trivial G-action. The value of $\underline{\mathbb{Z}}$ on a finite G-set $\mathcal{O}$ is the group of functions

$$\underline{\mathbb{Z}}(\mathcal{O}) = \hom^G(\mathcal{O}, \mathbb{Z}) = \hom(\mathcal{O}/G, \mathbb{Z}).$$

The restriction maps are given by (pre-)composition, and the transfer maps are defined by summing over the fibers. For $K \subset H \subset G$, the transfer map associated by $\underline{\mathbb{Z}}$ to

$$G/K \to G/H$$

is the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by the index of K in H.

Definition 2.45. Suppose that S is a G-set, and write $\mathbb{Z}\{S\}$ for the free abelian group generated by S. The Mackey functor $\underline{\mathbb{Z}}\{S\}$ is given by

$$\underline{\mathbb{Z}}\{S\}(\mathcal{O}) = \hom^G(\mathcal{O}, \mathbb{Z}\{S\}),$$

with restriction given by composition and transfer given by summing over the fibers. A *permutation Mackey functor* is a Mackey functor of the form $\underline{\mathbb{Z}}\{S\}$.

One easily checks that $\underline{\mathbb{Z}}\{S\}$ is the Mackey functor

$$\underline{\pi}_0 H\underline{\mathbb{Z}} \wedge S_+.$$

The following is immediate from the definition

Lemma 2.46. *If M is a permutation Mackey functor then the restriction mapping $M(G/H) \to M(G)$ gives an isomorphism*

$$M(G/H) \to M(G)^H$$

of $M(G/H)$ with the H-invariant part of $M(G)$. Therefore a map $M \to M'$ of permutation Mackey functors is an isomorphism if and only if $M(G) \to M'(G)$ is an isomorphism. □

Every G-set S receives a functorial map from a free G-set, namely $G \times S$. One can give $G \times S$ the product of the left action on G and the trivial action on S, or the diagonal action. In the first case, the equivariant map $G \times S \to S$ is the projection, and in the second case it is the action. The two are equivariantly isomorphic by the shearing map $(g, s) \mapsto (g, g\,s)$. The group G acts through equivariant automorphisms of $G \times S \to S$ via $g \cdot (x, s) = (xg^{-1}, gs)$ in the first case, and $g \cdot (x, s) = (xg^{-1}, s)$ in the second.

Lemma 2.47. *If M is a permutation Mackey functor and S is a finite G-set then the restriction map along the action map $G \times S \to S$ gives an isomorphism*

$$M(S) \to M(G \times S)^G$$

□

We will write

$$H^u_n(X; \mathbb{Z}) \text{ and } H_u^n(X; \mathbb{Z})$$

for the ordinary, non-equivariant homology and cohomology groups of the underlying spectrum $i_0^* X$. Of course there are isomorphisms

$$\begin{aligned} H^u_n(X; \mathbb{Z}) &\approx H^G_n(G_+ \wedge X; \underline{\mathbb{Z}}) \\ H_u^n(X; \mathbb{Z}) &\approx H_G^n(G_+ \wedge X; \underline{\mathbb{Z}}). \end{aligned}$$

2.6.3. *$RO(G)$-graded homotopy groups.* In addition to the Mackey functor homotopy groups $\underline{\pi}_* X$ there are the $RO(G)$ graded homotopy groups $\pi_\star X$ defined by

$$\pi^G_V X = [S^V, X]^G \qquad V \in RO(G).$$

Here $RO(G)$ is the Grothendieck group of real representations of G. The use of $\star$ for the wildcard symbol in $\pi_\star$ is taken from Hu-Kriz [26]. The $RO(G)$-graded homotopy groups are also part of a Mackey functor $\underline{\pi}_\star(X)$ defined by

$$\underline{\pi}_V X(B) = [S^V \wedge B_+, X]^G.$$

As with $\mathbb{Z}$-graded homotopy groups, we'll use the abbreviation

$$\pi^H_V X = (\underline{\pi}_V X)(G/H).$$

There are a few distinguished elements of $RO(G)$-graded homotopy groups we'll need. For a representation V of G we let

$$a_V \in \pi^G_{-V} S^0$$

be the equivariant map

$$a_V : S^0 \to S^V \tag{2.48}$$

corresponding to the inclusion $\{0\} \subset V$. The element a_V is the equivariant Euler class of V. If V contains a trivial representation then $a_V = 0$. For two representations V and W one has

$$a_{V\oplus W} = a_V a_W \in \pi^G_{-V-W} S^0.$$

If V is an *oriented* representation of G of dimension d, there is, in the homotopy category, a unique map

$$u_V : S^d \to H\underline{\mathbb{Z}} \wedge S^V \tag{2.49}$$

with the property that $i_0^* u_V$ is the generator of the non-equivariant homology group $H^u_d(S^V; \mathbb{Z})$ corresponding to the orientation of V (see Example 3.3 below). We regard u_V as an element of the $RO(G)$-graded group

$$\pi^G_{d-V} H\underline{\mathbb{Z}}.$$

If V and W are two oriented representations of G, and $V \oplus W$ is given the direct sum orientation, then

$$u_{V\oplus W} = u_V u_W.$$

Among other things this implies that the class u_V is stable in V in the sense that $u_{V+1} = u_V$.

For any V, the representation $V \oplus V$ has a canonical orientation, and so there's always a class

$$u_{V\oplus V} \in \pi^G_{2d-2V} H\underline{\mathbb{Z}}.$$

When V is oriented this class can be identified, up to sign, with u_V^2.

The classes a_V and u_V behave well with respect to the norm. The following result is a simple consequence of the fact (2.12) that $NS^V = S^{\operatorname{ind} V}$.

Lemma 2.50. *Suppose that V is a d-dimensional representation of a subgroup $H \subset G$. Then*

$$\begin{aligned} Na_V &= a_{\operatorname{ind} V} \\ u_{\operatorname{ind} d} \cdot Nu_V &= u_{\operatorname{ind} V}, \end{aligned}$$

where $\operatorname{ind} V = \operatorname{ind}_H^G V$ *is the induced representation and d is the trivial representation.* □

It is sometimes useful to think of the second identity above as

$$Nu_V = u_{(\operatorname{ind} V - \operatorname{ind} d)} = u_{\operatorname{ind}(V-d)},$$

even though the symbol $u_{\operatorname{ind}(V-d)}$ has no defined meaning.

In the case $G = C_{2^n}$ the sign representation of dimension 1 will play an important role. We'll denote this representation σ, or σ_G if more than one cyclic group is under consideration.

3. Some computations in equivariant cohomology

We now turn to some computations which will play an important role later in the paper.

3.1. **Elementary connectivity results and the gap.** We begin with the cohomology groups

$$H^*(S^{\rho_G-1}; M)$$

where $\rho_G - 1$ is the reduced real regular representation of G, of dimension $|G| - 1$.

To calculate these groups we need an equivariant cell decomposition of S^{ρ_G-1}. Since S^{ρ_G-1} is the mapping cone of the map

$$S(\rho_G - 1) \to \mathrm{pt}$$

from the unit sphere in $(\rho_G - 1)$ it suffices to construct an equivariant cell decomposition of $S(\rho_G - 1)$. Write $g = |G|$. Think of $\mathbb{R}^G$ as the vector space with basis the elements of G. The boundary of the standard simplex in this space is equivariantly homeomorphic to $S(\rho_G - 1)$. The simplicial decomposition of this simplex is not an equivariant cell decomposition, but the barycentric subdivision is. Thus $S(\rho_G - 1)$ is homeomorphic to the geometric realization of the poset of non-empty proper subsets of G. This leads to the complex

$$M(G/G) \to M(S_1) \to M(S_2) \to \cdots \to M(S_{g-1}) \tag{3.1}$$

in which S_k is the G-set of flags $F_0 \subset \cdots \subset F_{k-1} \subset G$ of proper inclusions of subsets of G, with G acting by translation. The coboundary map is the alternating sum of the restriction maps derived by omitting one of the sets in a flag.

Lemma 3.2. *For any Mackey functor M, the group*

$$\pi^G_{\rho_G-1} HM = H^0_G(S^{\rho_G-1}; M)$$

is given by

$$\bigcap_{H \subsetneq G} \ker M(G/G) \to M(G/H).$$

Proof: This follows from the complex (3.1). The set S_0 is the set of non-empty proper subsets $S \subset G$. The stabilizer of a flag $S \subset G$ is the set of elements $g \in G$ for which $g \cdot S = S$. Any proper subgroup H of G occurs as the stabilizer of itself, regarded as a flag in S_0. Thus the kernel of

$$M(G/G) \to M(S_0)$$

is the group asserted to be $H^0_G(S^{\rho_G-1}; M)$. □

There are some simplifications that arise when the Mackey functor M is $\underline{\mathbb{Z}}$. Suppose that Y is a G-CW spectrum, with n-skeleton denoted $Y^{(n)}$. Then by definition $Y^{(n)}/Y^{(n-1)} = B_+ \wedge S^n$ for some discrete G-set B. The Mackey functor $\pi_n H\underline{\mathbb{Z}} \wedge Y^{(n)}/Y^{(n-1)}$ is then the permutation Mackey functor $\mathbb{Z}[B]$, and associates to $\mathcal{O}$ the group of equivariant functions

$$\mathcal{O} \to C^{\mathrm{cell}}_n Y.$$

The entire Mackey functor chain complex for $H\mathbb{Z} \wedge Y$ is encoded in the cellular chain complex for Y associated with a G-CW decomposition, equipped with the action of G. The equivariant homology group $H^G_*(Y; \underline{\mathbb{Z}})$ are just the homology groups of the complex

$$\hom_G(G/G, C^{\mathrm{cell}}_*(Y)) = C^{\mathrm{cell}}_*(Y)^G$$

of G-invariant cellular chains. Similarly the equivariant cohomology groups $H^*_G(Y;\mathbb{Z})$ are given by the cohomology groups of the complex

$$C^*_{\text{cell}}(Y)^G$$

of equivariant cochains. The equivariant homology and cohomology groups depend only on the equivariant chain homotopy type of these complexes.

This discussion also shows that the cohomology group $H^*_G(X;\underline{\mathbb{Z}})$ are isomorphic to the cohomology groups

$$H^*(X/G;\mathbb{Z})$$

of the orbit space.

Example 3.3. Suppose that V is a representation of G of dimension d, and consider the equivariant cellular chain complex

$$C^{\text{cell}}_d S^V \to C^{\text{cell}}_{d-1} S^V \to \cdots \to C^{\text{cell}}_0 S^V,$$

associated to an equivariant cell decomposition of S^V. The homology groups are those of the sphere S^V, and so in particular the kernel of

$$C^{\text{cell}}_d S^V \to C^{\text{cell}}_{d-1} S^V$$

is isomorphic, as a G-module, to $H^u_d(S^V;\mathbb{Z})$. If V is orientable then the G-action is trivial, and one finds that the restriction map

$$H^G_d(S^V;\underline{\mathbb{Z}}) \to H^u_d(S^V;\mathbb{Z})$$

is an isomorphism. A choice of orientation gives an equivariant isomorphism

$$H^u_d(S^V;\mathbb{Z}) \approx \mathbb{Z}.$$

Thus when V is oriented there is a unique isomorphism

$$H^G_d(S^V;\underline{\mathbb{Z}}) \approx \mathbb{Z}$$

extending the non-equivariant isomorphism given by the orientation.

Example 3.4. Suppose that G is not the trivial group. In §4.6.2 we will encounter

$$\pi^G_{\rho_G-2} H\underline{\mathbb{Z}} \approx H^1_G(S^{\rho_G-1};\underline{\mathbb{Z}}).$$

The G-space S^{ρ_G-1} is the unreduced suspension of the unit sphere $S(\rho_G-1)$, and so the orbit space is also a suspension. If $|G|>2$ then $S(\rho_G-1)$ is connected, hence so is the orbit space. If $G=C_2$, then $S(\rho_G-1)\approx G$ and the orbit space is still connected. In all cases then, the unreduced suspension S^{ρ_G-1}/G is simply connected. Thus

$$\pi^G_{\rho_G-2} H\underline{\mathbb{Z}} \approx H^2_G(S^{\rho_G};\underline{\mathbb{Z}}) \approx H^1_G(S^{\rho_G-1};\underline{\mathbb{Z}}) = 0.$$

In fact, the same argument shows for $n>0$ the orbit space $S^{n(\rho_G-1)}/G$ is simply connected, and hence

$$H^0_G(S^{n(\rho_G-1)};\underline{\mathbb{Z}}) = H^1_G(S^{n(\rho_G-1)};\underline{\mathbb{Z}}) = 0$$

or, equivalently

$$\pi^G_{n(\rho_G-1)} H\underline{\mathbb{Z}} = \pi^G_{n(\rho_G-1)-1} H\underline{\mathbb{Z}} = 0.$$

Building on this, we have

Proposition 3.5. *Let G be any non-trivial finite group and $n \ge 0$ an integer. Except in case $G = C_3$, $i = 2$, $n = 1$ the groups*

$$H^i_G(S^{n\rho_G}; \underline{\mathbb{Z}})$$

are zero for $0 < i < 4$. In the exceptional case one has

$$H^2_G(S^{\rho_{C_3}-1}; \underline{\mathbb{Z}}) = \mathbb{Z}.$$

This result constitutes the computational part of the Gap Theorem, and contains the Cell Lemma as a special case.

Proof: Since

$$H^i_G(S^{n\rho_G}; \underline{\mathbb{Z}}) \approx H^{i-n}_G(S^{n(\rho_G-1)}; \underline{\mathbb{Z}})$$

Example 3.4 reduces us to the case $n = 1$. In case $n = 1$, the only group not covered by Example 3.4 is

$$H^2_G(S^{\rho_G-1}; \underline{\mathbb{Z}})$$

which is isomorphic to

$$H^2(S^{\rho_G-1}/G; \mathbb{Z}).$$

Since the orbit space S^{ρ_G-1}/G is simply connected, this group is torsion free. It therefore suffices to show that

$$H^2(S^{\rho_G-1}/G; \mathbb{Q}) = 0.$$

But since G is finite, this group is just the G-invariant part of

$$H^2(S^{\rho_G-1}; \mathbb{Q})$$

which is zero since G does not have order 3. When G does have order 3 the group is $\mathbb{Q}$. The claim follows. $\square$

3.2. **Equivariant chains on S^V.** Now suppose that $G = C_{2^n}$, and that V is a representation of G. For $k < n$ write G_{2^k} for the unique quotient of G of order 2^k. In this section we will describe a method for determining the complex of equivariant chains on S^V up to chain homotopy equivalence. This leads to a simple method of computing the equivariant homology and cohomology groups of S^V with coefficients in any permutation Mackey functor.

Embed $C_2 \subset C_{2^n}$ as the unique subgroup of order 2, and write

$$V = V_+ \oplus V_-$$

with V_+ the subspace of vectors invariant under C_2 and V_- the elements on which C_2 is acting through its sign representation. Since $C_2 \subset C_{2^n}$ is central, this splitting is direct sum of representations of G and there is an equivariant decomposition

$$S^V \approx S^{V_+} \wedge S^{V_-}.$$

The group G acts on S^{V_+} through its quotient $G_{2^{n-1}}$. Let's suppose that we have already handled the case of $G_{2^{n-1}}$, and so have identified a convenient equivariant cell decomposition of S^{V_+}. The group G acts freely on S^{V_-} away from the invariant $S^0 \subset S^{V_-}$. Since the orbit space S^{V_-}/G is connected we may give it a cell decomposition whose zero skeleton is the image of this invariant S^0. Pulling this back we get an equivariant cell decomposition of S^{V_-} with zero skeleton the invariant S^0, and all other cells free of dimension greater than zero. Since the smash product of any equivariant cell with a free cell is free, smashing this with S^{V_+} leads to an equivariant cell decomposition for S^V in which the $\dim V_+$-skeleton is S^{V_+} and the remaining cells are free and have dimensions ranging from $(1+\dim V_+)$ to $\dim V$. This leads to the following result.

Proposition 3.6. *Let V be a representation of $G = C_{2^n}$, and $V_i \subset G$ the subspace of vectors invariant under the unique subgroup of index 2^i, so that there is a filtration*

$$V_0 \subset V_1 \subset \cdots \subset V_n = V$$

of V by representations of G. Write $d_i = \dim V_i$. There is an equivariant cell decomposition of S^V whose d_i-skeleton is S^{V_i} and in which $S^{V_{i+1}}$ is constructed from S^{V_i} by adding cells which are free over the quotient $G_{2^{i+1}}$ and have dimensions ranging from $(1+d_i)$ to d_{i+1}. □

Fix an equivariant cell decomposition of S^V of the kind guaranteed by the above proposition, and let $C_*^{\text{cell}}(S^V)$ denote the corresponding complex of equivariant cellular chains. The kernel of

$$C_{d_i}^{\text{cell}}(S^{V_i}) \to C_{d_i-1}^{\text{cell}}(S^{V_i})$$

is the group $H_{d_i}(S^{V_i};\mathbb{Z}) = \mathbb{Z}$, with the group G acting through a character. We denote this representation $\mathbb{Z}(V_i)$. The portion of the cellular chain

$$C_{d_{i+1}}^{\text{cell}}(S^V) \to \cdots \to C_{d_i+1}^{\text{cell}}(S^V)$$

is thus a complex of free modules over the quotient group G_{i+1} in which the cokernel of the rightmost map is $\mathbb{Z}(V_i)$ and the kernel of the leftmost map is $\mathbb{Z}(V_{i+1})$. It follows that

$$\mathbb{Z}(V_{i+1}) \to C_{d_{i+1}}^{\text{cell}}(S^V) \to \cdots \to C_{d_i+1}^{\text{cell}}(S^V)$$

is a truncation of a projective resolution of $\mathbb{Z}(V_i)$ over G_{2^i}, and so determined up to chain homotopy equivalence by the property just described. We may therefore replace the portion

$$\mathbb{Z}(V_{i+1}) \to C_{d_{i+1}}^{\text{cell}}(S^V) \to \cdots \to C_{d_i+1}^{\text{cell}}(S^V) \to \mathbb{Z}(V_i)$$

of the complex of equivariant cellular chains with the minimal complex

$$\mathbb{Z}(V_{i+1}) \to \mathbb{Z}[G_{2^{i+1}}] \to \cdots \to \mathbb{Z}[G_{2^{i+1}}] \to \mathbb{Z}(V_i).$$

Concatenating these we find that the complex of cellular chains on S^V is chain homotopy equivalent to one of the form

$$\mathbb{Z}[G_{2^k}] \to \cdots \to \mathbb{Z}[G_{2^k}] \to \cdots \to \mathbb{Z}[G_2] \to \cdots \to \mathbb{Z}[G_2] \to \mathbb{Z}.$$

This complex may be used to compute the equivariant homology and cohomology of S^V with coefficients in any permutation Mackey functor.

For example, when $G = C_8$ and $V = \rho_8$, the complex of equivariant cellular chains is constructed by concatenating

$$\begin{gathered} \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}_\pm \\ \mathbb{Z}[G_4] \to \mathbb{Z}[G_4] \to \mathbb{Z}_\pm \\ \mathbb{Z}[G_2] \to \mathbb{Z}, \end{gathered}$$

in which $\mathbb{Z}_\pm$ is the sign representation, to give

$$\mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_8] \to \mathbb{Z}[G_4] \to \mathbb{Z}[G_4] \to \mathbb{Z}[G_2] \to \mathbb{Z},$$

with the $\mathbb{Z}$ on the right in dimension 1. Passing to invariants gives

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

from which one can easily read off the homology groups $H_*^{C_8}(S^{\rho_8};\underline{\mathbb{Z}})$. To calculate the cohomology groups $H^*_{C_8}(S^{\rho_8};\underline{\mathbb{Z}})$ one studies the complex of equivariant maps to $\mathbb{Z}$. It is

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z}.$$

For $G = C_8$ and $V = 2\rho_8$ the complex of invariants works out to be

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{8} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z},$$

with the rightmost $\mathbb{Z}$ in dimension 2. The equivariant cohomology of $S^{2\rho_8}$ is given by the cohomology of the cochain complex cohomology is

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z}.$$

We will return to this kind of computation in much greater detail in a later paper. We end this section with the following result, which plays an important role in the proof of the Reduction Theorem.

Proposition 3.7. *Let $G = C_{2^n}$. For any G-spectrum X, the $RO(G)$-graded homotopy groups of $\tilde{E}\mathcal{P} \wedge X$ are given by*

$$\pi_\star^G(\tilde{E}\mathcal{P} \wedge X) = a_\sigma^{-1}\pi_\star^G(X).$$

The homotopy groups of the commutative algebra $\Phi^G H\underline{\mathbb{Z}}$ are given by

$$\pi_*^G(\Phi^G H\underline{\mathbb{Z}}) = \mathbb{Z}/2[b],$$

where $b = u_{2\sigma}a_\sigma^{-2} \in \pi_2(\Phi^G H\underline{\mathbb{Z}}) \subset a_\sigma^{-1}\pi_\star^G H\underline{\mathbb{Z}}$.

Proof: As mentioned in Remark 2.32, the space $\tilde{E}\mathcal{P}$ can be identified with

$$\lim_{n\to\infty} S^{n\sigma}.$$

The method described above (or in this case, the usual equivariant cell decomposition of S^m with the antipodal action) gives a complex of equivariant chains on $\lim_{n\to\infty} S^{n\sigma}$

$$\cdots \to \mathbb{Z}[G_2] \to \cdots \to \mathbb{Z}[G_2] \to \mathbb{Z}[G_2] \to \mathbb{Z}.$$

The complex of invariants is

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

from which it follows that

$$\pi_k \Phi^G H\underline{\mathbb{Z}} = \begin{cases} 0 & k < 0 \text{ or odd} \\ \mathbb{Z}/2 & k \geq 0 \text{ and even.} \end{cases}$$

That the non-zero element in π_{2n} is b^n is immediate from the definition of b. $\square$

4. The slice filtration

The slice filtration is an equivariant analogue of the Postnikov tower, to which it reduces in the case of the trivial group. In this section we introduce the slice filtration and establish some of its basic properties. We work for the most part with a general finite group G, though our application to the Kervaire invariant problem involves only the case $G = C_{2^n}$. While the situation for general G exhibits many remarkable properties, the reader should regard as exploratory the apparatus of definitions at this level of generality.

From now until the end of §11 our focus will be on homotopy theory. Though it will not appear in the notation, all spectra should be replaced by cofibrant or fibrant approximations where appropriate.

4.1. **Slice cells.**

4.1.1. *Slice cells and their dimension.* For a subgroup $K \subseteq G$ let ρ_K denote its regular representation, and write

$$\widehat{S}(m, K) = G_+ \underset{K}{\wedge} S^{m\rho_K} \qquad m \in \mathbb{Z}.$$

Definition 4.1. The set of *slice cells* is

$$\mathcal{A} = \{\widehat{S}(m, K), \Sigma^{-1}\widehat{S}(m, K) \mid m \in \mathbb{Z}, K \subseteq G\}.$$

This brings two notions of "cell" into the story. The slice cells, and the more usual equivariant cells of the form $G/H_+ \wedge D^m$, used to manufacture G-CW spectra. We'll always refer to the traditional equivariant cells as "G-cells" in order to easily distinguish them from the "slice cells" which are our main focus.

Definition 4.2. A slice cell is *regular* if it is of the form $\widehat{S}(m, K)$.

Definition 4.3. A slice cell is *induced* if it is of the form

$$G_+ \underset{H}{\wedge} \widehat{S},$$

where $\widehat{S}$ is a slice cell for H and $H \subset G$ is a proper subgroup. It is *free* if H is the trivial group. A slice cell is *isotropic* if it is not free.

Since

$$[G_+ \underset{H}{\wedge} S, X]^G = [S, i_H^* X]^H \qquad \text{and}$$
$$[X, G_+ \underset{H}{\wedge} S]^G = [i_H^* X, S]^H,$$

induction on $|G|$ usually reduces claims about cells to the case of those which are not induced. The slice cells which are not induced are those of the form $S^{m\rho_G}$ and $S^{m\rho_G - 1}$.

Definition 4.4. The *dimension* of a slice cell is defined by

$$\dim \widehat{S}(m, K) = m|K|$$
$$\dim \Sigma^{-1}\widehat{S}(m, K) = m|K| - 1.$$

In other words the dimension of a slice cell is that of its underlying spheres.

Remark 4.5. Not every suspension of a slice cell is a slice cell. Typically, the spectrum $\Sigma^{-2}\widehat{S}(m, K)$ will *not* be a slice cell, and will *not* exhibit the properties of a slice cell of dimension $\dim \widehat{S}(m, K) - 2$.

The following is immediate from the definition.

Proposition 4.6. *Let $H \subset G$ be a subgroup. If $\widehat{S}$ is a G-slice cell of dimension d, then $i_H^* \widehat{S}$ is a wedge of H-slice cells of dimension d. If $\widehat{S}$ is an H-slice cell of dimension d then $G_+ \underset{H}{\wedge} \widehat{S}$ is a G-slice cell of dimension d.* □

The regular slice cells behave well under the norm.

Proposition 4.7. *Let $H \subset G$ be a subgroup. If $\widehat{W}$ is a wedge of regular H-slice cells, then $N_H^G \widehat{W}$ is a wedge of regular G-slice cells.*

Proof: The wedges of regular H-slice cells are exactly the indexed wedges (in the sense of §2.3.2) of spectra of the form $S^{m\rho_K}$ for $K \subset H$, and $m \in \mathbb{Z}$. Since regular representations induce to regular representations, the identity (2.12) and the distribution formula (Proposition A.37) show that the norm of such an indexed wedge is an indexed wedge of $S^{m\rho_K}$ with $K \subseteq G$ and $m \in \mathbb{Z}$. The claim follows. □

4.1.2. *Slice positive and slice null spectra.* Underlying the theory of the Postnikov tower is the notion of "connectivity" and the class of $(n-1)$-connected spectra. In this section we describe the slice analogues of these ideas. There is a simple relationship between "connectivity" and "slice-positivity" which we will describe in detail in §4.4.

Definition 4.8. A G-spectrum Y is *slice n-null*, written
$$Y < n \quad \text{or} \quad Y \le n-1$$
if for every slice cell $\widehat{S}$ with $\dim \widehat{S} \ge n$ the G-space
$$\mathcal{S}_G(\widehat{S}, Y)$$
is equivariantly contractible. A G-spectrum X is *slice n-positive*, written
$$X > n \quad \text{or} \quad X \ge n+1$$
if
$$\mathcal{S}_G(X, Y)$$
is equivariantly contractible for every Y with $Y \le n$.

We will use the terms *slice-positive* and *slice-null* instead of "slice 0-positive" and "slice 0-null." The full subcategory of $\mathcal{S}^G$ consisting of X with $X > n$ will be denoted $\mathcal{S}^G_{>n}$ or $\mathcal{S}^G_{\ge n+1}$. Similarly, the full subcategory of $\mathcal{S}^G$ consisting of X with $X < n$ will be denoted $\mathcal{S}^G_{<n}$ or $\mathcal{S}^G_{\le n-1}$.

Remark 4.9. The category $\mathcal{S}^G_{>n}$ is the smallest full subcategory of $\mathcal{S}^G$ containing the slice cells $\widehat{S}$ with $\dim \widehat{S} > n$ and possessing the following properties:

i) If X is weakly equivalent to an object of $\mathcal{S}^G_{>n}$, then X is in $\mathcal{S}^G_{>n}$.
ii) Arbitrary wedges of objects of $\mathcal{S}^G_{>n}$ are in $\mathcal{S}^G_{>n}$.
iii) If $X \to Y \to Z$ is a cofibration sequence and X and Y are in $\mathcal{S}^G_{>n}$ then so is Z.
iv) If $X \to Y \to Z$ is a cofibration sequence and X and Z are in $\mathcal{S}^G_{>n}$ then so is Y.

More briefly, these properties are that $\mathcal{S}^G_{>n}$ is closed under weak equivalences, homotopy colimits (properties ii) and iii)), and extensions.

For $n = 0, -1$, the notions of slice n-null and slice n-positive are familiar.

Proposition 4.10. *For a G-spectrum X the following hold*

i) $X \geq 0 \iff X$ *is* (-1)*-connected, i.e.* $\underline{\pi}_k X = 0$ *for* $k < 0$*;*

ii) $X \geq -1 \iff X$ *is* (-2)*-connected, i.e.* $\underline{\pi}_k X = 0$ *for* $k < -1$*;*

iii) $X < 0 \iff X$ *is* 0 *co-connected, i.e.* $\underline{\pi}_k X = 0$ *for* $k \geq 0$*;*

iv) $X < -1 \iff X$ *is* (-1) *co-connected, i.e.* $\underline{\pi}_k X = 0$ *for* $k \geq -1$*;*

Proof: These are all straightforward consequences of the fact that S^0 is a slice cell of dimension 0, and S^{-1} is a slice cell of dimension (-1). $\square$

Remark 4.11. It is not the case that if $Y > 0$ then $\pi_0 Y = 0$. In Proposition 4.14 we will see that the fiber F of $S^0 \to H\underline{\mathbb{Z}}$ has the property that $F > 0$. On the other hand $\underline{\pi}_0 F$ is the augmentation ideal of the Burnside ring. Proposition 4.45 below gives a characterization of slice-positive spectra.

The classes of slice n-null and slice n-positive spectra are preserved under change of group.

Proposition 4.12. *Suppose $H \subset G$, that X is a G-spectrum and Y is an H-spectrum. The following implications hold*

$$\begin{aligned} X > n &\implies i_H^* X > n \\ X < n &\implies i_H^* X < n \\ Y > n &\implies G_+ \underset{H}{\wedge} Y > n \\ Y < n &\implies G_+ \underset{H}{\wedge} Y < n. \end{aligned}$$

Proof: The second and third implications are straightforward consequences of Proposition 4.6. The fourth implication follows from the Wirthmüller isomorphism and Proposition 4.6, and the first implication is an immediate consequence of the fourth. $\square$

We end this section with a mild simplification of the condition that a spectrum be slice n-null.

Lemma 4.13. *For a G-spectrum X, the following are equivalent*

i) $X < n$*;*

ii) *For all slice cells* $\widehat{S}$ *with* $\dim \widehat{S} \geq n$, $[\widehat{S}, X]^G = 0$.

Proof: The first condition trivially implies the second. We prove that the second implies the first by induction on $|G|$. By the induction hypothesis we may assume that the G-space

$$\mathcal{S}_G(\widehat{S}, X)$$

is contractible for all induced slice cells $\widehat{S}$ with $\dim \widehat{S} \geq n$, and that for all slice cells $\widehat{S}$ with $\dim \widehat{S} \geq n$, and all proper $H \subset G$, the space

$$\mathcal{S}_G(\widehat{S}, X)^H$$

is contractible. We therefore also know that the G-space

$$\mathcal{S}_G(T \wedge \widehat{S}, X)$$

is contractible, for all slice cells $\widehat{S}$ with $\dim \widehat{S} \geq n$ and all (-1)-connected G CW-spectra T built entirely from induced G-cells. We must show that for each $t \geq 0$, the groups

$$\begin{aligned} &[S^t \wedge S^{m\rho_G - 1}, X]^G \qquad m|G| - 1 \geq n \\ &[S^t \wedge S^{m\rho_G}, X]^G \qquad m|G| \geq n \end{aligned}$$

are zero. They are zero by assumption when $t = 0$. For $t > 0$, the first case is a special case of the second, since $S^1 \wedge S^{m\rho_G - 1}$ is a slice cell of dimension $m|G|$. Let T be the homotopy fiber of the map

$$S^t \subset S^{t\rho_G},$$

and consider the exact sequence

$$[S^{t\rho_G} \wedge S^{m\rho_G}, X]^G \to [S^t \wedge S^{m\rho_G}, X]^G \to [T \wedge S^{m\rho_G}, X]^G.$$

The leftmost group is zero since $S^{t\rho_G} \wedge S^{m\rho_G}$ is a slice cell of dimension $(t + m)|G| \geq n$. The rightmost group is zero by the induction hypothesis since T is (-1)-connected and built entirely from induced cells. It follows from exactness that the middle group is zero. □

4.2. **The slice tower.** Let $P^n X$ be the Bousfield localization, or Dror Farjoun nullification ([12, 22]) of X with respect to the class $\mathcal{S}^G_{>n}$, and $P_{n+1}X$ the homotopy fiber of $X \to P^n X$. Thus, by definition, there is a functorial fibration sequence

$$P_{n+1}X \to X \to P^n X.$$

The functor $P^n X$ can be constructed as the colimit of a sequence of functors

$$W_0 X \to W_1 X \to \cdots .$$

The $W_i X$ are defined inductively starting with $W_0 X = X$, and taking $W_k X$ to be the cofiber of

$$\bigvee_I \Sigma^t \widehat{S} \to W_{k-1} X,$$

in which the indexing set I is the set of maps $\Sigma^t \widehat{S} \to W_{k-1} X$ with $\widehat{S} > n$ a slice cell and $t \geq 0$. By Lemma 4.13 the functors P^n can also be formed using the analogous construction using only slice cells themselves, and not their suspensions.

Proposition 4.14. *A spectrum X is slice n-positive if an only if it admits (up to weak equivalence) a filtration*

$$X_0 \subset X_1 \subset \cdots$$

whose associated graded spectrum $\bigvee X_k / X_{k-1}$ is a wedge of slice cells of dimension greater than n. For any spectrum X, $P_{n+1}X$ is slice n-positive.

Proof: This follows easily from the construction of $P^n X$ described above. □

The map $P_{n+1}X \to X$ is characterized up to a contractible space of choices by the properties

i) for all X, $P_{n+1}X \in \mathcal{S}^G_{>n}$;

ii) for all $A \in \mathcal{S}^G_{>n}$ and all X, the map $\mathcal{S}_G(A, P_{n+1}X) \to \mathcal{S}_G(A, X)$ is a weak equivalence of G-spaces.

In other words, $P_{n+1}X \to X$ is the "universal map" from an object of $\mathcal{S}^G_{>n}$ to X. Similarly $X \to P^nX$ is the universal map from X to a slice $(n+1)$-null G-spectrum Z. More specifically

iii) the spectrum P^nX is slice $(n+1)$-null;
iv) for any slice $(n+1)$-null Z, the map

$$\mathcal{S}_G(P^nX, Z) \to \mathcal{S}_G(X, Z)$$

is a weak equivalence.

These conditions lead to a useful recognition principle.

Lemma 4.15. *Suppose X is a G-spectrum and that*

$$\tilde{P}_{n+1} \to X \to \tilde{P}^n$$

is a fibration sequence with the property that $\tilde{P}^n \le n$ and $\tilde{P}_{n+1} > n$. Then the canonical maps $\tilde{P}_{n+1} \to P_{n+1}X$ and $P^nX \to \tilde{P}^n$ are weak equivalences.

Proof: We show that the map $X \to \tilde{P}^n$ satisfies the universal property of P^nX. Suppose that $Z \le n$, and consider the fibration sequence of G-spaces

$$\mathcal{S}_G(\tilde{P}^n, Z) \to \mathcal{S}_G(X, Z) \to \mathcal{S}_G(\tilde{P}_{n+1}, Z)$$

The rightmost space is contractible since $\tilde{P}_{n+1} > n$, so the map $\mathcal{S}_G(\tilde{P}^n, Z) \to \mathcal{S}_G(X, Z)$ is a weak equivalence. □

The following consequence of Lemma 4.15 is used in the proof of the Reduction Theorem.

Corollary 4.16. *Suppose that $X \to Y \to Z$ is a cofibration sequence, and that the mapping cone of $P^nX \to P^nY$ is slice $(n+1)$-null. Then both*

$$P^nX \to P^nY \to P^nZ$$

and

$$P_{n+1}X \to P_{n+1}Y \to P_{n+1}Z$$

are cofibration sequences.

Proof: Consider the diagram

$$\begin{tikzcd}
P_{n+1}X \arrow[r] \arrow[d] & P_{n+1}Y \arrow[r] \arrow[d] & \tilde{P}_{n+1}Z \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
P^nX \arrow[r] & P^nY \arrow[r] & \tilde{P}^nZ
\end{tikzcd}$$

in which the rows and columns are cofibration sequences. By construction, $\tilde{P}_{n+1}Z$ is slice n-positive (Remark 4.9). If $\tilde{P}^nZ \le n$ then the left column satisfies the condition of 4.15, and the result follows. □

Since $\mathcal{S}^G_{>n} \subset \mathcal{S}^G_{>n-1}$, there is a natural transformation

$$P^nX \to P^{n-1}X.$$

Definition 4.17. The *slice tower* of X is the tower $\{P^n X\}_{n\in\mathbb{Z}}$. The spectrum $P^n X$ is the n^{th} *slice section* of X.

When considering more than one group, we will write $P^n X = P^n_G X$ and $P_n X = P^G_n X$.

Let $P^n_n X$ be the fiber of the map

$$P^n X \to P^{n-1} X.$$

Definition 4.18. The *n-slice* of a spectrum X is $P^n_n X$. A spectrum X is an *n-slice* if $X = P^n_n X$.

The spectrum $P_{n+1} X$ is analogous to the n-connected cover of X, and for two values of n they coincide. The following is a straightforward consequence of Proposition 4.10.

Proposition 4.19. *For any spectrum X, $P_0 X$ is the (-1)-connected cover of X and $P_{-1} X$ is the (-2)-connected cover of X. The (-1)-slice of X is given by*

$$P^{-1}_{-1} X = \Sigma^{-1} H\underline{\pi}_{-1} X.$$

□

The formation of slice sections and therefore of the slices themselves behave well with respect to change of group.

Proposition 4.20. *The functor P^n commutes with both restriction to a subgroup and left induction. More precisely, given $H \subset G$ there are natural weak equivalences*

$$i^*_H (P^n_G X) \to P^n_H (i^*_H X)$$

and

$$G_+ \underset{H}{\wedge} (P^n_H X) \to P^n_G (G_+ \underset{H}{\wedge} X).$$

Proof: This is an easy consequence of Lemma 4.15 and Proposition 4.12. □

Remark 4.21. When G is the trivial group the slice cells are just ordinary cells and the slice tower becomes the Postnikov tower. It therefore follows from Proposition 4.20 that the tower of non-equivariant spectra underlying the slice tower is the Postnikov tower.

4.3. **Multiplicative properties of the slice tower.** The slice filtration does not quite have the multiplicative properties one might expect. In this section we collect a few results describing how things work. One important result is Corollary 4.31 asserting that the slice sections of a (-1)-connected commutative or associative algebra are (-1)-connected commutative or associative algebras. We'll show in §4.7 show that for the group $G = C_{2^n}$ the slice filtration does behave in the expected way for the special class of "pure" spectra, defined in §4.6.2.

Lemma 4.22. *Smashing with $S^{m\rho_G}$ gives a bijection of the set of slice cells $\widehat{S}$ with* $\dim \widehat{S} = k$ *and those with* $\dim \widehat{S} = k + m|G|$.

Proof: Since the restriction of ρ_G to $K \subset G$ is $|G/K|\rho_K$ there is an identity

$$S^{\rho_G} \wedge (G_+ \underset{K}{\wedge} S^{m\rho_K}) \approx G_+ \underset{K}{\wedge} (S^{\rho_G} \wedge S^{m\rho_K}) \approx G_+ \underset{K}{\wedge} S^{(|G/K|+m)\rho_K}.$$

The result follows easily from this. □

Corollary 4.23. *Smashing with $S^{m\rho_G}$ gives an equivalence*

$$\mathcal{S}^G_{\geq n} \to \mathcal{S}^G_{\geq n+m|G|}\,.$$

$\square$

Corollary 4.24. *The natural maps*

$$S^{m\rho_G} \wedge P_{k+1}X \to P_{k+m|G|+1}\left(S^{m\rho_G} \wedge X\right)$$
$$S^{m\rho_G} \wedge P^k X \to P^{k+m|G|}\left(S^{m\rho_G} \wedge X\right)$$

are weak equivalences. $\square$

Proposition 4.25. *If $X \geq n$, $Y \geq m$, and n is divisible by $|G|$ then $X \wedge Y \geq n+m$.*

Proof: By smashing X with $S^{(-n/|G|)\rho_G}$ and using Corollary 4.24 we may assume $n = 0$. Suppose that $Z < m$. Since $Y \geq m$, the zero space of function spectrum Z^Y is contractible. It follows that

$$\mathcal{S}_G(X \wedge Y, Z) \approx \mathcal{S}_G(X, Z^Y)$$

is contractible and so $X \wedge Y \geq m$. $\square$

Definition 4.26. A map $X \to Y$ is a *P^n-equivalence* if $P^n X \to P^n Y$ is an equivalence. Equivalently, $X \to Y$ is a P^n-equivalence if for every $Z < n$, the map

$$\mathcal{S}_G(Y, Z) \to \mathcal{S}_G(X, Z)$$

is a weak equivalence.

Lemma 4.27. *If the homotopy fiber F of $f : X \to Y$ is in $\mathcal{S}^G_{>n}$, then f is a a P^n equivalence.*

Proof: Immediate from the fibration sequence

$$\mathcal{S}_G(Y, Z) \to \mathcal{S}_G(X, Z) \to \mathcal{S}_G(F, Z).$$

$\square$

Remark 4.28. The converse of the above result is not true. For instance, $* \to S^0$ is a P^{-1}-equivalence, but the fiber S^{-1} is not in $\mathcal{S}^G_{>-1}$.

Lemma 4.29. i) *If $Y \to Z$ is a P^n-equivalence and $X \geq 0$, then $X \wedge Y \to X \wedge Z$ is a P^n-equivalence;*

ii) *For $X_1, \ldots, X_k \in \mathcal{S}^G_{\geq 0}$, the map*

$$X_1 \wedge \cdots \wedge X_k \to P^n X_1 \wedge \cdots \wedge P^n X_k$$

is a P^n-equivalence.

Proof: Since $P_{n+1}X$ and $P_{n+1}Y$ are both slice n-positive the vertical map in the square below are P^n-equivalences by Lemmas 4.27 and 4.25

$$\begin{array}{ccc} X \wedge Y & \longrightarrow & X \wedge Z \\ \downarrow & & \downarrow \\ X \wedge P^n Y & \longrightarrow & X \wedge P^n Z. \end{array}$$

The bottom row is a weak equivalence by assumption. It follows that the top row is a P^n-equivalence. The second assertion is proved by induction on k, the case $k=1$ being trivial. For the induction step consider

$$\begin{array}{ccc} X_1 \wedge \cdots \wedge X_{k-1} \wedge X_k & \longrightarrow & P^n X_1 \wedge \cdots \wedge P^n X_{k-1} \wedge X_k \\ & & \downarrow \\ & & P^n X_1 \wedge \cdots \wedge P^n X_{k-1} \wedge P^n X_k. \end{array}$$

The first map is a P^n-equivalence by the induction hypothesis and part i). The second map is a P^n-equivalence by part i). $\square$

Remark 4.30. Lemma 4.29 can be described as asserting that the functor

$$P^n : \{(-1)\text{-connected spectra}\} \to \left\{ \mathcal{S}^G_{>n}\text{-null spectra} \right\}$$

is weakly monoidal.

Corollary 4.31. *Let R be a (-1)-connected G-spectrum. If R is a (homotopy) commutative or (homotopy) associative algebra, then so is $P^n R$ for all n.* $\square$

4.4. **The slice spectral sequence.** The *slice spectral sequence* is the homotopy spectral of the slice tower. The main point of this section is to establish strong convergence of the slice spectral sequence, and to show that for any X the E_2-term is distributed in the gray region of Figure 1. We begin with some results relating the slice sections to Postnikov sections.

4.4.1. *Connectivity and the slice filtration.* Our convergence result for the slice spectral sequence depends on knowing how slice cells are constructed from G-cells. We will say that a space or spectrum X *decomposes* into the elements of a collection of spectra $\{T_\alpha\}$ if X is weakly equivalent to a spectrum $\tilde{X}$ admitting an increasing filtration

$$X_0 \subset X_1 \subset \dots$$

with the property that X_n/X_{n-1} is weakly equivalent to a wedge of T_α.

Remark 4.32. A G-spectrum X decomposes into a collection of spectra $\{G/H_+ \wedge S^m\}$, with H and m ranging through some indexing list, if and only if X is weakly equivalent a G-CW spectrum with G-cells of the form $G/H_+ \wedge D^m$, with H and m ranging through the same list.

Remark 4.33. To say that X decomposes into the elements of a collection of compact objects $\{T_\alpha\}$ means that X is in the smallest subcategory of $\mathcal{S}_G$ closed under weak equivalences, arbitrary wedges, and the formation of mapping cones and extensions (i.e., the properties listed in Remark 4.9).

Lemma 4.34. *Let $\widehat{S} \in \mathcal{A}$ be a slice cell. If $\dim \widehat{S} = n \geq 0$, then $\widehat{S}$ decomposes into the spectra $G/H_+ \wedge S^k$ with $\lfloor n/|G| \rfloor \leq k \leq n$. If $\dim \widehat{S} = n < 0$ then $\widehat{S}$ decomposes into $G/H_+ \wedge S^k$ with $n \leq k \leq \lfloor n/|G| \rfloor$.*

Proof: We start with $\widehat{S} = S^{m\rho_G}$, $m \geq 0$. Using the technique described in §3.2 one sees that $\widehat{S}$ is the suspension spectrum of a G-CW complex whose m-skeleton is S^m and whose remaining G-cells range in dimension from $(m+1)$ to $m|G|$. So the result is clear in this case. Desuspending, we find that $\widehat{S} = \Sigma^{-1}S^{m\rho_G}$, of dimension $m|G|-1$, admits a G-CW decomposition with cells ranging in dimension from

$$m-1 = \left\lfloor \frac{m|G|-1}{|G|} \right\rfloor$$

to $m|G|-1 = n$. For $m < 0$, Spanier-Whitehead duality gives an equivariant cell decomposition of $S^{m\rho_G}$ into cells whose dimensions range from $m|G|$ to m and of $\Sigma^{-1}S^{m\rho_G}$ into cells whose dimensions range from $n = m|G|-1$ to $m-1 = \lfloor n/|G| \rfloor$. Finally, the case in which $\widehat{S}$ is induced from a subgroup $K \subset G$ is proved by left inducing its K-equivariant cell decomposition. $\square$

Corollary 4.35. *Let $Y \in \mathcal{S}^G_{\geq n}$. If $n \geq 0$, then Y can be decomposed into the spectra $G/H_+ \wedge S^m$ with $m \geq \lfloor n/|G| \rfloor$. If $n \leq 0$ then Y can be decomposed into $G/H_+ \wedge S^m$ with $m \geq n$.*

Proof: The class of G-spectra Y which can be decomposed into $G/H_+ \wedge S^m$ with $m \geq \lfloor n/|G| \rfloor$ is closed under weak equivalences, homotopy colimits, and extensions. By Lemma 4.34 it contains the slice cells $\widehat{S}$ with $\dim \widehat{S} \geq n$. It therefore contains all $Y \in \mathcal{S}^G_{\geq n}$ by Remark 4.9. A similar argument handles the case $n < 0$. $\square$

Proposition 4.36. *Write $g = |G|$.*

i) *If $n \geq 0$ and $k > n$ then $(G/H)_+ \wedge S^k > n$.*

ii) *If $m \leq -1$ and $k \geq m$ then $(G/H)_+ \wedge S^k \geq (m+1)g - 1$.*

iii) *If $Y \geq n$ with $n \geq 0$, then $\underline{\pi}_i Y = 0$ for $i < \lfloor n/g \rfloor$.*

iv) *If $Y \geq n$ with $n \leq 0$, then $\underline{\pi}_i Y = 0$ for $i < n$.*

Proof: We start with the first assertion. We will prove the claim by induction on $|G|$, the case of the trivial group being obvious. Using Proposition 4.12 we may assume by induction that $(G/H)_+ \wedge S^k > n$ when $k > n \geq 0$ and $H \subset G$ is a proper subgroup. This implies that if T is an equivariant CW-spectrum built from G-cells of the form $(G/H)_+ \wedge S^k$ with $k > n$ and $H \subset G$ a proper subgroup, then $T > n$. The homotopy fiber of the natural inclusion

$$S^k \to S^{k\rho_G}$$

is such a T. We need to show that $S^k > n$. Since $S^{k\rho_G} \geq k|G| > n$ the fibration sequence

$$T \to S^k \to S^{k\rho_G}$$

exhibits S^k as an extension of two slice n-positive spectra, making it slice n-positive. The second assertion is trivial for $k \geq 0$ since in that case $G/H_+ \wedge S^k \geq 0$ and $(m+1)g - 1 \leq -1$. The case $k \leq -1$ is handled by writing

$$(G/H)_+ \wedge S^k = \Sigma^{-1}(G/H)_+ \wedge S^{(k+1)\rho_G} \wedge S^{-(k+1)(\rho_G - 1)}.$$

Since $-(k+1) \geq 0$, the spectrum $S^{-(k+1)(\rho_G-1)}$ is a suspension spectrum and so

$$(G/H)_+ \wedge S^k \geq (k+1)g - 1 \geq (m+1)g - 1.$$

The third and fourth assertions are immediate from Corollary 4.35. $\square$

Remark 4.37. We've stated part ii) of Proposition 4.36 in the form in which it is most clearly proved. When it comes up, it is needed as the implication that for $n < 0$,

$$k \geq \lfloor (n+1)/g \rfloor \implies G/H_+ \wedge S^k > n.$$

To relate these, write $m = \lfloor (n+1)/g \rfloor$, so that

$$m + 1 > (n+1)/g$$

and by part ii) of Proposition 4.36

$$G/H_+ \wedge S^k \geq (m+1)g - 1 > n.$$

4.4.2. *The spectral sequence.* The slice spectral sequence is the spectral sequence associated to the tower of fibration $\{P^n X\}$, and it takes the form

$$E_2^{s,t} = \pi_{t-s}^G P_t^t X \implies \pi_{t-s}^G X.$$

It can be regarded as a spectral sequence of Mackey functors, or of individual homotopy groups. We have chosen our indexing so that the display of the spectral sequence is in accord with the classical Adams spectral sequence: the $E_r^{s,t}$-term is placed in the plane in position $(t-s, s)$. The situation is depicted in Figure 1. The differential d_r maps $E_r^{s,t}$ to $E_r^{s+r,t+r-1}$, or in terms the display in the plane, the group in position $(t-s, s)$ to the group in position $(t-s-1, s+r)$.

The following is an immediate consequence of Proposition 4.36. As there, we write $g = |G|$.

Theorem 4.38. *Let X be a G-spectrum. The Mackey functor homotopy groups of $P^n X$ satisfy*

$$\underline{\pi}_k P^n X = 0 \textit{ for } \begin{cases} k > n & \textit{if } n \geq 0 \\ k \geq \lfloor (n+1)/g \rfloor & \textit{if } n < 0 \end{cases}$$

and the map $X \to P^n X$ induces an isomorphism

$$\underline{\pi}_k X \xrightarrow{\approx} \underline{\pi}_k P^n X \textit{ for } \begin{cases} k < \lfloor (n+1)/g \rfloor & \textit{if } n \geq 0 \\ k \leq n & \textit{if } n < 0. \end{cases}$$

Thus for any X,

$$\varinjlim_n P^n X$$

is contractible, the map

$$X \to \varprojlim_n P^n X$$

is a weak equivalence, and for each k, the map

$$\{\underline{\pi}_k(X)\} \to \{\underline{\pi}_k P^n X\}$$

from the constant tower to the slice tower of Mackey functors is a pro-isomorphism. □

Corollary 4.39. *If M is an n-slice then*

$$\underline{\pi}_k M = 0$$

if $n \geq 0$ and k lies outside of the region $\lfloor n/g \rfloor \leq k \leq n$, or if $n < 0$ and k lies outside of the region $n \leq k < \lfloor (n+1)/g \rfloor$. □

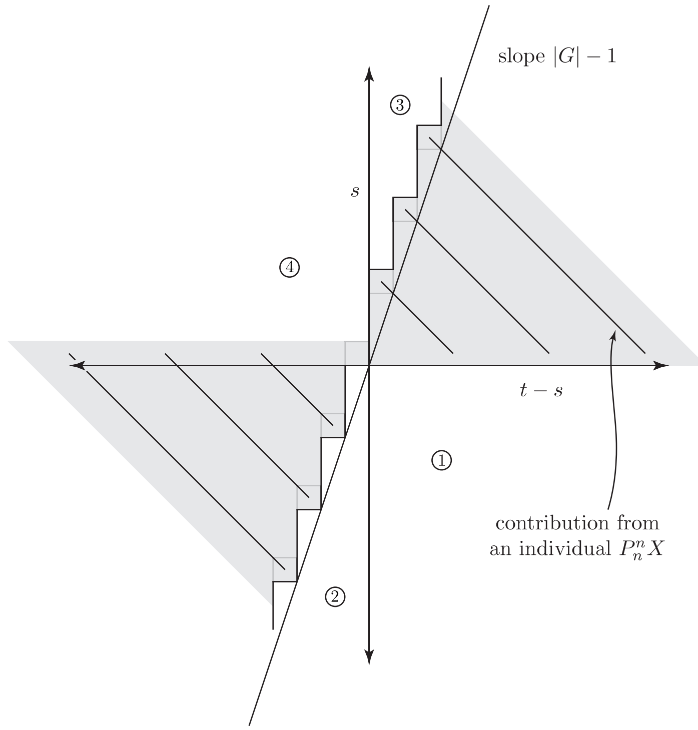

FIGURE 1. The slice spectral sequence

Theorem 4.38 gives the strong convergence of the slice spectral sequence, while Corollary 4.39 shows that the E_2-term vanishes outside of a restricted range of dimensions. The situation is depicted in Figure 1. The homotopy groups of individual slices lie along lines of slope -1, and the groups contributing to $\pi_* P^n X$ lie to the left of a line of slope -1 intersecting the $(t-s)$-axis at $(t-s)=n$. All of the groups outside the gray region are zero. The vanishing in the regions labeled 1-4 correspond to the four parts of Proposition 4.36.

Proposition 4.36 also gives a relationship between the Postnikov tower and the slice tower.

Corollary 4.40. *If X is an $(n-1)$-connected G-spectrum with $n \geq 0$ then $X \geq n$.*

Proof: The assumption on X means it is weakly equivalent to a G-CW spectrum having cells $G/H_+ \wedge S^m$ only in dimensions $m \geq n$. By part i) of Proposition 4.36 these cells are in $\mathcal{S}^G_{\geq n}$. □

We end this section with an application. The next result says that if a tower looks like the slice tower, then it is the slice tower.

Proposition 4.41. *Suppose that $X \to \{\tilde{P}^n\}$ is a map from X to a tower of fibrations with the properties*

i) *the map $X \to \varprojlim \tilde{P}^n$ is a weak equivalence;*

ii) *the spectrum* $\varinjlim_n \tilde{P}^n$ *is contractible;*

iii) *for all* n*, the fiber of the map* $\tilde{P}^n \to \tilde{P}^{n-1}$ *is an* n*-slice.*

Then $\tilde{P}^n$ *is the slice tower of* X*.*

Proof: We first show that $\tilde{P}^n$ is slice $(n+1)$-null. We will use the criteria of Lemma 4.13. Suppose that $\widehat{S}$ is a slice cell with $\dim \widehat{S} > n$. By condition iii), the maps

$$[\widehat{S}, \tilde{P}^n]^G \to [\widehat{S}, \tilde{P}^{n-1}]^G \to [\widehat{S}, \tilde{P}^{n-2}]^G \to \cdots$$

are all monomorphisms. Since $\widehat{S}$ is finite, the map

$$\varinjlim_{k \le n} [\widehat{S}, \tilde{P}^k]^G \to [\widehat{S}, \varinjlim_{k \le n} \tilde{P}^k]^G$$

is an isomorphism. It then follows from assumption ii) that $[\widehat{S}, \tilde{P}^n]^G = 0$. This shows that $\tilde{P}^n$ is slice $(n+1)$-null. Now let $\tilde{P}_{n+1}$ be the homotopy fiber of the map $X \to \tilde{P}^n$. By Lemma 4.15, the result will follow if we can show $\tilde{P}_{n+1} > n$. By assumption iii), for any $N > n+1$, the spectrum

$$\tilde{P}_{n+1} \cup C\tilde{P}_N$$

admits a finite filtration whose layers are m-slices, with $m \ge n+1$. It follows that

$$\tilde{P}_{n+1} \cup C\tilde{P}_N > n.$$

In view of the cofibration sequence

$$\tilde{P}_n \to \tilde{P}_{n+1} \to \tilde{P}_{n+1} \cup C\tilde{P}_N,$$

to show that $\tilde{P}_{n+1} > n$ it suffices to show that $\tilde{P}_N > n$ for any $N > n$.

Let Z be any slice $(n+1)$-null spectrum. We need to show that the space

$$\mathcal{S}_G(\tilde{P}_N, Z)$$

is contractible. We do this by studying the Mackey functor homotopy groups of the spectra involved, and appealing to an argument using the usual equivariant notion of connectivity. By Theorem 4.38, there is an integer m with the property that for $k > m$,

$$\underline{\pi}_k Z = 0.$$

By Corollary 4.39 and assumption iii), for $N \gg 0$ and any $N' > N$,

$$\underline{\pi}_k \tilde{P}_N \cup C\tilde{P}_{N'} = 0, \quad k \le m.$$

Since $\operatorname{ho}\varprojlim_{N'} \tilde{P}_{N'}$ is contractible by assumption i), this implies that for $N \gg 0$

$$\underline{\pi}_k \tilde{P}_N, \quad k \le m.$$

Thus for $N \gg 0$, $\tilde{P}_N$ is m-connected in the usual sense and so

$$\mathcal{S}_G(\tilde{P}_N, Z)$$

is contractible. $\square$

4.5. **The $RO(G)$-graded slice spectral sequence.** Often the most useful information about certain homotopy groups is not gotten directly from the slice tower, but from smashing the slice tower with another fixed spectrum. This is especially true when considering an $RO(G)$-graded homotopy group

$$\pi_V^G X = [S^V, X]^G = [S^0, X \wedge S^{-V}]^G.$$

Rather than work with the slice tower for $X \wedge S^{-V}$ we'll work with the slice tower for X.

Definition 4.42. Let V be a virtual representation of G, of virtual dimension d. The *slice spectral sequence for* $\pi_{*+V}^G X$ is the spectral sequence derived from the tower of fibrations

$$\{S^{-V} \wedge P^{n+d} X\}.$$

We write the $RO(G)$-graded slice spectral sequence as

$$E_2^{s,t+V} = \pi_{V+t-s}^G P_{d+t}^{d+t} X \implies \pi_{V+t-s} X,$$

with

$$d_r : E_2^{s,t+V} \to E_2^{s+r,t+(r-1)+V}.$$

We've chosen this indexing convention in part to retain some familiar properties of the distribution of groups in the classical Adams spectral sequence. For example, suppose that V is the trivial virtual representation of dimension d. Then the slice spectral sequence for $\pi_{V+*} X$ is gotten from the slice spectral sequence for X by simply shifting the display d units to the left. The vanishing regions shown in Figure 1 do not necessarily hold for the $RO(G)$-graded slice spectral sequence. But by Corollary 4.24 they do hold as stated when $V = m\rho_G$. In case V is trivial the vanishing regions are just shifted along the $(t-s)$-axis.

Our indexing convention amounts roughly to thinking of the t-index as an element of $RO(G)$, which enters only through its dimension when written as an index of a slice. The authors have found the mnemonic "V goes with t" to be a helpful reminder.

4.6. **Special slices.** In this section we investigate special slices of spectra, and introduce the notion of a *spectrum with cellular slices*, and of a *pure* G-spectrum Our main result (Proposition 4.56) asserts that a map $X \to Y$ of G-spectra with cellular slices is a weak equivalence if and only if the underlying map of non-equivariant spectra is. This result plays an important role in the proof of the Reduction Theorem in §7. We also include material useful for investigating the slices of more general spectra.

4.6.1. *Slice positive spectra,* 0*-slices and* (-1)*-slices.* In this section we will describe methods for determining the slices of spectra, and introduce a convenient class of equivariant spectra. Our first results make use of the isotropy separation sequence (§2.5.2) obtained by smashing with the cofibration sequence of pointed G-spaces

$$E\mathcal{P}_+ \to S^0 \to \tilde{E}\mathcal{P}.$$

The space $E\mathcal{P}_+$ is an equivariant CW-complex built from G-cells of the form $(G/H)_+ \wedge S^n$ with $H \subset G$ a proper subgroup. It follows that if W is a pointed G-space whose H-fixed points are contractible for all proper $H \subset G$, then $\mathcal{T}_G(E\mathcal{P}_+, W)$ is contractible.

Lemma 4.43. *Fix an integer d. If X is a G-spectrum with the property that $i_H^* X > d$ for all proper $H \subset G$, then $E\mathcal{P}_+ \wedge X > d$.*

Proof: Suppose that $Z \leq d$. Then

$$\mathcal{S}_G(E\mathcal{P}_+ \wedge X, Z) \approx \mathcal{T}_G(E\mathcal{P}_+, \mathcal{S}_G(X, Z)).$$

By the assumption on X, the G-space $\mathcal{S}_G(X, Z)$ has contractible H fixed points for all proper $H \subset G$. The Lemma now follows from the remark preceding its statement. □

Lemma 4.44. *Write $g = |G|$. The suspension spectrum of $\tilde{E}\mathcal{P}$ is in $\mathcal{S}^G_{\geq g-1}$.*

Proof: The map $\tilde{E}\mathcal{P} \wedge S^0 \to \tilde{E}\mathcal{P} \wedge S^{\rho_G - 1}$ is a weak equivalence. The suspension spectrum of $\tilde{E}\mathcal{P}$ is in $\mathcal{S}^G_{\geq 0}$, since it is (-1)-connected (Proposition 4.10). So $\tilde{E}\mathcal{P} \wedge S^{\rho_G - 1} \geq g - 1$ by Proposition 4.25. □

Proposition 4.45. *A G-spectrum X is slice 0-positive if and only if it is (-1) connected and $\pi_0^u X = 0$ (i.e.,the non-equivariant spectrum $i_0^* X$ underlying X is 0-connected).*

Proof: The only if assertion follows from the fact that the slice cells of positive dimension are (-1)-connected and have 0-connected underlying spectra. The "if" assertion is proved by induction on $|G|$, the case of the trivial group being trivial. We may therefore assume X is (-1)-connected and has the property that $i_H^* X > 0$ for all proper $H \subset G$. Now consider the isotropy separation sequence for X

$$E\mathcal{P}_+ \wedge X \to X \to \tilde{E}\mathcal{P} \wedge X.$$

The leftmost term is slice-positive by Lemma 4.43, and the rightmost term is by Propositions 4.10 and 4.25, and Lemma 4.44. It follows that X is slice-positive. □

Example 4.46. Suppose that $f : S \to S'$ a surjective map of G-sets. Proposition 4.45 implies that the suspension spectrum of the mapping cone of f is slice positive. This implies that if HM is an Eilenberg-MacLane spectrum which is a zero slice then for every surjective $S \to S'$ the map $M(S') \to M(S)$ is a monomorphism. The proposition below shows that this is also a sufficient condition.

Proposition 4.47. i) *A spectrum X is a (-1)-slice if and only if it is of the form $X = \Sigma^{-1} HM$, with M an arbitrary Mackey functor.*

ii) *A spectrum X is a 0-slice if and only if it is of the form HM with M a Mackey functor all of whose restriction maps are monomorphisms.*

Remark 4.48. The condition on M in ii) is that if $S \to S'$ is a surjective map of finite G-sets then $M(S') \to M(S)$ is a monomorphism. This is equivalent to requiring that for all $K \subset H \subset G$ the restriction mapping $M(G/H) \to M(G/K)$ is a monomorphism. Let G act on $G \times S$ and $G \times S'$ through its left action on G. Then $G \times S \to G \times S'$ has a section, and the condition is also equivalent to requiring that the map $M(S') \to M(G \times S')$ induced by the action mapping $G \times S' \to S'$, be a monomorphism.

Proof: The first assertion is immediate from Proposition 4.19, as is the fact that a 0-slice is an Eilenberg-MacLane spectrum. For the second assertion, suppose that M is a Mackey functor, and consider the sequence

$$P_1 HM \to HM \to P^0 HM.$$

Since $P_1 HM \geq 0$ it is (-1)-connected, and so $P_1 HM$ is an Eilenberg-MacLane spectrum. For convenience, write

$$M' = \underline{\pi}_0 P_1 HM$$
$$M'' = \underline{\pi}_0 P^0 HM.$$

There is a short exact sequence

$$M' \rightarrowtail M \twoheadrightarrow M''.$$

Suppose that S is any finite G-set and consider the following diagram

$$\begin{tikzcd} M'(S) \arrow[r] \arrow[d] & M(S) \arrow[r] \arrow[d] & M''(S) \arrow[d] \\ M'(G\times S) \arrow[r] & M(G\times S) \arrow[r] & M''(G\times S) \end{tikzcd}$$

in which the rows are short exact, and the vertical maps are induced by the action mapping, as in Remark 4.48. The bottom right arrow is an isomorphism since $i_0^* HM \to i_0^* P^0 HM$ is an equivalence. Thus $M'(G\times S) = 0$ (this also follows from Proposition 4.45). The claim now follows from a simple diagram chase. □

Remark 4.49. The second assertion of Proposition 4.47 can also be deduced directly from Lemma 3.2.

Corollary 4.50. *If $X = HM$ is a zero slice and $\pi_0^u X = 0$ then X is contractible.* □

Corollary 4.51. *The (-1)-slice of S^{-1} is $\Sigma^{-1} H\underline{A}$. The zero slice of S^0 is $H\underline{\mathbb{Z}}$.*

Proof: The first assertion follows easily from Part i) of Proposition 4.47. For the second assertion note that the $S^0 \to H\underline{A}$ is a P^0-equivalence, so the zero slice of S^0 is $P^0 H\underline{A}$. Consider the fibration sequence

$$H\underline{I} \to H\underline{A} \to H\underline{\mathbb{Z}},$$

in which $\underline{I} = \ker \underline{A} \to \underline{\mathbb{Z}}$ is the augmentation ideal. The leftmost term is > 0 by Proposition 4.45, and the rightmost term is ≤ 0 by Proposition 4.47. The claim now follows from Lemma 4.15. □

Corollary 4.52. *For $K \subseteq G$, the $m|K|$-slice of $\widehat{S}(m,K)$ is*

$$H\underline{\mathbb{Z}} \wedge \widehat{S}(m,K)$$

and the $(m|K|-1)$-slice of $\Sigma^{-1}\widehat{S}(m,K)$ is

$$H\underline{A} \wedge \Sigma^{-1}\widehat{S}(m,K).$$

Proof: Using the fact that $G_+ \underset{K}{\wedge} (-)$ commutes with the formation of the slice tower (Proposition 4.20) it suffices to consider the case $K = G$. The result then follows from Corollaries 4.24 and 4.51. □

4.6.2. *Cellular slices, isotropic and pure spectra.*

Definition 4.53. A d-slice M is *cellular* if $M \approx H\underline{\mathbb{Z}} \wedge \widehat{W}$, where $\widehat{W}$ is a wedge of slice cells of dimension d. A cellular slice is *isotropic* if $\widehat{W}$ can be written as a wedge of slice cells, none of which is free (i.e., of the form $G_+ \wedge S^n$). A cellular slice is *pure* if $\widehat{W}$ can be written as a wedge of regular slice cells (those of the form $\widehat{S}(m, K)$, and not $\Sigma^{-1}\widehat{S}(m, K)$).

Definition 4.54. A G-spectrum X *has cellular slices* if $P_n^n X$ is cellular for all n, and is *isotropic* or *pure* if its slices are isotropic or pure.

Lemma 4.55. *Suppose that $f : X \to Y$ is a map of cellular d-slices and $\pi_d^u f$ is an isomorphism. Then f is a weak equivalence.*

Proof: The proof is by induction on $|G|$. If G is the trivial group, the result is obvious since X and Y are Eilenberg-MacLane spectra. Now suppose we know the result for all proper $H \subset G$, and consider the map of isotropy separation sequences

$$\begin{tikzcd}
E\mathcal{P}_+ \wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tilde{E}\mathcal{P} \wedge X \arrow[d] \\
E\mathcal{P}_+ \wedge Y \arrow[r] & Y \arrow[r] & \tilde{E}\mathcal{P} \wedge Y.
\end{tikzcd}$$

By the induction hypothesis, the left vertical map is a weak equivalence. If d is not congruent to 0 or -1 modulo $|G|$ then the rightmost terms are contractible, since every slice cell of dimension d is induced. Smashing with $S^{m\rho_G}$ for suitable m, we may therefore assume $d = 0$ or $d = -1$. Smashing with S^1 in case $d = -1$ we reduce to the case $d = 0$ and therefore assume that $X = HM_0$ and $Y = HM_1$ with M_0 and M_1 permutation Mackey functors. The result then follows from Lemma 2.46. $\square$

Proposition 4.56. *Suppose that X and Y have cellular slices. If $f : X \to Y$ has the property that $\pi_*^u f$ is an isomorphism. Then f is a weak equivalence.*

Proof: It suffices to show that for each d the induced map of slices

$$P_d^d X \to P_d^d Y \tag{4.57}$$

is a weak equivalence. Since the map of ordinary spectra underlying the slice tower is the Postnikov tower, the map satisfies the conditions of Lemma 4.55, and the result follows. $\square$

For certain slices, the condition on Y in Proposition 4.56 can be dropped.

Lemma 4.58. *Suppose that $f : X \to Y$ is a map of 0-slices and X is cellular. If $\pi_0^u f$ is an isomorphism then f is an equivalence.*

Proof: Write $X = HM$ and $Y = HM'$, and let S be a finite G-set. Consider the diagram

$$\begin{tikzcd}
M(S) \arrow[r, "\therefore\sim"] \arrow[d, "\sim"'] & M'(S) \arrow[d, "\text{mono}"] \\
M(G \times S)^G \arrow[r, "\sim"'] & M'(G \times S)^G
\end{tikzcd}$$

in which the vertical maps are the restriction to the G-invariant parts. The bottom arrow is an isomorphism by assumption. The vertical maps are monomorphisms by Proposition 4.47. The left vertical map is an isomorphism since M is a permutation Mackey functor (Lemma 2.46). The result follows. $\square$

Proposition 4.59. *Suppose that $f : X \to Y$ is a map of d-slices, X is cellular and $d \not\equiv -1 \mod p$ for any prime p dividing $|G|$. If $\pi^u_d X \to \pi^u_d Y$ is an isomorphism then f is a weak equivalence.*

Proof: Let C be the mapping cone of f. We know that $C \geq d$. We will show that

$$[\widehat{S}, C]^G = 0$$

for all slice cells $\widehat{S}$ with $\dim \widehat{S} \geq d$. This will show (Lemma 4.13) that $C < d$ and hence must be contractible since its identity map is null. The assertion is obvious when G is the trivial group. By induction on $|G|$ we may assume $\widehat{S}$ is not induced. If d is divisible by $|G|$ we may smash with $S^{-d/|G|\,\rho_G}$ and reduce to the case $d = 0$ which is Lemma 4.14. It remains to show that $\pi^G_{m\rho_G} C = 0$ when $m|G| \geq d$ and that $\pi^G_{m\rho_G - 1} C = 0$ when $m|G| - 1 \geq d$. Since

$$d \not\equiv 0, -1 \mod |G|,$$

the conditions in fact implies $m|G| - 1 > d$. So we are in the situation $m|G| - 1 > d$ and we need to show that both $\pi^G_{m\rho_G} C$ and $\pi^G_{m\rho_G - 1} C$ are zero. The exact sequence

$$\pi^G_{m\rho_G} Y \to \pi^G_{m\rho_G} C \to \pi^G_{m\rho_G - 1} X$$

gives the vanishing of $\pi^G_{m\rho_G} C$. For the remaining case consider the exact sequence

$$\pi^G_{m\rho_G - 1} Y \to \pi^G_{m\rho_G - 1} C \to \pi^G_{m\rho_G - 2} X \to \pi^G_{m\rho_G - 2} Y.$$

As above, the left group vanishes since Y is a d-slice and $S^{m\rho_G - 1} > d$. Lemma 4.60 below implies that the left vertical map in

$$\begin{array}{ccc} \pi^G_{m\rho_G - 2} X & \longrightarrow & \pi^G_{m\rho_G - 2} Y \\ \downarrow & & \downarrow \\ \pi^u_{mg - 2} X & \xrightarrow[\approx]{} & \pi^u_{mg - 2} Y \end{array}$$

is monomorphism, and therefore so is the top horizontal map. Thus $\pi^G_{m\rho_G - 1} C = 0$ by exactness. $\square$

Lemma 4.60. *Suppose $\widehat{S}$ is a slice cell of dimension d. If $m|G| - 1 > d$ then the restriction mapping*

$$\pi^G_{m\rho_G - 2} H\underline{\mathbb{Z}} \wedge \widehat{S} \to \pi^u_{mg - 2} H\underline{\mathbb{Z}} \wedge \widehat{S}$$

is a monomorphism.

Proof: When G is trivial the map is an isomorphism. By induction on $|G|$ we may therefore assume G is not the trivial group and that $\widehat{S}$ is not induced, in which case $\widehat{S} = S^{k\rho_G}$ or $\widehat{S} = S^{k\rho_G - 1}$. Note that

$$S^{m\rho_G - 2} = S^{(m-1)\rho_G - 1} \wedge S^{\rho_G - 1} \geq (m-1)|G| - 1 > (m-2)|G|$$

so that both $\pi^G_{m\rho_G-2}H\underline{\mathbb{Z}}\wedge S^{k\rho_G}$ and $\pi^G_{m\rho_G-2}H\underline{\mathbb{Z}}\wedge S^{k\rho_G-1}$ are zero unless $k=m-1$. The group $\pi^G_{m\rho_G-2}H\underline{\mathbb{Z}}\wedge S^{(m-1)\rho_G-1}$ is zero since it is isomorphic to

$$\pi^G_{m\rho_G-1}H\underline{\mathbb{Z}}\wedge S^{(m-1)\rho_G}$$

and $S^{m\rho_G-1}\geq m|G|-1>(m-1)|G|$. This leaves the group

$$\pi^G_{m\rho_G-2}H\underline{\mathbb{Z}}\wedge S^{(m-1)\rho_g}\approx \pi^G_{\rho_G-2}H\underline{\mathbb{Z}}$$

whose triviality was established in Example 3.4. □

4.6.3. *The special case of $G=C_{2^n}$.* In this section we record some results which were used in an earlier approach to the main results of this paper, but are no longer. We include them here because they provide useful tools for investigating slices of various spectra. Throughout this section the group G will be cyclic group C_{2^n}.

Suppose that X is a G-spectrum with the property that $\pi^u_d X$ is a free abelian group. In §5.3 we will define a *refinement of* $\pi^u_d X$ to be a map

$$c:\widehat{W}\to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension d, with the property that the summands in $i^*_0\widehat{W}$ represent a basis of $\pi^u_d X$.

Proposition 4.61. *If $\widehat{W}\to X$ is a refinement of $\pi^u_{2k}X$ then the canonical map*

$$H\underline{\mathbb{Z}}\wedge\widehat{W}\to P^{2k}_{2k}X$$

is an equivalence.

Proof: Since $\widehat{W}\geq 2k$ the map $\widehat{W}\to X$ lifts to $P_{2k}\widehat{W}\to P_{2k}X$. Applying the functor P^{2k} and using Corollary 4.52 leads to a map

$$H\underline{\mathbb{Z}}\wedge\widehat{W}\to P^{2k}_{2k}X$$

which, since the slice tower refines the Postnikov tower, is an equivalence of underlying non-equivariant spectra. The result now follows from Proposition 4.59. □

Proposition 4.61 gives some control over the even slices of a C_{2^n}-spectrum X. The odd slices are something of a different story, and getting at them requires some knowledge of the equivariant homotopy theory of X. Note that by Proposition 4.47 any Mackey functor can occur in an odd slice. On the other hand, only special ones can occur in even slices.

Corollary 4.62. *If $\widehat{S}$ is a slice cell of odd dimension d, then for any X,*

$$[\widehat{S},X]^G=[\widehat{S},P^d_dX]^G.$$

Proof: Since the formation of P^d_dX commutes with the functors i^*_H, induction on $|G|$ reduces us to the case when $\widehat{S}$ is not an induced cell. So we may assume $\widehat{S}=S^{m\rho_G-1}$. Smashing $\widehat{S}$ and X with $S^{-m\rho_G}$, and using Corollary 4.24 reduces to the case $m=0$, which is given by Proposition 4.19. □

The situation most of interest to us in this paper is when the odd slices are contractible. Proposition 4.63 below gives a useful criterion.

Proposition 4.63. *For a G spectrum X and an odd integer d, the following are equivalent:*

i) *The d-slice of X is contractible;*

ii) *For every slice cell $\widehat{S}$ of dimension d, $[\widehat{S}, X]^G = 0$.*

Proof: By Corollary 4.62 (which requires d to be odd), there is an isomorphism

$$[\widehat{S}, X]^G = [\widehat{S}, P^d_d X]^G.$$

By Lemma 4.13, the vanishing of this group implies that $P^d_d X < d$ and hence must be contractible, since it is also $\geq d$. □

Corollary 4.64. *Suppose that d is odd. If $X \to Y \to Z$ is a cofibration sequence, and the d-slices of X and Z are contractible, then the d-slice of Y is contractible.*

Proof: This is immediate from Proposition 4.63 and the long exact sequence of homotopy classes of maps. □

Remark 4.65. Using the slice spectral sequence one can easily show that a pure spectrum always admits a refinement of homotopy groups. Thus the results above say that a spectrum X is *pure* if and only if the even homotopy groups admit an equivariant refinement, and the "slice homotopy groups" $\pi^H_{m\rho_H - 1} X$ are all zero whenever $H \subseteq G$ is non-trivial.

4.7. **Further multiplicative properties of the slice filtration.** In this section we show that the slice filtration has the expected multiplicative properties for pure spectra. Our main result is Proposition 4.66 below. It has the consequence that if X and Y are pure spectra, and $E^{s,t}_r(\,-\,)$ is the slice spectral sequence, then there is a map of spectral sequences

$$E^{s,t}_r(X) \otimes E^{s',t'}_r(Y) \to E^{s+s',t+t'}_r(X \wedge Y)$$

representing the pairing $\underline{\pi}_* X \wedge \underline{\pi}_* Y \to \underline{\pi}_*(X \wedge Y)$. In other words, multiplication in the slice spectral sequence of pure spectra behaves in the expected manner. We leave the deduction of this property from Proposition 4.66 to the reader.

Proposition 4.66. *If $X \geq n$ is pure and $Y \geq m$ has cellular slices, then $X \wedge Y \geq n + m$.*

Proof: We need to show that $P^{n+m-1}(X \wedge Y)$ is contractible. By Lemma 4.29 the map

$$X \wedge Y \to P^{n+m-1} X \wedge P^{n+m-1} Y$$

is a P^{n+m-1}-equivalence, so we may reduce to the case in which the slice filtrations of X and Y are finite. That case in turn reduces to the situation in which

$$\begin{aligned} X &= H\underline{\mathbb{Z}} \wedge \widehat{S}(m, K) \\ Y &= H\underline{\mathbb{Z}} \wedge \widehat{S}' \end{aligned}$$

in which $\widehat{S}'$ is any slice cell. By induction on $|G|$ the assertion further reduces to the case in which neither $\widehat{S}$ nor $\widehat{S}'$ is induced. Thus we may assume

$$\begin{aligned} X &= H\underline{\mathbb{Z}} \wedge S^{k\rho_G} \\ Y &= H\underline{\mathbb{Z}} \wedge S^{\ell\rho_G} \quad \text{or} \quad H\underline{\mathbb{Z}} \wedge \Sigma^{-1} S^{\ell\rho_G}, \end{aligned}$$

in which case the result follows from Proposition 4.25. □

5. The complex cobordism spectrum

From here forward we specialize to the case $G = C_{2^n}$, and for convenience localize all spectra at the prime 2. Write

$$g = |G|,$$

and let $\gamma \in G$ be a fixed generator.

5.1. **The spectrum $MU^{(G)}$.** We now introduce our equivariant variation on the complex cobordism spectrum by defining

$$MU^{(G)} = N_{C_2}^{G} MU_{\mathbb{R}},$$

where $MU_{\mathbb{R}}$ is the C_2-equivariant *real bordism* spectrum of Landweber [30] and Fujii [16] (and further studied by Araki [2] and Hu-Kriz [26]). In §B.7 we will give a construction of $MU_{\mathbb{R}}$ as a commutative algebra in $\mathcal{S}_{C_2}$. The norm is taken along the unique inclusion $C_2 \subset G$. Since the norm is symmetric monoidal, the spectrum $MU^{(G)}$ is an equivariant commutative ring spectrum. For $H \subset G$ the unit of the restriction-norm adjunction (Proposition 2.13) gives a canonical commutative algebra map

$$MU^{(H)} \to i_H^* MU^{(G)}. \tag{5.1}$$

It will be convenient to employ the shorthand notation

$$i_1^* = i_{C_2}^*$$

for the restriction map $\mathcal{S}_G \to \mathcal{S}_{C_2}$ induced by the unique inclusion $C_2 \subset G$. Choosing the elements $\{\gamma, \dots, \gamma^{(g/2)-1}\}$ for coset representatives of $C_2 \subset G$, we may write $MU^{(G)}$ as the indexed smash product

$$MU^{(G)} = \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbb{R}}. \tag{5.2}$$

Restricting to C_2 gives a C_2-equivariant smash product decomposition

$$i_1^* MU^{(G)} = \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbb{R}}. \tag{5.3}$$

5.2. **Real bordism, real orientations and formal groups.** We begin by reviewing work of Araki [2] and Hu-Kriz [26] on real bordism.

5.2.1. *The formal group.* Consider $\mathbf{CP}^n$ and $\mathbf{CP}^\infty$ as pointed C_2-spaces under the action of complex conjugation, with $\mathbf{CP}^0$ as the base point. The fixed point spaces are $\mathbf{RP}^n$ and $\mathbf{RP}^\infty$. There are homeomorphisms

$$\mathbf{CP}^n / \mathbf{CP}^{n-1} \equiv S^{n\rho_2}, \tag{5.4}$$

and in particular an identification $\mathbf{CP}^1 \equiv S^{\rho_2}$.

Definition 5.5 (Araki [2])**.** Let E be a C_2-equivariant homotopy commutative ring spectrum. A *real orientation* of E is a class $\bar{x} \in \tilde{E}_{C_2}^{\rho_2}(\mathbf{CP}^\infty)$ whose restriction to

$$\tilde{E}_{C_2}^{\rho_2}(\mathbf{CP}^1) = \tilde{E}_{C_2}^{\rho_2}(S^{\rho_2}) \approx E_{C_2}^0(\mathrm{pt})$$

is a unit. A *real oriented spectrum* is a C_2-equivariant ring spectrum E equipped with a real orientation.

If $(E, \bar{x})$ is a real oriented spectrum and $f: E \to E'$ is an equivariant multiplicative map, then the composite

$$f_*(\bar{x}) \in (E')^{\rho_2}(\mathbf{CP}^\infty)$$

is a real orientation of E'. We will often not distinguish in notation between $\bar{x}$ and $f_*\bar{x}$.

Example 5.6. The zero section $\mathbf{CP}^\infty \to MU(1)$ is an equivariant equivalence, and defines a real orientation

$$\bar{x} \in MU_{\mathbb{R}}^{\rho_2}(\mathbf{CP}^\infty),$$

making $MU_{\mathbb{R}}$ into a real oriented spectrum.

Example 5.7. From the map

$$MU_{\mathbb{R}} \to i_1^* MU^{(G)}$$

provided by (5.1), the spectrum $i_1^* MU^{(G)}$ gets a real orientation which we'll also denote

$$\bar{x} \in (MU^{(G)})^{\rho_2}(\mathbf{CP}^\infty).$$

Example 5.8. If $(H, \bar{x}_H)$ and $(E, \bar{x}_E)$ are two real oriented spectra then $H \wedge E$ has two real orientations given by

$$\bar{x}_H = \bar{x}_H \otimes 1 \text{ and } \bar{x}_E = 1 \otimes \bar{x}_E.$$

The following result of Araki follows easily from the homeomorphisms (5.4).

Theorem 5.9 (Araki [2]). *Let E be a real oriented cohomology theory. There are isomorphisms*

$$E^\star(\mathbf{CP}^\infty) \approx E^\star[\![\bar{x}]\!]$$
$$E^\star(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \approx E^\star[\![\bar{x} \otimes 1, 1 \otimes \bar{x}]\!]$$

Because of Theorem 5.9, the map $\mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ classifying the tensor product of the two tautological line bundles defines a formal group law over $\pi_\star^G E$. Using this, much of the theory relating formal groups, complex cobordism, and complex oriented cohomology theories works for C_2-equivariant spectra, with $MU_{\mathbb{R}}$ playing the role of MU. For information beyond the discussion below, see [2, 26].

Remark 5.10. A real orientation $\bar{x}$ corresponds to a *coordinate* on the corresponding formal group. Because of this we will use the terms interchangeably, preferring "coordinate" when the discussion predominantly concerns the formal group, and "real orientation" when it is the spectrum which is under focus.

The standard formulae for formal groups give elements in the $RO(C_2)$-graded homotopy groups $\pi_\star^{C_2} E$ of real oriented E. For example, there is a map from the Lazard ring to $\pi_\star^{C_2} E$ classifying the formal group law. Using Quillen's theorem to identify the Lazard ring with the complex cobordism ring this map can be written as

$$MU_* \to \pi_\star^{C_2} E.$$

It sends MU_{2n} to $\pi_{n\rho_2}^{C_2} E$. When $E = MU_{\mathbb{R}}$ this splits the forgetful map

$$\pi_{n\rho_2}^{C_2} MU_{\mathbb{R}} \to \pi_{2n}^u MU_{\mathbb{R}} = \pi_{2n} MU, \tag{5.11}$$

which is therefore surjective. A similar discussion applies to iterated smash products of $MU_{\mathbb{R}}$ giving

Proposition 5.12. *For every $m > 0$, the above construction gives a ring homomorphism*

$$\pi_*^u \bigwedge^m MU_{\mathbb{R}} \to \bigoplus_j \pi_{j\rho_2} \bigwedge^m MU_{\mathbb{R}} \tag{5.13}$$

splitting the forgetful map

$$\bigoplus_j \pi_{j\rho_2}^{C_2} \bigwedge^m MU_{\mathbb{R}} \to \pi_*^u \bigwedge^m MU_{\mathbb{R}}. \tag{5.14}$$

In particular, (5.14) *is a split surjection.* □

It is a result of Hu-Kriz[26] that (5.14) is in fact an isomorphism. This result, and a generalization to $MU^{(G)}$ can be recovered from the slice spectral sequence.

The class

$$\bar{x}_H \in H_{C_2}^{\rho_2}(\mathbf{CP}^\infty; \underline{\mathbb{Z}}_{(2)})$$

corresponding to $1 \in H_{C_2}^0(\mathrm{pt}, \underline{\mathbb{Z}}_{(2)})$ under the isomorphism

$$H_{C_2}^{\rho_2}(\mathbf{CP}^\infty; \underline{\mathbb{Z}}_{(2)}) \approx H_{C_2}^{\rho_2}(\mathbf{CP}^2; \underline{\mathbb{Z}}_{(2)}) \approx H_{C_2}^0(\mathrm{pt}, \underline{\mathbb{Z}}_{(2)})$$

defines a real orientation of $H\underline{\mathbb{Z}}_{(2)}$. As in Example 5.8, the classes $\bar{x}$ and $\bar{x}_H$ give two orientations of $E = H\underline{\mathbb{Z}}_{(2)} \wedge MU_{\mathbb{R}}$. By Theorem 5.9 these are related by a power series

$$\begin{aligned} \bar{x}_H &= \log_F(\bar{x}) \\ &= \bar{x} + \sum_{i>0} \bar{m}_i \bar{x}^{i+1}, \end{aligned}$$

with

$$\bar{m}_i \in \pi_{i\rho_2}^{C_2} H\underline{\mathbb{Z}}_{(2)} \wedge MU_{\mathbb{R}}.$$

This power series is the *logarithm* of F. Similarly, the invariant differential on F gives classes $(n+1)\bar{m}_n \in \pi_{n\rho_2}^{C_2} MU_{\mathbb{R}}$. The coefficients of the formal sum give

$$\bar{a}_{ij} \in \pi_{(i+j-1)\rho_2}^{C_2} MU_{\mathbb{R}}.$$

If $(E, \bar{x}_E)$ is a real oriented spectrum then $E \wedge MU_{\mathbb{R}}$ has two orientations

$$\begin{aligned} \bar{x}_E &= \bar{x}_E \otimes 1 \\ \bar{x}_R &= 1 \otimes \bar{x}. \end{aligned}$$

These two orientations are related by a power series

$$\bar{x}_R = \sum \bar{b}_i x_E^{i+1} \tag{5.15}$$

defining classes

$$\bar{b}_i = \bar{b}_i^E \in \pi_{i\rho_2}^{C_2} E \wedge MU_{\mathbb{R}}.$$

The power series (5.15) is an isomorphism over $\pi_\star^{C_2} E \wedge MU_{\mathbb{R}}$

$$F_E \to F_R$$

of the formal group law for $(E, \bar{x}_E)$ with the formal group law for $(MU_{\mathbb{R}}, \bar{x})$.

Theorem 5.16 (Araki [2])**.** *The map*

$$E_\star[\bar{b}_1, \bar{b}_2, \dots] \to \pi_\star^{C_2} E \wedge MU_{\mathbb{R}}$$

is an isomorphism. □

Araki's theorem has an evident geometric counterpart. For each j choose a map

$$S^{j\rho_2} \to E \wedge MU_{\mathbb{R}}$$

representing $\bar{b}_j$. As in §2.4, let

$$S[\bar{b}_j] = \bigvee_{k\geq 0} S^{k\cdot j\rho_2}$$

be the free associative algebra on $S^{j\rho_2}$ and

$$S[\bar{b}_j] \to E \wedge MU_{\mathbb{R}}$$

the homotopy associative algebra map extending (5.34). Using the multiplication map, smash these together to form a map of spectra

$$E[\bar{b}_1, \bar{b}_2, \dots] \to E \wedge MU^{(G)}, \tag{5.17}$$

where

$$E[\bar{b}_1, \bar{b}_2, \dots] = E \wedge \operatorname{holim}_{\overrightarrow{k}} S[\bar{b}_1] \wedge S[\bar{b}_2] \wedge \cdots \wedge S[\bar{b}_k].$$

The map on $RO(C_2)$-graded homotopy groups induced by (5.17) is the isomorphism of Araki's theorem. This proves

Corollary 5.18. *If E is a real oriented spectrum then there is a weak equivalence*

$$E \wedge MU_{\mathbb{R}} \approx E[\bar{b}_1, \bar{b}_2, \dots].$$

$\square$

Remark 5.19. If E is an associative (homotopy associative) algebra then (5.17) is a map of associative (homotopy associative) algebras.

As in §2.4, write

$$S^0[\bar{b}_1, \bar{b}_2, \dots] = \operatorname{holim}_{\overrightarrow{k}} S^0[\bar{b}_1] \wedge S^0[\bar{b}_2] \wedge \cdots \wedge S^0[\bar{b}_k],$$

and

$$S^0[G\cdot \bar{b}_1, G\cdot \bar{b}_1, \dots] = N_{C_2}^G S^0[\bar{b}_1, \bar{b}_2, \dots].$$

Using Proposition 4.7 one can easily check that $S^0[G\cdot \bar{b}_1, G\cdot \bar{b}_2, \dots]$ is a wedge of isotropic regular slice cells. Finally, let

$$MU^{(G)}[G\cdot \bar{b}_1, G\cdot \bar{b}_2, \dots] = MU^{(G)} \wedge S^0[G\cdot \bar{b}_1, G\cdot \bar{b}_1, \dots]$$

Corollary 5.20. *For $H \subset G$ of index 2, there is an equivalence of associative algebras*

$$i_H^* MU^{(G)} \approx MU^{(H)}[H\cdot \bar{b}_1, H\cdot \bar{b}_2, \dots].$$

Proof: Apply $N_{C_2}^H$ to the decomposition of Corollary 5.18 with $E = MU_{\mathbb{R}}$. $\square$

If E is a real oriented spectrum with formal group law F_E, then over $\pi_\star^{C_2} E \wedge MU^{(G)}$ there is the formal group law F_E of E, the formal group law F with coordinate $\bar{x}$ coming from

$$MU_{\mathbb{R}} \to i_1^* MU^{(G)} \approx \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbb{R}},$$

and the formal group laws F^{γ^j} with coordinate $\gamma^j\bar{x}$ associated to the other inclusions $MU_{\mathbb{R}} \to i_1^* MU^{(G)}$. There is also a diagram of isomorphisms

$$\begin{tikzcd}
 & & F_E \arrow[dll, "\bar{b}"'] \arrow[dl] \arrow[dr, "\gamma^{g/2-1}\bar{b}"] & \\
F \arrow[r] & F^{\gamma} \arrow[r] & \cdots \arrow[r] & F^{\gamma^{g/2-1}}
\end{tikzcd}$$

The isomorphism $\bar{b}$ is given by

$$\bar{x} = \bar{b}(\bar{x}_E) = \bar{x}_E + \sum \bar{b}_j \bar{x}_E^{j+1},$$

and the others are gotten by applying γ^j

$$\gamma^j \bar{x} = \gamma^j \bar{b}(\bar{x}_E) = \bar{x}_E + \sum \gamma^j \bar{b}_j \bar{x}_E^{j+1}.$$

Write

$$G \cdot \bar{b}_i$$

for the sequence

$$\bar{b}_i, \gamma\bar{b}_i, \dots, \gamma^{g/2-1}\bar{b}_i.$$

Using the decomposition (5.3) and iterating Araki's theorem gives

Proposition 5.21. *There is an isomorphism*

$$\pi_\star^{C_2} E \wedge i_1^* MU^{(G)} \approx E_\star[G \cdot \bar{b}_1, G \cdot \bar{b}_2, \dots].$$

□

5.2.2. *The unoriented cobordism ring.* Passing to geometric fixed points from

$$\bar{x} : \mathbf{CP}^\infty \to \Sigma^{\rho_2} MU_{\mathbb{R}}$$

gives the canonical inclusion

$$a : \mathbf{RP}^\infty = MO(1) \to \Sigma MO,$$

defining the MO Euler class of the tautological line bundle. There are isomorphisms

$$\begin{aligned} MO^*(\mathbf{RP}^\infty) &\approx MO^*[\![a]\!] \\ MO^*(\mathbf{RP}^\infty \times \mathbf{RP}^\infty) &\approx MO^*[\![a \otimes 1, 1 \otimes a]\!] \end{aligned}$$

and the multiplication map $\mathbf{RP}^\infty \times \mathbf{RP}^\infty \to \mathbf{RP}^\infty$ gives a formal group law over MO_*. By Quillen [40], it is the universal formal group law F over a ring of characteristic 2 for which $F(a, a) = 0$.

As described by Quillen [42, Page 53], the formal group can be used to give convenient generators for the unoriented cobordism ring. Let

$$e \in H^1(\mathbf{RP}^\infty; \mathbb{Z}/2)$$

be the $H\mathbb{Z}/2$ Euler class of the tautological line bundle. Over $\pi_* H\mathbb{Z}/2 \wedge MO$ there is a power series relating e and the image of the class a

$$e = \ell(a) = a + \sum \alpha_n a^{n+1}.$$

Lemma 5.22. *The composite series*

$$(a + \sum \alpha_{2^j-1} a^{2^j})^{-1} \circ \ell(a) = a + \sum_{j>0} h_j a^{j+1} \tag{5.23}$$

has coefficients in $\pi_* MO$. The classes h_j with $j+1 = 2^k$ are zero. The remaining h_j are polynomial generators for the unoriented cobordism ring

$$\pi_* MO = \mathbb{Z}/2[h_j, j \neq 2^k - 1]. \tag{5.24}$$

Proof: The assertion that $h_j = 0$ for $j+1 = 2^k$ is straightforward. Since the sequence

$$\pi_* MO \to \pi_* H\mathbb{Z}/2 \wedge MO \rightrightarrows \pi_* H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \wedge MO \tag{5.25}$$

is a split equalizer, to show that the remaining h_j are in $\pi_* MO$ it suffices to show that they are equalized by the parallel maps in (5.25). This works out to showing that the series (5.23) is invariant under substitutions of the form

$$e \mapsto e + \sum \zeta_m e^{2^m}, \tag{5.26}$$

The series (5.23) is characterized as the unique isomorphism of the formal group law for unoriented cobordism with the additive group, having the additional property that the coefficients of a^{2^k} are zero. This condition is stable under the substitutions (5.26). The last assertion follows from Quillen's characterization of $\pi_* MO$. □

Remark 5.27. Recall the real orientation $\bar{x}$ of $i_1^* MU^{(G)}$ of Example 5.7. Applying the $RO(G)$-graded cohomology norm (§2.3.3) to $\bar{x}$, and then passing to geometric fixed points, gives a class

$$\Phi^G N(\bar{x}) \in MO^1(\mathbf{RP}^\infty).$$

One can easily check that $\Phi^G N(\bar{x})$ coincides with the MO Euler class a defined at the beginning of this section. Similarly one has

$$\Phi^G N(x_H) = e.$$

Applying $\Phi^G N$ to $\log_{\bar{F}}$ and using the fact that it is a ring homomorphism (Proposition 2.43) gives

$$e = a + \sum \Phi^G N(\bar{m}_k) a^{k+1}.$$

It follows that

$$\Phi^G N(\bar{m}_k) = \alpha_k.$$

5.3. **Refinement of homotopy groups.** We begin by focusing on a simple consequence of Proposition 5.12.

Proposition 5.28. *For every $m > 1$, every element of*

$$\pi_{2k}\left(\bigwedge^m MU\right)$$

can be refined to an equivariant map

$$S^{k\rho_2} \to \bigwedge^m MU_{\mathbb{R}}.$$

□

This result expresses an important property of the C_2-spectra given by iterated smash products of $MU_{\mathbb{R}}$. Our goal in this section is to formulate a generalization to the case $G = C_{2^n}$.

Definition 5.29. Suppose X is a G-spectrum with the property that $\pi_k^u X$ is a free abelian group. A *refinement of* $\pi_k^u X$ is a map

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension k, with the property that the summands in $i_0^* \widehat{W}$ represent a basis of $\pi_k^u X$. A *refinement of the homotopy groups of* X (or a *refinement of homotopy of* X) is a map

$$\widehat{W} = \bigvee \widehat{W}_k \to X$$

whose restriction to each $\widehat{W}_k$ is a refinement of π_k^u.

The splitting (5.13) used to prove Proposition 5.28 is multiplicative. This too has an important analogue.

Definition 5.30. Suppose that R is an equivariant associative algebra. A *multiplicative refinement of homotopy* is an associative algebra map $\widehat{W} \to R$ which, when regarded as a map of G-spectra is a refinement of homotopy.

Proposition 5.31. *For every $m \geq 1$ there exists a multiplicative refinement of homotopy*

$$\widehat{W} \to \bigwedge^m MU^{(G)},$$

with $\widehat{W}$ a wedge of regular isotropic slice cells.

Two ingredients form the proof of Proposition 5.31. The first, Lemma 5.32 below, is a description of $\pi_*^u MU^{(G)}$ as a G-module. The computation is of interest in its own right, and is used elsewhere in this paper. It is proved in §5.4. The second is the classical description of $\pi_*^u(\bigwedge^m MU^{(G)})$, $m > 1$, as a $\pi_*^u MU^{(G)}$-module.

Lemma 5.32. *There is a sequence of elements $r_i \in \pi_{2i}^u MU^{(G)}$ with the property that*

$$\pi_*^u MU^{(G)} = \mathbb{Z}_2[G \cdot r_1, G \cdot r_2, \dots],$$

in which $G \cdot r_i$ stands for the sequence

$$(r_i, \dots \gamma^{\frac{g}{2}-1} r_i)$$

of length $g/2$.

For example, Lemma 5.32 asserts

$$\begin{aligned} \pi_* MU &= \mathbb{Z}_{(2)}[r_1, r_2, \dots] \\ \pi_*^u MU^{(C_4)} &= \pi_* MU \wedge MU = \mathbb{Z}_{(2)}[r_1, \gamma r_1, r_2, \gamma r_2, \dots] \\ \pi_*^u MU^{(C_8)} &= \pi_* \bigwedge^4 MU = \mathbb{Z}_{(2)}[r_1, \dots, \gamma^3 r_1, r_2, \dots] \end{aligned}$$

Over $\pi_*^u MU^{(G)} \wedge MU^{(G)}$, there are two formal group laws, F_L and F_R coming from the canonical orientations of the left and right factors. There is also a canonical isomorphism between them, which can be written as

$$x_R = \sum b_j x_L^{j+1}.$$

As in the statement of Proposition 5.21 write

$$G \cdot b_i$$

for the sequence

$$b_i, \gamma b_i, \dots, \gamma^{g/2-1} b_i.$$

The following result is a standard computation in complex cobordism. See for example [45].

Lemma 5.33. *The ring* $\pi_*^u MU^{(G)} \wedge MU^{(G)}$ *is given by*

$$\pi_*^u MU^{(G)} \wedge MU^{(G)} = \pi_*^u MU^{(G)}[G \cdot b_1, G \cdot b_2, \dots].$$

For $m > 1$,

$$\pi_*^u \bigwedge^m MU^{(G)} = \pi_*^u MU^{(G)} \wedge \bigwedge^{m-1} MU^{(G)}$$

is the polynomial ring

$$\pi_*^u MU^{(G)}[G \cdot b_i^{(j)}],$$

with

$$\begin{aligned} i &= 1, 2, \dots, \quad \textit{and} \\ j &= 1, \dots, m-1. \end{aligned}$$

The element $b_i^{(j)}$ *is the class* b_i *arising from the* j^{th} *factor of* $MU^{(G)}$ *in* $\bigwedge^{m-1} MU^{(G)}$. □

Proof of Proposition 5.31, assuming Lemma 5.32: This is a straightforward application of the method of polynomial algebras of §2.4. To keep the notation simple we begin with the case $m = 1$. Choose a sequence $r_i \in \pi_{2i}^u MU^{(G)}$ with the property described in Lemma 5.32. Let

$$\bar{r}_i : S^{i\rho_2} \to i_1^* MU^{(G)}, \tag{5.34}$$

be a representative of the image of r_i under the splitting (5.13). Since $MU^{(G)}$ is a commutative algebra, the method of polynomial algebras can be used to construct an associative algebra map

$$S^0[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots] \to MU^{(G)}, \tag{5.35}$$

Using Proposition 4.7 one can easily check that $S^0[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots]$ is a wedge of regular isotropic G-slice cells. Using Lemma 5.32 one then easily checks that (5.35) is multiplicative refinement of homotopy. The case $m \geq 1$ is similar, using in addition Lemma 5.33 and the collection $\{r_i, b_i(j)\}$. □

Remark 5.36. The structure of $\pi_*^u MU^{(G)}$ and the fact that it admits a refinement of its homotopy groups actually determines the slice cells involved in the refinement. To describe them explicitly, notice that the monomials in the $\{\gamma^i r_j\} \subset \pi_*^u MU^{(G)}$ are permuted, up to sign, by the action of G. For instance, the G-orbit through r_1 consists of the elements

$$G \cdot r_1 = \{\pm r_1, \dots, \pm \gamma^{\frac{g}{2}-1} r_1\},$$

while the orbit through r_i^2 is half as large

$$G \cdot r_1^2 = \{r_1^2, \dots, \gamma^{\frac{g}{2}-1} r_1^2\}.$$

Let's call the list of monomials which occur up to sign in a G-orbit a *mod* 2 *orbit of monomials* and indicate it with absolute value signs. For instance the mod 2 orbit of monomials containing b_1 is the set

$$|G\cdot r_1| = \{r_1,\dots,\gamma^{\frac{g}{2}-1}r_1\}. \tag{5.37}$$

The slice cells refining the homotopy groups of $MU^{(G)}$ correspond to the mod 2 orbits in the monomials in $\{\gamma^i r_j\}$. For example the mod 2 orbit (5.37) is refined by a map from the slice cell

$$G_+ \underset{C_2}{\wedge} S^{\rho_2}.$$

The monomial $r_1\cdots\gamma^{\frac{g}{2}-1}r_1$ is itself a mod 2-orbit and corresponds to the slice cell

$$S^{\rho_G}.$$

A similar remark applies to the refinement of

$$\pi_*^u \bigwedge^m MU^{(G)}.$$

The series of examples below explicitly describe the refinement of $\pi_*^u MU^{(G)}$ for $G=C_8$, and $*\le 4$.

Example 5.38. In π_0^u we have the monomial 1, corresponding to an equivariant map $S^0\to MU^{(G)}$. So $\widehat{W}_0=S^0$.

Example 5.39. The group π_2^u consists of the single mod 2 orbit

$$\{x_1,\gamma x_1,\gamma^2x_1,\gamma^3x_1\}$$

and so the homotopy in this dimension refines with

$$\widehat{W}_2 = G_+ \underset{Z/2}{\wedge} S^{\rho_2} \leftrightarrow \{x_1,\gamma x_1,\gamma^2x_1,\gamma^3x_1\}.$$

Example 5.40. The group $\pi_4^u MU^{(G)}$ has rank 14, and decomposes into the mod 2 orbits

$$\{x_1,\gamma x_1,\gamma^2x_1,\gamma^3x_1\},\quad \{x_1\gamma(x_1),\gamma(x_1)\gamma^2(x_1),\gamma^2(x_1)\gamma^3(x_1),\gamma^3(x_1)x_1\},$$
$$\{x_2,\gamma x_2,\gamma^2x_2,\gamma^3x_2\},\quad \{x_1\,\gamma^2x_1,\gamma x_1\,\gamma^3x_1\}.$$

The homotopy in this dimension refines to an equivariant map $\widehat{W}_4\to MU^{(G)}$ with $\widehat{W}_4$ the wedge of

$$\begin{aligned}
G_+ \underset{C_2}{\wedge} S^{2\rho_2} &\leftrightarrow \{x_1,\gamma x_1,\gamma^2x_1,\gamma^3x_1\}\\
G_+ \underset{C_2}{\wedge} S^{2\rho_2} &\leftrightarrow \{x_1\gamma(x_1),\gamma(x_1)\gamma^2(x_1),\gamma^2(x_1)\gamma^3(x_1),\gamma^3(x_1)x_1\}\\
G_+ \underset{C_2}{\wedge} S^{2\rho_2} &\leftrightarrow \{x_2,\gamma x_2,\gamma^2x_2,\gamma^3x_2\}\\
G_+ \underset{C_4}{\wedge} S^{\rho_4} &\leftrightarrow \{x_1\,\gamma^2(x_1),\gamma(x_1)\,\gamma^3(x_1)\}.
\end{aligned}$$

5.4. **Algebra generators for $\pi_*^u MU^{(G)}$.** In this section we will describe convenient algebra generators for $\pi_*^u MU^{(G)}$. Our main results are Proposition 5.45 (giving a criterion for a sequence of elements r_i to "generate" $\pi_*^u MU^{(G)}$ as a G-algebra, as in Lemma 5.32) and Corollary 5.48 (specifying a particular sequence of r_i). Proposition 5.45 directly gives Lemma 5.32.

We remind the reader that the notation $H_*^u X$ refers to the homology groups $H_*(i_0^*X)$ of the non-equivariant spectrum underlying X.

5.4.1. *A criterion for a generating set.* Let $m_i \in H_{2i}MU$ be the coefficient of the universal logarithm. As in Proposition 5.21, using the identification (5.3)

$$i_1^* MU^{(G)} = \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbb{R}}$$

one has

$$H_*^u MU^{(G)} = \mathbb{Z}_{(2)}[\gamma^j m_k],$$

where

$$k = 1, 2, \dots,$$
$$j = 0, \dots, g/2 - 1.$$

By definition, the action of G on $H_*^u MU^{(G)}$ is given by

$$\gamma \cdot \gamma^j m_k = \begin{cases} \gamma^{j+1} m_k & j < g/2 - 1 \\ (-1)^k m_k & j = g/2 - 1. \end{cases} \tag{5.41}$$

Let

$$I = \ker \pi_*^u MU^{(G)} \to \mathbb{Z}_{(2)}$$
$$I_H = \ker H_*^u MU^{(G)} \to \mathbb{Z}_{(2)}$$

denote the augmentation ideals, and

$$Q_* = I/I^2$$
$$QH_* = I_H/I_H^2$$

the modules of indecomposable, with Q_{2m} and QH_{2m} indicating the homogeneous parts of degree $2m$ (the odd degree parts are zero). The module QH_* is the free abelian group with basis $\{\gamma^j m_k\}$, and from Milnor [39], one knows that the Hurewicz homomorphism gives an isomorphism

$$Q_{2k} \to QH_{2k}$$

if $2k$ is not of the form $2(2^\ell - 1)$, and an exact sequence

$$Q_{2(2^\ell-1)} \rightarrowtail QH_{2(2^\ell-1)} \twoheadrightarrow \mathbb{Z}/2 \tag{5.42}$$

in which the rightmost map is the one sending each $\gamma^j m_k$ to 1.

Formula (5.41) implies that the G-module QH_{2k} is the module induced from the sign representation of C_2 if k is odd and from the trivial representation if k is even.

Lemma 5.43. *Let $r = \sum a_j \gamma^j m_k \in QH_{2k}$. The unique G-module map*

$$\mathbb{Z}_{(2)}[G] \to QH_{2k}$$
$$1 \mapsto r$$

factors through a map

$$\mathbb{Z}_{(2)}[G]/(\gamma^{g/2} - (-1)^k) \to QH_{2k}$$

which is an isomorphism if and only if $\sum a_j \equiv 1 \mod 2$.

Proof: The factorization is clear, since $\gamma^{g/2}$ acts with eigenvalue $(-1)^k$ on QH_{2k}. Use the unique map $\mathbb{Z}_{(2)}[G] \to QH_{2k}$ sending 1 to m_k to identify QH_{2k} with $A = \mathbb{Z}_{(2)}[G]/(\gamma^{g/2} - (-1)^k)$. The main assertion is then that an element $r = \sum a_j \gamma^j \in A$ is a unit if and only if $\sum a_j \equiv 1 \mod 2$. Since A is a finitely generated free module over the Noetherian local ring $\mathbb{Z}_{(2)}$, Nakayama's lemma implies that the map $A \to A$ given by multiplication by r is an isomorphism if and only if it is after reduction modulo 2. So r is a unit if and only if it is after reduction modulo 2. But $A/(2) = \mathbb{Z}/2[\gamma]/(\gamma^{g/2} - 1)$ is a local ring with nilpotent maximal ideal $(\gamma - 1)$. The residue map

$$A/(2) \to A/(2, \gamma - 1) = \mathbb{Z}/2$$

sends $\sum a_j \gamma^j m_k$ to $\sum a_j$. The result follows. □

Lemma 5.44. *The G-module $Q_{2(2^\ell-1)}$ is isomorphic to the module induced from the sign representation of C_2. For $y \in QH_{2(2^\ell-1)}$, the unique G-map*

$$\begin{aligned} \mathbb{Z}_{(2)}[G] &\to QH_{2(2^\ell-1)} \\ 1 &\mapsto y \end{aligned}$$

factors through a map

$$A = \mathbb{Z}_{(2)}[G]/(\gamma^{g/2} + 1) \to Q_{2(2^\ell-1)}$$

which is an isomorphism if and only if $y = (1-\gamma)r$ where $r \in QH_{2(2^\ell-1)}$ satisfies the condition $\sum a_j = 1 \mod 2$ of Lemma 5.43.

Proof: Identify $QH_{2(2^\ell-1)}$ with A by the map sending 1 to $m_{2^\ell-1}$. In this case A is isomorphic to $\mathbb{Z}_{(2)}[\zeta]$, with ζ a primitive $(g/2)^{\text{th}}$ root of unity, and in particular is an integral domain. Under this identification, the rightmost map in (5.42) is the quotient of A by the principal ideal $(\zeta - 1)$. Since A is an integral domain, this ideal is a rank 1 free module generated by any element of the form $(1-\gamma)r$ with $r \in A$ a unit. The result follows. □

This discussion proves

Proposition 5.45. *Let*

$$\{r_1, r_2, \dots\} \subset \pi_*^u MU^{(G)}$$

be any sequence of elements whose images

$$s_k \in QH_{2k}$$

have the property that for $j \neq 2^\ell - 1$, $s_j = \sum a_j \gamma^j m_k$ with

$$\sum a_j \equiv 1 \mod 2,$$

and $s_{2^\ell-1} = (1-\gamma)\left(\sum a_j \gamma^j m_k\right)$, with

$$\sum a_j \equiv 1 \mod 2.$$

Then the sequence

$$\{r_1, \dots \gamma^{\frac{g}{2}-1} r_1, r_2, \dots, \gamma^{\frac{g}{2}-1} r_2, \dots\}$$

generates the ideal I, and so

$$\mathbb{Z}_{(2)}[r_1, \dots \gamma^{\frac{g}{2}-1} r_1, r_2, \dots, \gamma^{\frac{g}{2}-1} r_2, \dots] \to \pi_*^u MU^{(G)}$$

is an isomorphism. □

5.4.2. *Specific generators.* We now use the action of G on $i_1^* MU^{(G)}$ to define specific elements $\bar{r}_i \in \pi^{C_2}_{i\rho_2} MU^{(G)}$ refining a sequence satisfying the condition of Proposition 5.45.

Write

$$\bar{F}(\bar{x}, \bar{y})$$

for the formal group law over $\pi^{C_2}_{\star} MU^{(G)}$, and

$$\log_{\bar{F}}(\bar{x}) = \bar{x} + \sum_{i>0} \bar{m}_k \bar{x}^{k+1}$$

for its logarithm. This defines elements

$$\bar{m}_k \in \pi^{C_2}_{k\rho_2} H\underline{\mathbb{Z}}_{(2)} \wedge MU^{(G)}.$$

We define the elements

$$\bar{r}_k \in \pi^{C_2}_{k\rho_2} MU^{(G)}$$

to be the coefficients of the unique strict isomorphism of $\bar{F}$ with the 2-typification of $\bar{F}^{\gamma}$. The Hurewicz images

$$\bar{r}_k \in \pi^{C_2}_{k\rho_2} H\underline{\mathbb{Z}}_{(2)} \wedge MU^{(G)}$$

are given by the power series identity

$$\sum \bar{r}_k \bar{x}^{k+1} = \Big(\bar{x} + \sum \gamma(\bar{m}_{2^\ell-1}) \bar{x}^{2^\ell}\Big)^{-1} \circ \log_{\bar{F}}(\bar{x}). \tag{5.46}$$

Modulo decomposables this becomes

$$\bar{r}_k \equiv \begin{cases} \bar{m}_k - \gamma \bar{m}_k & k = 2^\ell - 1 \\ \bar{m}_k & \text{otherwise.} \end{cases} \tag{5.47}$$

This proves

Corollary 5.48. *The classes $r_k = i_0^* \bar{r}_k$ defined above satisfy the condition of Proposition 5.45.* □

These are the specific generators with which we shall work. Though it does not appear in the notation, the classes $\bar{r}_i$ depend on the group G. In §9 we will need to consider the classes $\bar{r}_i$ for a group G and for a subgroup $H \subset G$. We will then use the notation

$$\bar{r}_i^H \text{ and } \bar{r}_i^G$$

to distinguish them.

The following result establishes an important property of these specific $\bar{r}_k$.

Proposition 5.49. *For all k*

$$\Phi^G N(\bar{r}_k) = h_k \in \pi_k MO,$$

where the h_k are the classes defined in §5.2.2. In particular, the set

$$\{\Phi^G N(\bar{r_k}) \mid k \neq 2^\ell - 1\}$$

is a set of polynomial algebra generators of $\pi_* MO$, and for all ℓ

$$\Phi^G N(\bar{r}_{2^\ell-1}) = h_{2^\ell-1} = 0.$$

Proof: From Remark 5.27 we know that

$$\begin{aligned}\Phi^G N\bar{x} &= a\\ \Phi^G N\bar{x}_H &= e\\ \Phi^G N\bar{m}_n &= \alpha_n.\end{aligned}$$

Corollary 2.15 implies that

$$\Phi^G N\gamma\bar{m}_n = \Phi^G N\bar{m}_n,$$

so we also know that

$$\Phi^G N\gamma\bar{m}_n = \alpha_n.$$

Since the Hurewicz homomorphism

$$\begin{tikzcd} \pi_*\Phi^G MU^{(G)} \arrow[r] \arrow[d, "\approx"'] & \pi_*\Phi^G(H\underline{\mathbb{Z}}_{(2)} \wedge MU^{(G)}) \arrow[d, "\approx"] \\ \pi_* MO \arrow[r] & \pi_* H\mathbb{Z}/2[b] \wedge MO \end{tikzcd}$$

is a monomorphism, we can calculate $\Phi^G N\bar{r}_k$ using (5.46). Applying $\Phi^G N$ to (5.46), and using the fact that it is a ring homomorphism gives

$$\begin{aligned} a+&\sum(\Phi^G N\bar{r}_k)a^{k+1}\\ &= \left(a+\sum(\Phi^G N\gamma\bar{m}_{2^\ell-1})a^{2^\ell}\right)^{-1}\circ\left(a+\sum(\Phi^G N\bar{m}_k)a^{k+1}\right)\\ &= \left(a+\sum\alpha_{2^\ell-1}a^{2^\ell}\right)^{-1}\circ\left(a+\sum\alpha_k a^{k+1}\right). \end{aligned}$$

But this is the identity defining the classes h_k. □

In addition to

$$h_k = \Phi^G N(\bar{r}_k) \in \pi_k\Phi^G MU^{(G)} = \pi_k MO$$

there are some important classes f_k attached to these specific $\bar{r}_k$.

Definition 5.50. Set

$$f_k = a_{\bar{\rho}_G}^k N\bar{r}_k \in \pi_k^G MU^{(G)},$$

where $\bar{\rho}_G = \rho_G - 1$ is the reduced regular representation.

The relationship between these classes is displayed by the following commutative diagram.

$$\begin{tikzcd} & S^k \arrow[dl, "a_{\bar{\rho}_G}^k"'] \arrow[d, "f_k"] \arrow[dr, "h_k"] & \\ S^{k\rho_G} \arrow[r, "N\bar{r}_k"] & MU^{(G)} \arrow[r] & \tilde{E}\mathcal{P}\wedge MU^{(G)} \end{tikzcd}$$

6. The Slice Theorem and the Reduction Theorem

Using the method of polynomial algebras one can show the Slice Theorem and the Reduction Theorem to be equivalent. In §6.1 we formally state the Reduction Theorem, and assuming it, prove the Slice Theorem. In §6.2 we establish a converse, for associative algebras R which are pure and which admit a multiplicative refinement of homotopy by a polynomial algebra. Both assertions are used in the proof of the Reduction Theorem in §7.

6.1. **From the Reduction Theorem to the Slice Theorem.** We now state the Slice Theorem, using the language of §4.6.2.

Theorem 6.1 (Slice Theorem)**.** *The spectrum $MU^{(G)}$ is an isotropic pure spectrum.*

For the proof of the slice theorem, let

$$A = S^0[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots] \to MU^{(G)}$$

be the multiplicative refinement of homotopy constructed in §5.3 using the method of polynomial algebras, and the specific generators of §5.4.2. Let J be the left G-set defined by

$$J = \coprod_i G/C_2.$$

As described in §2.4, the spectrum A is the indexed wedge

$$A = \bigvee_{f \in \mathbb{N}_0^J} S^{\rho_f},$$

in which ρ_f is the unique multiple of the regular representation of the stabilizer group of f having dimension

$$\dim f = 2 \sum_{j \in J} j\, f(j).$$

As in Example 2.18, let

$$M_d \subset A$$

be the monomial ideal consisting of the indexed wedge of the S^{ρ_f} with $\dim f \geq d$. Then $M_{2d-1} = M_{2d}$, and the M_{2d} fit into a sequence

$$\cdots \hookrightarrow M_{2d+2} \hookrightarrow M_{2d} \hookrightarrow M_{2d-2} \hookrightarrow \cdots .$$

The quotient

$$M_{2d}/M_{2d+2}$$

is the indexed wedge

$$\widehat{W}_{2d} = \bigvee_{\dim f = 2d} S^{\rho_f} \tag{6.2}$$

on which A is acting through the multiplicative map $A \to S^0$ (Examples 2.18 A.42). The G-spectrum (6.2) is a wedge of regular isotropic slice cells of dimension $2d$.

Replace $MU^{(G)}$ with a cofibrant A-module, and form

$$K_{2d} = MU^{(G)} \underset{A}{\wedge} M_{2d}.$$

The K_{2d} fit into a sequence

$$K_{2d+2} \hookrightarrow K_{2d} \hookrightarrow \cdots .$$

Lemma 6.3. *The sequences*

$$K_{2d+2} \to K_{2d} \to K_{2d}/K_{2d+2}$$

$$K_{2d}/K_{2d+2} \to MU^{(G)}/K_{2d+2} \to MU^{(G)}/K_{2d}$$

are weakly equivalent to cofibration sequences. There is an equivalence

$$K_{2d}/K_{2d+2} \approx R(\infty) \wedge \widehat{W}_{2d} \tag{6.4}$$

in which

$$R(\infty) = MU^{(G)} \underset{A}{\wedge} S^0.$$

Proof: The first assertion is immediate from the fact that $M_{2d+2} \to M_{2d}$ and $M_{2d}/M_{2d+2} \to A/M_{2d+2}$ are inclusions of wedge summands, hence h-cofibrations, and from the fact that $MU^{(G)} \underset{A}{\wedge} (\,-\,)$ preserves h-cofibrations since it is a topological left adjoint. The second assertion follows from the associativity of the smash product

$$MU^{(G)} \underset{A}{\wedge} (M_{2d}/M_{2d+1}) \approx (MU^{(G)} \underset{A}{\wedge} S^0) \wedge \widehat{W}_{2d} \approx R(\infty) \wedge \widehat{W}_{2d}.$$

This completes the proof. □

The Thom map

$$MU^{(G)} \to H\underline{\mathbb{Z}}_{(2)}$$

factors uniquely through an $MU^{(G)}$-module map

$$R(\infty) \to H\underline{\mathbb{Z}}_{(2)}.$$

The following important result will be proved in §7.3.

Theorem 6.5 (The Reduction Theorem)**.** *The map*

$$R(\infty) \to H\underline{\mathbb{Z}}_{(2)}$$

is a weak equivalence.

The case $G = C_2$ of the Reduction Theorem is Proposition 4.9 of Hu-Kriz[26]. Its analogue in motivic homotopy theory appears in unpublished work of the second author and Morel.

To deduce the Slice Theorem from Theorem 6.5 we need two simple lemmas.

Lemma 6.6. *The spectrum K_{2d+2} is slice $2d$-positive.*

Proof: The class of left A-modules M for which $M \underset{A}{\wedge} M_{2d+2} > 2d$ is closed under homotopy colimits and extensions. It contains every module of the form $\Sigma^k G/H_+ \wedge A$, with $k \geq 0$. Since A is (-1)-connected this means it contains every (-1)-connected cofibrant A-module. In particular it contains the cofibrant replacement of $MU^{(G)}$. □

Lemma 6.7. *If Theorem 6.5 holds then $MU^{(G)}/K_{2d+2} \leq 2d$.*

Proof: This is easily proved by induction on d, using the fact that

$$R(\infty) \wedge \widehat{W}_{2d} \to MU^{(G)}/K_{2d+2} \to MU^{(G)}/K_{2d}.$$

is weakly equivalent to a cofibration sequence (Lemma 6.3). □

Proof of the Slice Theorem assuming the Reduction Theorem: It follows from the fibration sequence

$$K_{2d+2} \to MU^{(G)} \to MU^{(G)}/K_{2d+2},$$

Lemmas 6.6 and 6.7 above, and Lemma 4.15 that

$$P^{2d+1}MU^{(G)} \approx P^{2d}MU^{(G)} \approx MU^{(G)}/K_{2d+2}.$$

Thus the odd slices of $MU^{(G)}$ are contractible and the $2d$-slice is weakly equivalent to

$$R(\infty) \wedge \widehat{W}_{2d} \approx H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}.$$

This completes the proof. $\square$

6.2. **A converse.** The arguments of the previous section can be reversed. Suppose that R is a (-1)-connected associative algebra which we know in advance to be pure, and that $A \to R$ is a multiplicative refinement of homotopy, with

$$A = S^0[G \cdot \bar{x}_1, \dots]$$

a polynomial algebra having the property that $|\bar{x}_i| > 0$ for all i. Note that this implies that $\pi_0^u R = \mathbb{Z}$ and that $P_0^0 R = H\underline{\mathbb{Z}}$. Let $M_{d+1} \subset A$ be the monomial ideal consisting of the slice cells in A of dimension $> d$, write

$$\tilde{P}_{d+1} R = M_{d+1} \underset{A}{\wedge} R$$

and

$$\tilde{P}^d R = R/\tilde{P}_{d+1} R \approx (A/M_{d+1}) \underset{A}{\wedge} R.$$

Then the $\tilde{P}^d R$ form a tower. Since $M_{d+1} > d$ and $R \geq 0$ (Proposition 4.19), the spectrum $\tilde{P}_{d+1} R$ is slice d-positive. There is therefore a map

$$\tilde{P}^d R \to P^d R, \tag{6.8}$$

compatible with variation in d.

Proposition 6.9. *The map* (6.8) *is a weak equivalence. The tower* $\{\tilde{P}^d R\}$ *is the slice tower for* R.

By analogy with the slice tower, write $\tilde{P}^d_{d'} R$ for the homotopy fiber of the map

$$\tilde{P}^d R \to \tilde{P}^{d'-1} R,$$

when $d' \leq d$.

We start with a lemma concerning the case $d = 0$.

Lemma 6.10. *Let* $n \geq 0$. *If the map*

$$\tilde{P}^0 R \to P^0 R$$

becomes an equivalence after applying P^n, *then for every* $d \geq 0$ *the map*

$$\tilde{P}^d_d R \to P^d_d R$$

becomes an equivalence after applying P^{d+n}.

Proof: Write $\widehat{W}_d = M_d/M_{d+1}$. Then there are equivalences

$$\tilde{P}^d_d R \approx \widehat{W}_d \underset{A}{\wedge} R \approx \widehat{W}_d \wedge (S^0 \underset{A}{\wedge} R) \approx \widehat{W}_d \wedge \tilde{P}^0_0 R.$$

Since $A \to R$ is a refinement of homotopy and R is pure, the analogous map

$$\widehat{W}_d \wedge P^0_0 R \to P^d_d R$$

is also a weak equivalence. Now consider the following diagram

$$\begin{tikzcd}
\widehat{W}_d \wedge P^n(\tilde{P}^0_0 R) \arrow[r, "\sim"] \arrow[d] & \widehat{W}_d \wedge P^n(P^0_0 R) \arrow[d] \\
P^{d+n}(\widehat{W}_d \wedge \tilde{P}^0_0 R) \arrow[r] \arrow[d, "\sim"'] & P^{d+n}(\widehat{W}_d \wedge P^0_0 R) \arrow[d, "\sim"] \\
P^{d+n}(\tilde{P}^d_d R) \arrow[r] & P^{d+n}(P^d_d R)
\end{tikzcd}$$

The top map is an equivalence by assumption. The bottom vertical maps are the result of applying P^{d+n} to the weak equivalences just described. Since $\widehat{W}_d$ is a wedge of regular slice cells of dimension d, Corollary 4.24 implies that the upper vertical maps are weak equivalences. It follows that the bottom horizontal map is a weak equivalence as well. □

Proof of Proposition 6.9: We will show by induction on k that for all d, the map

$$P^{d+k}(\tilde{P}^d R) \to P^{d+k}(P^d R)$$

is a weak equivalence. By the strong convergence of the slice tower (Theorem 4.38) this will give the result. The induction starts with $k = 0$ since $\tilde{P}_{d+1} R > d$ and so $R \to \tilde{P}^d R$ is a P^d-equivalence. For the induction step, suppose we know the result for some $k > 0$, and consider

$$\begin{tikzcd}
P^{d+k} \tilde{P}^d_d R \arrow[r] \arrow[d, "\sim"'] & P^{d+k}(\tilde{P}^d R) \arrow[r] \arrow[d, "\sim"'] & P^{d+k}(\tilde{P}^{d-1} R) \arrow[d] \\
P^{d+k}(P^d_d R) \arrow[r] & P^{d+k}(P^d R) \arrow[r] & P^{d+k}(P^{d-1} R)
\end{tikzcd}$$

The bottom row is a cofibration sequence since it can be identified with

$$P^d_d R \to P^d R \to P^{d-1} R.$$

The middle vertical map is a weak equivalence by the induction hypothesis, and the left vertical map is a weak equivalence by the induction hypothesis and Lemma 6.10. It follows that the cofiber of the upper left map is weakly equivalent to $P^{d+k}(P^{d-1}R)$ and hence is $(d+k+1)$-slice null (in fact d slice null). The top row is therefore a cofibration sequence by Corollary 4.16, and so the rightmost vertical map is a weak equivalence. This completes the inductive step, and the proof. □

7. The Reduction Theorem

We will prove the Reduction Theorem by induction on $g = |G|$. The case in which G is the trivial group follows from Quillen's results. We may therefore assume that we are working with a non-trivial group G and that the Reduction Theorem is known for all proper subgroups of G. In the first subsection below we collect some consequences of this induction hypothesis. The proof of the induction step is in §7.3.

7.1. **Consequences of the induction hypothesis.** This next result holds for general G.

Lemma 7.1. *Suppose that X is pure spectrum and $\widehat{W}$ is a wedge of regular slice cells. Then $\widehat{W} \wedge X$ is pure. If X is pure and isotropic and $\widehat{W}$ is regular isotropic, then $\widehat{W} \wedge X$ is pure and isotropic.*

Proof: Using Proposition 4.20 one reduces to the case in which $\widehat{W} = S^{m\rho_G}$. In that case the claim follows from Corollary 4.24. $\square$

Proposition 7.2. *Suppose $H \subset G$ has index 2. If the Slice Theorem holds for H then the spectrum $i_H^* MU^{(G)}$ is an isotropic pure spectrum.*

Proof: This is an easy consequence of Corollary 5.20, which gives an associative algebra equivalence

$$i_H^* MU^{(G)} \approx MU^{(H)}[H \cdot \bar{b}_1, H \cdot \bar{b}_2, \dots].$$

This shows that $i_H^* MU^{(G)}$ is a wedge of smash products of even dimensional isotropic slice cells with $MU^{(H)}$, and hence (by Lemma 7.1) an isotropic pure spectrum since $MU^{(H)}$ is. $\square$

Proposition 7.3. *Suppose $H \subset G$ has index 2. If the Slice Theorem holds for H then the map*

$$i_H^* R(\infty) \to i_H^* H\mathbb{Z}_{(2)}$$

is an equivalence.

Proof: By Proposition 7.2 we know that $i_H^* MU^{(G)}$ is pure. The claim then follows from Proposition 6.9. $\square$

7.2. **Certain auxiliary spectra.** Our proof of the Reduction Theorem will require certain auxiliary spectra. For an integer $k > 0$ we define

$$R(k) = MU^{(G)}/(G \cdot \bar{r}_1, \dots, G \cdot \bar{r}_k) = MU^{(G)} \underset{A}{\wedge} A'$$

where

$$A = S^0[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots]$$
$$A' = S^0[G \cdot \bar{r}_{k+1}, G \cdot \bar{r}_{k+2}, \dots].$$

The spectrum $R(k)$ is a right A'-module, and as in the case of $MU^{(G)}$ described in §6, the filtration of A' by the "dimension" monomial ideals leads to a filtration of $R(k)$ whose associated graded spectrum is

$$R(\infty) \wedge A'.$$

Thus the reduction theorem also implies that $R(k)$ is a pure isotropic spectrum. By the results of the previous section, the induction hypothesis implies that $i_H^* R(k)$ is pure and isotropic.

We know from Proposition 4.61 that when m is even, the slice $P_m^m R(k)$ is given by

$$P_m^m R(k) \approx H\mathbb{Z}_{(2)} \wedge \widehat{W}_m$$

where $\widehat{W} \subset A'$ is the summand consisting of the wedge of slice cells of dimension m. When m is odd the above discussion implies that $T \wedge P_m^m R(k)$ is contractible for any G-CW spectrum T built entirely from induced cells. In particular, the equivariant homotopy groups of $E\mathcal{P}_+ \wedge R(k)$ may be investigated by smashing the slice tower of $R(k)$ with $E\mathcal{P}_+$, and we will do so in §7.3, where we will exploit some very elementary aspects of the situation.

7.3. **Proof of the Reduction Theorem.** As mentioned at the beginning of the section, our proof of the Reduction Theorem is by induction on $|G|$, the case of the trivial group being a result of Quillen. We may therefore assume that G is non-trivial, and that the result is known for all proper subgroups $H \subset G$. By Proposition 7.3 this implies that the map

$$R(\infty) \to H\underline{\mathbb{Z}}_{(2)}$$

becomes a weak equivalence after applying i_H^*.

For the induction step we smash the map in question with the isotropy separation sequence (2.29)

$$\begin{tikzcd} E\mathcal{P}_+ \wedge R(\infty) \arrow[r] \arrow[d, "f"'] & R(\infty) \arrow[r] \arrow[d, "g"] & \tilde{E}\mathcal{P} \wedge R(\infty) \arrow[d, "h"] \\ E\mathcal{P}_+ \wedge H\underline{\mathbb{Z}}_{(2)} \arrow[r] & H\underline{\mathbb{Z}}_{(2)} \arrow[r] & \tilde{E}\mathcal{P} \wedge H\underline{\mathbb{Z}}_{(2)}. \end{tikzcd}$$

By the induction hypothesis, the map f is an equivalence. It therefore suffices to show that the map h is, and that, as discussed in Remark 2.33, is equivalent to showing that

$$\pi_*^G h : \pi_* \Phi^G R(\infty) \to \pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)} \tag{7.4}$$

is an isomorphism.

We first show that the two groups are abstractly isomorphic.

Proposition 7.5. *The ring* $\pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)}$ *is given by*

$$\pi_* \Phi^G H\underline{\mathbb{Z}}_{(2)} = \mathbb{Z}/2[b],$$

with

$$b = u_{2\sigma} a_\sigma^{-2} \in \pi_2 \Phi^G H\underline{\mathbb{Z}}_{(2)} \subset a_\sigma^{-1} \pi_\star^G H\underline{\mathbb{Z}}_{(2)}.$$

The groups $\pi_* \Phi^G R(\infty)$ *are given by*

$$\pi_n \Phi^G R(\infty) = \begin{cases} \mathbb{Z}/2 & n \geq 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The first assertion is a restatement of Proposition 3.7. For the second we will make use of the monoidal geometric fixed point functor Φ_M^G. The main technical issue is to take care that at key points in the argument we are working with spectra X for which $\Phi^G X$ and $\Phi_M^G X$ are weakly equivalent.

Recall the definition

$$R(\infty) = MU_c^{(G)} \underset{A}{\wedge} S^0,$$

where for emphasis we've written $MU_c^{(G)}$ as a reminder that $MU^{(G)}$ has been replaced by a cofibrant A-module (see §2.4). Proposition B.95 implies that $R(\infty)$ is cofibrant, so there is an isomorphism

$$\pi_*\Phi^G R(\infty) \approx \pi_*\Phi_M^G R(\infty)$$

(Proposition B.88). For the monoidal geometric fixed point functor, Proposition B.95 gives an isomorphism

$$\Phi_M^G(R(\infty)) = \Phi_M^G(MU_c^{(G)} \underset{A}{\wedge} S^0) \approx \Phi_M^G MU_c^{(G)} \underset{\Phi_M^G A}{\wedge} S^0.$$

We next claim that there are associative algebra *isomorphisms*

$$\Phi_M^G A \approx S^0[\Phi^G N\bar{r}_1, \Phi^G N\bar{r}_2, \dots] \approx S^0[\Phi^{C_2}\bar{r}_1, \Phi^{C_2}\bar{r}_2, \dots].$$

For the first, decompose A into an indexed wedge, and use Proposition B.79. For the second use the fact that the monoidal geometric fixed point functor distributes over wedges, and for V and W representations of C_2 can be computed in terms of the isomorphisms

$$\Phi_M^G(N_{C_2}^G(S^{-W} \wedge S^V)) \approx \Phi_M^G(S^{-\operatorname{ind}_{C_2}^G W} \wedge S^{\operatorname{ind}_{C_2}^G V}) \approx \Phi_M^{C_2}(S^{-W} \wedge S^V).$$

By Proposition B.89, $\Phi_M^G MU_c^{(G)}$ is a cofibrant $\Phi_M^G A$-module, and so

$$\Phi_M^G MU_c^{(G)} \underset{\Phi_M^G A}{\wedge} S^0 \approx \Phi_M^G MU^{(G)}/(\Phi_M^G N\bar{r}_1, \Phi_M^G N\bar{r}_2, \dots).$$

Since $MU_c^{(G)}$ is a cofibrant A-module, and the polynomial algebra A has the property that $S^{-1} \wedge A$ is cofibrant, the spectrum underlying $MU_c^{(G)}$ is cofibrant (Corollary B.94). This means that

$$\Phi_M^G MU_c^{(G)}$$

and

$$\Phi^G MU_c^{(G)} \sim \Phi^G MU^{(G)} \sim MO$$

are related by a functorial zig-zag of weak equivalences (Proposition B.88). Putting all of this together, we arrive at the equivalence

$$\Phi^G R(\infty) \sim MO/(\Phi^{C_2}\bar{r}_1, \Phi^{C_2}\bar{r}_2, \dots).$$

By Proposition 5.49

$$\Phi^G \bar{r}_i = \begin{cases} h_i & i \neq 2^k - 1 \\ 0 & i = 2^k - 1. \end{cases}$$

From this is an easy matter to compute $\pi_* MO/(\Phi^G\bar{r}_1, \Phi^G\bar{r}_2, \dots)$ using the cofibration sequences described at the end of §2.4.3. The outcome is as asserted. $\square$

Before going further we record a simple consequence of the above discussion which will be used in §9.1.

Proposition 7.6. *The map*

$$\pi_*\Phi^G MU^{(G)} = \pi_* MO \to \pi_*\Phi^G H\underline{\mathbb{Z}}_{(2)}$$

is zero for $* > 0$. $\square$

A simple multiplicative property reduces the problem of showing that (7.4) is an isomorphism to showing that it is surjective in dimensions which are a power of 2.

Lemma 7.7. *If for every $k \geq 1$, the class $b^{2^{k-1}}$ is in the image of*

$$\pi_{2^k}\Phi^G MU^{(G)}/(G\cdot \bar{r}_{2^k-1}) \to \pi_{2^k}\Phi^G H\underline{\mathbb{Z}}_{(2)}, \tag{7.8}$$

then (7.4) *is surjective, hence an isomorphism.*

Proof: By writing

$$R(\infty) = MU^{(G)}/(G\cdot \bar{r}_1) \underset{MU^{(G)}}{\wedge} MU^{(G)}/(G\cdot \bar{r}_2) \underset{MU^{(G)}}{\wedge} \cdots$$

we see that if for every $k \geq 1$, $b^{2^{k-1}}$ is in the image of (7.8), then all products of the $b^{2^{k-1}}$ are in the image of

$$\pi_*\Phi^G R(\infty) \to \pi_*\Phi^G H\underline{\mathbb{Z}}_{(2)}. \tag{7.9}$$

Hence every power of b is in the image of (7.9). □

In view of Lemma 7.7, the Reduction Theorem follows from

Proposition 7.10. *For every $k \geq 1$, the class $b^{2^{k-1}}$ is in the image of*

$$\pi_{2^k}\Phi^G(MU^{(G)}/(G\cdot \bar{r}_{2^k-1})) \to \pi_{2^k}\Phi^G(H\underline{\mathbb{Z}}_{(2)}).$$

To simplify some of the notation, write

$$c_k = 2^k - 1$$

and

$$M_k = MU^{(G)}/(G\cdot \bar{r}_{c_k}).$$

Since $S^{c_k\rho_G}$ is obtained from S^{c_k} by attaching induced cells, the restriction map

$$\pi^G_{c_k\rho_G+1}\tilde{E}\mathcal{P}\wedge M_k \to \pi^G_{c_k+1}\tilde{E}\mathcal{P}\wedge M_k$$

is an isomorphism (Remark 2.34). The element of interest in this group (the one hitting $b^{2^{k-1}}$) arises from the class

$$N\bar{r}_{c_k} \in \pi^G_{c_k\rho_G} MU^{(G)}$$

and the fact that it is zero for two reasons in $\pi^G_{c_k\rho_G}\tilde{E}\mathcal{P}\wedge M_k$ (it has been coned off in the formation of M_k, and it is zero in $\pi^G_{c_k\rho_G}\tilde{E}\mathcal{P}\wedge MU^{(G)} = \pi_{c_k}MO$ by Proposition 5.49). We make this more precise and prove Proposition 7.10 by chasing the class $N\bar{r}_{c_k}$ around the sequences of equivariant homotopy groups arising from the diagram

$$\begin{tikzcd}
E\mathcal{P}_+\wedge MU^{(G)} \arrow[r] \arrow[d] & MU^{(G)} \arrow[r] \arrow[d] & \tilde{E}\mathcal{P}\wedge MU^{(G)} \arrow[d] \\
E\mathcal{P}_+\wedge M_k \arrow[r] \arrow[d] & M_k \arrow[r] \arrow[d] & \tilde{E}\mathcal{P}\wedge M_k \arrow[d] \\
E\mathcal{P}_+\wedge H\underline{\mathbb{Z}}_{(2)} \arrow[r] & H\underline{\mathbb{Z}}_{(2)} \arrow[r] & \tilde{E}\mathcal{P}\wedge H\underline{\mathbb{Z}}_{(2)}.
\end{tikzcd} \tag{7.11}$$

We start with the top row. By Proposition 5.49 the image of $N\bar{r}_{c_k}$ in

$$\pi^G_{c_k\rho_G}\tilde{E}\mathcal{P}\wedge MU^{(G)} \approx \pi^G_{c_k}\tilde{E}\mathcal{P}\wedge MU^{(G)} \approx \pi_{c_k}MO$$

is zero. There is therefore a class

$$y_k \in \pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge MU^{(G)}$$

lifting $N\bar{r}_{c_k}$. The key computation, from which everything follows is

Proposition 7.12. *The image under*

$$\pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge MU^{(G)} \to \pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge H\underline{\mathbb{Z}}_{(2)},$$

of any choice of y_k above, is non-zero.

Proof of Proposition 7.10 assuming Proposition 7.12: We continue chasing around the diagram (7.11). By construction the image of y_k in $\pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge M_k$ maps to zero in $\pi^G_{c_k\rho_G} M_k$. It therefore comes from a class

$$\tilde{y}_k \in \pi^G_{c_k\rho_G+1} \tilde{E}\mathcal{P} \wedge M_k.$$

The image of $\tilde{y}_k$ in $\pi^G_{c_k\rho_G+1} \tilde{E}\mathcal{P} \wedge H\underline{\mathbb{Z}}_{(2)}$ is non-zero since it has a non-zero image in

$$\pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge H\underline{\mathbb{Z}}_{(2)}$$

by Proposition 7.12. Now consider the commutative square below, in which the horizontal maps are the isomorphisms (Remark 2.34) given by restriction along the fixed point inclusion $S^{2^k} \subset S^{c_k\rho_G+1}$:

$$\begin{tikzcd}
\pi^G_{c_k\rho_G+1} \tilde{E}\mathcal{P} \wedge M_k \arrow[r, "\approx"] \arrow[d] & \pi^G_{2^k} \tilde{E}\mathcal{P} \wedge M_k \arrow[d] \\
\pi^G_{c_k\rho_G+1} \tilde{E}\mathcal{P} \wedge H\underline{\mathbb{Z}}_{(2)} \arrow[r, "\approx"'] & \pi^G_{2^k} \tilde{E}\mathcal{P} \wedge H\underline{\mathbb{Z}}_{(2)}.
\end{tikzcd}$$

The group on the bottom right is cyclic of order 2, generated by $b^{2^{k-1}}$. We've just shown that the image of $\tilde{y}_k$ under the left vertical map is non-zero. It follows that the right vertical map is non-zero and hence that $b^{2^{k-1}}$ is in its image. □

The remainder of this section is devoted to the proof of Proposition 7.12.

The advantage of Proposition 7.12 is that it entirely involves G-spectra which have been smashed with $E\mathcal{P}_+$, and which (as discussed in §7.2) therefore fall under the jurisdiction of the induction hypothesis. In particular, the map

$$E\mathcal{P}_+ \wedge MU^{(G)} \to E\mathcal{P}_+ \wedge H\underline{\mathbb{Z}}_{(2)} \tag{7.13}$$

can be studied by smashing the slice tower of $MU^{(G)}$ with $E\mathcal{P}_+$.

We can cut down some the size of things by making use of the spectra introduced in §7.2. Factor (7.13) as

$$E\mathcal{P}_+ \wedge MU^{(G)} \to E\mathcal{P}_+ \wedge R(c_k-1) \to E\mathcal{P}_+ \wedge H\underline{\mathbb{Z}}_{(2)},$$

and replace y_k with its image

$$y_k \in \pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge R(c_k-1).$$

Lemma 7.14. *For $0 < m < c_k g$,*

$$\pi_{c_k\rho_G} E\mathcal{P}_+ \wedge P_m^m R(c_k - 1) = 0.$$

There is an exact sequence

$$\begin{tikzcd} \pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge P_{c_k g} R(c_k - 1) \arrow[r] & \pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge R(c_k - 1) \arrow[d] \\ & \pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge H\underline{\mathbb{Z}}_{(2)} = \mathbb{Z}/2. \end{tikzcd}$$

Proof: Because of the induction hypothesis, we know that the spectrum

$$E\mathcal{P}_+ \wedge P_m^m R(c_k - 1)$$

is contractible when m is odd, and that when m is even it is equivalent to

$$E\mathcal{P}_+ \wedge H\mathbb{Z} \wedge \widehat{W}_m,$$

where $\widehat{W} \subset S^0[G \cdot \bar{r}_{c_k}, \dots]$ is the summand consisting of the wedge of slice cells of dimension m. Since $1 < m < c_k g$ all of these cells are induced. This implies that the map

$$E\mathcal{P}_+ \wedge H\mathbb{Z} \wedge \widehat{W}_m \to H\mathbb{Z} \wedge \widehat{W}_m$$

is an equivalence, since

$$E\mathcal{P}_+ \to S^0$$

is an equivalence after restricting to any proper subgroup of G. But

$$\pi^G_{c_k\rho_G} H\mathbb{Z} \wedge \widehat{W}_m = \pi^G_0 H\mathbb{Z} \wedge S^{-c_k\rho_G} \wedge \widehat{W}_m = 0$$

since

$$H\mathbb{Z} \wedge S^{-c_k\rho_G} \wedge \widehat{W}_m$$

is an $(m - c_k g)$-slice and $m - c_k g < 0$. This proves the first assertion. It implies that the map

$$\pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge P_1 R(c_k - 1)$$

is surjective. The induction hypothesis implies that $P_0^0 R(c_k - 1) = H\mathbb{Z}_{(2)}$, and so the second assertion follows from the exact sequence of the fibration

$$E\mathcal{P}_+ \wedge P_1 R(c_k - 1) \to E\mathcal{P}_+ \wedge R(c_k - 1) \to E\mathcal{P}_+ \wedge P_0^0 R(c_k - 1).$$

$\square$

The exact sequence in Lemma 7.14 converts the problem of showing that y_k has non-zero image in $\pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge H\underline{\mathbb{Z}}_{(2)}$ to showing that it is not in the image of

$$\pi^G_{c_k\rho_G} P_{c_k g} R(c_k - 1).$$

We now isolate a property of this image that is not shared by y_k. Recall that γ is a fixed generator of G.

Proposition 7.15. *The image of*

$$\pi^G_{c_k\rho_G} E\mathcal{P}_+ \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k\rho_G} R(c_k - 1) \xrightarrow{i_0^*} \pi^u_{c_k g} R(c_k - 1)$$

is contained in the image of $(1 - \gamma)$.

The class y_k does not have the property described in Proposition 7.15. Its image in $\pi^u_{c_k g} R(c_k - 1)$ is $i_0^* N \bar{r}_{c_k}$ which generates a sign representation of G occurring as a summand of $\pi^u_{c_k g} R(c_k - 1)$. Thus once Proposition 7.15 is proved the proof of the Reduction Theorem is complete.

The proof of Proposition 7.15 is in two steps. First it is shown (Corollary 7.18) that the image of

$$\pi^G_{c_k \rho_G} E\mathcal{P}_+ \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} R(c_k - 1)$$

is contained in the image of the transfer map

$$\pi^H_{c_k \rho_G} R(c_k - 1) \to \pi^G_{c_k \rho_G} R(c_k - 1)$$

from the subgroup $H \subset G$ of index 2. We then show (Lemma 7.19) that the image of the transfer map in $\pi^u_{c_k g} R(c_k - 1)$ is in the image of $(1 - \gamma)$. We now turn to these steps.

Lemma 7.16. *Let $M \geq 0$ be a G-spectrum. The image of*

$$\pi^G_0 E\mathcal{P}_+ \wedge M \to \pi^G_0 M$$

is the image of the transfer map

$$\pi^H_0 M \to \pi^G_0 M$$

where $H \subset G$ is the subgroup of index 2.

Proof: Since M is (-1)-connected (Proposition 4.10) the cell decomposition of $E\mathcal{P}_+$ implies that $\pi^G_0 C_{2+} \wedge M \to \pi^G_0 E\mathcal{P}_+ \wedge M$ is surjective. The composite

$$\pi^G_0 C_{2+} \wedge M \to \pi^G_0 E\mathcal{P}_+ \wedge M \to \pi^G_0 M$$

is the transfer. □

Corollary 7.17. *The image of*

$$\pi^G_{c_k \rho_G} E\mathcal{P}_+ \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} P_{c_k g} R(c_k - 1)$$

is contained in the image of the transfer map.

Proof: This follows from Lemma 7.16 above, after the identification

$$\pi^G_{c_k \rho_G} P_{c_k g} R(c_k - 1) \approx \pi^G_0 S^{-c_k \rho_G} \wedge P_{c_k g} R(c_k - 1)$$

and the observation that

$$S^{-c_k \rho_G} \wedge P_{c_k g} R(c_k - 1) \approx P_0(S^{-c_k \rho_G} \wedge R(c_k - 1))$$

is ≥ 0. □

Corollary 7.18. *The image of*

$$\pi^G_{c_k \rho_G} E\mathcal{P}_+ \wedge P_{c_k g} R(c_k - 1) \to \pi^G_{c_k \rho_G} R(c_k - 1)$$

is contained in the image of the transfer map.

Proof: Immediate from Corollary 7.17 and the naturality of the transfer. □

The remaining step is the special case $X = P_{c_k g} R(c_k - 1)$, $V = c_k \rho_G$ of the next result.

Lemma 7.19. *Let X be a G-spectrum, V a virtual representation of G of virtual dimension d, and $H \subset G$ the subgroup of index 2. Write $\epsilon \in \{\pm 1\}$ for the degree of*

$$\gamma : i_0^* S^V \to i_0^* S^V.$$

The image of

$$\pi_V^H X \xrightarrow{\mathrm{Tr}} \pi_V^G X \to \pi_d^u X$$

is contained in the image of

$$(1+\epsilon\gamma) : \pi_d^u X \to \pi_d^u X.$$

Proof: Consider the diagram

$$\begin{tikzcd} \pi_V^G(C_{2+} \wedge X) \arrow[r] \arrow[d] & \pi_V^G X \arrow[d] \\ \pi_d^u(C_{2+} \wedge X) \arrow[r] & \pi_d^u X, \end{tikzcd}$$

in which the map of the top row is induced by the projection $C_{2+} \to S^0$. By the Wirthmüller isomorphism, the term in the upper left is isomorphic to $\pi_V^H X$ and the map of the top row can be identified with the transfer map. The non-equivariant identification

$$C_{2+} \approx S^0 \vee S^0$$

gives an isomorphism of groups of non-equivariant stable maps

$$[C_{2+} \wedge S^V, X] \approx [S^V, X] \oplus [S^V, X],$$

and so an isomorphism of the group in the lower left hand corner with

$$\pi_d^u X \oplus \pi_d^u X$$

under which the generator $\gamma \in G$ acts as

$$(a, b) \mapsto (\epsilon\gamma b, \epsilon\gamma a).$$

The map along the bottom is $(a, b) \mapsto a + b$. Now the image of the left vertical map is contained in the set of elements invariant under γ which, in turn, is contained in the set of elements of the form

$$(a, \epsilon\gamma a).$$

□

Proof of Proposition 7.15: As described after its statement, Proposition 7.15 is a consequence of Corollary 7.18 and Lemma 7.19. □

8. The Gap Theorem

The proof of the Gap Theorem was sketched in the introduction, and various supporting details were scattered throughout the paper. We collect the story here for convenient reference.

Given the Slice Theorem, the Gap Theorem is a consequence of the following special case of Proposition 3.5

Proposition 8.1. *Suppose that $G = C_{2^n}$ is a non-trivial group, and $m \geq 0$. Then*

$$H^i_G(S^{m\rho_G}; \underline{\mathbb{Z}}_{(2)}) = 0 \qquad \text{for } 0 < i < 4.$$

□

Lemma 8.2 (The Cell Lemma)**.** *Let $G = C_{2^n}$ for some $n > 0$. If $\widehat{S}$ is an isotropic slice cell of even dimension, then the groups $\pi^G_k H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}$ are zero for $-4 < k < 0$.*

Proof: Suppose that

$$\widehat{S} = G_+ \underset{H}{\wedge} S^{m\rho_H}$$

with $H \subset G$ non-trivial. By the Wirthmüller isomorphism

$$\pi^G_k H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} \approx \pi^H_k H\underline{\mathbb{Z}}_{(2)} \wedge S^{m\rho_H},$$

so the assertion is reduced to the case $\widehat{S} = S^{m'\rho_G}$ with G non-trivial. If $m' \geq 0$ then $\pi^G_k H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = 0$ for $k < 0$. For the case $m' < 0$ write $i = -k$, $m = -m' > 0$, and

$$\pi^G_k H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = H^i_G(S^{m\rho_G}; \underline{\mathbb{Z}}_{(2)}).$$

The result then follows from Proposition 8.1. □

Theorem 8.3. *If X is pure and isotropic, then*

$$\pi^G_i X = 0 \qquad -4 < i < 0.$$

Proof: Immediate from the slice spectral sequence for X and the Cell Lemma. □

Corollary 8.4. *If Y can be written as a directed homotopy colimit of isotropic pure spectra, then*

$$\pi^G_i X = 0 \qquad -4 < i < 0.$$

□

Theorem 8.5 (The Gap Theorem)**.** *Let $G = C_{2^n}$ with $n > 0$ and $D \in \pi_{\ell\rho_G} MU^{(G)}$ be any class. Then for $-4 < i < 0$*

$$\pi^G_i D^{-1} MU^{(G)} = 0.$$

Proof: The spectrum $D^{-1}MU^{(G)}$ is the homotopy colimit

$$\underset{\overrightarrow{j}}{\operatorname{holim}}\, \Sigma^{-j\,\ell\rho_G} MU^{(G)}.$$

By the Slice Theorem, $MU^{(G)}$ is pure and isotropic. But then the spectrum

$$\Sigma^{-j\,\ell\rho_G} MU^{(G)}$$

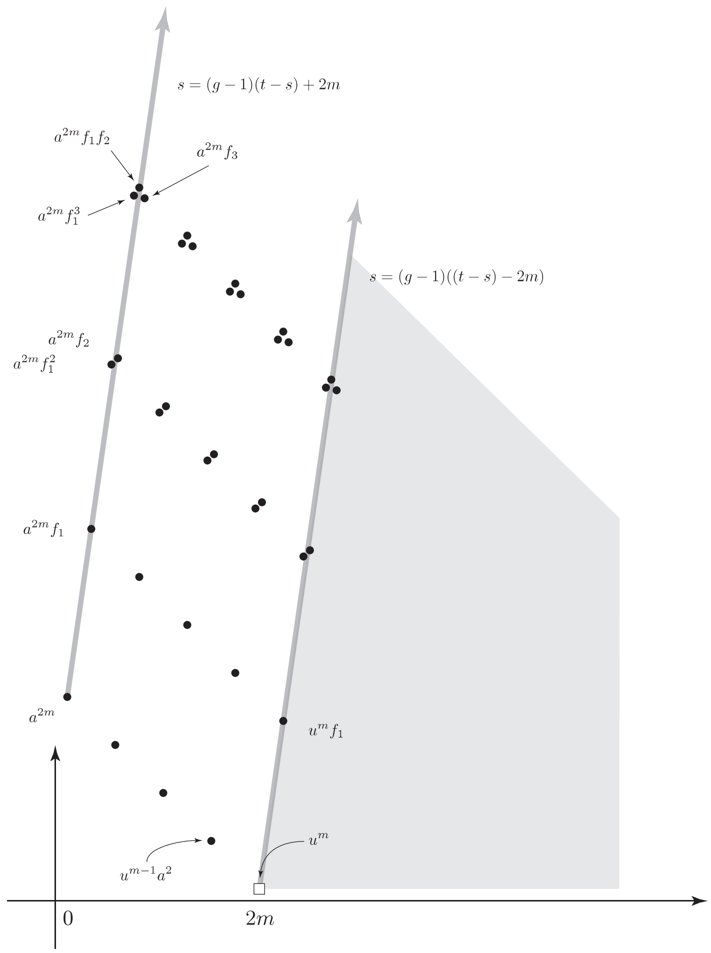

FIGURE 2. The slice spectral sequence for $\Sigma^{2m\sigma}MU^{(G)}$

is also pure and isotropic, since for any X

$$P_m^m \Sigma^{\rho_G} X \approx \Sigma^{\rho_G} P_{m-g}^{m-g} X$$

by Corollary 4.24. The result then follows from Corollary 8.4. $\square$

9. The Periodicity Theorem

In this section we will describe a general method for producing periodicity results for spectra obtained from $MU^{(G)}$ by inverting suitable elements of $\pi_\star^G MU^{(G)}$. The Periodicity Theorem (Theorem 9.15) used in the proof of Theorem 1.1 is a special case. The proof relies on a small amount of computation of $\pi_\star^G MU^{(G)}$.

9.1. **The $RO(G)$-graded slice spectral sequence for $MU^{(G)}$.** Let $\sigma = \sigma_G$ be the real sign representation of G. The key issues surrounding the periodicity results reduce to questions about the $RO(G)$-graded homotopy groups

$$\pi_{p+q\sigma}^G MU^{(G)} \quad p, q \in \mathbb{Z}.$$

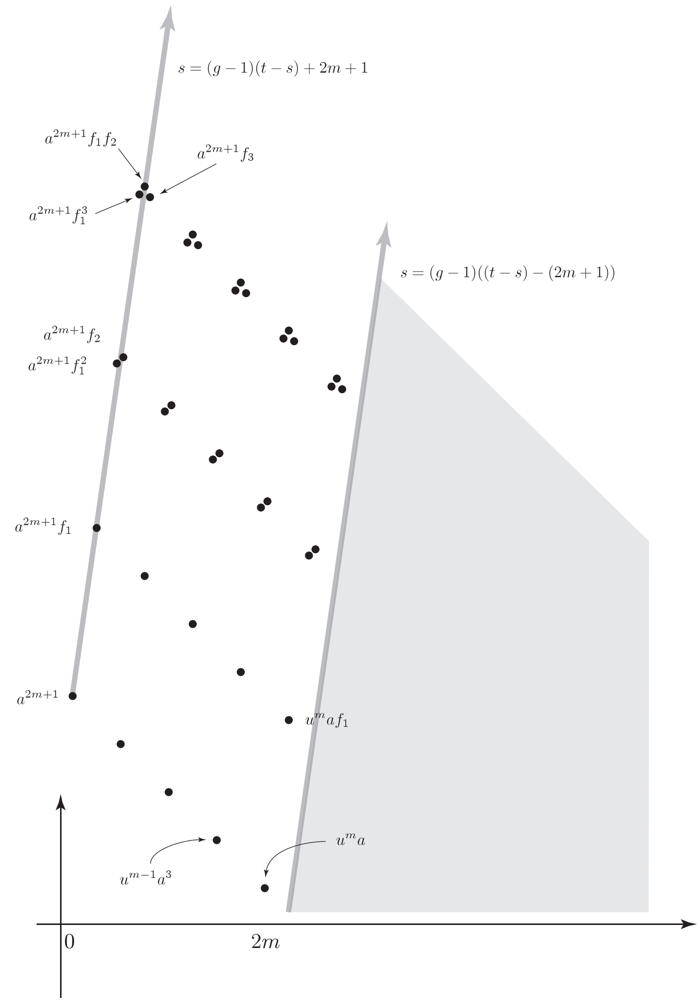

FIGURE 3. The slice spectral sequence for $\Sigma^{(2m+1)\sigma} MU^{(G)}$

We study these groups using the $RO(G)$-graded slice spectral sequence, with the conventions described in §4.5.

Certain elements play an important role. The classes

$$f_i \in \pi_i^G MU^{(G)}$$

described in Definition 5.50 are represented at the E_2-term of the slice spectral sequence by elements we will also call

$$f_i \in E_2^{i(g-1),ig} = \pi_i^G P_{ig}^{ig} MU^{(G)}$$

given by

$$S^i \xrightarrow{a_{\bar\rho}^i} S^{i\rho_G} \xrightarrow{N\bar r_i} P_{ig}^{ig} MU^{(G)}.$$

By construction, the classes f_i are permanent cycles.

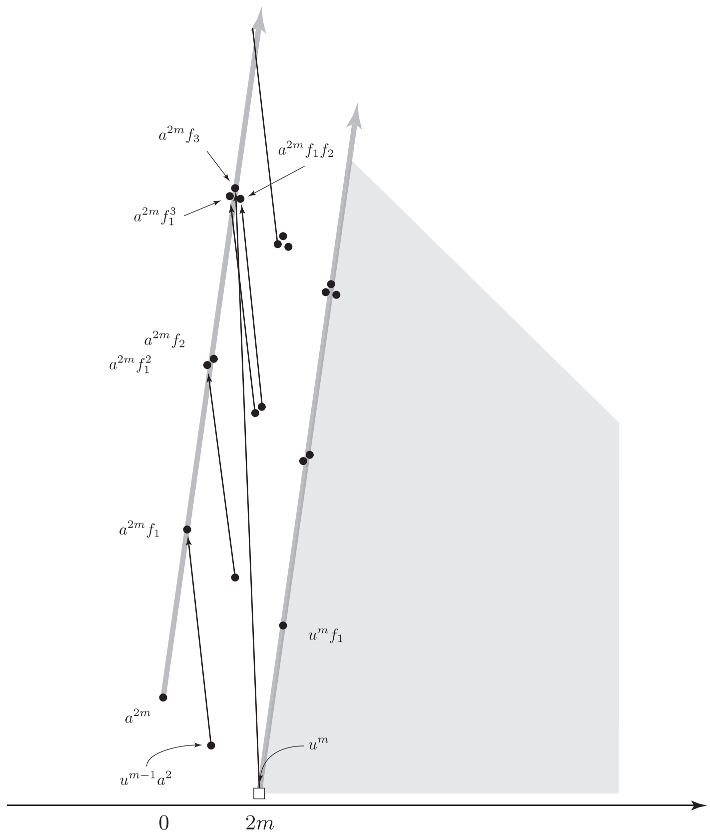

FIGURE 4. Differentials on u^m

We also need the Hurewicz image of the classes a_σ defined by (2.48)

$$S^0 \xrightarrow{a_\sigma} S^\sigma \to S^\sigma \wedge MU^{(G)}. \tag{9.1}$$

We won't distinguish in notation between the classes a_σ and the composite (9.1). The class a_σ is represented in the E_2-term of the slice spectral sequence by

$$S^0 \to S^\sigma \wedge P_0^0 MU^{(G)} = S^\sigma \wedge H\underline{\mathbb{Z}}_{(2)}.$$

We denote this representing element

$$a \in E_2^{1,1-\sigma} = \pi_{-\sigma}^G P_0^0 MU^{(G)}.$$

With our conventions it is displayed on the $(t-s,s)$-plane in position $s=1$, $t-s=0$ (see Figure 2).

Finally, there is the class

$$u = u_{2\sigma} \in E_2^{0,2-2\sigma} = \pi_{2-2\sigma}^G H\underline{\mathbb{Z}}_{(2)} = \pi_{2-2\sigma}^G P_0^0 MU^{(G)},$$

defined by (2.49).

Taking products defines a map

$$\mathbb{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \to \bigoplus_{s,t,k\geq 0} E_2^{s,t-k\sigma}. \tag{9.2}$$

Proposition 9.3. *The map* (9.2) *is an isomorphism for*

$$s \geq (g-1)((t-s)-k).$$

Proof: The proof makes use of Lemmas 9.4, 9.5, 9.6, and 9.7 below. By Lemma 9.4 the only contributions to the E_2-term in the region $s \geq (g-1)(t-k-s)$ come from the summands $H\underline{\mathbb{Z}}_{(2)} \wedge S^{m\rho_G}$ occurring in the even slices of $MU^{(G)}$. These are the summands indexed by the monomials in the r_i invariant up to sign under the action of G, or in other words, the ones which refine to monomials in the elements $N\bar{r}_i$. The result is then straightforward application of Lemmas 9.5, 9.6 and 9.7. □

We have used

Lemma 9.4. *Suppose that $\widehat{S}$ is an even dimensional slice cell of dimension $d \geq 0$. Then for $k \geq 0$,*

$$\pi_t^G S^{k\sigma} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} = 0 \text{ for } \begin{cases} t < \frac{d}{g} & \text{always} \\ t < \frac{2d}{g} + k & \text{if } \widehat{S} \text{ is induced.} \end{cases}$$

Proof: The first assertion follows from part iii) of Proposition 4.36. For the second, suppose that $\widehat{S} = G_+ \underset{H}{\wedge} \widehat{S}'$, with $\widehat{S}'$ an even dimensional slice cell of dimension d for a proper subgroup $H \subset G$. Since H is proper, the restriction of σ to H is trivial, and so

$$\pi_t^G S^{k\sigma} \wedge H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S} \approx \pi_t^H S^k \wedge H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}' \approx \pi_{t-k}^H H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{S}'.$$

Using this, the second assertion then follows from the first, applied to the group H. □

Lemma 9.5. *The map "multiplication by $a_{\bar{\rho}_G}^k$" ($\bar{\rho}_G = \rho_G - 1$) is an isomorphism*

$$\pi_t^G H\underline{\mathbb{Z}}_{(2)} \wedge S^{m\sigma+k} \to \pi_t^G H\underline{\mathbb{Z}}_{(2)} \wedge S^{m\sigma+k\rho_G}$$

for $t < m+k$ and an epimorphism for $t = m+k$.

Proof: This is immediate from the fact that $S^{m\sigma+k\rho_G}$ is constructed from $S^{m\sigma+k}$ by attaching equivariant cells $G_+ \underset{H}{\wedge} D^p$ with $p > m+k$. □

The next two results are straightforward computations, using the standard equivariant cell decomposition of $S^{d\sigma}$, and the technique described in §3.

Lemma 9.6. *Suppose that $\ell \geq 0$. Then*

$$\pi_t^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = 0$$

unless $t = 2j$ with $0 \leq j \leq \ell$. One has

$$\pi_{2\ell}^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = \mathbb{Z}_{(2)}$$

generated by $u_{2\ell\sigma} = u_{2\sigma}^{\ell}$*, and for* $0 \le j < \ell$*,*

$$\pi_{2j}^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = \mathbb{Z}/2$$

generated by $a_\sigma^{2\ell-2j} u_{2\sigma}^j$. □

Lemma 9.7. *Suppose that* $\ell \ge 0$*. Then*

$$\pi_t^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{(2\ell+1)\sigma}) = 0$$

unless $t = 2j$ *with* $0 \le j \le \ell$*. In that case*

$$\pi_{2j}^G(H\underline{\mathbb{Z}}_{(2)} \wedge S^{2\ell\sigma}) = \mathbb{Z}/2$$

generated by $a_\sigma^{2\ell-2j+1} u_{2\sigma}^j$. □

Remark 9.8. It follows from Proposition 9.3 that the groups $E_2^{s,t-k\sigma}$ are zero for $s > (g-1)(t-s)$, and that what lies on the "vanishing line"

$$s = (g-1)(t-s)$$

is the algebra

$$\mathbb{Z}_{(2)}[a, f_i]/(2a, 2f_i).$$

In Proposition 5.49 it was shown that the kernel of the map

$$\mathbb{Z}_{(2)}[a_\sigma, f_i]/(2a, 2f_i) \to \pi_\star^G MU^{(G)} \to \pi_\star^G \Phi^G MU^{(G)} = \pi_* MO[a_\sigma^{\pm 1}]$$

is the ideal $(2, f_1, f_3, f_7, \dots)$. The only possible non-trivial differentials into the vanishing line must therefore land in this ideal.

Having described the E_2-term in the range of interest, we now turn to some differentials. The case $G = C_2$ of the following result appears in unpublished work of Araki and in Hu-Kriz [26].

Theorem 9.9 (Slice Differentials Theorem)**.** *In the slice spectral sequence for* $\pi_\star^G MU^{(G)}$ *the differentials* $d_i u^{2^{k-1}}$ *are zero for* $i < r = 1 + (2^k - 1)g$*, and*

$$d_r u^{2^{k-1}} = a^{2^k} f_{2^k-1}.$$

Proof: The simplest way to check that the index r is the correct one for the asserted differential is to recall that a has s-filtration 1, and that f_i has s-filtration $i(g-1)$. The differential goes from s-filtration 0 to s-filtration

$$2^k + (2^k - 1)(g-1) = (2^k - 1)g + 1.$$

We'll establish the differential by induction on k, and refer the reader to Figure 2. Assume the result for $k' < k$. Then what's left in the range $s \ge (g-1)(t-s-k)$ after the differentials assumed by induction is the sum of two modules over $\mathbb{Z}_{(2)}[f_i]/(2f_i)$. One is generated by a^{2^k} and is free over the quotient ring

$$\mathbb{Z}/2[f_i]/(f_1, f_3, \dots, f_{2^{k-1}-1}).$$

The other is generated by $u^{2^{k-1}}$. Since the differential must take its value in the ideal $(2, a, f_1, f_3, \dots)$, the next (and only) possible differential on $u^{2^{k-1}}$ is the one asserted in the theorem. So all we need do is show that the classes $u^{2^{k-1}}$ do not survive the spectral sequence. For this it suffices to do so after inverting a. Consider the map

$$a_\sigma^{-1} \pi_\star^G MU^{(G)} \to a_\sigma^{-1} \pi_\star^G H\underline{\mathbb{Z}}_{(2)}.$$

We know the $\mathbb{Z}$-graded homotopy groups of both sides, since they can be identified with the homotopy groups of the geometric fixed point spectrum. If $u^{2^{k-1}}$ is a permanent cycle, then the class $a^{-2^k}u^{2^{k-1}}$ is as well, and represents a class with non-zero image in $\pi_*^G\Phi^G H\underline{\mathbb{Z}}_{(2)}$. This contradicts Proposition 7.6. □

Remark 9.10. After inverting a_σ, the differentials described in Theorem 9.9 describe completely the $RO(G)$-graded slice spectral sequence. The spectral sequence starts from

$$\mathbb{Z}/2[f_i, a^{\pm 1}, u].$$

The class $u^{2^{k-1}}$ hits a unit multiple of f_{2^k-1}, and so the E_∞-term is

$$\mathbb{Z}/2[f_i, i \neq 2^k-1][a^{\pm 1}] = MO_*[a^{\pm 1}]$$

which we know to be the correct answer since $\Phi^G MU^{(G)} = MO$. This also shows that the class $u^{2^{k-1}}$ is a permanent cycle modulo $(\bar{r}_{2^k-1})$. This fact corresponds to the main computation in the proof of Theorem 6.5 (which, of course we used in the above proof). The logic can be reversed, and in [26] the results are established in the reverse order (for the group $G = C_2$).

Write

$$\bar{\mathfrak{d}}_k = N\bar{r}_{2^k-1} \in \pi^G_{(2^k-1)\rho_G} MU^{(G)},$$

and note that with this notation

$$f_{2^k-1} = a_{\bar\rho}^{2^k-1}\bar{\mathfrak{d}}_k.$$

Corollary 9.11. *In the $RO(G)$-graded slice spectral sequence for $\bar{\mathfrak{d}}_k^{-1}MU^{(G)}$, the class u^{2^k} is a permanent cycle.*

Proof: Because of the vanishing regions in the slice spectral sequence for

$$\pi_\star^G \bar{\mathfrak{d}}_k^{-1} MU^{(G)},$$

the differentials described in Theorem 9.9 are the last possible differentials on the u^{2^j}, even after inverting $\bar{\mathfrak{d}}_k$. The result thus follows from Theorem 9.9 if we can show that

$$a^{2^{k+1}} f_{2^{k+1}-1} = 0$$

before the $E_{r'}$-term of the slice spectral sequence, where $r' = 1 + (2^{k+1}-1)g$. Unpacking the notation we find that

$$f_{2^{k+1}-1}\bar{\mathfrak{d}}_k = a_{\bar\rho}^{2^{k+1}-1}\bar{\mathfrak{d}}_{k+1}\bar{\mathfrak{d}}_k = f_{2^k-1}a_{\bar\rho}^{2^k}\bar{\mathfrak{d}}_{k+1}$$

and so

$$d_r u^{2^{k-1}} \cdot \epsilon = a^{2^{k+1}} f_{2^{k+1}-1},$$

where

$$r = 1 + (2^k-1)g < r'$$

and

$$\epsilon = a^{2^k} a_{\bar\rho}^{2^k}\bar{\mathfrak{d}}_{k+1}\bar{\mathfrak{d}}_k^{-1}.$$

□

9.2. **Periodicity theorems.** We now turn to our main periodicity theorem. As will be apparent to the reader, the technique can be used to get a much more general result. We have chosen to focus on a case which contains the result needed for the proof of Theorem 1.1, and yet can be stated for general $G = C_{2^n}$. In order to formulate it we need to consider all of the spectra $MU^{(H)}$ for $H \subseteq G$, and we'll need to distinguish some of the important elements of the homotopy groups we've specified. We use (5.1) to map

$$\underline{\pi}_\star MU^{(H)} \to \underline{\pi}_\star MU^{(G)},$$

and will make all of our computations in $\underline{\pi}_\star MU^{(G)}$. Let

$$\bar{r}_i^H \in \pi^{C_2}_{i\rho_2} MU^{(H)} \subset \pi^{C_2}_{i\rho_2} MU^{(G)}$$

be the element defined in §5.4.2,

$$\bar{\mathfrak{d}}_k^H = N_{C_2}^H(\bar{r}^H_{2^k-1}) \in \pi^H_{(2^k-1)\rho_H} MU^{(G)},$$

and let Δ_k^H be the element of the E_2-term of the slice spectral sequence for $MU^{(G)}$ given by

$$\Delta_k^H = u_{2(2^k-1)\rho_H}(\bar{\mathfrak{d}}_k^G)^2.$$

Finally, in addition to $g = |G|$ we will write $h = |H|$ for $H \subseteq G$.

Theorem 9.12. *Let $D \in \pi^G_{\ell\rho_G} MU^{(G)}$ be any class whose image in $\pi^H_\star MU^{(G)}$ is divisible by $\bar{\mathfrak{d}}^H_{g/h}$, for all $0 \neq H \subseteq G$. In the slice spectral sequence for $\pi^G_\star D^{-1}MU^{(G)}$ the element $u_{2\rho_G}^{2^{g/2}}$ is a permanent cycle.*

Proof: For simplicity write

$$\sigma_h = \sigma_H$$
$$\rho_h = \rho_H$$

for the sign and regular representations of a subgroup H. By Corollary 9.11, for each nontrivial subgroup $H \subseteq G$, the class $u_{2\sigma_h}^{2^{g/h}}$ is a permanent cycle, and therefore so is the class $u_{2\sigma_h}^{2^{g/2}}$. Starting with

$$u_{2\sigma_2}^{2^{g/2}} = u_{2\rho_2}^{2^{g/2}},$$

norm up to C_4, multiply by $u_{4\sigma_4}^{2^{g/2}} = (u_{2\sigma_4}^{2^{g/2}})^2$ and use Lemma 2.50 to conclude that

$$u_{4\sigma_4}^{2^{g/2}} N u_{2\rho_2}^{2^{g/2}} = u_{2\rho_4}^{2^{g/2}}$$

is a permanent cycle. Norming and continuing we find that

$$u_{2\rho_G}^{2^{g/2}}$$

is a permanent cycle, as claimed. □

Corollary 9.13. *In the situation of Theorem 9.12 the class*

$$(\Delta_1^G)^{2^{g/2}} = u_{2\rho_G}^{2^{g/2}} (\bar{\mathfrak{d}}_1^G)^{2\cdot 2^{g/2}} \tag{9.14}$$

is a permanent cycle. Any class in $\pi^G_{2\cdot g\cdot 2^{g/2}} D^{-1}MU^{(G)}$ represented by (9.14) restricts to a unit in $\pi^u_* D^{-1}MU^{(G)}$.

Proof: The fact that (9.14) is a permanent cycle is immediate from Theorem 9.12. Since the slice tower refines the Postnikov tower, the restriction of an element in the $RO(G)$-graded group $\pi^G_\star D^{-1}MU^{(G)}$ to $\pi^u_* D^{-1}MU^{(G)}$ is determined entirely by any representative at the E_2-term of the slice spectral sequence. Since $u_{2\rho_G}$ restricts to 1, the restriction of any representative of (9.14) is equal to the restriction of $(\bar{\mathfrak{d}}^G_1)^{2\cdot 2^{g/2}}$, which is a unit since $\bar{\mathfrak{d}}^G_1$ divides D. □

This gives

Theorem 9.15. *With the notation of Theorem 9.12, if M is any equivariant $D^{-1}MU^{(G)}$-module, then multiplication by $(\Delta^G_1)^{2^{g/2}}$ is a weak equivalence*

$$\Sigma^{2\cdot g\cdot 2^{g/2}} i_0^* M \to i_0^* M$$

and hence an isomorphism

$$(\Delta^G_1)^{2^{g/2}} : \pi_* M^{hG} \to \pi_{*+2\cdot g\cdot 2^{g/2}} M^{hG}.$$

□

For example, in the case of $G = C_2$ the groups $\pi_*(MU^{(G)})^{hG}$ are periodic with period $2 * 2 * 2 = 8$ and for $G = C_4$ there is a periodicity of $2 * 4 * 2^2 = 32$. For $G = C_8$ we have a period of $2 * 8 * 2^4 = 256$.

Corollary 9.16 (The Periodicity Theorem)**.** *Let $G = C_8$, and*

$$D = (N^{C_8}_{C_2}\bar{\mathfrak{d}}^{C_2}_4)\,(N^{C_8}_{C_4}\bar{\mathfrak{d}}^{C_4}_2)\,(\bar{\mathfrak{d}}^{C_8}_1) \in \pi^G_{19\rho_G} MU^{(G)}.$$

Then multiplication by $(\Delta^G_1)^{16}$ gives an isomorphism

$$\pi_*(D^{-1}MU^{(G)})^{hG} \to \pi_{*+256}(D^{-1}MU^{(G)})^{hG}.$$

□

10. The Homotopy Fixed Point Theorem

Until now we haven't had occasion to refer to the function G-spectrum of maps from a pointed G-space S to a G-spectrum X, which exists as part of the completeness of $\mathcal{S}_G$ as a topological G-category. We will write X^S for this object, so that

$$\mathcal{S}^G(Z, X^S) = \mathcal{S}^G(Z \wedge S, X).$$

Definition 10.1. A G-spectrum X is *cofree* if the map

$$X \to X^{EG_+} \tag{10.2}$$

adjoint to the projection map $EG_+ \wedge X \to X$ is a weak equivalence.

If X is cofree then the map

$$\pi^G_* X \to \pi^G_* X^{EG_+} = \pi_* X^{hG}$$

is an isomorphism. The main result of this section (Theorem 10.8) asserts that any module over $D^{-1}MU^{(G)}$ is cofree.

The map (10.2) is an equivalence of underlying spectra, and hence becomes an equivalence after smashing with any G CW-spectrum built entirely out of free G-cells. In particular, the map

$$EG_+ \wedge X \xrightarrow{\sim} EG_+ \wedge (X^{EG_+}) \tag{10.3}$$

is an equivariant equivalence. One exploits this, as in [8], by making use of the pointed G-space $\tilde{E}G$ defined by the cofibration sequence

$$EG_+ \to S^0 \to \tilde{E}G. \tag{10.4}$$

Lemma 10.5. *For a G-spectrum X, the following are equivalent:*

i) *For all non-trivial $H \subseteq G$, the spectrum $\Phi^H X$ is contractible.*

ii) *The map $EG_+ \wedge X \to X$ is a weak equivalence.*

iii) *The G-spectrum $\tilde{E}G \wedge X$ is contractible.*

Proof: The equivalence of the second and third conditions is immediate from the cofibration sequence defining $\tilde{E}G$. Since EG_+ is built from free G-cells, condition ii) implies condition i). For $H \subseteq G$ non-trivial, we have

$$\Phi^H(\tilde{E}G \wedge X) \approx \Phi^H(\tilde{E}G) \wedge \Phi^H(X) \approx S^0 \wedge \Phi^H(X).$$

Since the non-equivariant spectrum underlying $\tilde{E}G$ is contractible, condition i) therefore implies that $\Phi^H \tilde{E}G \wedge X$ is contractible for *all* $H \subset G$. But this means that $\tilde{E}G \wedge X$ is contractible (Proposition 2.36). □

Corollary 10.6. *If R is an equivariant ring spectrum satisfying the equivalent conditions of Lemma 10.5 then any module over R is cofree.*

The condition of Corollary 10.6 requires R to be an equivariant ring spectrum in the weakest sense, that R possesses a unital multiplication (not necessarily associative) in $\operatorname{ho} \mathcal{S}^G$. Similarly, the "module" condition is also one taking place in the homotopy category.

Proof: Let M be an R-module. Consider the diagram

$$\begin{tikzcd} EG_+ \wedge M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \tilde{E}G \wedge M \arrow[d] \\ EG_+ \wedge M^{EG_+} \arrow[r] & M^{EG_+} \arrow[r] & \tilde{E}G \wedge M^{EG_+} \end{tikzcd} \tag{10.7}$$

obtained by smashing $M \to M^{EG_+}$ with the sequence (10.4). The fact that R satisfies the condition i) of Lemma 10.5 implies that any R-module M' does since $\Phi^H(M)$ is a retract of $\Phi^H(R \wedge M) \approx \Phi^H(R) \wedge \Phi^H(M)$. Thus both M and M^{EG_+} satisfy the conditions of Lemma 10.5, and the terms on the right in (10.7) are contractible. The left vertical arrow is the weak equivalence (10.3). It follows that the middle vertical arrow is a weak equivalence. □

Turning to our main purpose, we now consider a situation similar to the one in §9.2, and fix a class

$$D \in \pi^G_{\ell\rho_G} MU^{(G)}$$

with the property that for all non-trivial $H \subseteq G$ the restriction of D to $\pi^H_* MU^{(G)}$ is divisible by $\bar{\mathfrak{d}}^H_k$ for some k which may depend on H.

Theorem 10.8 (Homotopy Fixed Point Theorem)**.** *Any module M over $D^{-1}MU^{(G)}$ is cofree, and so*

$$\pi^G_* M \to \pi_* M^{hG}$$

is an isomorphism.

Proof: We will show that $D^{-1}MU^{(G)}$ satisfies condition i) of Lemma 10.5. The result will then follow from Corollary 10.6. Suppose that $H \subseteq G$ is non-trivial. Then

$$\Phi^H(D^{-1}MU^{(G)}) \approx \Phi^H(D)^{-1}\Phi^H(MU^{(G)}).$$

But D is divisible by $\bar{\mathfrak{d}}_k^H$, and so $\Phi^H(D)$ is divisible by

$$\Phi^H(\bar{\mathfrak{d}}_k^H) = \Phi^H(N_{C_2}^H(\bar{r}_{2^k-1}^H))y = \Phi^{C_2}(\bar{r}_{2^k-1}^H)$$

which is zero by Proposition 5.49. This completes the proof. □

Corollary 10.9. *In the situation of Corollary 9.16, the map "multiplication by Δ_1^G" gives an isomorphism*

$$\pi_*^G(D^{-1}MU^{(G)}) \to \pi_{*+256}^G(D^{-1}MU^{(G)}).$$

Proof: In the diagram

$$\begin{tikzcd}
\pi_*^G(D^{-1}MU^{(G)}) \arrow[r] \arrow[d] & \pi_{*+256}^G(D^{-1}MU^{(G)}) \arrow[d] \\
\pi_*(D^{-1}MU^{(G)})^{hG} \arrow[r] & \pi_{*+256}^G(D^{-1}MU^{(G)})^{hG}
\end{tikzcd}$$

the vertical maps are isomorphisms by Theorem 10.8, and the bottom horizontal map is an isomorphism by Corollary 9.16. □

11. The Detection Theorem

11.1. θ_j **in the Adams-Novikov spectral sequence.** Browder's theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2),$$

where $\mathcal{A}$ denotes the mod 2 Steenrod algebra. This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \mathrm{Ext}_{BP_*(BP)}^{2,6i-2j}(BP_*, BP_*)$$

for certain values of of i and j. When $j = 1$, it is customary to omit it from the notation. The definition of these elements can be found in [44, Chapter 5].

Here are the first few of these in the relevant bidegrees.

bidegree of θ_2 :	$\beta_{2/2}$
bidegree of θ_3 :	$\beta_{4/4}$ and β_3
bidegree of θ_4 :	$\beta_{8/8}$ and $\beta_{6/2}$
bidegree of θ_5 :	$\beta_{16/16}$, $\beta_{12/4}$ and β_{11}

and so on. In the bidegree of θ_j, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1\alpha_{2^j-1}$. According to [52], [44, Corollary 5.4.5], a basis for

$$\mathrm{Ext}_{BP_*(BP)}^{2,2^{j+1}}(BP_*, BP_*)$$

for $j > 0$ is given by

$$\{\alpha_1\alpha_{2^j-1}\} \cup \left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \le k < j/2\right\}, \tag{11.1}$$

where $c(j,k) = 2^{j-1-2k}(1+2^{2k+1})/3$. For $j = 1$, the second set is empty. For $j > 1$, $\alpha_1\alpha_{2^j-1}$ supports a nontrivial d_3, but we do not need this fact. None of these elements is divisible by 2. The element $\beta_{c(j,k)/2^{j-1-2k}}$ is represented by the chromatic fraction

$$\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}};$$

no correction terms are needed.

We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for Ω.

Theorem 11.2 (The Detection Theorem)**.** *Let*

$$u \in \operatorname{Ext}^{2,2^{j+1}}_{BP_*(BP)}(BP_*, BP_*)$$

be any element whose image in $\operatorname{Ext}^{2,2^{j+1}}_{\mathcal{A}}(\mathbb{Z}/2,\mathbb{Z}/2)$ *is* h_j^2 *with* $j \ge 6$*. Then the image of* u *in* $H^2(C_8; \pi_*\Omega_{\mathbb{O}})$ *is nonzero.*

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, *the theory of formal A-modules*, where A is the ring of integers in a suitable field.

11.2. **Formal A-modules.** Recall the a formal group law over a ring R is a power series

$$F(x,y) = x + y + \sum_{i,j>0} a_{i,j}x^iy^j \in R[[x,y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m, and we get a homomorphism τ from $\mathbb{Z}$ to $End(F)$, the endomorphism ring of F.

If the ground ring R is an algebra over the p-local integers $\mathbb{Z}_{(p)}$ or the p-adic integers $\mathbb{Z}_p$, then we can make sense of $[m](x)$ for m in $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$.

Now suppose R is an algebra over a larger ring A, such as the ring of integers in a number field or a finite extension of the p-adic numbers. We say that the formal group law F is a *formal A-module* if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$

The theory of formal A-modules is well developed. Lubin-Tate [33] used them to do local class field theory, and a good reference for the theory is Hazewinkel's book [21, Chapter 21].

The example of interest to us is $A = \mathbb{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. The maximal ideal of A is generated by $\pi = \zeta_8 - 1$, and π^4 is a unit multiple of 2. There is a formal A-module F over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_F(F(x, y)) = \log_F(x) + \log_F(y)$$

(see [21, 24.5.2 and 25.3.16]) where

$$\log_F(x) = x + \sum_{k>0} \frac{w^{2^k-1}x^{2^k}}{\pi^k}. \tag{11.3}$$

11.3. $\pi_* MU^{(4)}$ **and** R_***.** What does all this have to do with our spectrum $\Omega_{\mathbb{O}} = D^{-1}MU^{(G)}$ where $G = C_8$? Recall that

$$\bar{\mathfrak{d}}_k^H \quad = \quad N_2^h \overline{r}_{2^k-1}^H \in \pi^H_{(2^k-1)\rho_H} MU^{(H)} \qquad \text{for } h = |H|$$

where H is a nontrivial subgroup of G. This can be mapped into $\pi^H_{(2^k-1)\rho_H} MU^{(G)}$ using (5.1). Equivalently, we can map $\overline{r}_{2^k-1}^H \in \pi_{(2^k-1)\rho_{C_2}} MU^{(H)}$ itself into $\pi_{(2^k-1)\rho_{C_2}} MU^{(G)}$ in the same way and apply N_2^h there. Then we have

$$\begin{aligned} D &= N_2^8 \left(\bar{\mathfrak{d}}_4^{C_{2n}}\right) N_4^8 \left(\bar{\mathfrak{d}}_2^{C_4}\right) \overline{\Delta}_1^{C_8} \\ &= N_2^8 \left(\overline{r}_{15}^{C_2}\right) N_4^8 \left(N_2^4 \overline{r}_3^{C_4}\right) N_2^8 \left(\overline{r}_1^{C_8}\right) \\ &= N_2^8 \left(\overline{r}_{15}^{C_2} \overline{r}_3^{C_4} \overline{r}_1^{C_8}\right) \\ &\in \pi^G_{19\rho_G} MU^{(G)}. \end{aligned}$$

We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\bar{\mathfrak{d}}$. They are the smallest ones that satisfy the second part of the following.

Lemma 11.4. *The classifying homomorphism $\lambda : \pi_* MU \to R_*$ for F factors through $\pi_* MU^{(4)}$ in such a way that*

(i) *the homomorphism $\lambda^{(4)} : \pi_* MU^{(4)} \to R_*$ is equivariant, where C_8 acts on $\pi_* MU^{(4)}$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8.*

(ii) *The element $i_0^* D \in \pi_* MU^{(4)}$ that we invert to get $i_0^* \Omega_{\mathbb{O}}$ goes to a unit in R_*.*

We will prove this later.

11.4. **The proof of the Detection Theorem.** It follows from Lemma 11.4 that we have a map

$$H^*(C_8; \pi_*(i_0^* D)^{-1} MU^{(4)}) \to H^*(C_8; R_*).$$

The source here is the E_2-term of the homotopy fixed point spectral sequence for M, and the target is easy to calculate. We will use it to prove the Detection Theorem above by showing that the image of x in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero.

We will calculate with BP-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, M)$ (for a $BP_*(BP)$-comodule M) by $\mathrm{Ext}^{s,t}(M)$.

The Hopf algebroid associated with $H^*(C_8; R_*)$ has the form $(R_*, R_*(C_8))$, where $R_*(C_8)$ denotes the ring of R_*-valued functions on C_8. Its left unit sends R_* to the set of constant functions, and the right unit is determined by the group action on R_* via the formula

$$\eta_R(r)(\gamma) = \gamma(r) \qquad \text{for } r \in R_* \text{ and } \gamma \in C_8.$$

This map is A-linear and C_8 has a generator γ for which $\eta_R(w)(\gamma^k) = \zeta_8^k w$.

We identify the coproduct

$$\Delta : R_*(C_8) \to R_*(C_8) \otimes_{R_*} R_*(C_8)$$

by composing it with the isomorphism

$$R_*(C_8) \otimes_{R_*} R_*(C_8) \to R_*(C_8 \times C_8)$$

given by

$$(f_1 \otimes f_2)(\gamma_1, \gamma_2) = f_1(\gamma_1)\gamma_1 f_2(\gamma_2),$$

where the factor $\gamma_1 f_2(\gamma_2)$ refers to the action of C_8 on R_*. The resulting composite

$$\delta : R_*(C_8) \to R_*(C_8 \times C_8)$$

is defined by $(\delta f)(\gamma_1, \gamma_2) = f(\gamma_1\gamma_2)$.

There is a map to this Hopf algebroid from $BP_*(BP)$ in which t_n maps to an R_*-valued function on C_8 (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{i\geq 0}{}^{F} t_i(\zeta)(\zeta x)^{2^i}$$

for each root ζ, where t_0 is the constant function with value 1. (For an analogous description of the t_i as functions on the full Morava stabilizer group, see the last two paragraphs of the proof of [44, Theorem 6.2.3].) Taking the log of each side we get

$$\begin{aligned} \zeta \sum_{n\geq 0} \frac{w^{2^n-1}x^{2^n}}{\pi^n} &= \sum_{i,j\geq 0} \frac{w^{2^i-1}}{\pi^i} t_i(\zeta)^{2^i}(\zeta x)^{2^{i+j}} \\ \zeta\frac{w^{2^n-1}}{\pi^n} &= \zeta^{2^n} \sum_{0\leq i\leq n} \frac{w^{2^i-1}}{\pi^i} t_{n-i}(\zeta)^{2^i} \end{aligned}$$

For $n = 0$ this gives $t_0(\zeta) = 1$ as expected. For $n = 1$ we get

$$\begin{aligned} \frac{\zeta w}{\pi} &= \zeta^2\left(t_1(\zeta) + \frac{w}{\pi}\right) \\ t_1(\zeta) &= \frac{w(1-\zeta)}{\pi\zeta} \in R_* \end{aligned} \tag{11.5}$$

This a unit when ζ is a primitive eighth root of unity.

Lemma 11.6. *Let*

$$b_{1,j-1} = \frac{1}{2}\sum_{0<i<2^j}\binom{2^j}{i}\left[t_1^i | t_1^{2^j-i}\right] \in \mathrm{Ext}^{2,2^{j+1}}(BP_*)$$

Its image in $H^{2,2^{j+1}}(C_8; R_*)$ is nontrivial for $j \geq 2$.

This element is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2; see [44, Theorem 5.4.6(a)].

Proof. Let $\gamma \in C_8$ be the generator with $\gamma(w) = \zeta_8 w$. Then $H^*(C_8; R_*)$ is the cohomology of the cochain complex of $R_*[C_8]$-modules

$$R_* \xrightarrow{\gamma-1} R_* \xrightarrow{\text{Trace}} R_* \xrightarrow{\gamma-1} \cdots \tag{11.7}$$

where Trace is multiplication by $1+\gamma+\cdots+\gamma^7$.

The cohomology groups $H^s(C_8; R_*)$ for $s > 0$ are periodic in s with period 2. We have

$$\begin{aligned}
H^0(C_8; R_{2m}) &= \ker(\zeta_8^m - 1) \\
&= \begin{cases} A & \text{for } m \equiv 0 \bmod 8 \\ 0 & \text{otherwise} \end{cases} \\
H^1(C_8; R_{2m}) &= \ker(1+\zeta_8^m+\cdots+\zeta_8^{7m})/\operatorname{im}(\zeta_8^m - 1) \\
&= \begin{cases} w^m A/(\pi) & \text{for } m \text{ odd} \\ w^m A/(\pi^2) & \text{for } m \equiv 2 \bmod 4 \\ w^m A/(2) & \text{for } m \equiv 4 \bmod 8 \\ 0 & \text{for } m \equiv 0 \bmod 8 \end{cases} \\
H^2(C_8; R_{2m}) &= \ker(\zeta_8^m - 1)/\operatorname{im}(1+\zeta_8^m+\cdots+\zeta_8^{7m}) \\
&= \begin{cases} w^m A/(8) & \text{for } m \equiv 0 \bmod 8 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Note that the A-modules occurring above are

$$\begin{aligned}
A/(\pi) &\cong \mathbb{Z}/2 \\
A/(\pi^2) &\cong \mathbb{Z}/2[\pi]/(\pi^2) \\
A/(2) &\cong \mathbb{Z}/2[\pi]/(\pi^4) \\
A/(8) &\cong \mathbb{Z}/8[\pi]/(1+(1+\pi)^4)
\end{aligned}$$

We also have a map

$$\operatorname{Ext}^{*,*}_{BP_*(BP)}(BP_*, BP_*/2) \to H^*(C_8; R_*/(2))$$

Reducing the complex of (11.7) modulo (2) makes the trace map trivial, so for $t \geq 0$ we have

$$H^{2t}(C_8; R_{2m}/2) = \begin{cases} \pi^3 A/(2) & \text{for } m \text{ odd} \\ \pi^2 A/(2) & \text{for } m \equiv 2 \bmod 4 \\ A/(2) & \text{for } m \equiv 0 \bmod 4 \end{cases}$$

$$H^{2t+1}(C_8; R_{2m}/2) = \begin{cases} A/(\pi) & \text{for } m \text{ odd} \\ A/(\pi^2) & \text{for } m \equiv 2 \bmod 4 \\ A/(2) & \text{for } m \equiv 0 \bmod 4 \end{cases}$$

For $j \geq 0$, the image of the class $[t_1^{2^j}] \in \operatorname{Ext}^{1,2^{j+1}}_{BP_*(BP)} BP_*/2$ in $H^1(C_8; R_{2^{j+1}}/(2))$ is a unit in $A/(2) = \mathbb{Z}/2[\pi]/(\pi^4)$ since the function t_1 is not divisible by π by (11.5).

For $j \geq 2$, consider the following diagram.

$$\begin{tikzcd}
\mathrm{Ext}^{1,2^{j+1}}(BP_*) \arrow[r] \arrow[d, "\lambda"] & \mathrm{Ext}^{1,2^{j+1}}(BP_*/(2)) \arrow[r, "\delta"] \arrow[d, "\lambda"] & \mathrm{Ext}^{2,2^{j+1}}(BP_*) \arrow[d, "\lambda"] \\
H^1(C_8; R_{2^j}) \arrow[r] \arrow[d, equal] & H^1(C_8; R_{2^j}/(2)) \arrow[r, "\delta'"] \arrow[d, equal] & H^2(C_8; R_{2^j}) \arrow[d, equal] \\
0 \arrow[r] & A/(2) \arrow[r] & A/(8)
\end{tikzcd}$$

Here δ and δ' are the evident connecting homomorphisms, $\lambda : BP_* \to R_*$ is the classifying map for our formal A-module and the rows are exact. $\delta([t_1^{2^j}]) = b_{1,j-1}$, and $\lambda([t_1^{2^j}])$ is a unit, so $\lambda(b_{1,j-1})$ has the desired property. $\square$

To finish the proof of the Detection Theorem we need to show that $\alpha_1\alpha_{2^j-1}$ and the other βs in the same bidegree map to zero. We will do this for $j \geq 6$. The appropriate Ext group was described in (11.1). Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements $\beta_{c(j,k)/2^{j-1-2k}}$ with $k > 0$ map to zero.

Lemma 11.8. *The classifying map $BP_* \to R_*$ for the formal A-module of (11.3) sends v_i to a unit multiple (with the unit in A) of $\pi^{4-i}w^{2^i-1}$ for $1 \leq i \leq 4$.*

Proof. The logarithm of the formal group law over BP_* is

$$x + \sum_{n>0} \ell_n x^{2^n}$$

where the relation between the ℓ_ns and Hazewinkel's v_ns is given recursively by

$$2\ell_n = \sum_{0 \leq i < n} \ell_i v_{n-i}^{2^i}$$

Hence under the classifying map $\ell_n \mapsto w^{2^n-1}/\pi^n$ we find that

$$\begin{aligned}
v_1 &\mapsto \left(-\pi^3 - 4\pi^2 - 6\pi - 4\right) w &&= \pi^3 w \cdot \mathrm{unit} \\
v_2 &\mapsto \left(4\pi^3 + 11\pi^2 + 6\pi - 6\right) w^3 &&= \pi^2 w^3 \cdot \mathrm{unit} \\
v_3 &\mapsto \left(40\pi^3 + 166\pi^2 + 237\pi + 100\right) w^7 &&= \pi w^7 \cdot \mathrm{unit} \\
v_4 &\mapsto \left(-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707\right) w^{15} &&= w^{15} \cdot \mathrm{unit},
\end{aligned}$$

where each unit is in A. $\square$

We can define a valuation $||\cdot||$ on R_* by setting $||\pi|| = 1/4$ (so $||2|| = 1$) and $||w|| = 0$. We can define one on $BP_*(BP)$ with $||t_n|| = 0$ for all $n > 0$ as long as $||v_n||$ decreases monotonically as n increases. For compatibility with the valuation on R_*, we need $||v_n||$ to be no more than $||\lambda(v_n)||$. The lemma above implies that these conditions are met by defining

$$||v_n|| = \max(0, (4-n)/4).$$

The valuation extends in an obvious way to the chromatic modules M^n, such as

$$M^2 = v_2^{-1} BP_*/(2^\infty, v_1^\infty).$$

We can also extend this valuation to $BP_*(BP)$ by setting $||t_n|| = 0$. From there we can extend it to the cobar complex and to the chromatic cobar complex defined in [44, 5.1.10]. Thus we get valuations on the groups

$$\mathrm{Ext}^0_{BP_*(BP)}\left(M^2\right) \longrightarrow \mathrm{Ext}^2_{BP_*(BP)}\left(BP_*\right) \longrightarrow H^2(C_8; R_*)$$

The left group contains the chromatic fractions $\beta_{i/j}$. The homomorphisms cannot lower (but may raise) this valuation. We will show that the valuation of the relevant chromatic fractions is ≥ 3. This valuation is a lower bound on the one in $H^*(C_8; R_*)$, where every group has exponent at most 8. Hence a valuation ≥ 3 means the β-element has trivial image.

Hence for $k \geq 1$ and $j \geq 6$ we have

$$\begin{aligned}
||\beta_{c(j,k)/2^{j-1-2k}}|| &\geq \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right| \\
&= \frac{c(j,k)}{2} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\
&= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\
&= (2^{j-1} - 7\cdot 2^{j-3-2k})/3 - 1 \\
&\geq 5.
\end{aligned}$$

This means $\beta_{c(j,k)/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero.

We have to make a similar computation with the element $\alpha_1\alpha_{2^j-1}$. We have

$$\begin{aligned}
||\alpha_{2^j-1}|| &\geq \left|\left|\frac{v_1^{2^j-1}}{2}\right|\right| \\
&= \frac{3(2^j-1)}{4} - 1 \\
&\geq \frac{21}{4} - 1 > 4 \quad \text{for } j \geq 3.
\end{aligned}$$

This completes the proof of the Detection Theorem modulo Lemma 11.4.

11.5. **The proof of Lemma 11.4.** To prove the first part, consider the following diagram for an arbitrary ring K.

$$\begin{tikzcd}
 & MU_*(MU) \arrow[d, equal] & \\
\pi_*MU \arrow[ur, "\eta_L"] \arrow[dr, "\lambda_1"'] & \pi_*MU^{(2)} \arrow[d, dotted, "\lambda^{(2)}"'] & \pi_*MU \arrow[ul, "\eta_R"'] \arrow[dl, "\lambda_2"] \\
 & K &
\end{tikzcd}$$

The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a strict isomorphism between F_1 and F_2.

Similarly consider the diagram

$$\begin{tikzcd}
 & & \pi_*MU^{(4)} \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*MU \arrow[urr, "\eta_1"] \arrow[drr, "\lambda_1"'] & \pi_*MU \arrow[ur, "\eta_2"'] \arrow[dr, "\lambda_2"] & & \pi_*MU \arrow[ul, "\eta_3"] \arrow[dl, "\lambda_3"'] & \pi_*MU \arrow[ull, "\eta_4"'] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}$$

where the homomorphisms η_j are unit maps corresponding to the four smash product factors of $MU^{(4)}$. The existence of $\lambda^{(4)}$ is equivalent to that of strict isomorphisms between the formal group laws

$$F_1 \to F_2 \to F_3 \to F_4.$$

Now suppose that $K = R_*$ and each λ_j classifies the formal A-module given by (11.3). Then we have the required isomorphisms, so $\lambda^{(4)}$ exists. The inclusions η_j are related by the action of C_8 on $\pi_*MU^{(4)}$ via

$$\gamma\eta_j = \eta_{j+1} \qquad \text{for } 1 \leq j \leq 3$$

and $\gamma\eta_4$ differs from η_1 by the $(-1)^i$ in dimension $2i$. The λ_j can be chosen to satisfy a similar relation to the C_8-action on R_*. It follows that $\lambda^{(4)}$ is equivariant with respect to the C_8-actions on its source and target. This proves the first part of the Lemma.

For the second part of Lemma 11.4, recall that

$$D = N_2^8(\overline{r}_{15}^{C_2}\overline{r}_3^{C_4}\overline{r}_1^{C_8}).$$

The norm sends products to products, and $N(x)$ is a product of conjugates of x under the action of C_8. Hence its image in R_* is a unit multiple of that of a power of x, so it suffices to show that each of the three elements $\overline{r}_{15}^{C_2}$, $\overline{r}_3^{C_4}$ and $\overline{r}_1^{C_8}$ maps to a unit in R_*.

The generators $\overline{r}_i^H$ are defined by (5.46), which we rewrite as

$$\bar{x} + \sum_{i>0} \bar{m}_i \bar{x}^{i+1} \quad = \quad \left(\bar{x} + \sum_{k>0} \gamma_H(\bar{m}_{2^k-1})\bar{x}^{2^k}\right) \circ \left(\bar{x} + \sum_{i>0} \overline{r}_i^H \bar{x}^{i+1}\right)$$

where $\gamma_H = \gamma^{8/h}$ denotes a generator of $H \subset G$ and γ is a generator of C_8. Note here that the $\bar{m}_i$ are independent of the choice of subgroup H. For our purposes we can replace this by the corresponding equation in underlying homotopy, namely

$$x + \sum_{i>0} m_i x^{i+1} \quad = \quad \left(x + \sum_{k>0} \gamma^{8/h}(m_{2^k-1})x^{2^k}\right) \circ \left(x + \sum_{i>0} r_i^H x^{i+1}\right)$$

Applying the homomorphism $\lambda^{(4)} : \pi_*MU^{(4)} \to R_*$, we get

$$\begin{aligned} & x + \sum_{k>0} \frac{w^{2^k-1}}{\pi^k} x^{2^k} \\ = & \left(x + \sum_{j>0} \frac{\zeta^{8/h} w^{2^j-1}}{\pi^j} x^{2^j}\right) \circ \left(x + \sum_{i>0} \lambda^{(4)}(r_i^H) x^{i+1}\right). \end{aligned} \tag{11.9}$$

For brevity, let $s_{H,i} = \lambda^{(4)}(r_i^H)$ and

$$f_H(x) = x + \sum_{i>0} s_{H,i} x^{i+1},$$

so (11.9) reads

$$\begin{aligned} x + \sum_{k>0} \frac{w^{2^k-1}}{\pi^k} x^{2^k} &= \left(x + \sum_{j>0} \frac{\zeta^{8/h} w^{2^j-1}}{\pi^j} x^{2^j} \right) \circ f_H(x) \\ &= f_H(x) + \sum_{j>0} \frac{\zeta^{8/h} w^{2^j-1}}{\pi^j} f_H(x)^{2^j} \end{aligned} \tag{11.10}$$

We can solve (11.10) directly for $s_{H,2^k-1}$ for various H and k. Doing so gives

$$\begin{aligned} s_{C_2,1} &= \left(-\pi^3 - 4\pi^2 - 6\pi - 4\right) w \;=\; \pi^3 \cdot \text{unit} \cdot w \\ s_{C_2,3} &= \left(-4\pi^3 - 5\pi^2 + 14\pi + 26\right) w^3 \;=\; \pi^2 \cdot \text{unit} \cdot w^3 \\ s_{C_2,7} &= \left(-6182\pi^3 - 21426\pi^2 - 22171\pi - 1052\right) w^7 \\ &= \pi \cdot \text{unit} \cdot w^7 \\ s_{C_2,15} &= \big(306347134\pi^3 - 3700320563\pi^2 \\ &\qquad -15158766469\pi - 16204677587\big) w^{15} \\ &= \text{unit} \cdot w^{15} \\ s_{C_4,1} &= (-\pi - 2)\, w \;=\; \pi \cdot \text{unit} \cdot w \\ s_{C_4,3} &= \left(8\pi^3 + 26\pi^2 + 25\pi - 1\right) w^3 \;=\; \text{unit} \cdot w^3 \\ s_{C_8,1} &= -w, \end{aligned}$$

where each unit is in A. (Recall that π^4 is a unit multiple of 2.)

Hence the images under $\lambda^{(4)}$ of $r_1^{C_2}$, $r_3^{C_2}$, $r_7^{C_2}$, and $r_1^{C_4}$ are not units. For this reason, smaller subscripts of $\bar{\mathfrak{d}}$ in the definition of D would not work. On the other hand, the images of $r_{15}^{C_2}$, $r_3^{C_4}$, and $r_1^{C_8}$ are units as required. Thus we have shown that each factor of $i_0^* D$ and hence $i_0^* D$ itself maps to a unit in R_*, thereby proving the lemma. ☐

Appendix A. The category of equivariant orthogonal spectra

In this appendix we recall the definition and some basic properties of the theory of equivariant orthogonal spectra. For further details and references the reader is referred to Mandell-May [36] and to Mandell-May-Schwede-Shipley [37].

One of the reasons we have chosen to use equivariant orthogonal spectra is that it has many convenient category theoretic properties. These are independent of the homotopy theory of equivariant orthogonal spectra, and so we make two passes through the theory, one focused on the category theory, and the other on the homotopy theory.

Our main new innovation is the theory of the norm (§A.4). Most of the category theoretic aspects apply to any symmetric monoidal category, and things work out much cleaner at that level of generality.

A.1. **Category theory preliminaries.**

A.1.1. *Symmetric monoidal categories.*

Definition A.1. A *symmetric monoidal category* is a category $\mathcal{V}$ equipped with a functor

$$\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V},$$

a unit object $\mathbf{1} \in \mathcal{V}$, a natural associativity isomorphism

$$a_{ABC} : (A \otimes B) \otimes C \approx A \otimes (B \otimes C)$$

a natural commutativity isomorphism

$$s_{AB} : A \otimes B \approx B \otimes A$$

and a unit isomorphism

$$\iota_A : \mathbf{1} \otimes A \approx A.$$

This data is required to satisfy the associative and commutative coherence axioms, as well as the strict symmetry axiom.

The two coherence axioms express that all of the ways one might get from one iterated tensor product to another using the associativity and commutativity transformations coincide. The strict symmetry axiom is that the square of the commutativity transformation is the identity map. See MacLane MacL:CatWork, or Borceux [6, §6.1].

Even though it requires six pieces of data to specify a symmetric monoidal category we will usually indicate one with a triple $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$.

A symmetric monoidal category is *closed* if for each A, the functor $A \otimes (\,-\,)$ has a right adjoint $(\,-\,)^A$, which one can think of as an "internal hom." Note that

$$\mathcal{V}(\mathbf{1}, X^A) \approx \mathcal{V}(A, X)$$

so that one can recover the usual hom from the internal hom.

A.1.2. *Sifted colimits, commutative and associative algebras.* In a closed symmetric monoidal category, the monoidal product commutes with colimits in each variable. It follows easily that the iterated monoidal product

$$X \mapsto X^{\otimes n}$$

commutes with all colimits over indexing categories I for which the diagonal $I \to I^n$ is *final* in the sense of [35, §IX.3]. If $I \to I \times I$ is final, then for all $n \geq 2$, $I \to I^n$ is also final.

Definition A.2. A category I is *sifted* the diagonal embedding $I \to I \times I$ is final.

Equivalently (see [17]), a small category I is sifted if and only if the formation of colimits over I in sets commutes with finite products.

Definition A.3. A *sifted colimit* is a colimit over a sifted category.

Examples of sifted colimits include reflexive coequalizers and directed colimits. In fact the class of sifted colimits is essentially the smallest class of colimits containing reflexive coequalizers and directed colimits. See, for example [1] (references: Gabriel and Ulmer, , Lurie)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a closed symmetric monoidal category.

Definition A.4. An *associative algebra* in $\mathcal{V}$ is an object A equipped with a multiplication map $A \otimes A \to A$ which is unital and associative. A *commutative algebra* is an associative algebra for which the multiplication map is commutative.

The categories of associative and commutative algebras (and algebra maps) in $\mathcal{V}$ are denoted $\mathbf{ass}\,\mathcal{V}$ and $\mathbf{comm}\,\mathcal{V}$, respectively.

The following straightforward result holds more generally for algebras over any operad. The existence of colimits in the algebra categories is proved by expressing any algebra as a reflexive coequalizer of a diagram of free algebras. There is an even more general result for algebras over a triple [6, Proposition 4.3.1]

Proposition A.5. *Suppose that $\mathcal{V}$ is a closed symmetric monoidal category. The forgetful functors*

$$\mathbf{ass}\,\mathcal{V} \to \mathcal{V}$$
$$\mathbf{comm}\,\mathcal{V} \to \mathcal{V}$$

create limits. If $\mathcal{V}$ is cocomplete these functors have left adjoints

$$X \mapsto T(X) = \coprod_{n\geq 0} X^{\otimes n}$$
$$X \mapsto \mathrm{Sym}(X) = \coprod_{n\geq 0} X^{\otimes n}/\Sigma_n,$$

the categories $\mathbf{ass}\,\mathcal{V}$ and $\mathbf{comm}\,\mathcal{V}$ are cocomplete, and the "free" functors above commute with all sifted colimits. □

A *left module* over an associative algebra A is an object M equipped with a unital and associative left multiplication

$$A \otimes M \to M.$$

Similarly a *right module* is an object N equipped with a unital, associative right multiplication $N \otimes A \to N$. Given a left A-module M and a right A-module N one defines $N \underset{A}{\otimes} M$ by the (reflexive) coequalizer

$$N \otimes A \otimes M \rightrightarrows N \otimes M \to N \underset{A}{\otimes} M$$

When A is commutative, a left A-module can be regarded as a right A-module by the action

$$M \otimes A \xrightarrow{\text{flip}} A \otimes M \to M.$$

Using this, the formation $M \underset{A}{\otimes} N$ makes the category of left A-modules into a symmetric monoidal category.

A.1.3. *Enriched categories.* In this section we briefly describe the basic notions of enriched categories.. The reader is referred to [27] or [6, Ch. 6] for further details.

Suppose that $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ is a symmetric monoidal category.

Definition A.6. A $\mathcal{V}$-category $\mathcal{C}$ consists of a collection $\mathrm{ob}\,\mathcal{C}$ called the *objects* of $\mathcal{C}$, for each pair $X, Y \in \mathrm{ob}\,\mathcal{C}$ a *morphism object* $\mathcal{C}(X,Y) \in \mathrm{ob}\,\mathcal{V}_0$, for each X an *identity morphism* $\mathbf{1} \to \mathcal{C}(X,X)$ and for each triple X, Y, Z of objects of $\mathcal{C}$ a *composition law*

$$\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z).$$

This data is required to satisfy the usual unit and associativity properties.

As is customary, we write $X \in \mathcal{C}$ rather than $X \in \operatorname{ob}\mathcal{C}$. Most of the notions of ordinary category theory carry through in the context of enriched categories, once formulated without reference to "elements" of mapping objects. For example a functor $F : \mathcal{C} \to \mathcal{D}$ of $\mathcal{V}$-categories consists of a function

$$F : \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$$

and for each pair of objects $X, Y \in \mathcal{C}$ a map

$$F : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$$

compatible with the unit and composition. A natural transformation between two functors F and G is a function assigning to each $X \in \mathcal{C}$ a map $T_X : \mathbf{1} \to \mathcal{D}(FX, GX)$ which for every X, Y makes the diagram

$$\begin{tikzcd} \mathcal{C}(X,Y) \arrow[r, "T_Y \otimes F"] \arrow[d, "G \otimes T_X"] & \mathcal{D}(FY, GY) \otimes \mathcal{D}(FX, FY) \arrow[d] \\ \mathcal{D}(GX, GY) \otimes \mathcal{D}(FX, GX) \arrow[r] & \mathcal{D}(FX, GY) \end{tikzcd} \tag{A.7}$$

commute.

There is an ordinary category $\mathcal{C}_0$ underlying the enriched category $\mathcal{C}$. The objects of $\mathcal{C}_0$ are the objects of $\mathcal{C}$, and one defines

$$\mathcal{C}_0(X, Y) = \mathcal{V}_0(\mathbf{1}, \mathcal{C}(X, Y)).$$

If $\mathcal{V}$ itself underlies a $\mathcal{W}$-enriched category, then any $\mathcal{V}$-category $\mathcal{C}$ has an underlying $\mathcal{W}$-category, whose underlying ordinary category if $\mathcal{C}_0$.

When $\mathcal{V}$ is is a closed symmetric monoidal category, the internal hom defines an enrichment of $\mathcal{V}$ over itself, with underlying category $\mathcal{V}_0$.

When $\mathcal{V}$ is closed, a natural transformation $F \to G$ can be described as a map

$$\mathbf{1} \to \prod_{X \in \mathcal{C}} \mathcal{D}(FX, GX)$$

which equalizes the two arrows

$$\prod_{X \in \mathcal{C}} \mathcal{D}(FX, GX) \rightrightarrows \prod_{X, Y \in \mathcal{C}} \mathcal{D}(FX, GY)^{\mathcal{C}(X,Y)}. \tag{A.8}$$

describing the two ways of going around (A.7).

We will write $\mathfrak{Cat}_{\mathcal{V}}$ for the 2-category of $\mathcal{V}$-categories, and denote the category of enriched functors $\mathcal{C} \to \mathcal{D}$ as $\mathfrak{Cat}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})_0$. When $\mathcal{V}$ is closed and contains enough products, the category $\mathfrak{Cat}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})_0$ underlies an enriched category $\mathfrak{Cat}_{\mathcal{V}}(\mathcal{C}, \mathcal{D})$ in which the object of natural transformations from F to G is given by the equalizer of (A.8).

A.2. **Equivariant orthogonal spectra.**

A.2.1. *Equivariant spaces.* Let $\mathcal{T}$ be the category of pointed, compactly generated weak Hausdorff spaces. The category $\mathcal{T}$ is symmetric monoidal under the smash product, with unit S^0. A *topological* category is a category enriched over $(\mathcal{T}, \wedge, S^0)$.

Now suppose that G is a group. Let $\mathcal{T}_G = (\mathcal{T}^G, \wedge, S^0)$ be the topological symmetric monoidal category of spaces with a left G-action and spaces of equivariant maps. The category $\mathcal{T}_G$ is closed, with internal mapping spaces $\mathcal{T}_G(X, Y) = Y^X$ given by the space of non-equivariant maps, with the conjugation action of G.

A word about notation. The expression "category of G-spaces" can reasonably refer to three objects, depending on what is meant by a map. It can be an ordinary category, a category enriched over topological spaces, or a category in which the hom objects are the G-spaces of non-equivariant maps. As indicated above (and following the conventions of enriched categories) we will use $\mathcal{T}_G$ to denote the category enriched over G-spaces, with $\mathcal{T}_G(X, Y)$ denoting the G-space of non-equivariant maps, and $\mathcal{T}^G$ for the *topological* category of G-spaces, and spaces of equivariant maps. Rather than $\mathcal{T}^G(X, Y)$ we will use $\mathcal{T}_G(X, Y)^G$ for the space of equivariant maps.

We will be making use of categories enriched over $\mathcal{T}^G$. As in [36], we will refer to them as *topological G-categories* (or just *G-categories* for short). Let $\mathfrak{Cat}_G$ denote the collection of topological G-categories, and write $\mathfrak{Cat}_G(\mathcal{C}, \mathcal{D})$ for the enriched category of functors and left G-spaces of natural transformations. The symbol $\mathfrak{Cat}_G(\mathcal{C}, \mathcal{D})^G$ will denote the topological category of functors and spaces of equivariant natural transformations.

A.2.2. *The basic indexing categories.* For a real orthogonal representation of V of G let $O(V)$ be the orthogonal group of non-equivariant orthogonal maps $V \to V$. The group G acts on $O(V)$ by conjugation, and the group of fixed points is the orthogonal group of equivariant maps. Given orthogonal representations V and W, we define $O(V, W)$ to be the Stiefel manifold of orthogonal embeddings of V into W, with the conjugation action of G. The G fixed points in $O(V, W)$ are the equivariant orthogonal embeddings. The group $O(W)$ acts transitively on $O(V, W)$ on the left. A choice of embedding $V \to W$ identifies $O(V, W)$ with the homogeneous space $O(W)/O(W - V)$.

Definition A.9. The category $\mathscr{I}_G$ is the topological G-category whose objects are finite dimensional real orthogonal representations of G, and with morphism space $\mathscr{I}_G(V, W)$ the Thom complex

$$\mathscr{I}_G(V, W) = \mathrm{Thom}(O(V, W); W - V)$$

of the "complementary bundle" $W - V$ over $O(V, W)$.

We will denote the topological category underlying $\mathscr{I}_G$ with the symbol $\mathscr{I}^G$. Thus $\mathscr{I}^G(V, W) = \mathscr{I}_G(V, W)^G$.

The space $\mathscr{I}_G(V, W)$ is a G-space. As a set it is

$$\bigvee_{V \to W} S^{W-V}.$$

When $\dim W < \dim V$ it reduces to the one point space $*$. When $\dim W \geq \dim V$ one can get a convenient description by choosing an orthogonal G-representation U with $\dim U + \dim V = \dim W$ (for example the trivial representation). With this choice one has

$$\mathscr{I}_G(V, W) = O(V \oplus U, W)_+ \underset{O(U)}{\wedge} S^U.$$

The fixed point space $\mathscr{I}_G(V, W)^G$ is given by

$$\mathscr{I}_G(V, W)^G = \mathscr{I}(V^G, W^G) \wedge O(V^\perp, W^\perp)^G_+, \tag{A.10}$$

in which V^G denotes space of invariant vectors in V, and $V^\perp$ its orthogonal complement. The space $O(V^\perp, W^\perp)^G$ in turn decomposes into the product

$$\prod_\alpha O(V_\alpha, W_\alpha)$$

in which α is running through the set of non-trivial irreducible representations of G and V_α (respectively W_α) indicate the α-isotypical part of V (respectively W).

When G is the trivial group we will denoted the category $\mathscr{I}_G$ simply by $\mathscr{I}$. For any G there is an inclusion $\mathscr{I} \subset \mathscr{I}_G$ identifying $\mathscr{I}$ with the full subcategory objects with a trivial G-action. There is also a forgetful functor $\mathscr{I}_G \to \mathscr{I}$ which refines in the evident manner to a functor from $\mathscr{I}_G$ to the G-category of objects in $\mathscr{I}$ equipped with a G-action. One can easily check that this is an equivalence. For later reference, we single this statement out.

Proposition A.11. *The forgetful functor described above gives an equivalence of $\mathscr{I}_G$ with the topological G-category of objects in $\mathscr{I}$ equipped with a G-action. Passage to fixed points gives an equivalence of $\mathscr{I}^G$ with the topological category of objects in $\mathscr{I}$ equipped with a G-action.* □

Proposition A.11 plays an important role in establishing one of the basic properties of the norm (Proposition A.52).

We denote by $\mathscr{I}_{\mathcal{B}G}$ the topological G-category of objects in $\mathscr{I}$ equipped with a G-action, and $\mathscr{I}^{\mathcal{G}G}$ its underlying topological category. Proposition A.11 asserts equivalences

$$\mathscr{I}_G \xrightarrow{\approx} \mathscr{I}_{\mathcal{B}G}$$
$$\mathscr{I}^G \xrightarrow{\approx} \mathscr{I}^{\mathcal{B}G}.$$

The notation is explained more fully in §2.3.2. Briefly, $\mathcal{B}G$ is the category with one object whose monoid of self maps is the group G. For any (enriched) category $\mathcal{C}$ we use $\mathcal{C}^{\mathcal{B}G}$ for the (enriched) category of functors and natural transformations, and $\mathcal{C}_{\mathcal{B}G} = \mathfrak{Cat}_G(\mathcal{B}G, \mathcal{C})$ for the (enriched) G-category whose objects are functors $\mathcal{B}G \to \mathcal{C}$ and with morphisms the G-objects of un-natural transformations.

A.2.3. *Orthogonal spectra.*

Definition A.12. An *orthogonal G-spectrum* is a functor

$$\mathscr{I}_G \to \mathcal{T}_G$$

of topological G-categories.

Informally, an orthogonal spectrum X consists of a collection of spaces $\{X_V\}$, and for each $V \to W$ a G-equivariant map

$$S^{W-V} \wedge X_V \to X_W.$$

These maps are required to be compatible with composition in $\mathscr{I}_G$, and to vary continuously in $V \to W$. More formally, one has equivariant maps

$$\mathrm{Thom}(O(V, W); S^{W-V}) \wedge X_V \to X_W$$

compatible with composition.

Definition A.13. The *topological G-category of orthogonal G-spectra* is the category

$$\mathcal{S}_G = \mathfrak{Cat}_G(\mathscr{I}_G, \mathcal{T}_G).$$

The *(topological) category of G-spectra* is

$$\mathcal{S}^G = \mathfrak{Cat}_G(\mathscr{I}_G, \mathcal{T}_G)^G.$$

Since the category $\mathscr{I}_G$ is equivalent to a small category, there are no set-theoretic issues in the enriched categories of orthogonal G-spectra just described.

We will use the notation

$$\mathcal{S} = \mathfrak{Cat}_G(\mathscr{I}, \mathcal{T})$$

to denote the category $\mathcal{S}_G$ for the case of the trivial group.

The (G-)category of orthogonal G-spectra is complete and cocomplete (in the sense of enriched categories). Both limits and colimits in $\mathcal{S}^G$ are computed object-wise:

$$(\varinjlim X^\alpha)_V = \varinjlim X^\alpha_V$$
$$(\varprojlim X^\alpha)_V = \varprojlim X^\alpha_V.$$

Certain orthogonal G-spectra play fundamental role. For $V \in \mathscr{I}_G$ let

$$S^{-V} : \mathscr{I}_G \to \mathcal{T}_G$$

be the functor co-represented by V. By the Yoneda lemma

$$\mathcal{S}_G(S^{-V}, X) = X_V.$$

For a pointed G-space A let $S^{-V} \wedge A$ be the orthogonal G-spectrum with

$$\big(S^{-V} \wedge A\big)_W = \big(S^{-V}\big)_W \wedge A.$$

Again, by Yoneda,

$$\mathcal{S}_G(S^{-V} \wedge A, X) = \mathcal{T}_G(A, X_V).$$

It also follows from the Yoneda lemma that every X is functorially expressed as a reflexive coequalizer

$$\bigvee_{V,W} \mathscr{I}_G(V,W)_+ \wedge S^{-W} \wedge X_V \rightrightarrows \bigvee_V S^{-V} \wedge X_V \to X. \tag{A.14}$$

We call this the *canonical presentation* of X and for ease of typesetting, sometimes indicate it as

$$X = \varinjlim_V S^{-V} \wedge X_V. \tag{A.15}$$

A.2.4. *Smash product.* The symmetric monoidal structures on $\mathscr{I}_G$ and $\mathcal{T}_G$ combine to give $\mathcal{S}_G$ a symmetric monoidal structure, denoted $\wedge$. The smash product of two orthogonal G-spectra X and Y is defined to be the left Kan extension of

$$(V,W) \mapsto X_V \wedge Y_W : \mathscr{I}^G \times \mathscr{I}^G \to \mathcal{T}_G$$

along the map

$$\mathscr{I}^G \times \mathscr{I}^G \to \mathscr{I}^G$$

sending (V,W) to $V \oplus W$. The smash product is thus characterized by the fact that it commutes with enriched colimits in both variables, and satisfies

$$S^{-V} \wedge S^{-W} = S^{-(V\oplus W)}.$$

In terms of the canonical presentations

$$X = \varinjlim_{V} S^{-V} \wedge X_V$$
$$Y = \varinjlim_{W} S^{-W} \wedge Y_W$$

one has

$$\begin{aligned} X \wedge Y &= \varinjlim_{V} S^{-V} \wedge X_V \wedge Y \\ &= \varinjlim_{V} S^{-V} \wedge X_V \wedge \varinjlim_{W} S^{-W} \wedge Y_W \\ &= \varinjlim_{V,W} S^{-V\oplus W} \wedge X_V \wedge Y_W. \end{aligned}$$

The above expression is, of course, an abbreviation for the reflexive coequalizer diagram

$$\bigvee_{\substack{V_0,V_1,\\ W_0,W_1}} \mathscr{I}_G(V_0,V_1) \wedge \mathscr{I}_G(W_0,W_1) \wedge S^{-V_1\oplus W_1} \wedge X_{V_0} \wedge Y_{W_0} \rightrightarrows \bigvee_{V,W} S^{-V\oplus W} \wedge X_V \wedge Y_W.$$

Proposition A.16. *The category $\mathcal{S}_G$ is a closed symmetric monoidal category with respect to $\wedge$.* □

Smashing the canonical presentation of a general spectrum X with S^{-V} gives a presentation of $S^{-V} \wedge X$ as a (reflexive) coequalizer

$$\bigvee_{W_0,W_1} S^{-V\oplus W_1} \mathscr{I}_{W_0,W_1} \wedge X_{W_0} \rightrightarrows \bigvee_{W} S^{-V\oplus W} \wedge X_W \to S^{-V} \wedge X.$$

This is *not* the canonical presentation of $S^{-V} \wedge X$, but from it, one can read off the the formula of the following lemma

Lemma A.17. *If $\dim W < \dim V$, then $(S^{-V} \wedge X)_W = *$. If $\dim W \geq \dim V$, then there is a natural isomorphism*

$$(S^{-V} \wedge X)_W \approx O(V \oplus U, W)_+ \underset{O(U,U)}{\wedge} X_U$$

where U is any orthogonal G-representation with

$$\dim U + \dim V = \dim W.$$

□

A.2.5. *Variations on the indexing category.* There is a lot of flexibility in defining $\mathcal{S}_G$. In this section we describe a variation which is especially convenient for certain category theoretical properties, and will be used in our construction of the norm. We learned of the result below from Lars Hesselholt and Mark Hovey. It is due to Mandell-May ([36, Lemma V.1.5]).

Proposition A.18. *Let $i : \mathscr{I} \to \mathscr{I}_G$ be the inclusion of the full subcategory of trivial G-representations. The functors*

$$i^* : \mathfrak{Cat}_G(\mathscr{I}^G, \mathcal{T}_G) \to \mathfrak{Cat}_G(\mathscr{I}, \mathcal{T}_G)$$

and

$$i_* : \mathfrak{Cat}_G(\mathscr{I}, \mathcal{T}_G) \to \mathfrak{Cat}_G(\mathscr{I}^G, \mathcal{T}_G)$$

given by restriction and left Kan extension along i are inverse equivalences of enriched symmetric monoidal categories.

In other words the symmetric monoidal topological G-category $\mathcal{S}_G$ can simply be regarded as the symmetric monoidal topological G category, $\mathcal{S}_{\mathcal{B}G}$, of objects in $\mathcal{S}$ equipped with a G-action.

Remark A.19. This equivalence, while very useful for category theoretic properties of $\mathcal{S}_G$ is not so convenient for describing the homotopy theory of orthogonal G-spectra.

The proof of Proposition A.18 requires a simple technical lemma ([36, Lemma V.1.1]).

Lemma A.20. *Suppose that V and W are orthogonal G-representations with* $\dim V = \dim W$. *Then for any U*

$$O(V,U) \times O(V) \times O(W,V) \rightrightarrows O(V,U) \times O(W,V) \to O(W,U)$$

is a (reflexive) coequalizer in $\mathcal{T}^G$.

Proof: Since the forgetful functor $\mathcal{T}^G \to \mathcal{T}$ preserves colimits and reflects isomorphisms, it suffices to prove the result in $\mathcal{T}$, where it is obvious, since the coequalizer diagram can be split by choosing an orthogonal (non-equivariant) isomorphism of V with W. □

Proof: Since $\mathscr{I} \to \mathscr{I}_G$ is fully faithful, the composite

$$\mathfrak{Cat}_G(\mathscr{I}, \mathcal{T}_G) \to \mathfrak{Cat}_G(\mathscr{I}_G, \mathcal{T}_G) \to \mathfrak{Cat}_G(\mathscr{I}, \mathcal{T}_G)$$

is the identity, and so

$$\mathfrak{Cat}_G(\mathscr{I}, \mathcal{T}_G)(X,Y) \approx \mathfrak{Cat}_G(\mathscr{I}_G, \mathcal{T}_G)(i_*X, i_*Y) \approx \mathfrak{Cat}_G(\mathscr{I}, \mathcal{T}_G)(X, i^*i_*Y)$$

and the left Kan extension is fully faithful. To show essential surjectivity, let $W \in \mathscr{I}_G$ be any object, and let $V \in \mathscr{I}$ be a vector space of the same dimension as W. Define X by the coequalizer

$$(O(W,V) \times O(V))_+ \wedge S^{-V} \rightrightarrows O(W)_+ \wedge S^{-V} \to X.$$

Since $\mathscr{I}_G(W,V) = O(W,V)$, i_*X is given by the coequalizer of

$$(\mathscr{I}_G(W,V) \times \mathscr{I}_G(V,V))_+ \wedge S^{-V} \rightrightarrows \mathscr{I}_G(W,V)_+ \wedge S^{-V} \to i_*X.$$

There is thus a natural map

$$i_*X \to S^{-W}. \tag{A.21}$$

Evaluating at $U \in \mathscr{I}_G$ and using Lemma A.20 shows that (A.21) is an isomorphism. Thus S^{-W} is in the image of i_*. It then follows easily that i_* is essentially surjective.

Finally, the fact that i_* is symmetric monoidal is immediate from the fact that left Kan extensions commute. It follows that i^* is as well, since it is the inverse equivalence. □

A.2.6. *Equivariant commutative and associate algebras.* Using the notions described in §A.1.2 one can transport many algebraic structures to $\mathcal{S}_G$ using the symmetric monoidal smash product.

Definition A.22. A *G-equivariant commutative (associative) algebra* is a commutative (associative) algebra with unit in $\mathcal{S}_G$.

The conventions of §A.1.2 dictate that we refer to the *topological* categories of G-equivariant commutative and associative algebras as $\mathbf{comm}\,\mathcal{S}^G$ and $\mathbf{ass}\,\mathcal{S}^G$, and the corresponding G-equivariant categories $\mathbf{comm}\,\mathcal{S}_G$ and $\mathbf{ass}\,\mathcal{S}_G$ respectively. To ease some of the typesetting it will be convenient to employ the slightly abbreviated notation

$$\begin{aligned}\mathbf{Comm}_G &= \mathbf{comm}\,\mathcal{S}_G\\ \mathbf{Alg}_G &= \mathbf{ass}\,\mathcal{S}_G\end{aligned}$$

Thus both $\mathbf{Comm}_G$ and $\mathbf{Alg}_G$ are G-equivariant topological categories. As customary, we will write

$$\mathbf{Comm}_G(X,Y) \quad \text{and} \quad \mathbf{Alg}_G(X,Y)$$

for the G-spaces of non-equivariant maps, and

$$\mathbf{Comm}_G(X,Y)^G \quad \text{and} \quad \mathbf{Alg}_G(X,Y)^G$$

for the spaces of equivariant maps.

Since $\mathcal{S}_G$ is a closed symmetric monoidal category under $\wedge$, Proposition A.5 implies that both $\mathbf{Comm}_G$ and $\mathbf{Alg}_G$ are complete and cocomplete, and that the forgetful functors

$$\begin{aligned}\mathbf{Comm}_G &\to \mathcal{S}_G\\ \mathbf{Alg}_G &\to \mathcal{S}_G\end{aligned}$$

create (enriched) limits, sifted colimits, and have left adjoints

$$\begin{aligned}\mathrm{Sym} : \mathcal{S}_G &\to \mathbf{Comm}_G\\ T : \mathcal{S}_G &\to \mathbf{Alg}_G.\end{aligned}$$

Similarly, one has a categories of left and right modules over an associative algebra A. We will use the symbol $\mathcal{M}_A$ for the category of *left A-modules.* As described in §A.1.1, when A is commutative, the category $\mathcal{M}_A$ inherits a symmetric monoidal product $M \underset{A}{\wedge} N$ defined by the reflexive coequalizer diagram

$$M \wedge A \wedge N \rightrightarrows M \wedge N \to M \underset{A}{\wedge} N.$$

A.3. **Indexed monoidal products.**

A.3.1. *Covering categories and fiberwise constructions.* We begin with an example. Suppose that $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal category and that I is a finite set. Write C^I for the I-fold product of copies of $\mathcal{C}$. For notational purposes, and subsequent generalization it will be useful to think of an object of $\mathcal{C}^I$ as a functor $X : I \to \mathcal{C}$, with I regarded as a category with no non-identity morphisms. The iterated monoidal product

$$\otimes^I X = \bigotimes_{j \in I} Xi$$

defines a functor

$$\otimes^I : \mathcal{C}^I \to \mathcal{C}.$$

The functor $\otimes^I$ natural in isomorphisms in I (this is just the *symmetry* of the symmetric monoidal structure). In this section we make use of the notion of a *covering category* to exploit this naturality in a systematic way.

Let $\mathfrak{Sets}_{\text{iso}}$ be the groupoid of sets and isomorphisms. Suppose that J is a category, and that $P : J \to \mathfrak{Sets}_{\text{iso}}$ is a functor with the property that each Pj is finite. Then P defines a J-diagram of finite sets, and the iterated monoidal product defines for each j a functor

$$\otimes^{Pj} : \mathcal{C}^{Pj} \to \mathcal{C}. \tag{A.23}$$

These vary functorially in j. This functoriality is expressed most cleanly using the Grothendieck construction [20, §VI.8] (see also [34, p. 44] where the special case in which $\mathfrak{Cat}$ is replaced by $\mathfrak{Sets}$ is attributed to Yoneda).

Suppose that J is a category, and that $P : J \to \mathfrak{Cat}$ is a functor. The Grothendieck construction associates to P the category

$$I = \int_J P$$

of pairs (j, s) with $j \in J$ and $s \in P(j)$. The set of maps from (j, s) to (j', s') is the subset of $J(j, j')$ consisting of the $f : j \to j'$ having the property that $Pf(s) = s'$. By regarding a finite set as a category with no non-identity morphisms we get an embedding

$$\mathfrak{Sets}_{\text{iso}} \to \mathfrak{Cat}.$$

In this way the Grothendieck construction also applies to functors $P : J \to \mathfrak{Sets}_{\text{iso}}$.

A functor $p : I \to J$ arises from the Grothendieck construction of $P : J \to \mathfrak{Sets}_{\text{iso}}$ if and only if it is satisfies the following two conditions

i) for every morphism $f : i \to j$ in J, and every $a \in I$ with $pa = i$, there is a unique morphism g with domain a, and with $pg = f$;
ii) for every morphism $f : i \to j$ in J, and every $b \in I$ with $pb = j$, there is a unique morphism g with range b, and with $pg = f$.

If $p : I \to J$ satisfies the above conditions, then $Tj = p^{-1}(j)$ defines a functor from J to $\mathfrak{Sets}_{\text{iso}}$. This structure is analogous to the notion of a covering space, and we name it accordingly.

Definition A.24. A functor $I \to J$ satisfying properties *i*) and *ii*) is called a *covering category*.

A covering category $p : I \to J$ in which each of the fibers $p^{-1}J$ is finite will be called a *finite* covering category.

The aggregate of the functors (A.23) is a functor

$$p_*^{\otimes} : \mathcal{C}^I \to \mathcal{C}^J$$

given in terms of p by

$$p_*^{\otimes} X(j) = \bigotimes_{p(i)=j} X(i).$$

We will have much more to say about this in the next few sections. For now we focus on the general process that led to its construction.

Suppose we are given a formation of a category depending functorially on a set I, or in other words a functor

$$C : \mathfrak{Sets}_{\text{iso}} \to \mathfrak{Cat}.$$

Given a covering category $p : I \to J$ let $C(p)$ be the category obtained by applying the Grothendieck construction to the composite

$$J \to \mathfrak{Sets}_{\text{iso}} \to \mathfrak{Cat}$$

in which the first functor is the one classifying $I \to J$. We will say that $C(p)$ is constructed from C by *working fiberwise*. For example the category constructed from $C(S) = \mathcal{C}^S$ by working fiberwise is $\mathcal{C}^I$. The category constructed from the constant functor $C'(S) = \mathcal{C}$ is $\mathcal{C}^J$.

A natural transformation $C \to C'$ leads, via the same process, to a functor $C(p) \to C'(p)$ which we will also describe as being constructed by *working fiberwise*.

A.3.2. *Indexed monoidal products.*

Definition A.25. Let $p : I \to J$ be a finite covering category and $(\mathcal{C}, \otimes, \mathbf{1})$ a symmetric monoidal category. The *indexed monoidal product* (along p) is the functor

$$p_*^{\otimes} : \mathcal{C}^I \to \mathcal{C}^J$$

constructed fiberwise from the iterated monoidal product.

The properties of iterated monoidal products listed in the following proposition are straightforward.

Proposition A.26. *The functor $\otimes^I : \mathcal{C}^I \to \mathcal{C}$ is symmetric monoidal. If*

$$\otimes : \mathcal{C}^2 \to \mathcal{C}$$

commutes with colimits in each variable then so does $\otimes^I$. In this situation $\otimes^I$ commutes with sifted colimits. ☐

Applying Proposition A.26 fiberwise to a finite covering category $p : I \to J$ gives

Proposition A.27. *The indexed monoidal product $p_*^{\otimes} : \mathcal{C}^I \to \mathcal{C}^J$ is symmetric monoidal. If*

$$\otimes : \mathcal{C}^2 \to \mathcal{C}$$

commutes with colimits in each variable then $p_*^{\otimes}$ commutes with sifted colimits. ☐

Remark A.28. Though it plays no role in this paper, it can be useful to observe that the class of colimits preserved by $p_*^{\otimes}$ is slightly larger than the class of sifted colimits. For example $p_*^{\otimes}$ will commute with *objectwise reflexive coequalizers*, which are diagrams of the form

$$X \rightrightarrows Y \to Z$$

with the property that for each $j \in J$ there is a map $Yj \to Xj$ completing

$$Xj \rightrightarrows Yj$$

to a reflexive coequalizer diagram. The maps $Yj \to Xj$ are not required to be natural in j.

The following is also straightforward

Proposition A.29. *Suppose that $p : I \to J$ and $q : J \to K$ are covering categories. Then $q \circ p$ is a covering category, which is finite if p and q are. In that case there is a natural isomorphism*

$$q_*^{\otimes} \circ p_*^{\otimes} = (q \circ p)_*^{\otimes}.$$

arising from the symmetric monoidal structure. □

The following results are also proved by working fiberwise.

Proposition A.30. *Suppose that $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbf{1}_{\mathcal{D}})$ are symmetric monoidal categories, and that*

$$F : \mathcal{C} \to \mathcal{D}$$
$$T : FX \wedge FY \to F(X \otimes Y)$$
$$\phi : \mathbf{1}_{\mathcal{D}} \to F\mathbf{1}_{\mathcal{D}}$$

form a lax monoidal functor. If $p : I \to J$ is a finite covering category then T gives a natural transformation

$$p_*^T : p_*^{\wedge} \circ F^I \to F^J \circ p_*^{\otimes}$$

between the two ways of going around

$$\begin{tikzcd} \mathcal{C}^I \arrow[r, "F^I"] \arrow[d, "p_*^{\otimes}"'] & \mathcal{D}^I \arrow[d, "p_*^{\wedge}"] \\ \mathcal{C}^J \arrow[r, "F^J"'] & \mathcal{D}^J. \end{tikzcd}$$

If T is a natural isomorphism, then so is p^T. □

Suppose that $p : I \to J$ is a covering category, and $f : \tilde{J} \to J$ is a functor. Let $\tilde{I}$ be the "rigid pullback" category of pairs $(j', i) \in \tilde{J} \times I$ with $f(i') = p(j)$, and in which a morphism is a pair (g, g') with $f(g) = p(g')$. Then the functor $\tilde{p} : \tilde{I} \to \tilde{J}$ defined by $(j', i) \mapsto j'$ is a covering category.

Proposition A.31. *In the situation described above, if $p : I \to J$ is finite then the following commutes up to a natural isomorphism given by the symmetric monoidal structure*

$$\begin{tikzcd} \mathcal{C}^I \arrow[r] \arrow[d, "\tilde{p}_*^{\otimes}"'] & \mathcal{C}^{\tilde{I}} \arrow[d, "\tilde{p}_*^{\otimes}"] \\ \mathcal{C}^J \arrow[r, "f^*"'] & \mathcal{C}^{\tilde{J}} \end{tikzcd}$$

□

The categories I and J used in this paper arise from a left action of a group G on a finite set A. Given such an A, let $\mathcal{B}_A G$ be the category whose set of objects is A and in which a map $a \to a'$ is an element $g \in G$ with the property that $g\,a = a'$. When $A = \text{pt}$ we will abbreviate $\mathcal{B}_A G$ to just $\mathcal{B}G$. For any finite map $A \to B$ of G-sets, the corresponding functor

$$\mathcal{B}_A G \to \mathcal{B}_B G$$

is a covering category.

In the following series of examples we suppose $H \subset G$ is a subgroup, take $A = G/H$ to be the set of right H-cosets, and write $p : A \to \mathrm{pt}$ for the unique equivariant map. In this case the inclusion of the identity coset gives an equivalence

$$\mathcal{B}H \to \mathcal{B}_A G$$

and hence an equivalence of functor categories

$$\mathcal{C}^{\mathcal{B}_A G} \to \mathcal{C}^{\mathcal{B}H}.$$

Example A.32. Suppose $\mathcal{C}$ is the category of abelian groups, with $\oplus$ as the symmetric monoidal structure. Then $\mathcal{C}^{\mathcal{B}_A G}$ is equivalent to the category of left H-modules, and the functor $p_*^{\oplus}$ is left additive induction. If symmetric monoidal structure is taken to be the tensor product, then $p_*^{\otimes}$ is "norm induction."

Example A.33. Now take $(\mathcal{C}, \otimes, \mathbf{1})$ to be the category $(\mathcal{S}, \wedge, S^0)$ of orthogonal spectra. From the above, and the discussion following Proposition A.18, the category $\mathcal{S}^{\mathcal{B}_A G}$ is equivalent to the category of orthogonal H-spectra, and $\mathcal{S}^{\mathcal{B}G}$ is equivalent to the category of orthogonal G-spectra. In this case $p_*^{\wedge}$ defines a multiplicative transfer from orthogonal H-spectra to orthogonal G-spectra. This is the *norm.* It is discussed more fully in §A.4 and §B.2.

Remark A.34. When $\mathcal{C}$ has all colimits, and is pointed, in the sense that the tensor unit $\mathbf{1}$ is the initial object, the condition that $p : I \to J$ be finite may be dropped. Indeed if I is an infinite set and $\{Xi\}$ a collection of objects indexed by $i \in I$ one defines

$$\otimes^I Xi = \varinjlim_{I' \subset I \text{ finite}} \otimes^{I'} Xi$$

in which the transition maps associated to $I' \subset I''$ are given by tensoring with the unit

$$\otimes^{I'} Xi \approx \left(\otimes^{I'} Xi\right) \otimes \left(\otimes^{I'' \setminus I'} \mathbf{1}\right) \to \otimes^{I''} Xi.$$

The general case is derived from this by working fiberwise.

Remark A.35. The results of this section apply, with the obvious modifications, in the setting of enriched categories.

A.3.3. *Distributive laws.* Continuing with the same notation, we now assume that the category $\mathcal{C}$ comes equipped with two symmetric monoidal structures, $\otimes$ and $\oplus$, and that $\otimes$ distributes over $\oplus$ in the sense that there is a natural isomorphism

$$A \otimes (B \oplus C) \approx (A \otimes B) \oplus (A \otimes C)$$

compatible with all of the symmetries. Put differently, we are requiring that

$$A \otimes (\,-\,)$$

be a symmetric monoidal functor with respect to $\oplus$. In all of our examples, $\oplus$ will be the categorical coproduct, and $A \otimes (\,-\,)$ will commute with all colimits. Given $p : I \to J$ and $q : J \to K$ we can form

$$q_*^{\otimes} \circ p_*^{\oplus}.$$

Our goal is to express this in the form

$$q_*^{\otimes} \circ p_*^{\oplus} = r_*^{\oplus} \circ \pi_*^{\otimes}.$$

We start with the case in which K is the trivial category, and $p : I \to J$ is a map of finite sets. Let $\Gamma = \Gamma(I/J)$ be the set of sections $s : J \to I$ of p. Write

$\mathrm{ev} : J \times \Gamma \to I$ for the evaluation map, $\pi : J \times \Gamma \to \Gamma$ for the projection, and with an eye toward generalization, $r : \Gamma \to \{\mathrm{pt}\}$ for the unique map. The following lemma expresses the usual distributivity expansion

$$\bigotimes_{j\in J}\left(\bigoplus_{p(i)=j} Xi\right) \approx \bigoplus_{s\in\Gamma}\left(\bigotimes_{j\in J} X s(j)\right)$$

in functorial terms.

Lemma A.36. *The following diagram of functors commutes, up to a canonical natural isomorphism given by the symmetries of the symmetric monoidal structures*

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, "\mathrm{ev}^*"] \arrow[d, "p_*^{\oplus}"'] & \mathcal{C}^{J\times\Gamma} \arrow[d, "\pi_*^{\otimes}"] \\
\mathcal{C}^J \arrow[dr, "q_*^{\otimes}"'] & \mathcal{C}^{\Gamma} \arrow[d, "r_*^{\oplus}"] \\
& \mathcal{C}
\end{tikzcd}.$$

□

Working fiberwise, it is now a simple matter to deal with the more general case in which $I \to J$ and $J \to K$ are covering categories. Let Γ be the category of pairs (k, s), with $k \in K$ and s a section of $(p\circ q)^{-1}k \to p^{-1}k$. A morphism $(k, s) \to (k, s')$ in Γ is a map $f : k \to k'$ making the following diagram commute

$$\begin{tikzcd}
(p\circ q)^{-1}k \arrow[r, "I_f"] & (p\circ q)^{-1}k' \\
p^{-1}k \arrow[u, "s"] \arrow[r, "J_f"'] & p^{-1}k' \arrow[u, "s'"']
\end{tikzcd}.$$

Write $\Gamma \underset{K}{\times} J$ for the fiber product,

$$\mathrm{ev} : \Gamma \underset{K}{\times} J \to I$$

for the "evaluation," and $\pi : \Gamma \underset{K}{\times} J \to J$ for the projection. By naturality in I and J in Lemma A.36 we have

Proposition A.37. *The following diagram of functors commutes, up to a canonical natural isomorphism given by the symmetries of the symmetric monoidal structures*

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, "\mathrm{ev}^*"] \arrow[d, "p_*^{\oplus}"'] & \mathcal{C}^{J\underset{K}{\times}\Gamma} \arrow[d, "\pi_*^{\otimes}"] \\
\mathcal{C}^J \arrow[dr, "q_*^{\otimes}"'] & \mathcal{C}^{\Gamma} \arrow[d, "r_*^{\oplus}"] \\
& \mathcal{C}^K
\end{tikzcd}$$

□

This formula is use in showing that the norm of a wedge of regular slice cells is a wedge of regular slice cells (Proposition 4.7), in the construction of monomial ideals (§A.3.6), and in describing the structure of equivariant polynomial algebras and their monomial ideals (§2.4).

A.3.4. *Indexed monoidal products and pushouts.* The homotopy theoretic properties of the norm depend on a formula for the indexed monoidal product of a pushout. We describe here the absolute case, and leave it to the reader to work out the fiber-wise analogue. Suppose that $(\mathcal{C}, \otimes)$ is a closed symmetric monoidal category with finite colimits, and let I be a finite set. For $X \in \mathcal{C}^I$ write $X^{\otimes I}$ for the iterated monoidal product. Suppose we are given a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{C}^I$. We wish to express $Y^{\otimes I}$ as an iterated pushout. Since the coequalizer diagram

$$X \amalg A \amalg B \rightrightarrows X \amalg B \to Y$$

can be completed to a reflexive coequalizer, the sequence

$$(X \amalg A \amalg B)^{\otimes I} \rightrightarrows (X \amalg B)^{\otimes I} \to Y^{\otimes I}$$

is a coequalizer (Proposition A.26). Using the distributivity law of §A.3.3 this can be re-written as

$$\coprod_{I = I_0 \amalg I_1 \amalg I_2} X^{\otimes I_0} \otimes A^{\otimes I_1} \otimes B^{\otimes I_2} \rightrightarrows \coprod_{I = I_0 \amalg I_1} X^{\otimes I_0} \otimes B^{\otimes I_1} \to Y^{\otimes I}.$$

The horizontal arrows do not preserve the coproduct decompositions, but the sequence can be filtered by the cardinality of the exponent of B. Define $\operatorname{fil}_n Y$ by the coequalizer diagram

$$\coprod_{\substack{I = I_0 \amalg I_1 \amalg I_2 \\ |I_1| + |I_2| \le n}} X^{\otimes I_0} \otimes A^{\otimes I_1} \otimes B^{\otimes I_2} \rightrightarrows \coprod_{\substack{I = I_0 \amalg I_1 \\ |I_1| \le n}} X^{\otimes I_0} \otimes B^{\otimes I_1} \to \operatorname{fil}_n Y.$$

Thus $\operatorname{fil}_0 Y = X^{\otimes I}$ and $\operatorname{fil}_{|I|} Y = Y^{\otimes I}$. There is an evident coequalizer diagram

$$\coprod_{\substack{I = I_0 \amalg I_1 \amalg I_2 \\ |I_1| + |I_2| = n}} X^{\otimes I_0} \otimes A^{\otimes I_1} \otimes B^{\otimes I_2} \rightrightarrows \operatorname{fil}_{n-1} Y \coprod_{\substack{I = I_0 \amalg I_1 \\ |I_1| = n}} X^{\otimes I_0} \otimes B^{\otimes I_1} \to \operatorname{fil}_n Y,$$

which can be re-written as a pushout square

$$\begin{tikzcd} \displaystyle\coprod_{\substack{I = I_2 \amalg I_1 \amalg I_0 \\ |I_0| = |I| - n}} X^{\otimes I_0} \otimes A^{\otimes I_1} \otimes B^{\otimes I_2} \arrow[r] \arrow[d] & \displaystyle\coprod_{\substack{I = I_1 \amalg I_0 \\ |I_1| = n}} X^{\otimes I_0} \otimes B^{\otimes I_1} \arrow[d] \\ \operatorname{fil}_{n-1} Y \arrow[r] & \operatorname{fil}_n Y. \end{tikzcd}$$

The upper left term may be replaced by its effective quotient

$$\coprod_{|I_0| = n} X^{\otimes I_1} \otimes \partial_A B^{\otimes I_0}$$

in which $\partial_A B^{\otimes S}$ is defined by the coequalizer diagram

$$\coprod_{\substack{S=S_0\amalg S_1\amalg S_2\\ S_0\neq\emptyset}} A^{\otimes S_0}\otimes A^{\otimes S_1}\otimes B^{\otimes S_2} \rightrightarrows \coprod_{\substack{S=S_0\amalg S_1\\ S_0\neq\emptyset}} A^{\otimes S_0}\otimes B^{\otimes S_1}\to \partial_A B^{\otimes S},$$

leading to a pushout square

(A.38)
$$\begin{tikzcd}
\displaystyle\coprod_{\substack{I=I_0\amalg I_1\\ |I_1|=n}} X^{\otimes I_0}\otimes \partial_A B^{\otimes I_1} \arrow[r] \arrow[d] & \displaystyle\coprod_{\substack{I=I_0\amalg I_1\\ |I_1|=n}} X^{\otimes I_0}\otimes B^{\otimes I_1} \arrow[d] \\
\operatorname{fil}_{n-1} Y^{\otimes I} \arrow[r] & \operatorname{fil}_n Y^{\otimes I}.
\end{tikzcd}$$

The symbol $\partial_A B^{\otimes S}$ is motivated by the evident identity of pairs

$$(B,A)^{\otimes S} = (B^{\otimes S}, \partial_A B^{\otimes S}),$$

and especially the case in which $B=[0,1]^{n_s}$ is a cube and $A=\partial[0,1]^{n_s}$ its boundary. In that case

$$\partial_A B^S = \partial[0,1]^n \qquad n=\sum_{s\in S} n_s.$$

By working fiberwise, one obtains a similar iterated pushout describing $p_*^{\otimes}Y$. We leave this case to the reader.

A.3.5. *Commutative algebras and indexed monoidal products.* By Proposition A.5, if $\mathcal{C}$ is a co-complete closed symmetric monoidal category, then $\mathbf{comm}\,\mathcal{C}$ is cocomplete. Therefore the restriction functor $p^*:\mathbf{comm}\,\mathcal{C}^J\to\mathbf{comm}\,\mathcal{C}^I$ then has a left adjoint p_* given by left Kan extension.

Proposition A.39. *If $p: I\to J$ is a covering functor, the following diagram commutes*

$$\begin{tikzcd}
\mathbf{comm}\,\mathcal{C}^I \arrow[r] \arrow[d, "p_*"] & \mathcal{C}^I \arrow[d, "p_*^{\otimes}"] \\
\mathbf{comm}\,\mathcal{C}^J \arrow[r] & \mathcal{C}^J.
\end{tikzcd}$$

Proof: For a commutative algebra $A\in\mathbf{comm}\,\mathcal{C}^I$, and $j\in J$ the value of p_*A at j is calculated as the colimit over the category I/j of the restriction of p. Since $p: I\to J$ is a covering category, the category I/j is equivalent to the discrete category $p^{-1}j$, and so

$$(p_*A)j = \otimes^{p^{-1}j}A,$$

and the result follows. $\square$

A.3.6. *Monomial ideals.* Let I be a set and consider the polynomial algebra

$$A=\mathbb{Z}[x_i],\quad i\in I.$$

As an abelian group, it has a basis the monomials x^f, with

$$f: I\to\{0,1,2,\dots\}$$

a function taking the value zero on all but finitely many elements, and

$$x^f = \prod_{j \in J} x_j^{f(j)}.$$

The collection of such f is a monoid under addition, and we denote it $\mathbb{N}_0^I$. If $D \subset \mathbb{N}_0^I$ is a monoid ideal then the subgroup $M_D \subset A$ with basis $\{x^f \mid f \in D\}$ is an ideal. Such ideals are *monomial ideals* and they can be formed in any monoidal product of free associative algebras in any closed symmetric monoidal category.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a closed symmetric monoidal category. Fix a set I which, for simplicity, we will temporarily assume to be finite. Given $X \in \mathcal{C}^I$ let

$$TX = \coprod_{n \geq 0} X^{\otimes n}$$

be the free associative algebra generated by X. Write $A = p_*^{\otimes} TX \in \mathcal{C}$, where $p : I \to \mathrm{pt}$ is the unique map. Then A is an associative algebra in $\mathcal{C}$. The motivating example above occurs when $\mathcal{C}$ is the category of abelian groups and X is the constant diagram $Xi = \mathbb{Z}$.

Using Proposition A.37 the object A can be expressed as an indexed coproduct

$$A = \coprod_{f : I \to \mathbb{N}_0} X^{\otimes f}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and

$$X^{\otimes f} = \coprod_{i \in I} X(i)^{\otimes f(i)}.$$

The set

$$\mathbb{N}_0^I = \{f : I \to \mathbb{N}_0\}$$

is a commutative monoid under addition of functions. The multiplication map in A is the sum of the isomorphisms

$$X^{\otimes f} \otimes X^{\otimes g} \approx X^{\otimes (f+g)} \tag{A.40}$$

given by the symmetry of the monoidal product $\otimes$, and the isomorphism

$$X^{\otimes f(i)} \otimes X^{\otimes g(i)} \approx X^{\otimes (f(i)+g(i))}.$$

For a monoid ideal $D \subseteq \mathbb{N}_0^I$, set set

$$M_D = \coprod_{f \in D} X^{\otimes f}.$$

Then the formula (A.40) for the multiplication in A gives M_D the structure an ideal in A. If $D \subset D'$ then the evident inclusion $M_D \subset M_{D'}$ is an inclusion of ideals.

When $\mathcal{C}$ is pointed (in the sense that the initial object is isomorphic to the terminal object), the map

$$A \to A/M_D$$

is a map of associative algebras, where A/M_D is defined by the pushout diagram

$$\begin{tikzcd} M_D \arrow[r] \arrow[d] & A \arrow[d] \\ * \arrow[r] & A/M_D, \end{tikzcd}$$

in which $*$ is the terminal (and initial) object.

Definition A.41. The ideal $M_D \subset A$ is the *monomial ideal* associated to the monoid ideal D.

Example A.42. Suppose that $\dim : \mathbb{N}_0^I \to \mathbb{N}_0$ is any homomorphism. Given $d \in \mathbb{N}_0$ the set

$$\{f \mid \dim f \geq d\}$$

is a monoid ideal. We denote the corresponding monomial ideal M_d. The M_d form a decreasing filtration

$$\cdots \subset M_{d+1} \subset M_d \subset \cdots \subset M_1 \subset M_0 = A.$$

When $\mathcal{C}$ is pointed, the quotient

$$M_d/M_{d+1}$$

is isomorphic as an A bi-module to

$$A/M_1 \otimes \coprod_{\dim f=d} M^{(f)},$$

in which A act through its action on the left factor.

Remark A.43. The quotient module is defined by the pushout square

$$\begin{tikzcd} M_{d+1} \arrow[r] \arrow[d] & M_d \arrow[d] \\ * \arrow[r] & M_d/M_{d+1}. \end{tikzcd}$$

The pushout can be calculated in the category of left A-modules, A bimodules, or just in $\mathcal{C}$.

Remark A.44. All of this discussion is natural in I. Suppose that $I_0 \subset I_1$ is a map of finite sets, $X_1 : \mathcal{B}_{I_1} \to \mathcal{C}$ a diagram, and X_0 its restriction to I_0. Let A_0 and A_1 be the associative algebras constructed from the X_i as described above. If $D_1 \subset \mathbb{N}_0^{I_1}$ is a monoid ideal and D_0 its intersection with $\mathbb{N}_0^{I_0}$ then there is a commutative diagram

$$\begin{tikzcd} M_{D_0} \arrow[r] \arrow[d] & M_{D_1} \arrow[d] \, . \\ A_0 \arrow[r] & A_1 \end{tikzcd}$$

Using this, the construction of monomial ideas applies to the infinite sets I, by passing to the colimit over the finite subsets. As in the motivating example, when the set I is infinite the indexing monoid $\mathbb{N}_0^I$ is the set of finitely supported functions.

By working fiberwise, this entire discussion applies to the situation of a (possibly infinite) covering category $p : I \to J$. Associated to $X : I \to \mathcal{C}$ is

$$A = p_*^{\otimes} TX \in \mathbf{ass}\, \mathcal{C}^J = (\mathbf{ass}\, \mathcal{C})^J.$$

In case I/J is infinite, the algebra A is formed fiberwise by passing to the colimit from the finite monoidal products using the unit map, as described in Remark A.34. As an object of $\mathcal{C}^J$, the algebra A decomposes into

$$A = \coprod_{f \in \Gamma} X^{\otimes f}$$

where Γ is the set of sections of

$$\mathbb{N}_0^{I/J} \to J$$

with $\mathbb{N}_0^{I/J}$ formed from the Grothendieck construction applied to

$$j \mapsto \mathbb{N}_0^{I_j} \qquad (I_j = p^{-1}(j)).$$

The category $\mathbb{N}_0^{I/J}$ is a commutative monoid over J, and associated to any monoid ideal $D \subset \mathbb{N}_0^{I/J}$ over J, is a monomial ideal $M_D \subset A$.

The situation of interest in this paper (see §2.4) is when $I \to J$ is of the form

$$\mathcal{B}_J G \to \mathcal{B}G$$

associated to a G-set J, and the unique map $J \to \mathrm{pt}$. In this case $\mathbb{N}_0^{I/J}$ is the G-set $\mathbb{N}_0^J$ of finitely supported functions $J \to \mathbb{N}_0$. The relative monoid ideals are just the G-stable monoid ideals. A simple algebraic example arises in the case of a polynomial algebra $\mathbb{Z}[x_i]$ in which a group G is acting on the set indexing the variables.

A.4. **The norm.** We now specialize the discussion of §A.3 to the case $(\mathcal{C}, \otimes, \mathbf{1}) = (\mathcal{S}, \wedge, S^0)$ and define the *norm functor*.

Because of Proposition A.18 (and the discussion following it) we may identify the category of G-equivariant orthogonal spectra as the functor category $\mathcal{S}^{\mathcal{B}G}$. If $H \subset G$, then the functor

$$\mathcal{B}H \to \mathcal{B}_{G/H} G$$

sending the unique object to the coset H/H is an equivalence of categories. Take $\mathcal{B}_{G/H} G \to \mathcal{B}H$ to be any inverse, and write $p : \mathcal{B}_{G/H} G \to \mathcal{B}G$ for the functor corresponding to the G-map $G/H \to \mathrm{pt}$.

Definition A.45. The *norm functor* $N_H^G : \mathcal{S}^H \to \mathcal{S}^G$ is the composite

$$\begin{tikzcd} \mathcal{S}^{\mathcal{B}H} \arrow[r] \arrow[dr, "N_H^G"'] & \mathcal{S}^{\mathcal{B}_{G/H} G} \arrow[d, "p_*^{\wedge}"] \\ & \mathcal{S}^{\mathcal{B}G} \end{tikzcd}$$

By Proposition A.27 we have

Proposition A.46. *The functor N_H^G is symmetric monoidal and commutes with sifted colimits.* □

Remark A.47. By Remark A.28, the norm also commutes with the formation of coequalizer diagrams in $\mathcal{S}^H$ whose underlying non-equivariant diagram in $\mathcal{S}$ extends to a reflexive coequalizer.

Remark A.48. We have defined the norm on the topological categories of equivariant spectra. Since it is symmetric monoidal it naturally extends to a functor of enriched categories

$$N_H^G : \mathcal{S}_H \to \mathcal{S}_G$$

compatible with the norm on spaces (and, in fact, spectra) in the sense that it gives for every $X, Y \in \mathcal{S}_H$ a G-equivariant map

$$N_H^G(\mathcal{S}_H(X, Y)) \to \mathcal{S}_G(N_H^G X, N_H^G Y).$$

By Proposition A.39, on equivariant commutative algebras the norm is the left adjoint of the restriction functor.

Corollary A.49. *The following diagram commutes up to a natural isomorphism given by the symmetry of the smash product:*

$$\begin{tikzcd} \mathbf{Comm}_H \arrow[r] \arrow[d] & \mathcal{S}^H \arrow[d, "N_H^G"] \\ \mathbf{Comm}_G \arrow[r] & \mathcal{S}^G. \end{tikzcd}$$

The the left vertical arrow is the left adjoint to the restriction functor.

Remark A.50. Because of Corollary A.49 we will refer to the left adjoint to the restriction functor

$$\mathbf{Comm}_G \to \mathbf{Comm}_H$$

as the *commutative algebra norm*, and denote it

$$N_H^G : \mathbf{Comm}_H \to \mathbf{Comm}_G.$$

The (covariant) Yoneda embedding gives a functor

$$\begin{aligned} \mathscr{I} &\to \mathcal{S} \\ V &\mapsto S^{-V}. \end{aligned}$$

By definition of $\wedge$ this is a symmetric monoidal functor, and we are in the situation described in Proposition A.30. Thus if $p : I \to J$ is a covering category, there is a natural isomorphism between the two ways of going around

$$\begin{tikzcd} \mathscr{I}^I \arrow[r] \arrow[d, "p_*^{\oplus}"'] & \mathcal{S}^I \arrow[d, "p_*^{\wedge}"] \\ \mathscr{I}^J \arrow[r] & \mathcal{S}^J. \end{tikzcd} \tag{A.51}$$

Take $I = \mathcal{B}_{G/H}G$ and $J = \mathcal{B}G$. Then the functor category $\mathscr{I}^I$ is equivalent to the category $\mathscr{I}^H$ (Proposition A.11), and $\mathcal{S}^I$ is equivalent to $\mathcal{S}^H$ (Proposition A.18). By naturality, the functor

$$\mathscr{I}^H \to \mathcal{S}^H$$

corresponding to

$$\mathscr{I}^I \to \mathcal{S}^I$$

is just the covariant Yoneda embedding, and so sends an orthogonal H-representation V to S^{-V}. Similarly $\mathscr{I}^J$ is equivalent to $\mathscr{I}^G$, $\mathcal{S}^J$ is equivalent to the category of orthogonal G-spectra, and the functor between them sends an orthogonal G-representation W to W^{-W}. One easily checks (as in Example A.32) that the functor $p_*^{\oplus}$ corresponds to additive induction. We therefore have a commutative diagram

$$\begin{tikzcd} \mathscr{I}^H \arrow[r] \arrow[d, "\mathrm{ind}_H^G"'] & \mathcal{S}^H \arrow[d, "N_H^G"] \\ \mathscr{I}^G \arrow[r] & \mathcal{S}^G \end{tikzcd}$$

This proves

Proposition A.52. *Let V be a finite dimensional H-representation, and set $W = \operatorname{ind}_H^G V$. There is a natural isomorphism*

$$N_H^G S^{-V} \approx S^{-W}.$$

□

A.5. ***h*-cofibrations.** Suppose that $\mathcal{C}$ is a topological category.

Definition A.53. A map $i : A \to X$ in $\mathcal{C}$ is an *h-cofibration* if it has the homotopy extension property: given $f : X \to Y$ and a homotopy $h : A \otimes [0,1] \to Y$ with $h|_{A\otimes\{0\}} = f \circ i$ there is an extension of h to $H : X \otimes [0,1] \to Y$.

Example A.54. The mapping cylinder $A \to X \underset{A}{\cup} A \otimes [0,1]$ of any map $A \to X$ is an h-cofibration.

As is well-known, a map $i : A \to X$ is an h-cofibration if and only if

$$\operatorname{cyl} i = X \otimes \{0\} \underset{A\otimes\{0\}}{\cup} A \otimes [0,1] \to X \otimes [0,1]$$

is the inclusion of a retract.

Proposition A.55. *The class of h-cofibrations is stable under composition, and the formation of coproducts and cobase change. Given a sequence*

$$X_1 \xrightarrow{f_1} \cdots \to X_i \xrightarrow{f_i} X_{i+1} \to \cdots$$

in which each f_i is an h-cofibration, the map

$$X_1 \to \varinjlim_i X_i$$

is an h-cofibration. □

Proposition A.56. *Any topological functor L which is a continuous left adjoint preserves the class of h-cofibrations.* □

Now suppose that $\mathcal{C}$ has a symmetric monoidal structure $\otimes$ which is compatible with the cartesian product of spaces, in the sense that for spaces S and G, and objects $X, Y \in \mathcal{C}$ there is a natural isomorphism

$$(X \otimes S) \otimes (Y \otimes T) \approx (X \otimes Y) \otimes (S \times T)$$

compatible with the enrichment and the symmetric monoidal structures. Then given $i : A \to X$ we may form

$$i^{\otimes n} : A^{\otimes n} \to X^{\otimes n}$$

and regard it as a map in the category $\mathcal{C}^{\mathcal{B}\Sigma_n}$ of objects in $\mathcal{C}$ equipped with a Σ_n-action.

Proposition A.57. *If $A \to X$ is an h-cofibration in $\mathcal{C}$, then for any Z, $A \otimes Z \to X \otimes Z$ is an h-cofibration.* □

Proposition A.58. *If $i : A \to X$ is an h-cofibration then $i^{\otimes n}$ is an h-cofibration in $\mathcal{C}^{\mathcal{B}\Sigma_n}$*

Remark A.59. In the category of equivariant orthogonal spectra a version of this result appears in [37, Lemma 15.8] (where the reader is referred to [14, Lemma XII.2.3]).

Proof: The first step is to show that the diagonal inclusion

$$\mathrm{cyl}(A^{\otimes n} \to X^{\otimes n}) \to \mathrm{cyl}(A \to X)^{\otimes n} \tag{A.60}$$

is the inclusion of a Σ_n-equivariant retract. Granting this for the moment, one constructs a Σ_n-equivariant retraction of

$$\mathrm{cyl}(A^{\otimes n} \to X^{\otimes n}) \to X^{\otimes n} \otimes [0,1]$$

as the composition

$$\begin{aligned} X^{\otimes n} \otimes [0,1] \xrightarrow{1 \otimes \mathrm{diag}} X^{\otimes n} \otimes [0,1]^n \approx (X \otimes [0,1])^{\otimes n} \\ \to \mathrm{cyl}(A \to X)^{\otimes n} \to \mathrm{cyl}(A^{\otimes n} \to X^{\otimes n}). \end{aligned}$$

For the retraction of (A.60) start with the pushout square

$$\begin{tikzcd} A \otimes \{0\} \arrow[r] \arrow[d] & A \otimes [0,1] \arrow[d] \\ X \arrow[r] & \mathrm{cyl}(A \to X) \end{tikzcd}$$

and consider the last stage of the filtration of $\mathrm{cyl}(A \to X)^{\otimes n}$ constructed using the method of §A.3.4

$$\begin{tikzcd} \partial_A (A \otimes [0,1])^{\otimes n} \arrow[r] \arrow[d] & (A \otimes [0,1])^{\otimes n} \arrow[d] \\ \mathrm{fil}_{n-1}(\mathrm{cyl}(A \to X)^{\otimes n}) \arrow[r] & \mathrm{cyl}(A \to X)^{\otimes n}. \end{tikzcd} \tag{A.61}$$

Form the Σ_n-equivariant map

$$\mathrm{fil}_{n-1}(\mathrm{cyl}(A \to X)^{\otimes n}) \to X^{\otimes n} \to \mathrm{cyl}(A^{\otimes n} \to X^{\otimes n})$$

using the map $\mathrm{cyl}(A \to X) \to X$. To extend it to $\mathrm{fil}_n(\mathrm{cyl}(A \to X)^{\otimes n}) = \mathrm{cyl}(A \to X)^{\otimes n}$ note that the top row of (A.61) can be identified with the tensor product of the identity map of $A^{\otimes n}$ with

$$\partial_{\{0\}} I^n \to I^n.$$

This identification is compatible with the action of the symmetric group. The desired extension is then constructed using any Σ_n-equivariant retraction of I^n to the diagonal which takes $\partial_{\{0\}} I^n$ to $\{0\}$ □

Working fiberwise one concludes

Proposition A.62. *Suppose that $\mathcal{C}$ is as above, and $p : I \to J$ is a covering category. The indexed monoidal product*

$$p_*^{\otimes} : \mathcal{C}^I \to \mathcal{C}^J$$

preserves the class of h-cofibrations. □

We will make frequent use of the following result. To set it up, let I be a finite set with a G-action, and Σ_I the group of permutations of I. Then G acts on Σ_I by conjugation, and we may form $\Sigma_I \rtimes G$.

Appendix B. Homotopy theory of equivariant orthogonal spectra

We now turn to the homotopy theoretic aspects of equivariant orthogonal spectra. In order that the category of commutative rings be Quillen equivalent to E_∞ rings, we will work with the topological G-category $\mathcal{S}_G$, equipped with its *positive stable* model structure. This comes at a technical cost. The spectra underlying cofibrant commutative rings are almost never cofibrant spectra. This means that we need a result asserting that the norm of a cofibrant commutative ring is weakly equivalent to norm of a spectrum cofibrant approximation. This is the content of Proposition B.63, and is one of the main results of this appendix. We also need conditions to guarantee that the Mandell-May geometric fixed point functor assumes the correct homotopy type in the cases in which we need to make use of it. To meet these demands we go through the theory in some detail, and introduce special classes of maps and objects (*flat maps* (Definition B.15), *flat object* (Proposition B.55), *very flat objects* §B.4) in order to have a language for dealing with objects in model categories that are not cofibrant, but nevertheless have favorable homotopy properties. Lewis and Mandell have introduced a related notion of "almost cofibrant" in [32].

B.1. **Model structures.** The positive stable model structure is built in stages, beginning with the case of G-spaces.

B.1.1. *Spaces.* We make the category $\mathcal{T}^G$ into a model category by defining a map $X \to Y$ to be a *fibration* (*weak equivalence*) if for all $H \subseteq G$ the map of fixed point spaces $X^H \to Y^H$ is a Serre fibration (weak equivalence). The *cofibrations* are defined to be the maps having the left lifting property with respect to the acyclic fibrations. Equipped with these classes of maps $\mathcal{T}^G$ becomes a topological model category. The model structure is compactly generated, with

$$\mathcal{A} = \{(G/H)_+ \wedge S^{n-1}_+ \to (G/H)_+ \wedge D^n_+\} \tag{B.1}$$

generating the cofibrations, and

$$\mathcal{B} = \{(G/H)_+ \wedge I^n_+ \to (G/H)_+ \wedge I^{n+1}_+\} \tag{B.2}$$

generating the acyclic cofibrations.

Remark B.3. The proof that these classes define a model structure is straightforward. We point out one aspect, which plays a role in the discussion of geometric fixed points in §B.5. A *compact object* in $\mathcal{T}^G$ is an object A with the property that the map

$$\mathcal{T}^G(A, \varinjlim X_n) \to \varinjlim_n \mathcal{T}^G(A, X_n)$$

is an isomorphism when the sequence

$$\cdots \to X_{n-1} \to X_n \to \dots$$

is a sequence of *closed inclusions*. In order that the small object argument work, one needs to check that the maps formed by cobase change along the generating cofibrations are closed inclusions. This is a straightforward, but important point. For example, the formation of fixed points does not commute with cobase change in general, but it does commute with cobase change along a closed inclusion. It follows that in the model structure on $\mathcal{T}^G$, the formation of H-fixed points preserves the classes of cofibrations and acyclic cofibrations, and commutes with cobase change along a cofibration.

The smash product of G-spaces satisfies the *pushout product* and *monoid* axioms of [50] ([36, Proposition III.7.4]), making $\mathcal{T}_G = (\mathcal{T}^G, \wedge, S^0)$ into a *symmetric monoidal model category* in the sense of [50, Definition 3.1].

The pushout product axiom is the analogue of Quillen's SM7 [41], and asserts that if $A \to B$ and $K \to L$ are cofibrations, then the *corner map* from the pushout of

$$\begin{tikzcd} A \otimes K \arrow[r] \arrow[d] & A \otimes L \\ B \otimes K & \end{tikzcd}$$

to $B \otimes L$ is a cofibration which is a weak equivalence if one of $A \to B$ or $K \to L$ is. The monoid axiom states that if $A \to B$ is an acyclic cofibration, and X is any object, then

$$X \otimes A \to X \otimes B$$

is a weak equivalence, as is any cobase change, and (possibly transfinite) composition of such maps. Combined, they express the fact that the topological G-category $\mathcal{T}_G$ is an *enriched model category*, enriched over $\mathcal{T}^G$.

B.1.2. *The positive level model structure.* We now turn to an important intermediate model structure on $\mathcal{S}^G$.

Definition B.4. A map $X \to Y$ in $\mathcal{S}^G$ is a *positive level equivalence* (*positive level fibration*) if for each V with $\dim V^G > 0$, the map

$$X_V \to Y_V$$

is a weak equivalence (fibration) in $\mathcal{T}^G$.

Theorem B.5 ([36], Theorem III.2.10)**.** *The category $\mathcal{S}^G$ is a topological model category when equipped with the positive level fibrations and weak equivalences.*

The model structure on $\mathcal{S}^G$ given by Theorem B.5 is the *positive level model structure*. It is compactly generated, with the set

$$\{S^{-V} \wedge f \mid f \in \mathcal{A}, \quad \dim V^G > 0\} \tag{B.6}$$

generating the cofibrations, and

$$\{S^{-V} \wedge f \mid f \in \mathcal{B}, \quad \dim V^G > 0\} \tag{B.7}$$

generating the acyclic cofibrations. Here $\mathcal{A}$ and $\mathcal{B}$ are any sets generating the (acyclic) cofibrations on $\mathcal{T}^G$, for example those listed in (B.1) and (B.2).

Remark B.8. As discussed in Remark B.3, it is important to know that the positive level cofibrations are objectwise closed inclusions. The generating positive level cofibrations are objectwise closed inclusions. This is straightforward to check. The generating cofibrations are of the form

$$S^{-V} \wedge A \to S^{-V} \wedge B$$

where $A \to B$ is a cofibration of G-spaces. The value at W is

$$\mathscr{I}_G(V, W) \wedge A \to \mathscr{I}_G(V, W) \wedge B$$

which is a closed inclusion, since $A \to B$ is. Since coproducts and pushouts in $\mathcal{S}^G$ are computed objectwise (all limits and colimits are), this implies that all of the positive level cofibrations are closed inclusions.

Remark B.9. The "positive" condition in the model structure is due to Jeff Smith, and arose first in the theory of symmetric spectra. The choice is dictated by two requirements. One is that the forgetful functor from commutative algebras in $\mathcal{S}^G$ to $\mathcal{S}^G$ create a model category structure from the positive stable structure on $\mathcal{S}^G$ (Theorem B.29 below). The other is that the geometric fixed point functor (§B.5) preserve (acyclic) cofibrations. The first requirement could be met by taking "positive" to mean $\dim V > 0$. The second requires $\dim V^G > 0$, once one is using a positive model structure on $\mathcal{S}$.

B.1.3. *The positive stable model structure.* Suppose that X is an equivariant orthogonal spectrum. For an integer $k \in \mathbb{Z}$ and a subgroup $H \subset G$ define

$$\pi_k^H(X) = \varinjlim_V [S^{k+V}, X_V]^H,$$

in which the symbol $[\,-\,,\,-\,]^H$ denotes the set of H-equivariant homotopy classes of maps. As H varies, the groups $\pi_k^H X$ form a Mackey functor $\underline{\pi}_k X$. The collection of groups $\pi_k^H X$ are the *stable homotopy groups* of X, and they form the graded Mackey functor $\underline{\pi}_* X$.

Definition B.10. A *stable weak equivalence* in $\mathcal{S}^G$ is a map $X \to Y$ inducing an isomorphism of stable homotopy groups π_n^H for all $H \subseteq G$ and $n \in \mathbb{Z}$.

Definition B.11. The *positive stable model structure* on $\mathcal{S}^G$ is the localization of the positive level model structure at the stable weak equivalences.

The smash product of equivariant orthogonal spectra in the positive stable model structure satisfies the pushout product and monoid axioms of [50] ([36, Proposition III.7.4]) described at the end of §B.1.2, making $\mathcal{S}^G$ into a *symmetric monoidal model category*. The monoid axiom is proved by direct computation of stable homotopy groups, using Lemma A.17. Corollary 6.9 below shows that a slightly stronger condition actually holds. Because of this, we may regard the topological G-category $\mathcal{S}_G$ as an enriched model category, enriched over $\mathcal{T}^G$.

In this paper we will work in the topological G-category $\mathcal{S}_G$ in its (enriched) positive stable model structure. Unless it is likely to cause confusion, the terms "fibration," "cofibration," and "weak equivalence" to refer to the positive stable fibrations, cofibrations, and weak equivalences. As in any localization, the positive stable cofibrations are the same as the positive level cofibrations, and the positive stable acyclic fibrations are the same as the level acyclic fibrations. The acyclic cofibrations are the positive level cofibrations inducing an isomorphism of stable homotopy groups, while the fibrations are the maps having the left lifting property with respect to the acyclic cofibrations.

The positive stable model structure can also be described as the localization of the positive level model structure at the maps

$$S^{-(V\oplus W)} \wedge S^W \to S^{-V}. \tag{B.12}$$

From this it is easy to check that an orthogonal G-spectrum X is fibrant if and only if for each equivariant inclusion $V \subset W$ in which V contains a trivial representation, the map

$$X_V \to \Omega^{W-V} X_W$$

is a weak equivalence. It also follows that the positive stable model structure is compactly generated. To specify generators, factor (B.12) into a (positive level) cofibration followed by a positive level weak equivalence

$$S^{-(V\oplus W)} \wedge S^W \hookrightarrow \tilde{S}^{-V} \xrightarrow{\sim} S^{-V},$$

Since the cofibrations are unchanged under localization, the generators for the positive stable cofibrations can be taken to be the same as the generators for the positive level model structure. The set of generating acyclic cofibrations is the union of the set of generating acyclic cofibrations for the positive level model structure, and the corner maps formed by smashing the maps

$$S^{-(V\oplus W)} \wedge S^W \hookrightarrow \tilde{S}^{-V}$$

in which $\dim V^G > 0$ with the generating cofibrations.

B.1.4. *Flat maps.* The usual directive when dealing with a construction in a model category is to replace all of the objects with cofibrant-fibrant approximations, replace all the maps with cofibrations or fibrations, then form the construction, and determine the homotopy type of the outcome. In practice many of these steps turn out not to be necessary. For example the homotopy type of the cofiber of a map is realized by the mapping cone, no matter how pathological the spaces involved may be. The notion of a *flat map* and a *flat functor* was introduced in the unpublished manuscript [23] in order to incorporate this practice into the general theory. The dual notion was coined a "sharp map" by Charles Rezk, and used for a different purpose in [46].

Definition B.13. A functor $F : \mathcal{C} \to \mathcal{D}$ between model categories is *flat* if it preserves colimits and weak equivalences.

Typically the functor F will be a left adjoint, and so will automatically preserve colimits.

Example B.14. Proposition III.7.3 of [36] (see also [37, Proposition 12.3]) asserts that if $A \in \mathcal{S}^G$ is cofibrant, then the functor $A \wedge (-)$ is flat. We will refer to this by saying that cofibrant equivariant orthogonal spectra are flat.

Definition B.15. Suppose that $\mathcal{C}$ is a model category. A map $f : A \to X$ in $\mathcal{C}$ is *flat* if for every $A \to B$ and every weak equivalence $B \to B'$, the map

$$X \underset{A}{\cup} B \to X \underset{A}{\cup} B'$$

is a weak equivalence.

In other words a morphism f is flat if and only if "cobase change along f" preserves weak equivalences. Since cobase change is a left adjoint this is equivalent to the flatness of the cobase change functor.

Example B.16. A model category is *left proper* if and only if every cofibration is flat.

Proposition B.17. i) *Finite coproducts of flat maps are flat.*

ii) *Composites of flat maps are flat*

iii) *Any cobase change of a flat map is flat.*

iv) *A retract of an flat map is flat.*

□

Proposition B.18. *Suppose that*

$$\begin{tikzcd} X_1 \arrow[d] & A_1 \arrow[l] \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_2 & A_2 \arrow[l] \arrow[r] & Y_2 \end{tikzcd}$$

is a diagram in which $A_2 \to Y_2$ and both maps in the top row are flat. If the vertical maps are weak equivalences, then so is the map

$$X_1 \underset{A_1}{\cup} Y_1 \to X_2 \underset{A_2}{\cup} Y_2$$

of pushouts.

Proof: First suppose that $A_1 = A_2 = A$. Then

$$X_1 \underset{A}{\cup} Y_1 \to X_1 \underset{A}{\cup} Y_2$$

is a weak equivalence since $A_1 \to X_1$ is flat. The map $X_1 \to X_1 \underset{A}{\cup} Y_2$ is flat, since it is a cobase change of $A \to X$ along $A \to Y_2$. But this implies that

$$X_1 \underset{A}{\cup} Y_2 \to X_2 \underset{X_1}{\cup} \left(X_1 \underset{A}{\cup} Y_2\right) = X_2 \underset{A}{\cup} Y_2.$$

is a weak equivalence. Putting these together gives the result in this case.

For the general case, consider the following diagram

$$\begin{tikzcd} X_1 \arrow[d] & A_1 \arrow[l] \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_1 \underset{A_1}{\cup} A_2 \arrow[d] & A_2 \arrow[l] \arrow[r] \arrow[d, equal] & A_2 \underset{A_1}{\cup} Y_1 \arrow[d] \\ X_2 & A_2 \arrow[l] \arrow[r] & Y_2. \end{tikzcd}$$

The flatness of the maps $A_1 \to X_1$ and $A_1 \to Y_1$ implies that the upper vertical maps (hence all the vertical maps) are weak equivalences, and that the maps in the middle row are flat. It also implies that

$$A_1 \to X_1 \underset{A_1}{\cup} Y_1$$

is flat. Since $A_1 \to A_2$ is a weak equivalence, this means that

$$X_1 \underset{A_1}{\cup} Y_1 \to A_2 \underset{A_1}{\cup} \left(X_1 \underset{A_1}{\cup} Y_1\right)$$

is a weak equivalence. But this is the map from the pushout of the top row to the pushout of the middle row. By the case in which $A_1 = A_2$, the map from the pushout of the middle row to the pushout of the bottom row is a also a weak equivalence. This completes the proof. $\square$

Remark B.19. If the model category $\mathcal{C}$ has the property that every map can be factored into a flat map followed by a weak equivalence, then the above result holds with the assumption that only one of the maps in the top row is a weak equivalence.

Suppose for instance that it is the map $A_1 \to X_1$, and factor $A_1 \to Y_1$ into a flat map $A_1 \to Y_1'$ followed by a weak equivalence $Y_1' \to Y_1$ Now consider the diagram

$$\begin{tikzcd}
X_1 \arrow[d, equal] & A_1 \arrow[l] \arrow[r] \arrow[d, equal] & Y_1' \arrow[d] \\
X_1 \arrow[d] & A_1 \arrow[l] \arrow[r] \arrow[d] & Y_1 \arrow[d] \\
X_2 & A_2 \arrow[l] \arrow[r] & Y_2.
\end{tikzcd}$$

By Proposition B.18, the map from the pushout of the top row to the pushout of the middle row is a weak equivalence, as is the map from the pushout of the top row to the pushout of the bottom row. The map from the pushout of the middle row to the pushout of the bottom row is then a weak equivalence by the "two out of three" property of weak equivalences(Axiom M2). Note that this assumption holds if $\mathcal{C}$ is proper, ie, every cofibration in $\mathcal{C}$ is flat.

We now show that the h-cofibrations in $\mathcal{S}_G$ are flat, and that $\mathcal{S}_G$ is left proper.

Lemma B.20. *Every cofibration in $\mathcal{S}_G$ is an h-cofibration. If $X \to Y$ is an h-cofibration and Z is any equivariant orthogonal spectrum, then $Z \wedge X \to Z \wedge Y$ is an h-cofibration.*

Proof: The first assertion is [36, Lemma III.2.5], and follows from the fact that the generating cofibrations are h-cofibrations, and that any cobase change of an h-cofibration is an h-cofibration. The second is [36, Lemma III.7.1] and is an immediate consequence of the fact that $\mathcal{S}_G$ is a closed symmetric monoidal category. □

The following result is one reason h-cofibrations are such a useful class of maps. It is useful in establishing the basic homotopy theoretic properties of many constructions. The proof is straightforward (see [36, Theorem III.3.5]). One noteworthy point is that because of the stabilization, there is no non-degeneracy assumption required for the base point. Inclusions of wedge summands are always flat, and the stable homotopy groups of a wedge are always the direct sums of the stable homotopy groups of the wedge summands.

Lemma B.21. *An h-cofibration $X \to Y$ gives rise to a natural long exact sequence of stable homotopy groups*

$$\cdots \to \underline{\pi}_k X \to \underline{\pi}_k Y \to \underline{\pi}_k (Y/X) \to \underline{\pi}_{k-1} X \to \dots,$$

in which the map $\pi_k Y \to \pi_k Y/X$ is induced by the evident quotient map, and the connecting homomorphism $\underline{\pi}_k Y/X \to \underline{\pi}_{k-1} X$ is induced by the maps

$$Y/X \leftarrow Y \cup CX \to \Sigma X.$$

□

Corollary B.22. *The h-cofibrations in $\mathcal{S}_G$ are flat.*

Proof: Suppose that $A \to X$ is an h-cofibration, $A \to B$ is a map, and $B \to B'$ is a weak equivalence. Consider the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & X \underset{A}{\cup} B \arrow[r] \arrow[d] & (X \underset{A}{\cup} B)/B \arrow[d, "\approx"] \\ B' \arrow[r] & X \underset{A}{\cup} B' \arrow[r] & (X \underset{A}{\cup} B')/B' \end{tikzcd}$$

The left horizontal arrows are h-cofibrations since they are constructed by cobase change from $A \to X$. The fact that the middle arrow is a weak equivalence now follows from the long exact sequence of stable homotopy groups (Lemma B.21). □

Corollary B.23. *The positive stable model structure on $\mathcal{S}_G$ is left proper.* □

Since h-cofibrations in the category of compactly generated weak Hausdorff spaces are closed inclusions one has

Lemma B.24. *An h-cofibration is an objectwise closed inclusion.* □

B.1.5. *The canonical homotopy presentation.* Let

$$\cdots \subset V_n \subset V_{n+1} \subset \ldots$$

be an exhausting sequence of orthogonal G-representations, and consider the transition diagram

$$\begin{tikzcd} S^{-V_{n+1}} \wedge \mathscr{I}_G(V_n, V_{n+1}) \wedge X_n \arrow[r] \arrow[d] & S^{-V_{n+1}} \wedge X_{n+1} \\ S^{-V_n} \wedge X_n . & \end{tikzcd} \tag{B.25}$$

Write

$$W_n = V_{n+1} - V_n$$

for the orthogonal complement of V_n in V_{n+1}. The inclusion $V_n \subset V_{n+1}$ gives an embedding

$$S^{W_n} \to \mathscr{I}_G(V_n, V_{n+1}),$$

and so from (B.25) a diagram

$$\begin{tikzcd} S^{-(V_n \oplus W_n)} \wedge S^{W_n} \wedge X_{V_n} \arrow[r] \arrow[d] & S^{-V_{n+1}} \wedge X_{n+1} \\ S^{-V_n} \wedge X_n & \end{tikzcd}$$

Putting these together as n varies results in a system

$$A_0 \xleftarrow{\sim} B_0 \to A_1 \xleftarrow{\sim} B_1 \to A_2 \xleftarrow{\sim} B_2 \to A_3 \xleftarrow{\sim} B_3 \to \cdots. \tag{B.26}$$

The system (B.26) maps to X and a simple check of equivariant stable homotopy groups shows that the map from its homotopy colimit to X is a weak equivalence. Now for each n let C_n be the homotopy colimit of the portion

$$A_0 \xleftarrow{\sim} B_0 \to \cdots \to A_{n-1} \xleftarrow{\sim} B_{n-1} \to A_n \tag{B.27}$$

of (B.26). Then C_n is naturally weakly equivalent to $A_n = S^{-V_n} \wedge X_{V_n}$, and the C_n fit into a sequence

$$C_0 \to C_1 \to C_2 \to \dots \tag{B.28}$$

whose homotopy colimit coincides with that of (B.26). This gives the canonical homotopy presentation of X. One can functorially replace the sequence (B.28) with a weakly equivalent sequence of cofibrations between cofibrant-fibrant objects. The colimit of this sequence is naturally weakly equivalent to X. It will be cofibrant automatically, and fibrant since the model category $\mathcal{S}_G$ is compactly generated.

We write the canonical homotopy presentation of X as

$$X \approx \underset{\overrightarrow{V}}{\operatorname{holim}}\,(S^{-V} \wedge X_V)_{\mathrm{cf}}$$

with the subscript indicating a cofibrant-fibrant replacement.

B.1.6. *Rings and modules.*

Theorem B.29 ([36], Theorem III.8.1)**.** *The forgetful functor*

$$\mathbf{Comm}_G \to \mathcal{S}_G$$

creates an enriched topological model category structure on $\mathbf{Comm}_G$ *in which the fibrations and weak equivalences in* $\mathbf{Comm}_G$ *are the maps that are fibrations and weak equivalences in* $\mathcal{S}_G$*.*

Corollary B.30. *For* $H \subset G$*, the adjoint functors*

$$\mathbf{Comm}_H \leftrightarrows \mathbf{Comm}_G$$

form a Quillen pair.

Proof: The restriction functor obviously preserves the classes of fibrations and weak equivalences. □

Corollary B.31. *The norm functor on commutative algebras*

$$N_H^G : \mathbf{Comm}_H \to \mathbf{Comm}_G$$

is a left Quillen functor. It preserves the classes of cofibrations and acyclic cofibrations, hence weak equivalences between cofibrant objects.

Proof: This is immediate from Corollary B.30 and Proposition A.49. The assertion about weak equivalences is Ken Brown's lemma (see, for example [25, Lemma 1.1.12]). □

The category $\mathcal{M}_R$ of left modules over an equivariant associative algebra R as defined in §A.2.6. As pointed out there, when R is commutative, a left R-module can be regarded as a right R-module, and $\mathcal{M}_R$ becomes a symmetric monoidal category under the operation

$$M \underset{R}{\wedge} N. \tag{B.32}$$

Theorem B.33 ([36], Theorem III.7.6)**.** *The forgetful functor*

$$\mathcal{M}_R \to \mathcal{S}_G$$

creates a model structure on the category $\mathcal{M}_R$ in which the fibrations and weak equivalences are the maps which become fibrations and weak equivalences in $\mathcal{S}_G$. When R is commutative, the operation (B.32) *satisfies the pushout-product and monoid axioms making $\mathcal{M}_R$ into a symmetric monoidal model category.*

Corollary B.34. *Let $f : R \to S$ be a map of equivariant A_∞ algebras. The functors*

$$S \underset{R}{\wedge} (-) : \mathcal{M}_R \leftrightarrows \mathcal{M}_S : U$$

given by restriction and extension of scalars form a Quillen pair. If S is cofibrant as a left R-module, then the restriction functor is also a left Quillen functor.

Proof: Proposition B.33 implies that the restriction functor preserves fibrations and acyclic fibrations. This gives the first assertion. The second follows from the fact that the restriction functor preserves colimits, and the consequence of Proposition B.33 that the generating (acyclic) cofibrations for $\mathcal{M}_S$ are formed as the smash product of S with the generating (acyclic) cofibrations for $\mathcal{S}_G$. $\square$

The following result is [36, Proposition III.7.7].

Proposition B.35. *Suppose that R is an associative algebra, and M is a cofibrant right R-module. The functor $M \underset{R}{\wedge} (-)$ preserves weak equivalences.* $\square$

In other words, the functor $M \underset{R}{\wedge} (-)$ is flat if M is cofibrant.

B.2. **Homotopy properties of the norm.** Before stating our main result, we generalize the situation slightly. Given a G-set J, consider the category $\mathcal{S}^{\mathcal{B}_J G}$ of functors $\mathcal{B}_J G \to \mathcal{S}$. A choice of point t in each G-orbit of J gives an equivalence

$$\mathcal{S}^{\mathcal{B}_J G} \approx \prod_t \mathcal{S}^{H_t},$$

where H_t is the stabilizer of t. We give $\mathcal{S}^{\mathcal{B}_J G}$ the model structure corresponding to the product of the positive stable model structures under this equivalence. The model structure is independent of the chosen points in each orbit. We will refer to the model category $\mathcal{S}^{\mathcal{B}_J G}$ as the model category of *equivariant J-diagrams* of spectra.

The indexed smash product gives a functor

$$N^J : \mathcal{S}^{\mathcal{B}_J G} \to \mathcal{S}^G$$

which we denoted $X^{\wedge J}$. When $J = G/H$ this is the norm. Our main result is

Proposition B.36. *Suppose that J is a G-set. If $X \to Y$ is a cofibration of equivariant J-diagrams then the indexed smash product*

$$X^{\wedge J} \to Y^{\wedge J}$$

is an h-cofibration. It is a cofibration if X is cofibrant.

Remark B.37. Proposition B.36 asserts in particular that the norm takes cofibrant orthogonal H-spectra to cofibrant orthogonal G-spectra.

Proof: The first assertion is immediate from Proposition A.62. However, in order to establish the second assertion we need to describe a slightly different proof. We work by induction on $|J|$, and may therefore assume the result to be know for any $J_0 \subset J$ and any $H \subset G$ stabilizing J_0 as a subset. In particular, we may assume that if X is cofibrant, then $X^{\wedge J_0}$ is a cofibrant H-spectrum for any proper $J_0 \subset J$ and any $H \subseteq G$ stabilizing J_0 as a subset.

We consider the case in which $X \to Y$ arises from a pushout square of J-diagrams

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in which $A \to B$ is a generating cofibration. We will show in this case that $X^{\wedge J} \to Y^{\wedge J}$ is an h-cofibration, and is a cofibration if X is cofibrant. Since the formation of indexed smash products commutes with directed colimits and retracts, the proposition will then follow from the small object argument.

Give $Y^{\wedge J}$ the filtration described in §A.3.4. The successive terms are related by the pushout square

$$\begin{array}{ccc} \bigvee\limits_{\substack{J=J_0 \amalg J_1 \\ |J_1|=n}} X^{\wedge J_0} \wedge \partial_A B^{\wedge J_1} & \longrightarrow & \bigvee\limits_{\substack{J=J_0 \amalg J_1 \\ |J_1|=n}} X^{\wedge J_0} \wedge B^{\wedge J_1} \\ \downarrow & & \downarrow \\ \operatorname{fil}_{n-1} Y^{\wedge Y} & \longrightarrow & \operatorname{fil}_n Y^{\wedge J}. \end{array} \tag{B.38}$$

Consider an individual map

$$\partial_A B^{\wedge J_1} \to B^{\wedge J_1}, \tag{B.39}$$

regarded as a map of H-spectra, where $H \subset G$ is the stabilizer of J_1. Since $A \to B$ is a generating cofibration, it is of the form

$$S^{-V_j} \wedge S_+^{n_j - 1} \to S^{-V_j} \wedge D_+^{n_j}$$

in which each V_j contains a non-zero vector fixed by the isotropy group of $j \in J$. The map (B.39) is then

$$S^{-\oplus V_j} \wedge S(W)_+ \to S^{-\oplus V_j} \wedge D(W)_+ \tag{B.40}$$

in which the index j is running through J_0 and the W is the evident indexed sum of trivial representations. Since each representation V_j contains a non-zero invariant vector, so does the indexed sum $\oplus_j V_j$, and (B.40) is therefore a positive level cofibration. Thus the bottom row of (B.38) is an h-cofibration and hence so is $X^{\wedge J} \to Y^{\wedge J}$. This proves the first assertion. If X is cofibrant, then $X^{\wedge J_0}$ is cofibrant by induction, hence so is

$$X^{\wedge J_0} \wedge \partial_A B^{\wedge J_1} \to X^{\wedge J_0} \wedge B^{\wedge J_1}$$

by the pushout-product axiom. Since indexed wedges preserve cofibrations, the top row of (B.38) is then a cofibration and hence so is the bottom row. □

For later reference we highlight one fact that emerged in the proof of Proposition B.36.

Lemma B.41. *If $A \to B$ is a generating cofibration in $\mathcal{S}^{\mathcal{B}_J G}$ then*

$$\partial_A B^{\wedge J} \to B^{\wedge J}$$

is a generating cofibration in $\mathcal{S}^G$, and hence an h-cofibration. □

Proposition B.42. *The functor*

$$N_H^G : \mathcal{S}_H \to \mathcal{S}_G$$

takes weak equivalences between cofibrant objects to weak equivalences.

Proof: The result makes use of the Quillen functors

$$\mathrm{Sym}^H : \mathcal{S}_H \leftrightarrows \mathbf{Comm}_H : U_H$$
$$\mathrm{Sym}^G : \mathcal{S}_G \leftrightarrows \mathbf{Comm}_G : U_G,$$

and the commutative diagram of Proposition A.39 which gives a natural isomorphism

$$N_H^G \circ U_H \approx U_G \circ N_H^G. \tag{B.43}$$

Suppose that $f : X \to Y$ is a weak equivalence between cofibrant objects in $\mathcal{S}_H$. Then the map $\mathrm{Sym}^H X \to \mathrm{Sym}^H Y$ is a weak equivalence between cofibrant objects in $\mathbf{Comm}_H$ since Sym^H is a left Quillen functor. Since the norm on commutative rings is a left Quillen functor (Corollary B.31) the map

$$N_H^G \mathrm{Sym}^H X \to N_H^G \mathrm{Sym}^H Y$$

is also a weak equivalence of cofibrant objects. Now the functor U_G creates weak equivalences, so

$$U_G\, N_H^G\, \mathrm{Sym}^H X \to U_G\, N_H^G\, \mathrm{Sym}^H Y \tag{B.44}$$

is a weak equivalence in $\mathcal{S}_G$. By (B.43) this map can be identified with

$$N_H^G\, U_H\, \mathrm{Sym}^H X \to N_H^G\, U_H\, \mathrm{Sym}^H Y. \tag{B.45}$$

But the map $X \to Y$ is a retract of $U_H \mathrm{Sym}^H X \to U_H \mathrm{Sym}^H Y$, and so $N_H^G X \to N_H^G Y$ is a retract of the weak equivalence (B.45) and hence also a weak equivalence. □

B.3. **Symmetric powers.** The symmetric powers of a cofibrant orthogonal spectrum are rarely cofibrant. But they are nearly so. In this section we investigate some homotopy theoretic properties of symmetric powers. These results will be used to get at further homotopy theoretic properties of the norm, and to show that the spectra underlying cofibrant commutative rings have many properties in common with cofibrant spectra, even though they are not.

We eventually wish to study the norms of symmetric powers of spectra. In order to absorb this into the discussion, we work at the outset with the orbit spectrum of an indexed smash product, by a subgroup of a symmetry group of the indexing set. More specifically let I be a finite G-set, and Σ_I the group automorphisms of I with G acting by conjugation. Regarding $A \in \mathcal{S}^G$ as a constant diagram in $\mathcal{S}^I$ one may form $A^{\wedge I}$. The G-action on $A^{\wedge I}$ extends naturally to $\Sigma_I \rtimes G$. Our aim is to investigate the homotopy theoretic properties of the orbit G-spectrum

$$\mathrm{Sym}_\Sigma^I A = A^{\wedge I}/\Sigma$$

for $\Sigma \subset \Sigma_I$ a G-stable subgroup. When the indexing set I has a trivial G-action and the symmetry group is the full symmetry group of I this is the symmetric power.

Before turning to these (generalized) symmetric powers, we discuss some universal G-spaces.

B.3.1. *Equivariant principal bundles.* Let Σ be a group with a G-action.

Definition B.46. A *universal* Σ*-space* is a space $E_G\Sigma$ with the property that for each finite $\Sigma \rtimes G$-set S, the space of $\Sigma \rtimes G$-equivariant maps

$$S \to E_G\Sigma$$

is empty if some element of S is fixed by a non-identity element of Σ, and contractible otherwise.

The defining property characterizes a universal Σ-space up to $\Sigma \rtimes G$-equivariant weak homotopy equivalence. The space $E_G\Sigma$ is the total space of the universal G-equivariant principal Σ-bundle. It can be constructed as a G-CW complex, using cells of the form $(\Sigma \rtimes G/H) \times D^m$, with $H \subset \Sigma \rtimes G$, and $H \cap \Sigma = \{e\}$.

Let J be a finite G-set, and $E_G\Sigma$ a universal Σ-space.

Lemma B.47. *The space*

$$(E_G\Sigma)^J$$

equipped with the evident $\Sigma^J \rtimes G$*-action is a universal* Σ^J*-space.*

Proof: Let S be a finite $\Sigma^J \rtimes G$-set. Note that since Σ^J acts trivially on J, the set S has a element fixed by a non-trivial element of Σ^J if and only if $S \times J$ does. The result then follows easily from the identification of the space of maps

$$S \to (E_G\Sigma)^J$$

with the space of maps

$$S \times J \to E_G\Sigma.$$

□

B.3.2. *Symmetric powers.* We now turn to our analysis of symmetric powers.

Definition B.48. Suppose that Σ is a group with an action of G, and that $X \in \mathcal{S}_G$ is an object equipped with an action of $\Sigma \rtimes G$ extending the G-action. We will say that X is Σ*-free* if for each orthogonal G-representation W the Σ-action on X_W is free away from the base point.

We now fix a G-set I, and let Σ be any G-stable subgroup of Σ_I.

Lemma B.49. *If* $A \in \mathcal{S}_G$ *is cofibrant and* Z *is any* G*-spectrum then* $A^{\wedge I} \wedge Z$ *is* Σ*-free. The map*

$$(E_G\Sigma)_+ \underset{\Sigma}{\wedge} A^{\wedge I} \wedge Z \to \mathrm{Sym}^I_\Sigma(A) \wedge Z \tag{B.50}$$

is a level equivalence hence a weak equivalence.

Remark B.51. The proof of Lemma B.49 is nearly identical to that of [36, Lemma III.8.4]. We go through the details because the statement is slightly more general, and in order to correct a minor error in [36]. The statements of [36, Lemma III.8.4], and the related [36, Lemma IV.4.5] each contain a misleading typo. In both cases, the symbol $E\Sigma_i$ is used, whereas the object that should really be used is $E_G\Sigma_i$.

Before turning to the proof, we state a mild generalization of Lemma B.41 which we will need.

Lemma B.52. *Let I be any finite G-set, and $\Sigma \subset \Sigma_I$ a G-stable subgroup. If $S \to T$ is a generating cofibration in $\mathcal{S}_G$ then*
$$\partial_S T^{\wedge I} \to T^{\wedge I}$$
is a Σ-equivariant h-cofibration.

Proof: As spelled out in the proof of Proposition B.36, the map
$$\partial_S T^{\wedge I} \to T^{\wedge I}$$
is of the form
$$S^{-V^I} \wedge S(W)_+ \to S^{-V^I} \wedge D(W)_+,$$
with W a representation of $\Sigma \rtimes G$, and V a representation of G. The map $S(W) \to D(W)$ is an equivariant h-cofibration since it is the inclusion of the base into a cone. The claim then follows since smashing with S^{-V^I} preserves the formation of mapping cylinders. $\square$

Proof of Lemma B.49: For the first assertion, it suffices to show that if $S \to T$ is a generating cofibration,
$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ A \arrow[r] & B, \end{tikzcd}$$
is a pushout square, and $A^{\wedge I} \wedge Z$ is Σ_I-free then $B^{\wedge I} \wedge Z$ is Σ_I-free. We use the filtration described in §A.3.4 and consider the pushout square below

$$\begin{tikzcd} \displaystyle\bigvee_{\substack{I=I_0\amalg I_1\\ |I_1|=m}} A^{\wedge I_0} \wedge \partial_S T^{\wedge I_1} \wedge Z \arrow[r] \arrow[d] & \displaystyle\bigvee_{\substack{I=I_0\amalg I_1\\ |I_1|=m}} A^{\wedge I_0} \wedge T^{\wedge I_1} \wedge Z \arrow[d] \\ \operatorname{fil}_{m-1} B \wedge Z \arrow[r] & \operatorname{fil}_m B \wedge Z. \end{tikzcd} \tag{B.53}$$

Since $S \to T$ is a generating cofibration, the rightmost map in the top row is an h-cofibration (Lemma B.41) hence a closed inclusion. It therefore suffices to show that Σ_I acts freely away from the base point on the upper right term. Induction on $|I|$ reduces this to the case $m = |I|$, and the assertion is thus reduced to the case
$$A = S^{-V} \wedge (G/H)_+ \wedge D^k_+,$$
with V an orthogonal representation containing a non-zero invariant vector. It suffices to consider the case $A = S^{-V}$, since the other factors can be absorbed into Z. In this case $A^{\wedge I} = S^{-V^I}$. For an orthogonal G-representation W we have, by Lemma A.17,
$$(S^{-V^I} \wedge Z)_W = \begin{cases} * & \dim W < \dim V^I \\ O(V^I \oplus U, W)_+ \underset{O(U,U)}{\wedge} Z_U & \dim W \geq \dim V^I \end{cases}$$
in which U is any orthogonal G-representation with $\dim U + \dim V^I = \dim W$. In the first case there is nothing to prove. The second follows easily from the fact that

$O(V^I \oplus U, W)/O(U,U)$ is the Stiefel-manifold $O(V^I, W)$ which is Σ_I-free since V is non-zero and Σ_I acts trivially on W.

With one additional observation, a similar argument reduces the second assertion to the case $A = S^{-V}$. To spell it out, abbreviate (B.53) as

$$\begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

and form

$$\begin{tikzcd} (E_G\Sigma)_+ \underset{\Sigma}{\wedge} X \arrow[d, "\sim"'] & (E_G\Sigma)_+ \underset{\Sigma}{\wedge} V \arrow[l] \arrow[r, "\flat"] \arrow[d, "\sim"'] & (E_G\Sigma)_+ \underset{\Sigma}{\wedge} W \arrow[d, "\sim"] \\ X/\Sigma & V/\Sigma \arrow[l] \arrow[r, "\flat"] & W/\Sigma \end{tikzcd}$$

Using Lemma B.52 one checks that the rightmost maps in both rows are h-cofibrations, hence flat. Inductively assuming the vertical maps to be weak equivalences then implies that he map of pushouts is a weak equivalence (Remark B.19). This gives the reduction of the claim.

We now assume $A = S^{-V}$. As above, the map on W-spaces induced by (B.50) is the identity map of the terminal object if $\dim W < \dim V^{\oplus I}$ and otherwise the map of Σ-orbit spaces induced by

$$(E_G\Sigma)_+ \wedge O(V^I \oplus U, W)_+ \underset{O(U,U)}{\wedge} Z_U \to O(V^I \oplus U, W)_+ \underset{O(U,U)}{\wedge} Z_U,$$

where U is any G-representation with $\dim U = \dim W - \dim V^I$. The claim then follows from the fact that

$$E_G\Sigma \times O(V^I \oplus U, W) \to O(V^I, W)$$

is an equivariant homotopy equivalence compact Lie group

$$\mathscr{G} = (O(U,U) \times \Sigma) \rtimes G.$$

For this, note that both sides are $\mathscr{G}$-CW complexes, so it suffices to check that the map is a homotopy equivalence of H-fixed point spaces, for all $H \subset \mathscr{G}$. If the image of H in $\Sigma \rtimes G$ is not a subgroup of Σ then $E_G\Sigma^H$ is contractible and the claim holds. If the image of H is contained in Σ, then H is a subgroup of $O(U,U) \times \Sigma_n$, which acts freely on $O(V^n \oplus U, W)$, and both sides have empty H-fixed point spaces. $\square$

Lemma B.54. *If A is cofibrant and $\Sigma \subset \Sigma_I$ is any G-stable subgroup, then $(E_G\Sigma)_+ \underset{\Sigma}{\wedge} A^{\wedge I}$ is a cofibrant G-spectrum.*

Proof: By working through an equivariant cell decomposition of $E_G\Sigma$, we reduce to showing that

$$((\Sigma_I \rtimes G)/H)_+ \wedge S^{m-1}_+ \underset{\Sigma}{\wedge} A^{\wedge I} \to ((\Sigma_I \rtimes G)/H)_+ \wedge D^m_+ \underset{\Sigma}{\wedge} A^{\wedge I}$$

is a cofibration, where $H \subset \Sigma_I \rtimes G$ has the property that $H \cap \Sigma_I = \{e\}$. For this it suffices to show that

$$(\Sigma_I \rtimes G/H)_+ \underset{\Sigma}{\wedge} A^{\wedge I}$$

is cofibrant. This spectrum is an indexed wedge over the Σ-orbits $\mathcal{O} \subset (G \times \Sigma)/H$, the summand corresponding to $\mathcal{O}$ is the $G_{\mathcal{O}}$-spectrum

$$\mathcal{O}_{+} \underset{\Sigma}{\wedge} A^{\wedge I},$$

with $G_{\mathcal{O}} \subset G$ the subgroup of G preserving $\mathcal{O}$. Since $\mathcal{O}$ is a Σ-torsor, this is just the indexed smash product

$$A^{\wedge I'}$$

with $I' = \mathcal{O} \underset{\Sigma}{\times} I$, and is cofibrant by Proposition B.36. $\square$

Proposition B.55. *The symmetric powers of cofibrant G-spectra are flat: if $A \in \mathcal{S}_G$ is cofibrant and $X \to Y$ is a weak equivalence then*

$$\mathrm{Sym}^I_\Sigma(A) \wedge X \to \mathrm{Sym}^I_\Sigma(A) \wedge Y$$

is a weak equivalence.

Proof: Consider the diagram

$$\begin{tikzcd}
(E_G\Sigma)_+ \underset{\Sigma}{\wedge} A^{\wedge I} \wedge X \arrow[r] \arrow[d] & \mathrm{Sym}^I_\Sigma(A) \wedge X \arrow[d] \\
(E_G\Sigma)_+ \underset{\Sigma}{\wedge} A^{\wedge I} \wedge Y \arrow[r] & \mathrm{Sym}^I_\Sigma(A) \wedge Y.
\end{tikzcd}$$

Since $(E_G\Sigma)_+ \underset{\Sigma}{\wedge} A^{\wedge I}$ is cofibrant it is flat (Example B.14), and the left vertical map is a weak equivalence. The horizontal maps are weak equivalences by Lemma A.58. It follows that the right vertical map is a weak equivalence. $\square$

In the next section we will establish, among other things, the analogue of Proposition B.55 for cofibrant commutative algebras. As preparation, we require a generalization of the above results to the case of equivariant J-diagrams (§B.2).

We continue with our fixed finite G-set I and G-stable subgroup $\Sigma \subset \Sigma_I$. In addition we now choose a second finite G-set J, and give the category $\mathcal{S}^{\mathcal{B}_J G}$ of equivariant J-diagrams the model structure described in §B.2. Start with an equivariant J-diagram A and form $A^{\wedge I}$. Then $A^{\wedge I}$ is an equivariant J-diagram equipped with an action of Σ, and the group Σ^J of functions

$$J \to \Sigma$$

acts on the indexed smash product

$$(A^{\wedge I})^{\wedge J}.$$

Lemma B.56. *If $A \in \mathcal{S}^{\mathcal{B}_J G}$ is cofibrant and Z is any G-spectrum then $(A^{\wedge I})^{\wedge J}$ is Σ^J-free. The map*

$$E(\Sigma^J)_+ \underset{\Sigma^J}{\wedge} (A^{\wedge I})^{\wedge J} \wedge Z \to ((A^{\wedge I})^{\wedge J}/\Sigma^J) \wedge Z$$

is a level equivalence, hence a weak equivalence.

Proof: The proof is very similar to that of Lemma B.49. One reduces to the case

$$A = S^{-V}$$

where now V is an equivariant J-diagram

$$j \mapsto V_j$$

of (orthogonal) real vector spaces, having the property that each V_j has a non-zero vector fixed by the isotropy group of $j \in J$. Write

$$V^J = \bigoplus_j V_j$$
$$(V^I)^J = \bigoplus_j (V_j)^I,$$

so that

$$(A^{\wedge I})^{\wedge J} = S^{-(V^I)^J}.$$

As in the proof of Lemma B.49, the freeness follows from the fact that $O((V^I)^J \oplus U, W)/O(U,U)$ is the Stiefel-manifold $O((V^I)^J, W)$ which is Σ^J-free when V is non-zero, and Σ^J acts trivially on W, and U is any orthogonal G-representation with

$$\dim U + \dim (V^I)^J = \dim W.$$

The level equivalence assertion follows from the fact that

$$E_G(\Sigma^J) \times O((V^I)^J \oplus U, W) \to O((V^I)^J \oplus U, W)$$

is a weak equivalence of $\mathscr{G}$-CW complexes, hence a $\mathscr{G}$-homotopy equivalence for the compact Lie group

$$\mathscr{G} = (O(U,U) \times \Sigma) \rtimes G.$$

□

The next results assert that the norm of the symmetric powers of a cofibrant equivariant J-diagram have the "correct" homotopy type (Proposition B.57), and that, like cofibrant spectra, they are flat (Proposition B.58).

Proposition B.57. *If $A \in \mathcal{S}^{\mathcal{B}_J G}$ is cofibrant then*

$$E_G\Sigma_+ \underset{\Sigma}{\wedge} A^{\wedge I} \to \operatorname{Sym}^I_\Sigma A$$

is a cofibrant approximation. For any G-spectrum Z the map

$$(E_G\Sigma_+ \underset{\Sigma}{\wedge} A^{\wedge I})^{\wedge J} \wedge Z \to (\operatorname{Sym}^I_\Sigma A)^{\wedge J} \wedge Z$$

is a level equivalence, hence a weak equivalence.

Proof: The first assertion is just the combination of Lemmas B.49 and B.54 applied component-wise. For the second, use the fact that N^J is symmetric monoidal to write

$$(E_G\Sigma_+ \underset{\Sigma}{\wedge} A^{\wedge I})^{\wedge J} \approx (E_G\Sigma^J_+ \underset{\Sigma^J}{\wedge} A^{\wedge I})^{\wedge J}$$
$$(\operatorname{Sym}^I_\Sigma A)^{\wedge J} \approx (A^{\wedge I})^{\wedge J}/\Sigma^J,$$

and the equivalence

$$(E_G\Sigma)^J \approx E_G(\Sigma^J)$$

of Lemma B.47 to reduce the claim to Lemma B.56. □

Proposition B.58. *The norm of the symmetric powers of a cofibrant equivariant J-diagram are flat: if A is a cofibrant equivariant J-diagram and $X \to Y$ is a weak equivalence of orthogonal G-spectra, then*

$$(\mathrm{Sym}^I_\Sigma A)^{\wedge J} \wedge X \to (\mathrm{Sym}^I_\Sigma A)^{\wedge J} \wedge Y$$

is a weak equivalence.

Proof: As in the proof of Proposition B.55, consider the diagram

$$\begin{tikzcd}
(E_G\Sigma_+ \underset{\Sigma}{\wedge} A^{\wedge I})^{\wedge J} \wedge X \arrow[r] \arrow[d] & (\mathrm{Sym}^I_\Sigma A)^{\wedge J} \wedge X \arrow[d] \\
(E_G\Sigma_+ \underset{\Sigma}{\wedge} A^{\wedge I})^{\wedge J} \wedge Y \arrow[r] & (\mathrm{Sym}^I_\Sigma A)^{\wedge J} \wedge Y.
\end{tikzcd}$$

The object $(E_G\Sigma_+ \underset{\Sigma}{\wedge} A^{\wedge I})^{\wedge J}$ is cofibrant (hence flat by Example B.14) since $E_G\Sigma_+ \underset{\Sigma}{\wedge} A^{\wedge I}$ is (Lemma B.54), and since the norm sends cofibrant objects to cofibrant objects (Proposition B.36). It follows that the left vertical map is a weak equivalence. The horizontal maps are weak equivalences by Proposition B.57. This completes the proof. □

B.4. **More homotopy properties of the norm.** We continue investigating the category $\mathcal{S}^{\mathcal{B}_J G}$ with the model structure described in §B.2. For lack of a better term, let's call an equivariant J-diagram X *very flat* if it has the following property: for every finite map of G-sets $p : J' \to J$ every cofibrant approximation $\tilde{X} \to X$ and every orthogonal G-spectrum Z the map

$$(p^*\tilde{X})^{\wedge J'} \wedge Z \to (p^*X)^{\wedge J'} \wedge Z \tag{B.59}$$

is a weak equivalence. Throughout this section we will drop the p^* from the notation and simply write $X^{\wedge J'}$ in place of $(p^*X)^{\wedge J'}$.

Remark B.60. Since $\tilde{X}^{\wedge J'}$ is cofibrant (Proposition B.36) this property implies that $X^{\wedge J'}$ is flat.

Remark B.61. If (B.59) is a weak equivalence for one cofibrant approximation it is a weak equivalence for any cofibrant approximation.

Lemma B.62. *If A is a cofibrant equivariant J-diagram and $n \geq 0$, then* $\mathrm{Sym}^n A$ *is very flat.*

Proof: That A is very flat follows from the fact that cofibrant spectra are flat (Example B.14) and that the indexed smash product of cofibrant spectra are cofibrant (Proposition B.36). The assertion about the higher symmetric powers is an immediate consequence Proposition B.57. □

Our main goal is to establish the following result.

Proposition B.63. *If $R \in \mathcal{S}^{\mathcal{B}_J G}$ is cofibrant commutative ring, then R is very flat.*

First a straightforward lemma.

Lemma B.64. *Arbitrary wedges of very flat spectra are very flat. Smash products of very flat spectra are very flat. Filtered colimits of very flat G-diagrams along h-cofibrations are very flat.*

Proof: For the first assertion we will show that if X and Y are very flat, then so is $X \vee Y$. The general case is similar. By Remark B.61 it suffices to consider the case of a cofibrant approximation constructed by wedging together cofibrant approximations $\tilde{X} \to X$ and $\tilde{Y} \to Y$ of X and Y individually. We may also assume that $J' = J$ since J is arbitrary. Using the distributive law of §A.3.3, decompose the map

$$(\tilde{X} \vee \tilde{Y})^{\wedge J} \wedge Z \to (X \vee Y)^{\wedge J} \wedge Z$$

into the indexed wedge

$$\bigvee_{J = J_0 \amalg J_1} \tilde{X}^{\wedge J_0} \wedge \tilde{Y}^{\wedge J_1} \wedge Z \to \bigvee_{J = J_0 \amalg J_1} X^{\wedge J_0} \wedge Y^{\wedge J_1} \wedge Z.$$

The summands are indexed by the set-theoretic decompositions $J = J_0 \amalg J_1$, regarded as H-sets, with $H \subset G$ the subgroup stabilizing the decomposition. Since indexed wedges of weak equivalences are weak equivalences, it suffices to show that

$$\tilde{X}^{\wedge J_0} \wedge \tilde{Y}^{\wedge J_0} \wedge Z \to X^{\wedge J_0} \wedge Y^{\wedge J_1} \wedge Z \tag{B.65}$$

is a weak equivalence. Factoring (B.65) as

$$\tilde{X}^{\wedge J_0} \wedge \tilde{Y}^{\wedge J_1} \wedge Z \to X^{\wedge J_0} \wedge \tilde{Y}^{\wedge J_1} \wedge Z \to X^{\wedge J_0} \wedge Y^{\wedge J_1} \wedge Z$$

the claim then follows from the fact that X and Y are very flat.

The case of the smash product is more straightforward. We will consider the case of two factors, to which the general case reduces by induction. We will also assume that $J' = J$. By Remark B.61, we may work with a cofibrant approximation

$$\tilde{X} \wedge \tilde{Y} \to X \wedge Y$$

constructed by smashing together cofibrant approximations to X and Y. Using the fact that the norm is symmetric monoidal, write

$$(\tilde{X} \wedge \tilde{Y})^{\wedge J} \approx \tilde{X}^{\wedge J} \wedge \tilde{Y}^{\wedge J}$$
$$(X \wedge Y)^{\wedge J} \approx X^{\wedge J} \wedge Y^{\wedge J}$$

and factor

$$(\tilde{X} \wedge \tilde{Y})^{\wedge J} \wedge Z \to (X \wedge Y)^{\wedge J} \wedge Z$$

as

$$\tilde{X}^{\wedge J} \wedge \tilde{Y}^{\wedge J} \wedge Z \to X^{\wedge J} \wedge \tilde{Y}^{\wedge J} \wedge Z \to X^{\wedge J} \wedge Y^{\wedge J} \wedge Z.$$

The claim then follows from the fact that X and Y are very flat. The assertion about filtered colimits makes use of Proposition A.62, and is left to the reader. $\square$

The next lemma requires an awkward condition on a map of equivariant J-diagrams $S \to T$. If the indexed corner map

$$\partial_S T^{\wedge J} \to T^{\wedge J}$$

were known to be an h-cofibration whenever $S \to T$ is, the condition of being an h-cofibration would suffice. We have not found a proof of this, and are not sure it is true. We have settled instead for the following, which simply requires what is needed at a key point in the proof. We suppose that $S \to T$ is a map of J-diagrams with the property that for some cofibrant approximation $\tilde{S} \to \tilde{T}$ of $S \to T$ and for every $J' \subset J$, the maps

$$\partial_S T^{\wedge J'} \to T^{\wedge J'}$$
$$\partial_{\tilde{S}} \tilde{T}^{\wedge J'} \to \tilde{T}^{\wedge J'}$$

are G'-equivariant h-cofibration, where $G' \subset G$ is the largest subgroup stabilizing J'. When it comes up (in the proof of Proposition B.63) the map $S \to T$ will be of the form

$$X \wedge \partial_A \operatorname{Sym}^n B \to X \wedge \operatorname{Sym}^n B$$

with $A \to B$ a generating cofibration. That the condition holds in this case is straightforward to check, and is spelled out in Lemma B.72.

Lemma B.66. *Consider a pushout square*

(B.67)
$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in which $S \to T$ is a map of J-diagrams satisfying the condition described above. If T, T/S and X are very flat, then so is Y.

Proof: By Remark B.61 we may assume the cofibrant approximation $\tilde{Y} \to Y$ fits into a pushout square

(B.68)
$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & \tilde{T} \arrow[d] \\ \tilde{X} \arrow[r] & \tilde{Y} \end{tikzcd}$$

of cofibrant approximations to (B.67), in which $\tilde{S} \to \tilde{T}$ is the cofibrant approximation occurring in the condition we have assumed to be satisfied by $S \to T$. First consider the diagram

$$\begin{tikzcd} \partial_{\tilde{S}} \tilde{T}^{\wedge J'} \wedge Z \arrow[r] \arrow[d] & \tilde{T}^{\wedge J'} \wedge Z \arrow[r] \arrow[d] & (\tilde{T}/\tilde{S})^{\wedge J'} \wedge Z \arrow[d] \\ \partial_S T^{\wedge J'} \wedge Z \arrow[r] & T^{\wedge J'} \wedge Z \arrow[r] & (T/S)^{\wedge J'} \wedge Z \end{tikzcd}$$

The middle and rightmost vertical maps are weak equivalences since T and T/S are very flat. The left horizontal maps are h-cofibrations by assumption. It follows from the long exact sequence of stable homotopy groups that

(B.69)
$$\partial_{\tilde{S}} \tilde{T}^{\wedge J'} \wedge Z \to \partial_S T^{\wedge J'} \wedge Z$$

is a weak equivalence.

Turning to the main assertion, we now give $\tilde{Y}$ and Y the filtration described in §A.3.4. We will prove by induction on n that the map

(B.70)
$$\operatorname{fil}_n \tilde{Y}^{\wedge J'} \wedge Z \to \operatorname{fil}_n Y^{\wedge J'} \wedge Z$$

is a weak equivalence. The case $n = 0$ is the assertion that X is very flat, which true by assumption. For the inductive step, consider the diagram

$$\begin{tikzcd}
\operatorname{fil}_{n-1} \tilde{Y}^{\wedge J'} \wedge Z \arrow[d] & \bigvee_{\substack{J'=J_0'\amalg J_1' \\ |J_1'|=n}} \tilde{X}^{\wedge J_0'} \wedge \partial_{\tilde{A}} \tilde{B}^{\wedge J_1'} \wedge Z \arrow[l] \arrow[r] \arrow[d] & \bigvee_{\substack{J'=J_0'\amalg J_1' \\ |J_1'|=n}} \tilde{X}^{\wedge J_0'} \wedge \tilde{B}^{\wedge J_1'} \wedge Z \arrow[d] \\
\operatorname{fil}_{n-1} Y^{\wedge J'} \wedge Z & \bigvee_{\substack{J'=J_0'\amalg J_1' \\ |J_1'|=n}} X^{\wedge J_0'} \wedge \partial_{A} B^{\wedge J_1'} \wedge Z \arrow[l] \arrow[r] & \bigvee_{\substack{J'=J_0'\amalg J_1' \\ |J_1'|=n}} X^{\wedge J_0'} \wedge B^{\wedge J_1'} \wedge Z
\end{tikzcd}$$

The map from the pushout of the top row to the pushout of the bottom row is (B.70). The rightmost horizontal maps are h-cofibrations by assumption. The left vertical map is a weak equivalence by induction. The equivalence (B.69) implies that the middle vertical map is a weak equivalence. The right vertical map is a weak equivalence since X and B are very flat. The map (B.70) of pushouts is then a weak equivalence since h-cofibrations are flat (Corollary B.22). This completes the proof. $\square$

In order to get at the homotopy types of cofibrant commutative rings we need an expression decomposing cofibrations of commutative rings into simpler pieces. As a start, suppose that $A \to B$ is a map of G-spectra. We wish to investigate the structure of $\operatorname{Sym} B$ as a $\operatorname{Sym} A$-module. For this, define a filtration

$$\operatorname{fil}_m \operatorname{Sym} B = \bigvee_n \operatorname{fil}_m \operatorname{Sym}^n B$$

where the $\operatorname{fil}_m \operatorname{Sym}^n B$ is constructed by passing to Σ_n-orbits from the filtration defined in §A.3.4, and fits into a pushout square

$$\begin{tikzcd}
\operatorname{Sym}^{n-m} A \wedge \partial_A \operatorname{Sym}^m B \arrow[r] \arrow[d] & \operatorname{Sym}^{n-m} A \wedge \operatorname{Sym}^m B \arrow[d] \\
\operatorname{fil}_{m-1} \operatorname{Sym}^n (B) \arrow[r] & \operatorname{fil}_m \operatorname{Sym}^n B,
\end{tikzcd}$$

with

$$\partial_A \operatorname{Sym}^m B = \left(\partial_A B^{\wedge m}\right)/\Sigma_m.$$

Wedging over n one sees that the $\operatorname{fil}_m B$ are A-submodules, and that there is a pushout square of A-modules

$$\begin{tikzcd}
\operatorname{Sym} A \wedge \partial_A \operatorname{Sym}^m B \arrow[r] \arrow[d] & \operatorname{Sym} A \wedge \operatorname{Sym}^m B \arrow[d] \\
\operatorname{fil}_{m-1} \operatorname{Sym} B \arrow[r] & \operatorname{fil}_m \operatorname{Sym} B.
\end{tikzcd}$$

If a map $X \to Y$ of commutative rings is gotten by cobase change from the above $\operatorname{Sym} A \to \operatorname{Sym} B$,

$$\begin{tikzcd}
\operatorname{Sym} A \arrow[r] \arrow[d] & \operatorname{Sym} B \arrow[d] \\
X \arrow[r] & Y,
\end{tikzcd}$$

we can define a filtration of Y by X-modules by

$$\operatorname{fil}_m Y = X \underset{A}{\wedge} \operatorname{fil}_m \operatorname{Sym} B.$$

Evidently these $\operatorname{fil}_m Y$ are related by the pushout square of X-modules

$$\begin{tikzcd} X \wedge \partial_A \operatorname{Sym}^m B \arrow[r] \arrow[d] & X \wedge \operatorname{Sym}^m B \arrow[d] \\ \operatorname{fil}_{m-1} Y \arrow[r] & \operatorname{fil}_m Y. \end{tikzcd} \tag{B.71}$$

Working component-wise, the same filtration exists for commutative algebras in $\mathcal{S}^{\mathcal{B}_J G}$

proof of Proposition B.63: Since the class of very flat G-diagrams is closed under formation of filtered colimits along h-cofibrations (Lemma B.64), it suffices to show that if $A \to B$ is a generating cofibration in $\mathcal{S}^{\mathcal{B}_J G}$,

$$\begin{tikzcd} \operatorname{Sym} A \arrow[r] \arrow[d] & \operatorname{Sym} B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pushout square of commutative J-algebras and X is very flat, then Y is very flat. Give Y the filtration described above and work by induction on m. Since $\operatorname{fil}_0 Y = X$, the induction starts. For the inductive step, consider the pushout square (B.71). The J-spectra $\operatorname{Sym}^m B$ and

$$\operatorname{Sym}^m B / \partial_A \operatorname{Sym}^m B = \operatorname{Sym}^m (B/A)$$

are very flat by Lemma B.62. Since smash products of very flat J-spectra are very flat (Lemma B.64), both $X \wedge \operatorname{Sym}^m B$ and $X \wedge \operatorname{Sym}^m (B/A)$ are very flat. By Lemma B.72 below, the condition on the top row of (B.71) needed for Lemma B.66 is satisfied. Lemma B.66 then implies that $\operatorname{fil}_m Y$ is very flat. This completes the proof. □

We have used

Lemma B.72. *If $A \to B$ is a generating cofibration $\mathcal{S}^{\mathcal{B}_J G}$, X is any equivariant J-diagram, and $n > 0$, then*

$$X \wedge \partial_A \operatorname{Sym}^n B \to X \wedge \operatorname{Sym}^n B$$

satisfies the condition stated prior to Lemma B.66.

Proof: There are two conditions. One on the map, and one on a cofibrant approximation We start with the case of the map. The component of $A \to B$ with index $j \in J$ is of the form

$$S^{-V_j} \wedge S_+^{m_j - 1} \to S^{-V_j} \wedge D_+^{m_j},$$

and so

$$X \wedge \partial_A \operatorname{Sym}^n B \to X \wedge \operatorname{Sym}^n B$$

is gotten by passing to Σ_n-orbits from the map whose j-component is

$$X \wedge S_+^{-nV_j} \to X \wedge D_+^{-nV_j}.$$

The map required to be an h-cofibration ($\partial_S T^{\wedge J'} \to T^{\wedge J'}$) is then gotten by passing to $\Sigma_n^{J'}$-orbits from

$$X^{\wedge J'} \wedge S_+^{-n \oplus V_j} \to X^{\wedge J'} \wedge D_+^{-n \oplus V_j}.$$

But this $\Sigma_n^{J'} \rtimes G'$-spectrum is the smash product of the identity map of $X^{\wedge J'}$ with an h-cofibration of $\Sigma_n^{J'} \rtimes G'$-spaces and is therefore an h-cofibration For the cofibrant replacement assertion, choose a cofibrant approximation $\tilde{X} \to X$ and smash the generating cofibration with $(E_G \Sigma_n)_+$. The Σ_n-orbit spectrum of

$$\tilde{X} \wedge (E_G \Sigma_n)_+ \wedge S_+^{nV_i} \to \tilde{X} \wedge (E_G \Sigma_n)_+ \wedge D_+^{nV_i}$$

is a map of cofibrant objects (Lemma B.54), so the map required to be an h-cofibration ($\partial_{\tilde{S}} \tilde{T}^{\wedge J'} \to \tilde{T}^{\wedge J'}$) is then gotten by passing to $\Sigma_n^{J'}$-orbits from

$$\tilde{X}^{\wedge J'} \wedge (E_G \Sigma_n^{J'})_+ \wedge S_+^{\oplus nV_i} \to \tilde{X}^{\wedge J'} \wedge (E_G \Sigma_n^{J'})_+ \wedge D_+^{\oplus nV_i},$$

which, again, is the smash product of the identity map of an equivariant spectrum with an h-cofibration of equivariant spaces, hence an h-cofibration. □

B.5. **The monoidal geometric fixed point functor.** The geometric fixed point functor and its main properties were summarized in §2.5.2. In this section we describe the variation constructed in Mandell-May [36, §V.4]. We refer to the Mandell-May construction as the *monoidal geometric fixed point functor* and denote it Φ_M^G, in order not to confuse it with the usual geometric fixed point functor.

B.5.1. *Motivation and the definition.* For an orthogonal representation V of G let $V^G \subset V$ be the space of invariant vectors, and $V^\perp$ the orthogonal complement of V^G. Note that

$$\mathscr{I}_G(V,W)^G \approx \mathscr{I}(V^G, W^G) \wedge O(V^\perp, W^\perp)_+^G, \tag{B.73}$$

so that there is a canonical map

$$\mathscr{I}_G(V,W)^G \to \mathscr{I}(V^G, W^G),$$

given in terms of (B.73) by smashing the identity map with the map $O(V^\perp, W^\perp)^G \to \mathrm{pt}$.

We wish to define a functor Φ_M^G with the property that

$$\Phi_M^G(S^{-V} \wedge A) = S^{-V^G} \wedge A^G \tag{B.74}$$

and which commutes with colimits as far as is possible. A value needs to be assigned to the effect of Φ_M^G on the map

$$S^{-W} \wedge \mathscr{I}_G(V,W) \to S^{-V}.$$

The only obvious choice is to take

$$\Phi_M^G(S^{-W} \wedge \mathscr{I}_G(V,W)) \to \Phi_M^G(S^{-V})$$

to be the composite

$$S^{-W^G} \wedge \mathscr{I}_G(V,W)^G \to S^{-W^G} \wedge \mathscr{I}(V^G, W^G) \to \wedge S^{-V^G}. \tag{B.75}$$

If Φ_M^G actually *were* to commute with colimits, it would be determined by the specifications given by (B.74) and (B.75). Indeed, using the canonical presentation to write a general equivariant orthogonal spectrum X as a reflexive coequalizer

$$\bigvee_{V,W} S^{-W} \wedge \mathscr{I}_G(V,W)_+ \wedge X_V \rightrightarrows \bigvee_V S^{-V} \wedge X_V \to X,$$

the value of $\Phi^G_M(X)$ would be given by the reflexive coequalizer diagram

$$\bigvee_{V,W} S^{-W^G} \wedge \mathscr{I}_G(V,W)^G_+ \wedge X^G_V \rightrightarrows \bigvee_V S^{-V^G} \wedge X^G_V \to \Phi^G_M X. \tag{B.76}$$

We take this as the definition of $\Phi^G_M(X)$.

Definition B.77. The *monoidal geometric fixed point functor*

$$\Phi^G_M : \mathcal{S}_G \to \mathcal{S}$$

is the functor defined by the coequalizer diagram (B.76).

Remark B.78. In case $X = S^{-V} \wedge A$, the tautological presentation is a split coequalizer, and one recovers both (B.74) and (B.75).

A fundamental property of the usual geometric fixed point Φ^G is that for proper $H \subset G$, the spectrum $\Phi^G(G_+ \underset{H}{\wedge} X)$ is contractible. The monoidal geometric fixed point functor has this property on the nose.

Proposition B.79. *If $H \subset G$ is a proper subgroup and X an orthogonal H-spectrum, then the map*

$$\Phi^G_M(G_+ \underset{H}{\wedge} X) \to *$$

to the terminal object is an isomorphism.

Proof: For $W \in \mathscr{I}_G$, the W-space of $G_+ \underset{H}{\wedge} X$ is

$$(G_+ \underset{H}{\wedge} X)_W = G_+ \underset{H}{\wedge} X_{i^*_H W}. \tag{B.80}$$

If H is a proper subgroup of G the space of G-fixed points in (B.80) is just the basepoint. The result then follows from the definition of Φ^G_M. $\square$

There is a natural map

$$X^G \to \Phi^G_M X \tag{B.81}$$

from the fixed point spectrum of X to the monoidal geometric fixed point spectrum. To construct it note that the fixed point spectrum of X is computed term-wise, and so is given by the coequalizer diagram

$$\bigvee_{V,W \in \mathscr{I}} S^{-W} \wedge \mathscr{I}(V,W)_+ \wedge X^G_V \rightrightarrows \bigvee_{V \in \mathscr{I}} S^{-V} \wedge X^G_V \to X^G. \tag{B.82}$$

The map (B.81) is given by the evident inclusion of (B.82) into (B.76).

B.5.2. *Homotopy properties of Φ^G_M.* The functor Φ^G_M cannot commute with all colimits. However, since colimits of orthogonal G-spectra are computed object-wise, the definition implies that Φ^G_M commutes with whatever enriched colimits are preserved by the fixed point functor on G-spaces. This means that there is a functorial isomorphism

$$\Phi^G_M(X \wedge A) \approx \Phi^G_M(X) \wedge A^G \tag{B.83}$$

for each pointed G-space A, and that Φ^G_M commutes with the formation of wedges, directed colimits and cobase change along a closed inclusion. Because h-cofibrations and cofibrations are objectwise closed inclusion (Lemma B.24 and Remark B.8), this means that Φ^G_M has convenient homotopy theoretic properties.

The following is a compendium of some results in [36, §V.4]. Each assertion reduces to the corresponding claims on the appropriate generating (acyclic) cofibrations, where they are straightforward.

Proposition B.84. *The functor Φ^G_M preserves the classes of cofibrations, positive level acyclic cofibrations, and stable acyclic cofibrations.* □

Corollary B.85. *The functor Φ^G_M preserves the class of weak equivalences between cofibrant objects.*

Proof: Suppose $A \to B$ is a weak equivalence between cofibrant objects, and construct

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A_f \arrow[r] & B_f \end{tikzcd}$$

in which the vertical arrows are acyclic cofibrations with A_f and B_f fibrant. Then the bottom arrow is in fact a homotopy equivalence. Applying Φ^G_M gives

$$\begin{tikzcd} \Phi^G_M A \arrow[r] \arrow[d] & \Phi^G_M B \arrow[d] \\ \Phi^G_M A_f \arrow[r] & \Phi^G_M B_f . \end{tikzcd}$$

The vertical maps are weak equivalences between cofibrant objects by Proposition B.84. The bottom arrow is a homotopy equivalence since Φ^G_M is continuous. □

B.5.3. *Monoidal geometric fixed points and smash product.* The properties (B.74) and (B.75) give an identification

$$\Phi^G_M(S^{-V} \wedge A \wedge S^{-W} \wedge B) \approx \Phi^G_M(S^{-V} \wedge A) \wedge \Phi^G_M(S^{-W} \wedge B)$$

making the diagram

$$\begin{tikzcd} \Phi^G_M(S^{-V_1} \wedge \mathscr{I}_G(W_1, V_1)) \wedge \Phi^G_M(S^{-V_2} \wedge \mathscr{I}_G(W_2, V_2)) \arrow[r] \arrow[d] & \Phi^G_M(S^{-W_1}) \wedge \Phi^G_M(S^{-W_2}) \arrow[d] \\ \Phi^G_M(S^{-V_1} \wedge \mathscr{I}_G(W_1, V_1) \wedge S^{-V_2} \wedge \mathscr{I}_G(W_2, V_2)) \arrow[r] & \Phi^G_M(S^{-W_1} \wedge S^{-W_2}) \end{tikzcd}$$

commute. Applying Φ^G_M term-wise to the smash product of the tautological presentations of X and Y, and using the above identifications, gives a natural transformation

$$\Phi^G_M(X) \wedge \Phi^G_M(Y) \to \Phi^G_M(X \wedge Y), \tag{B.86}$$

making Φ^G_M lax monoidal. From the formula (B.74) this map is an isomorphism if $X = S^{-V} \wedge A$ and $Y = S^{-W} \wedge B$. This leads to

Proposition B.87 ([36], Proposition V.4.7)**.** *The functor Φ^G_M is weakly monoidal: the map* (B.86) *is a weak equivalence (in fact an isomorphism) if X and Y are cofibrant.*

Proof: The class of spectra X and Y for which (B.86) is an isomorphism is stable under smashing with a G-space, the formation of wedges, directed colimits, and cobase change along an objectwise closed inclusion. Since (B.86) is an isomorphism when $X = S^{-V} \wedge A$ and $Y = S^{-W} \wedge B$ this implies it is an isomorphism when X and Y are cofibrant. Since isomorphisms are weak equivalences, the result follows. □

B.5.4. *Relation with the geometric fixed point functor.* We now turn to the relationship between the monoidal geometric fixed point functor Φ^G_M and the actual geometric fixed point functor Φ^G. The main result is [36, Proposition V.4.17], which we describe below.

As described in §2.5.2, let $E\mathcal{P}$ be a G-CW complex with the property that for proper $H \subset G$, the fixed point space $E\mathcal{P}^H$ is contractible, while $E\mathcal{P}^G = \emptyset$. Let $\tilde{E}\mathcal{P}$ be the unreduced suspension of $E\mathcal{P}$, so that that $\tilde{E}\mathcal{P}^H$ is contractible if $H \neq G$, and $\tilde{E}^G = S^0$. We regard $\tilde{E}\mathcal{P}$ as a pointed G-space, with one of the cone points as base point. The *geometric fixed point spectrum* of X is defined by

$$\Phi^G X = ((\tilde{E}\mathcal{P} \wedge X)_{cf})^G,$$

where the subscript "cf" indicates cofibrant-fibrant replacement. For any X, the stable homotopy groups of $\Phi^G X$ can be computed without taking cofibrant-fibrant replacements:

$$\pi_* \Phi^G X = \pi_*^G \tilde{E}\mathcal{P} \wedge X.$$

The (pointed) inclusion $S^0 \to \tilde{E}\mathcal{P}$ gives a map

$$X \to \tilde{E}\mathcal{P} \wedge X$$

and hence a zig-zag

$$X \to \tilde{E}\mathcal{P} \wedge X \leftarrow (\tilde{E}\mathcal{P} \wedge X)_c \to (\tilde{E}\mathcal{P} \wedge X)_{cf}$$

(in which the subscript c indicates "cofibrant approximation). If X is already cofibrant, then so is $\tilde{E}\mathcal{P} \wedge X$ and there is just a map

$$X \to (\tilde{E}\mathcal{P} \wedge X)_{cf}.$$

Proposition B.88 ([36], Proposition V.4.17)**.** *If X is cofibrant, then the maps*

$$\Phi^G X = ((\tilde{E}\mathcal{P} \wedge X)_{cf})^G \to \Phi^G_M((\tilde{E}\mathcal{P} \wedge X)_{cf}) \leftarrow \Phi^G_M(X)$$

are weak equivalences.

Sketch of proof: The arrow on the left is easily shown to induce an isomorphism of stable homotopy groups using the canonical homotopy presentation. The right arrow is the composition of

$$\Phi^G_M(X) \to \Phi^G_M(\tilde{E}\mathcal{P} \wedge X)$$

which is easily checked to be an isomorphism using B.74 and the argument for Proposition B.87, and

$$\Phi^G_M(\tilde{E}\mathcal{P} \wedge X) \to \Phi^G_M((\tilde{E}\mathcal{P} \wedge X)_{cf}),$$

which is an acyclic cofibration by Proposition B.84. □

B.5.5. *The relative monoidal geometric fixed point functor.* The functor Φ^G_M can be formulated relative to an equivariant commutative or associative algebra R. As described below, care must be taken in using the theory in this way.

Because it is lax monoidal, the functor Φ^G_M gives a functor

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}$$

which is lax monoidal in case R is commutative.

Proposition B.89. *The functor*

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}$$

commutes with cobase change along a cofibration and preserves the classes of cofibrations and acyclic cofibrations.

Proof: This follows easily from the fact that the maps of equivariant orthogonal spectra underlying the generating cofibrations for $\mathcal{M}_R$ are h-cofibrations. □

Proposition B.90. *When R is commutative, the functor*

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}$$

is weakly monoidal, and in fact

$$\Phi^G_M(M) \underset{\Phi^G_M(R)}{\wedge} \Phi^G_M(N) \to \Phi^G_M(M \underset{R}{\wedge} N) \tag{B.91}$$

is an isomorphism if M and N are cofibrant. □

Proof: The proof is the same as that of Proposition B.87 once one knows that the class of modules M and N for which (B.91) is an isomorphism is stable under cobase change along a generating cofibration. This, in turn, is a consequence of the fact that both sides of (B.91) preserve h-cofibrations in each variable, since h-cofibrations are closed inclusions. The functor Φ^G_M does so since it commutes with the formation of mapping cylinders, and $M \underset{R}{\wedge} (-)$ does since $\mathcal{M}_R$ is a closed symmetric monoidal category. □

As promising as it looks, it is not so easy to make use of Proposition B.90. The trouble is that unless X is cofibrant, $\Phi^G_M(X)$ may not have the weak homotopy type of $\Phi^G(X)$. So in order to use Proposition B.90 one needs a condition guaranteeing that $M \underset{R}{\wedge} N$ is a cofibrant spectrum. The criterion of Proposition B.92 below was suggested to us by Mike Mandell.

Proposition B.92. *Suppose R is an associative algebra with the property that $S^{-1} \wedge R$ is cofibrant. If M is a cofibrant right R-module, and $S^{-1} \wedge N$ is a cofibrant left R-module, then*

$$M \underset{R}{\wedge} N$$

is cofibrant.

Proof: First note that the condition on R guarantees that for every representation U with $\dim U^G > 0$ and every cofibrant G-space T, the spectrum

$$S^{-U} \wedge R \wedge T \tag{B.93}$$

is cofibrant. Since the formation of $M \underset{R}{\wedge} N$ commutes with cobase change in both variables, the result reduces to the case $M = S^{-V} \wedge R \wedge X$ and $N = S^{-W} \wedge R \wedge Y$ with V and having a non-zero fixed point spaces, and X and Y cofibrant G-spaces. But in that case

$$M \underset{R}{\wedge} N \approx S^{-V \oplus W} \wedge R \wedge X \wedge Y$$

which is of the form (B.93), and hence cofibrant. □

Corollary B.94. *Suppose R is an associative algebra with the property that $S^{-1} \wedge R$ is cofibrant. If M is a cofibrant right R-module, then the equivariant orthogonal spectrum underlying M is cofibrant.*

Proof: Just take $N = R$ in Proposition B.92. □

The following result plays an important role in determining $\Phi^G R(\infty)$ (§7.3).

Proposition B.95. *Suppose that $R \to S^0$ is an associative algebra equipped with an associative algebra map to S^0. If M is a cofibrant right R-module, the $M \underset{R}{\wedge} S^0$ is a cofibrant spectrum, and the map*

$$\Phi^G_M(M) \underset{\Phi^G_M R}{\wedge} S^0 \to \Phi^G_M(M \underset{R}{\wedge} S^0)$$

is an isomorphism.

Proof: One easily reduces to the case $M = S^{-V} \wedge X \wedge R$, in which V is a representation with $V^G \neq 0$, and X is a cofibrant G-space. But this case is clear. □

B.6. **Geometric fixed points and the norm.**

Proposition B.96. *Suppose $H \subset G$. There is a natural transformation*

$$\Phi^H_M(\,-\,) \to \Phi^G_M \circ N_H^G(\,-\,)$$

which is a weak equivalence on cofibrant objects.

Proof: To construct the natural transformation, first note that there is a natural isomorphism

$$A^H \approx (N_H^G A)^G$$

for H-equivariant *spaces* A. Next note that for an orthogonal representation V of H, Proposition A.52 and the property (B.74) give isomorphisms

$$\Phi^G N_H^G S^{-V} \approx \Phi^G S^{-\operatorname{ind}_H^G V} \approx S^{-V^H} \approx \Phi^H S^{-V}.$$

The monoidal properties of Φ^G_M and the norm then combine to give an isomorphism

$$\Phi^H(S^{-V} \wedge A) \approx \Phi^G N_H^G(S^{-V} \wedge A) \tag{B.97}$$

which one easily checks to be compatible with the maps

$$S^{-V} \wedge \mathscr{I}_H(W, V) \to S^{-W}.$$

To construct the transformation, write a general H-spectrum X in terms of its canonical presentation

$$\bigvee_{V,W} S^{-W} \wedge \mathscr{I}_H(V, W) \wedge X_V \rightrightarrows \bigvee_V S^{-V} \wedge X_V \to X,$$

and apply (B.97) termwise to produce a diagram

$$\bigvee_{V,W} S^{-W^H} \wedge \mathscr{I}_H(V,W)^H \wedge X_V^H \rightrightarrows \bigvee_V S^{-V^H} \wedge X_V^H \to \Phi^G N_H^G X.$$

The coequalizer of the two arrows is, by definition, $\Phi_M^H(X)$. This gives the natural transformation.

For the assertion about weak equivalences, suppose $X \in \mathcal{S}_G$ is cofibrant. Since both functors preserve weak equivalences between cofibrant objects, we may assume X is both cofibrant and fibrant. Since both functors preserve directed colimits, and weak equivalences between cofibrant objects, the canonical homotopy presentation

$$X \approx \underset{\overrightarrow{V}}{\operatorname{holim}}\,(S^{-V} \wedge X_V)_{\mathrm{cf}}$$

reduces the claim to the case $(S^{-V} \wedge A)_{\mathrm{cf}}$. Again since both functors preserve weak equivalences between cofibrant objects, this in turn reduces to the case $S^{-V} \wedge A$. But in that case the natural transformation is the isomorphism (B.97). This completes the proof. $\square$

Using cofibrant approximations and the functor Φ^G one can get a slightly better result. Start with $X \in \mathcal{S}_H$ and let $X_c \to X$ be a cofibrant approximation. Now consider the diagram

(B.98)
$$\begin{tikzcd}
\Phi^H X_c \arrow[r, leftrightarrow, "\sim", "\text{zig zag}"'] \arrow[d, "\sim"'] & \Phi_M^H X_c \arrow[r, "\sim"] & \Phi_M^G N_H^G X_c \arrow[r, leftrightarrow, "\sim", "\text{zig zag}"'] & \Phi^G N_H^G X_c \arrow[d] \\
\Phi^H X & & & \Phi^G N_H^G X
\end{tikzcd}$$

The left vertical arrows are weak equivalences since the geometric fixed point functor preserves weak equivalences. The weak equivalences in the top row are given by Propositions B.88, B.36, and B.96. This gives for any X a functorial diagram mapping something weakly equivalent to $\Phi^H X$ to $\Phi^G N_H^G X$. It also gives

Proposition B.99. *Suppose that $X \in \mathcal{S}_H$ has the property that for some (hence any) cofibrant approximation $X_c \to X$ the map*

$$N_H^G X_c \to N_H^G X$$

is a weak equivalence. Then the functorial relationship between $\Phi^H X$ and $\Phi^G N_H^G X$ given by (B.98) *is a weak equivalence.* $\square$

Remark B.100. Proposition B.99 applies in particular when X is *very flat* in the sense of §B.4. By Proposition B.63 this means that if $R \in \mathcal{S}_H$ is a cofibrant commutative ring, then $\Phi^H R$ and $\Phi^G N_H^G R$ are related by a functorial zig-zag of weak equivalences. The case of interest to us is when $H = C_2$, $G = C_{2^n}$ and $R = MU_{\mathbb{R}}$. In this case $N_H^G R = MU^{(G)}$, and we get an equivalence

$$\Phi^G MU^{(G)} \approx \Phi^{C_2} MU_{\mathbb{R}} \approx MO.$$

Remark B.101. Proposition B.99 also applies to the suspension spectra of cofibrant H-spaces. Indeed, if X is a cofibrant H-space then $S^{-1} \wedge S^1 \wedge X \to X$ is a cofibrant approximation. Applying N_H^G leads to the map

$$S^{-V} \wedge S^V \wedge N_H^G(X) \to N_H^G(X)$$

with $V = \operatorname{ind}_H^G \mathbb{R}$, which is a weak equivalence (in fact a cofibrant approximation). This case is used to show that $\Phi^G \circ N_H^G$ is a ring homomorphism on the $RO(G)$-graded cohomology of G-spaces (Proposition 2.43).

B.7. **Real bordism.** In this section we give a construction of the real bordism spectrum $MU_{\mathbb{R}}$ as a commutative algebra in $\mathcal{S}_{C_2}$. As will be apparent to the reader, this construction owes a great deal to the Stefan Schwede's construction of MU in [48, Chapter 2]. We are indebted to Stefan for some very helpful correspondence concerning these matters.

Our goal is to construct a C_2-equivariant commutative ring $MU_{\mathbb{R}}$ admitting the canonical homotopy presentation

$$MU_{\mathbb{R}} \approx \operatorname{holim}_{\longrightarrow} S^{-\mathbb{C}^n} \wedge MU(n), \tag{B.102}$$

in which $MU(n)$ is the Thom complex of the universal bundle over $BU(n)$. The group C_2 is acting on everything by complex conjugation, so we could also write this expression as

$$MU_{\mathbb{R}} \approx \operatorname{holim}_{\longrightarrow} S^{-n\rho_2} \wedge MU(n). \tag{B.103}$$

The map

$$S^{-\mathbb{C}} \wedge MU(1) \to MU_{\mathbb{R}}$$

defines a real orientation. Everything we have used about $MU_{\mathbb{R}}$ follows easily from this. The trick is getting it to be a commutative ring.

The most natural construction of $MU_{\mathbb{R}}$ realizes the above in the category of *real spectra*, which is related to the category of C_2-equivariant orthogonal spectra by a multiplicative Quillen equivalence. We first construct $MU_{\mathbb{R}}$ as a real spectrum, and then transport it to $\mathcal{S}_{C_2}$ using this equivalence. As with the norm, the only real technical issue arises from the fact that the spectra underlying commutative algebras are rarely cofibrant.

B.7.1. *Real spectra.* In this section we describe the basics of *real spectra*. These results are more or less all a special case of the results of [37]. We go through the details since, as with orthogonal G-spectra, the generalized flatness of commutative rings play an important role (Proposition B.116).

For finite dimensional complex Hermitian vector spaces A and B let $U(A, B)$ be the Stiefel manifold of unitary embeddings $A \hookrightarrow B$. There is a natural Hermitian inner product on the complexification $V_{\mathbb{C}}$ of a real orthogonal vector space V, so there is a natural map

$$O(V, W) \to U(V_{\mathbb{C}}, W_{\mathbb{C}}).$$

The group C_2 acts on $U(V_{\mathbb{C}}, W_{\mathbb{C}})$ by complex conjugation, and the fixed point space is $O(V, W)$.

We can now describe the basic indexing category for real spectra.

Definition B.104. The category $\mathscr{I}_{\mathbb{R}}$ is the C_2-equivariant topological category whose objects are finite dimensional orthogonal real vector spaces V, and whose morphism space $\mathscr{I}_{\mathbb{R}}(V, W)$ is the Thom complex

$$\mathscr{I}_{\mathbb{R}}(V, W) = \operatorname{Thom}(U(V_{\mathbb{C}}, W_{\mathbb{C}}); W_C - V_{\mathbb{C}}).$$

The action of the group C_2 is by complex conjugation.

The orthogonal sum makes $\mathscr{I}_{\mathbb{R}}$ in a symmetric monoidal C_2-equivariant topological category.

Definition B.105. The ($\mathcal{T}^{C_2}$-)category $\mathcal{S}_{\mathbb{R}}$ of *real spectra* is the C_2-equivariant topological category of functors

$$\mathscr{I}_{\mathbb{R}} \to \mathcal{T}_{C_2}\,.$$

The underlying topological category is $\mathcal{S}^{\mathbb{R}}$.

We will write

$$V \mapsto X_{V_{\mathbb{C}}}$$

for a typical real spectrum X, and let $S^{-V_{\mathbb{C}}} \in \mathcal{S}_{\mathbb{R}}$ be the functor co-represented by $V \in \mathscr{I}_{\mathbb{R}}$. From the Yoneda lemma there is a natural isomorphism

$$\mathcal{S}_{\mathbb{R}}(S^{-V_{\mathbb{C}}}, X) = X_{V_{\mathbb{C}}}.$$

As with equivariant orthogonal spectra, every real spectrum X has a canonical presentation

$$\bigvee_{V,W\in\mathscr{I}_{\mathbb{R}}} S^{-W_{\mathbb{C}}} \wedge \mathscr{I}_{\mathbb{R}}(V,W) \wedge X_{W_{\mathbb{C}}} \rightrightarrows \bigvee_{V\in\mathscr{I}_{\mathbb{R}}} S^{-V} \wedge X_{V_{\mathbb{C}}} \to X. \tag{B.106}$$

The category of real spectra is a closed symmetric monoidal category under the smash product operation $X \wedge Y$. It is determined by the specification

$$S^{-V_{\mathbb{C}}} \wedge S^{-W_{\mathbb{C}}} = S^{-(V\oplus W)_{\mathbb{C}}}$$

and the fact that it commutes with colimits in each variable.

B.7.2. *Homotopy theory of real spectra.* A map $X \to Y$ in $\mathcal{S}_{\mathbb{R}}$ is a *positive level* equivalence (fibration) if for each $V \neq 0 \in \mathscr{I}_{\mathbb{R}}$, the map $X_{V_{\mathbb{C}}} \to Y_{V_{\mathbb{C}}}$ is a weak equivalence (fibration) in $\mathcal{T}_{C_2}$. A *positive level cofibration* in $\mathcal{S}_{\mathbb{R}}$ is a map having the left lifting property with respect to the class of positive level acyclic fibrations. These three classes of maps form the *positive level model structure* on $\mathcal{S}_{\mathbb{R}}$. The positive level cofibrations are generated by the maps

$$\{S^{-V} \wedge A \to S^{-V} \wedge B\}$$

in which $V \neq 0$, and $A \to B$ is running through a collection of generating cofibrations for $\mathcal{T}_{C_2}$ (for example the list (B.1)), and the positive level acyclic cofibrations are generated by a similar set, only with $A \to B$ running through generating acyclic cofibrations for $\mathcal{S}_{C_2}$ (for example (B.2)).

For $H \subset C_2$, and $X \in \mathcal{S}_{\mathbb{R}}$, we define

$$\pi_k^H(X) = \varinjlim_{V\in\mathscr{I}_{\mathbb{R}}} \pi_{k+V}^H X_V$$

and write $\underline{\pi}_k X$ for the corresponding Mackey functor. A *stable weak equivalence* in $\mathcal{S}_{\mathbb{R}}$ is a map $X \to Y$ inducing an isomorphism $\underline{\pi}_k X \to \underline{\pi}_k Y$ for all k.

Definition B.107. The *positive stable model structure* on $\mathcal{S}_{\mathbb{R}}$ is the localization of the positive level model structure at the stable equivalences.

Proposition B.108. *The forgetful functor from commutative algebras in*

$$\mathbf{comm}\,\mathcal{S}_{\mathbb{R}} \to \mathcal{S}_{\mathbb{R}}$$

creates an enriched model category on commutative algebras in $\mathcal{S}_{\mathbb{R}}$ in which a map of commutative algebras is a fibration or weak equivalence if and only if the underlying map of real spectra is. $\square$

B.7.3. *Real spectra and C_2-spectra.* Let

$$i : \mathscr{I}_{\mathbb{R}} \to \mathscr{I}_{C_2}$$

be the functor sending V to

$$V_{\rho_2} = V \otimes \rho_2.$$

Then the restriction functor

$$i^* : \mathcal{S}^{C_2} \to \mathcal{S}_{\mathbb{R}}$$

has both a left and right adjoint which we denote i_* and $i_!$ respectively. The left adjoint sends $S^{-V_{\mathbb{C}}}$ to $S^{-V_{\rho_2}}$, and is easily given in terms of the canonical presentation term-wise.

Since the functor i is symmetric monoidal, the left adjoint i_* is strongly symmetric monoidal.

The next result is also a special case of the results of [37].

Proposition B.109. *The functors*

$$i_* : \mathcal{S}_{\mathbb{R}} \leftrightarrows \mathcal{S}_{C_2} : i^*$$

form a Quillen equivalence. □

Proof: Since i_* is a left adjoint and

$$i_*(S^{-V_{\mathbb{C}}} \wedge A) = S^{-V_{\rho_2}} \wedge A$$

it is immediate that i_* sends the generating (acyclic) cofibrations to generating (acyclic) cofibrations, and hence is a left Quillen functor. It also follows that the functor i^* preserves weak equivalences. Using the fact that the sequence $\{\mathbb{R}^n \otimes \rho_2\}$ is exhausting, one can easily check that a map $X \to Y$ in $\mathcal{S}_{C_2}$ is a weak equivalence if and only if $i^*X \to i^*Y$ is. Since i^* is also a left adjoint, it preserves colimits. These facts mean that in order to show that i_* and i^* form a Quillen equivalence, it suffices to show that for each $0 \neq V \in \mathscr{I}_{\mathbb{R}}$, the map

$$S^{-V_{\mathbb{C}}} \to i^* S^{-V_{\rho_2}} \tag{B.110}$$

is a weak equivalence. For each $W \in \mathscr{I}_{\mathbb{R}}$, the $W_{\mathbb{C}}$-space of $S^{-V_{\mathbb{C}}}$ is

$$\mathscr{I}_{\mathbb{R}}(V, W) = \mathrm{Thom}(U(V_{\mathbb{C}}, W_{\mathbb{C}}); W_C - V_{\mathbb{C}})$$

and the W-space of $i^* S^{-V_{\rho_2}}$ is

$$\mathscr{I}_{C_2}(V_{\rho_2}, W_{\rho_2}) = \mathrm{Thom}(O(V_{\rho_2}, W_{\rho_2}); W_{\rho_2} - V_{\rho_2}).$$

We must therefore show that for each k, the map

$$\varinjlim_{W \in \mathscr{I}_{\mathbb{R}}} \underline{\pi}_{k+W_{\mathbb{C}}} \mathscr{I}_{\mathbb{R}}(V, W) \to \varinjlim_{W \in \mathscr{I}_{\mathbb{R}}} \underline{\pi}_{k+W_{\mathbb{C}}} \mathscr{I}_{C_2}(V_{\rho_2}, W_{\rho_2}) \tag{B.111}$$

is an isomorphism.

We may suppose that $\dim W > \dim V$. For a fixed W choose an orthogonal embedding $V \subset W$, write $W = V \oplus U$, and consider the diagram

$$\begin{tikzcd}
S^{U_{\mathbb{C}}} \arrow[r] \arrow[d, "\approx"'] & \mathscr{I}_{\mathbb{R}}(V, W) \arrow[d] \\
S^{U_{\rho_2}} \arrow[r] & \mathscr{I}_{C_2}(V_{\rho_2}, W_{\rho_2}).
\end{tikzcd}$$

The left vertical map is an equivariant isomorphism. A straightforward argument using the connectivity of Stiefel manifolds shows that for $\dim W \gg 0$ the horizontal maps are isomorphisms in both $\pi^u_{k+W_{\mathbb{C}}}$ and $\pi^{C_2}_{k+W_{\mathbb{C}}}$. It follows that the right vertical map is as well, and hence so is (B.111). $\square$

For later reference, we record one fact that emerged in the proof of Proposition B.109.

Lemma B.112. *A map $X \to Y$ of C_2-spectra is a weak equivalence if and only if the underlying map $i^*X \to i^*Y$ of real spectra is.* $\square$

Definition B.113. A real spectrum $X \in \mathcal{S}_{\mathbb{R}}$ is *i_*-flat* if it satisfies the following property: for every cofibrant approximation $\tilde{X} \to X$ and every $Z \in \mathcal{S}_{C_2}$ the map

$$i_*\tilde{X} \wedge Z \to i_*X \wedge Z \tag{B.114}$$

is a weak equivalence.

Remark B.115. As in Remark B.61, by continuity, if (B.114) is a weak equivalence for one cofibrant approximation it is a weak equivalence for any cofibrant approximation.

Proposition B.116. *If $R \in \mathcal{S}_{\mathbb{R}}$ is a cofibrant commutative algebra the R is i_*-flat.*

The proof of Proposition B.116 follows the argument for the proof of Proposition B.63. We begin with an analogue of Proposition B.62.

Lemma B.117. *If $A \in \mathcal{S}_{\mathbb{R}}$ is cofibrant, and $n \geq 1$, then $\operatorname{Sym}^n A$ is i_*-flat.*

Proof: Let $E_{C_2}\Sigma_n$ be the total space of the universal C_2-equivariant Σ_n-bundle. The proof of Lemma B.49 applies to show that $A^{\wedge n}$ is Σ_n-free, and the proof of Proposition B.57 shows that

$$(E_{C_2}\Sigma_n)_+ \underset{\Sigma_n}{\wedge} A^{\wedge n} \to \operatorname{Sym}^n A$$

is a cofibrant approximation. Since i_* is a continuous left adjoint, we may identify

$$i_*(E_{C_2}\Sigma_n)_+ \underset{\Sigma_n}{\wedge} A^{\wedge n} \to i_* \operatorname{Sym}^n A \tag{B.118}$$

with

$$(E_{C_2}\Sigma_n)_+ \underset{\Sigma_n}{\wedge} (i_*A)^{\wedge n} \to \operatorname{Sym}^n(i_*A). \tag{B.119}$$

Since i_* is a left Quillen functor, $i_*(A)$ is cofibrant, and Proposition B.57 implies that (B.119), hence (B.118) is a weak equivalence. $\square$

We also require an analogue of Lemma B.66, though the statement and proof are much simpler in this case, since i_* is a left adjoint.

Lemma B.120. *If $S \to T$ is an h-cofibration in $\mathcal{S}_{\mathbb{R}}$, and two of S, T, T/S are i_*-good, then so is the third.*

Proof: We may choose a map $\tilde{S} \to \tilde{T}$ of cofibrant approximations which is a cofibration, hence an h-cofibration. Then both $i_*\tilde{S} \wedge Z \to i_*\tilde{T} \wedge Z$ and $i_*S \wedge Z \to i_*T \wedge Z$ are h-cofibrations and the claim follows easily from the long exact sequence of homotopy groups. $\square$

Lemma B.121. *Consider a pushout square in $\mathcal{S}_{\mathbb{R}}$,*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{B.122}$$

in which $S \to T$ is an h-cofibration. If T, T/S and X are i_*-flat, then so is Y.

Proof: Since T and T/S are i_*-flat, so is S by Lemma B.120. We may choose cofibrant approximations of everything fitting into a pushout diagram

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & \tilde{T} \arrow[d] \\ \tilde{X} \arrow[r] & \tilde{Y} \end{tikzcd}$$

in which the top row is an h-cofibration. Now consider

$$\begin{tikzcd} i_*\tilde{X} \wedge Z \arrow[d] & i_*\tilde{S} \wedge Z \arrow[l] \arrow[r] \arrow[d] & i_*\tilde{T} \wedge Z \arrow[d] \\ i_*X \wedge Z & i_*S \wedge Z \arrow[l] \arrow[r] & i_*T \wedge Z \end{tikzcd}$$

The left horizontal maps are h-cofibrations, hence flat, and the vertical maps are weak equivalences by assumption. It follows that the map of pushouts is a weak equivalence. ☐

Proof of Proposition B.116: It suffices to show that if $A \to B$ is a generating cofibration in $\mathcal{S}_{\mathbb{R}}$,

$$\begin{tikzcd} \operatorname{Sym} A \arrow[r] \arrow[d] & \operatorname{Sym} B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pushout square of commutative algebras in $\mathcal{S}_{\mathbb{R}}$, and X is good, then Y is good. We induct over the filtration described in §B.4. Since $\operatorname{fil}_0 Y = X$, the induction starts. For the inductive step, consider the pushout square

$$\begin{tikzcd} X \wedge \partial_A \operatorname{Sym}^m B \arrow[r] \arrow[d] & X \wedge \operatorname{Sym}^m B \arrow[d] \\ \operatorname{fil}_{m-1} Y \arrow[r] & \operatorname{fil}_m Y, \end{tikzcd} \tag{B.123}$$

and assume that $\operatorname{fil}_{m-1} Y$ is i_*-flat. Both $\operatorname{Sym}^m B$ and

$$\operatorname{Sym}^m B/\partial_A \operatorname{Sym}^m B = \operatorname{Sym}^m(B/A)$$

are i_*-flat by Lemma B.117. Since smash products of i_*-flat spectra are i_*-flat, both $X \wedge \operatorname{Sym}^m B$ and $X \wedge \operatorname{Sym}^m(B/A)$ are i_*-flat. Since the top row of (B.123) is an h-cofibration, Lemma B.121 implies that $\operatorname{fil}_m Y$ is i_*-flat. This completes the inductive step, and the proof. ☐

Corollary B.124. *The functors i_* and i^* restrict to a Quillen equivalence*

$$i_* : \mathbf{comm}\,\mathcal{S}_{\mathbb{R}} \leftrightarrows \mathbf{comm}\,\mathcal{S}_{C_2} : i^*.$$

Proof: It is immediate from the definition of the model structures on $\mathbf{comm}\,\mathcal{S}_{\mathbb{R}}$ and $\mathbf{comm}\,\mathcal{S}_{C_2}$, and the fact that

$$i_* : \mathcal{S}_{\mathbb{R}} \leftrightarrows \mathcal{S}_{C_2} : i^*$$

is a Quillen pair, that

$$i^* : \mathbf{comm}\,\mathcal{S}_{C_2} \to \mathbf{comm}\,\mathcal{S}_{\mathbb{R}}$$

preserves the classes of fibrations and acyclic fibrations. It remains to show that if $A \in \mathbf{comm}\,\mathcal{S}_{\mathbb{R}}$ is cofibrant, then the composition

$$A \to i^* i_* A \to i^*(i_* A_f)$$

in which $i_* A \to i_* A_f$ is a fibrant replacement. Since i^* preserves weak equivalences (Lemma B.112) this is equivalent to showing that

$$A \to i^* i_* A$$

is a weak equivalence. Let $A' \to A$ be a cofibrant approximation in $\mathcal{S}_{\mathbb{R}}$, and consider the following diagram in $\mathcal{S}_{\mathbb{R}}$

$$\begin{tikzcd} A' \arrow[r, "\sim"] \arrow[d, "\sim"'] & i^* i_* A' \arrow[d, "\sim"] \\ A \arrow[r] & i^* i_* A. \end{tikzcd} \tag{B.125}$$

By Proposition B.116 the map $i_* A' \to i_* A$ is a weak equivalence. The rightmost arrow in (B.125) is therefore a weak equivalence since i^* preserves weak equivalences. The top arrow is a weak equivalence by Proposition B.109, and the left arrow is a weak equivalence by definition. This implies that the bottom arrow is a weak equivalence. □

B.7.4. *The real bordism spectrum.* For $V \in \mathscr{I}_{\mathbb{R}}$ let

$$MU(V_{\mathbb{C}}) = \mathrm{Thom}(BU(V_{\mathbb{C}}), V_{\mathbb{C}})$$

be the Thom complex of the bundle $EU(V_{\mathbb{C}}) \underset{U(V_{\mathbb{C}})}{\times} V_{\mathbb{C}}$ over $BU(V_{\mathbb{C}})$, equipped with the C_2-action of complex conjugation. We will take our model of $BU(V_{\mathbb{C}})$ to be the one given by Segal's construction [51], so that

$$V \mapsto \mathrm{Thom}(BU(V_{\mathbb{C}}), V_{\mathbb{C}}) \tag{B.126}$$

is a lax symmetric monoidal functor $\mathscr{I}_{\mathbb{R}} \to \mathcal{T}_{C_2}$, and so defines a commutative ring $\mathcal{MU}_{\mathbb{R}} \in \mathbf{comm}\,\mathcal{S}_{\mathbb{R}}$. Let $\mathcal{MU}'_{\mathbb{R}} \to \mathcal{MU}_{\mathbb{R}}$ be a cofibrant approximation to $\mathcal{MU}_{\mathbb{R}}$ in $\mathbf{comm}\,\mathcal{S}_{\mathbb{R}}$.

Definition B.127. The *real bordism spectrum* is the spectrum $MU_{\mathbb{R}} = i_* \mathcal{MU}'_{\mathbb{R}}$.

To get at the homotopy type of $MU_{\mathbb{R}}$, we examine the canonical homotopy presentation of $\mathcal{MU}_{\mathbb{R}}$ using the exhausting sequence $V_n = \mathbb{R}^n$. This gives a weak equivalence

$$\mathcal{MU}'_{\mathbb{R}} \sim \operatorname{holim}_{\longrightarrow} S^{-\mathbb{C}^n} \wedge MU(n) \tag{B.128}$$

in which $MU(n) = MU(\mathbb{C}^n)$. By Proposition B.116, there results a weak equivalence

$$\begin{aligned} MU_{\mathbb{R}} &\sim \underrightarrow{\operatorname{holim}}\, S^{-\mathbb{C}^n} \wedge MU(\mathbb{C}^n) \\ &\sim \underrightarrow{\operatorname{holim}}\, S^{-n\rho_2} \wedge MU(\mathbb{C}^n). \end{aligned} \tag{B.129}$$

In this presentation the universal real orientation of $MU_{\mathbb{R}}$ (Example 5.6) is given by the inclusion

$$S^{-\mathbb{C}} \wedge MU(1) \to MU_{\mathbb{R}}.$$

The next result summarizes some further consequences of the presentation (B.128).

Proposition B.130. i) *The non-equivariant spectrum underlying $MU_{\mathbb{R}}$ is the usual complex cobordism spectrum MU.*

ii) *The equivariant cohomology theory represented by $MU_{\mathbb{R}}$ coincides with the one studied in* [30, 16, 2, 26].

iii) *There is an equivalence*

$$\Phi^{C_2} MU_{\mathbb{R}} \approx MO.$$

iv) *The Schubert cell decomposition of Grassmannians leads to a cofibrant approximation of $MU_{\mathbb{R}}$ by a C_2-CW complex with one 0-cell (S^0) and the remaining cells of the form $e^{m\rho_2}$, with $m > 0$.*

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