CHAPTER 5

The Chromatic Spectral Sequence

The spectral sequence of the title is a mechanism for organizing the Adams–Novikov E_2-term and ultimately $\pi_*(S^0)$ itself. The basic idea is this. If an element x in the E_2-term, which we abbreviate by $\mathrm{Ext}(BP_*)$ (see 5.1.1), is annihilated by a power of p, say p^i, then it is the image of some $x' \in \mathrm{Ext}(BP_*/p^i)$ under a suitable connecting homomorphism. In this latter group one has multiplication by a suitable power of v_1 (depending on i), say v_1^m. x' may or may not be annihilated by some power of v_1^m, say v_1^{mj}. If not, we say x is v_1-*periodic*; otherwise x' is the image of some $x'' \in \mathrm{Ext}(BP_*/(p^i, v_1^{mj}))$ and we say it is v_1-*torsion.* In this new Ext group one has multiplication by v_2^n for some n. If x is v_1-torsion, it is either v_2-periodic or v_2-torsion depending on whether x'' is killed by some power of v_2^n. Iterating this procedure one obtains a complete filtration of the original Ext group in which the nth subgroup in the v_n-torsion and the nth subquotient is v_n-periodic. This is *the chromatic filtration* and it is associated with the *chromatic spectral sequence* of 5.1.8. The chromatic spectral sequence is like a spectrum in the astronomical sense in that it resolves stable homotopy into periodic components of various types.

Recently we have shown that this algebraic construction has a geometric origin, i.e., that there is a corresponding filtration of $\pi_*(S^0)$. The chromatic spectral sequence is based on certain inductively defined short exact sequences of comodules 5.1.5. In Ravenel [**?**] we show that each of these can be realized by a cofibration

$$N_n \to M_n \to N_{n+1}$$

with $N_0 = S^0$ so we get an inverse system

$$S^0 \leftarrow \Sigma^{-1} N_1 \leftarrow \Sigma^{-2} N_2 \leftarrow \cdots .$$

The filtration of $\pi_*(S^0)$ by the images of $\pi_*(\Sigma^{-n} N^n)$ is the one we want. Applying the Novikov Ext functor to this diagram yields the chromatic spectral sequence, and applying homotopy yields a geometric form of it. For more discussion of this and related problems see Ravenel [**?**].

The chromatic spectral sequence is useful computationally as well as conceptually. In 5.1.10 we introduce the chromatic cobar complex $CC(BP_*)$. Even though it is larger than the already ponderous cobar complex $C(BP_*)$, it is easier to work with because many cohomology classes (e.g., the Greek letter elements) have far simpler cocycle representatives in CC than in C.

In Section 1 the basic properties of the chromatic spectral sequence are given, most notably the change-of-rings theorem 5.1.14, which equates certain Ext groups with the cohomology of certain Hopf algebras $\Sigma(n)$, the nth Morava stabilizer algebra. This isomorphism enables one to compute these groups and was the original motivation for the chromatic spectral sequence. These computations will be the subject of the next chapter. Section 1 also contains various computations (5.1.20–5.1.22 and 5.1.24) which illustrate the use of the chromatic cobar complex.

In Section 2 we compute various Ext^1 groups (5.2.6, 5.2.11, 5.2.14, and 5.2.17) and recover as a corollary the Hopf invariant one theorem (5.2.8), which says almost all elements in the Adams spectral sequence $E_2^{1,*}$ are not permanent cycles. Our method of proof is to show they are not in the image of the Adams–Novikov $E_2^{1,*}$ after computing the latter.

In Section 3 we compute the v_1-periodic part of the Adams–Novikov spectral sequence and its relation to the J-homomorphism and the μ-family of Adams [**?**]. The main result is 5.3.7, and the resulting pattern in the Adams–Novikov spectral sequence for $p = 2$ is illustrated in 5.3.8.

In Section 4 we describe Ext^2 for all primes (5.4.5), referring to the original papers for the proofs, which we cannot improve upon. Corollaries are the nontriviality of γ_t, (5.4.4) and a list of elements in the Adams spectral sequence $E_2^{2,*}$ which cannot be permanent cycles (5.4.7). This latter result is an analog of the Hopf invariant one theorem. The Adams spectral sequence elements not so excluded include the Arf invariant and η_j families. These are discussed in 5.4.8–5.4.10.

In Section 5 we compile all known results about which elements in Ext^2 are permanent cycles, i.e., about the β-family and its generalizations. We survey the relevant work of Smith and Oka for $p \geq 5$, Oka and Toda for $p = 3$, and Davis and Mahowald for $p = 2$.

In Section 6 we give some fragmentary results on Ext^s for $s \geq 3$. We describe some products of α's and β's and their divisibility properties. We close the chapter by describing a possible obstruction to the existence of the δ-family.

Since the appearance of the first edition, many computations related to the chromatic spectral sequence have been made by Shimomura. A list of some of them can be found in Shimomura [**?**]. A description of the first three columns of the chromatic spectral sequence (meaning the rational, v_1- and v_2-periodic parts) for the sphere can be found in Shimomura and Wang [**?**] for $p = 2$, in Shimomura and Wang [**?**] for $p = 3$, and in Shimomura and Yabe [**?**] for $p \geq 5$. Analogous computations for the mod p Moore spectrum can be found in Shimomura [**?**] for $p = 2$, in Shimomura [**?**] for $p = 3$ and in Shimomura [**?**] for $p \geq 5$.

1. The Algebraic Construction

In this section we set up the chromatic spectral sequence converging to the Adams–Novikov E_2-term, and use it to make some simple calculations involving Greek letter elements (1.3.17 and 1.3.19). The chromatic spectral sequence was originally formulated by Miller, Ravenel, and Wilson [**?**]. First we make the following abbreviation in notation, which will be in force throughout this chapter: given a $BP_*(BP)$ comodule M (A1.1.2), we define

$$\mathrm{Ext}(M) = \mathrm{Ext}_{BP_*(BP)}(BP_*, M). \tag{5.1.1}$$

To motivate our construction recall the short exact sequence of comodules given by 4.3.2(c)

$$0 \to \Sigma^{2(p^n-1)} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0 \tag{5.1.2}$$

and let

$$\delta_n\colon \operatorname{Ext}^s(BP_*/I_{n+1}) \to \operatorname{Ext}^{s+1}(BP_*/I_n)$$

denote the corresponding connecting homomorphism.

5.1.3. DEFINITION. *For $t, n > 0$ let*

$$\alpha_t^{(n)} = \delta_0\delta_1\cdots\delta_{n-1}(v_n^t) \in \operatorname{Ext}^n(BP_*). \qquad \square$$

Here $\alpha^{(n)}$ stands for the nth letter of the Greek alphabet. The status of these elements in π_*^S is described in 1.3.11, 1.3.15, and 1.3.18. The invariant prime ideals in I_n in 5.1.2 can be replaced by invariant regular ideals, e.g., those provided by 4.3.3. In particular we have

5.1.4. DEFINITION. $\alpha_{sp^i/i+1} \in \operatorname{Ext}^{1,qsp^i}(BP_*)$ *(where $q = 2p-2$) is the image of $v_1^{sp^i}$ under the connecting homomorphism for the short exact sequence*

$$0 \to BP_* \xrightarrow{p^{i+1}} BP_* \to BP_*/(p^{i+1}) \to 0. \qquad \square$$

We will see below that for $p > 2$ these elements generate $\operatorname{Ext}^1(BP_*)$ (5.2.6) and that they are nontrivial permanent cycles in $\operatorname{im} J$. We want to capture all of these elements from a single short exact sequence; those of 5.1.4 are related by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r, "p^i"] \arrow[d, equal] & BP_* \arrow[r] \arrow[d, "p"] & BP_*/(p^i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r, "p^{i+1}"] & BP_* \arrow[r] & BP_*/(p^{i+1}) \arrow[r] & 0
\end{tikzcd}$$

Taking the direct limit we get

$$0 \to BP_* \to \mathbf{Q}\otimes BP_* \to \mathbf{Q}/\mathbf{Z}_{(p)}\otimes BP_* \to 0;$$

we denote these three modules by N^0, M^0, and N^1, respectively. Similarly, the direct limit of the sequences

$$0 \to BP_*/(p^{i+1}) \xrightarrow{v_1^{p^{i+j}}} \Sigma^{-qp^{i+j}} BP_*/(p^{i+1}) \to \Sigma^{-1p^{i+j}} BP_*/(p^{i+1}, v_1^{p^{i+j}}) \to 0$$

gives us

$$0 \to BP_*/(p^\infty) \to v_1^{-1}BP_*/(p^\infty) \to BP_*/(p^\infty, v_1^\infty) \to 0$$

and we denote these three modules by N^1, M^1, and N^2, respectively. More generally we construct short exact sequences

$$0 \to N^n \to M^n \to N^{n+1} \to 0 \tag{5.1.5}$$

inductively by $M^n = v_n^{-1}BP_* \otimes_{BP_*} N^n$. Hence N^n and M^n are generated as $\mathbf{Z}_{(p)}$-modules by fractions $\frac{x}{y}$ where $x \in BP_*$ for N^n and $v_n^{-1}BP_*$ for M^n and y is a monomial in the ideal $(pv_1\cdots v_{n-1})$ of the subring $\mathbf{Z}_{(p)}[v_1, \ldots, v_{n-1}]$ of BP_*. The BP_*-module structure is such that $vx/y = 0$ for $v \in BP_*$ if this fraction when reduced to lowest terms does not have its denominator in the above ideal. For example, the element $\frac{1}{p^i v_1^j} \in N^2$ is annihilated by the ideal (p^i, v_1^j).

5.1.6. LEMMA. 5.1.5 *is an short exact sequence of $BP_*(BP)$-comodules.*

PROOF. Assume inductively that N^n is a comodule and let $N' \subset N^n$ be a finitely generated subcomodule. Then N' is annihilated by some invariant regular ideal with n generators given by 4.3.3. It follows from 4.3.3 that multiplication by some power of v_n, say v_n^k, is a comodule map, so

$$v_n^{-1}N' = \varinjlim_{v_n^k} \Sigma^{-\dim v_n^{ki}} N'$$

is a comodule. Alternatively, N' is annihilated by some power of I_n, so multiplication by a suitable power of v_n is a comodule map by Proposition 3.6 of Landweber [**?**] and $v_n^{-1}N'$ is again a comodule. Taking the direct limit over all such N' gives us a unique comodule structure on M^n and hence on the quotient N^{n+1}. □

5.1.7. DEFINITION. *The* chromatic resolution *is the long exact sequence of comodules*

$$0 \to BP_* \to M^0 \xrightarrow{d_e} M^1 \xrightarrow{d_e} \cdots$$

obtained by splicing the short exact sequences of 5.1.5. □

The associated resolution spectral sequence (A1.3.2) gives us

5.1.8. PROPOSITION. *There is a chromatic spectral sequence converging to* $\mathrm{Ext}(BP_*)$ *with* $E_1^{n,s} = \mathrm{Ext}^s(M^n)$ *and* $d_r\colon E_r^{n,s} \to E_r^{n+r,s+1-r}$ *where* d_1 *is the map induced by* d_e *in* 5.1.7. □

5.1.9. REMARK. There is a chromatic spectral sequence converging to $\mathrm{Ext}(F)$ where F is any comodule which is flat as a BP_*-module, obtained by tensoring the resolution of 5.1.7 with F.

5.1.10. DEFINITION. *The* chromatic cobar complex $CC(BP_*)$ *is given by*

$$CC^u(BP_*) = \bigoplus_{s+n=u} C^s(M^n),$$

where $C(\ )$ *is the cobar complex of A1.2.11, with* $d(x) = d_e^*(x) + (-1)^n d_i(x)$ *for* $x \in C^s(M^n)$ *where* d_e^* *is the map induced by* d_e *in 5.1.7 (the external component of* d*) and* d_i *(the internal component) is the differential in the cobar complex* $C(M_n)$*.* □

It follows from 5.1.8 and A1.3.4 that $H(CC(BP_*)) = H(C(BP_*)) = \mathrm{Ext}(BP_*)$. The embedding $BP_* \to M^0$ induces an embedding of the cobar complex $C(BP_*)$ into the chromatic cobar complex $CC(BP_*)$. Although $CC(BP_*)$ is larger than $C(BP_*)$, we will see below that it is more convenient for certain calculations such as identifying the Greek letter elements of 5.1.3.

This entire construction can be generalized to BP_*/I_m as follows.

5.1.11. DEFINITION. *Let* $N_m^0 = BP_*/I_m$ *and define* BP_**-modules* N_m^n *and* M_m^n *inductively by short exact sequences*

$$0 \to N_m^n \to M_m^n \to N_m^{n+1} \to 0$$

where $M_m^n = v_{m+n}^{-1}BP_* \otimes_{BP_*} N_m^n$. □

Lemma 5.1.6 can be generalized to show that these are comodules. Splicing them gives an long exact sequence

$$0 \to BP_*/I_m \to M_m^0 \xrightarrow{d_e} M_m^1 \xrightarrow{d_e} \cdots$$

and a chromatic spectral sequence as in 5.1.8. Moreover BP_*/I_m can be replaced by any comodule L having an increasing filtration $\{F_iL\}$ such that each subquotient F_i/F_{i-1} is a suspension of BP_*/I_m, e.g., $L = BP_*/I_m^k$. We leave the details to the interested reader.

Our main motivation here, besides the Greek letter construction, is the computability of $\mathrm{Ext}(M_n^0)$; it is essentially the cohomology of the automorphism group of a formal group law of height n (1.4.3 and A2.2.17). This theory will be the subject of Chapter 6. We will state the first major result now. We have $M_n^0 = v_n^{-1}BP_*/I_n$, which is a comodule algebra (A1.1.2), so $\mathrm{Ext}(M_n^0)$ is a ring (A1.2.14). In particular it is a module over $\mathrm{Ext}^0(M_n^0)$. The following is an easy consequence of the Morava–Landweber theorem, 4.3.2.

5.1.12. PROPOSITION. *For $n > 0$, $\mathrm{Ext}^0(M_n^0) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$. We denote this ring by $K(n)_*$. [The case $n = 0$ is covered by 5.2.1, so it is consistent to denote $\mathbf{Q}$ by $K(0)_*$.]* □

5.1.13. DEFINITION. *Make $K(n)_*$ a BP_*-module by defining multiplication by v_i to be trivial for $i \neq n$. Then let $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$.* □

$\Sigma(n)$, the nth Morava stabilizer algebra, is a Hopf algebroid which will be closely examined in the next chapter. It has previously been called $K(n)_*K(n)$, e.g., in Miller, Ravenel, and Wilson [**?**], Miller and Ravenel [**?**], and Ravenel [**?, ?**]. $K(n)_*$ is also the coefficient ring of the nth Morava K-theory; see Section 4.2 for references. We have changed our notation to avoid confusion with $K(n)_*(K(n))$, which is $\Sigma(n)$ tensored with a certain exterior algebra.

The starting point of Chapter 6 is

5.1.14. CHANGE-OF-RINGS THEOREM (Miller and Ravenel [**?**]).

$$\mathrm{Ext}(M_n^0) = \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*). \qquad \square$$

We will also show (6.2.10)

5.1.15. MORAVA VANISHING THEOREM. *If $(p-1) \nmid n$ then $\mathrm{Ext}^s(M_n^0) = 0$ for $s > n^2$.* □

Moreover this Ext satisfies a kind of Poincaré duality, e.g.,

$$\mathrm{Ext}^s(M_n^0) = \mathrm{Ext}^{n^2-s}(M_n^0),$$

and it is essentially the cohomology of a certain n stage nilpotent Lie algebra of rank n^2. If we replace $\Sigma(n)$ with a quotient by a sufficiently large finitely generated subalgebra, then this Lie algebra becomes abelian and the Ext [even if $(p-1)$ divides n] becomes an exterior algebra over $K(n)_*$ on n^2 generators of degree one.

To connect these groups with the chromatic spectral sequence we have

5.1.16. LEMMA. *There are short exact sequences of comodules*

$$0 \to M_{m+1}^{n-1} \xrightarrow{j} \Sigma^{\dim v_m} M_m^n \xrightarrow{v_m} M_m^n \to 0$$

and Bockstein spectral sequences converging to $\mathrm{Ext}(M_m^n)$ with

$$E_1^{s,*} = \mathrm{Ext}^s(M_{m+1}^{n-1}) \otimes P(a_m)$$

where multiplication by a_m in the Bockstein spectral sequence corresponds to division by v_m in $\mathrm{Ext}(M_m^n)$. d_r is not a derivation but if $d_r(a_m^r x) = y \neq 0$ then $d_r(a_m^{r+i}x) = v_m^i y$.

PROOF. The spectral sequence is that associated with the increasing filtration of M_m^n defined by $F_i M_m^n = \ker v_m^i$ (see A1.3.9). Then $E^0 M_m^n = M_{m+1}^{n-1} \otimes P(a_m)$. $\square$

Using 5.1.16 n times we can in principle get from $\mathrm{Ext}(M_n^0)$ to $\mathrm{Ext}(M_0^n) = \mathrm{Ext}(M^n)$ and hence compute the chromatic E_1-term (5.1.8). In practice these computations can be difficult.

5.1.17. REMARK. We will not actually use the Bockstein spectral sequence of 5.1.16 but will work directly with the long exact sequence

$$\to \mathrm{Ext}^s(M_{m+1}^{n-1}) \xrightarrow{j} \mathrm{Ext}^s(M_m^n) \xrightarrow{v_m} \mathrm{Ext}^s(\Sigma^{-2p^m+2} M_m^n) \xrightarrow{\delta} \mathrm{Ext}^{s+1}(M_{m+1}^{n-1}) \to \cdots$$

by induction on s. Given an element $x \in \mathrm{Ext}(M_{m+1}^{n-1})$ which we know not to be in $\operatorname{im}\delta$, we try to divide $j(x)$ by v_m as many times as possible. When we find an $x' \in \mathrm{Ext}(M_m^n)$ with $v_m^r x' = j(x)$ and $\delta(x') = y \neq 0$ then we will know that $j(x)$ cannot be divided any further by v_m. Hence δ serves as reduction mod I_{m+1}. This state of affairs corresponds to $d_r(a_m^r x) = y$ in the Bockstein spectral sequence of 5.1.16. We will give a sample calculation with δ below (5.1.20).

We will now make some simple calculations with the chromatic spectral sequence starting with the Greek letter elements of 5.1.3. The short exact sequence of 5.1.2 maps to that of 5.1.5, i.e., we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & BP_*/I_n \arrow[r, "v_n"] \arrow[d, "i"] & \Sigma^{-\dim v_n} BP_*/I_n \arrow[r] \arrow[d] & \Sigma^{-\dim v_n} BP_*/I_{n+1} \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & N^n \arrow[r] & M^n \arrow[r] & N^{n+1} \arrow[r] & 0
\end{tikzcd}$$

with

$$i(v_{n+1}^t) = \frac{v_{n+1}^t}{p v_1 \cdots v_n}.$$

Hence $\alpha_t^{(n)}$ can be defined as the image of $i(v_n^t)$ under the composite of the connecting homomorphisms of 5.1.5, which we denote by $\alpha\colon \mathrm{Ext}^0(N^n) \to \mathrm{Ext}^n(BP_*)$. On the other hand, the chromatic spectral sequence has a bottom edge homomorphism

$$\begin{tikzcd}
\mathrm{Ext}^0(M^n) \arrow[r, equal] & E_1^{n,0} & & \\
\mathrm{Ext}^0(N^n) \arrow[r] \arrow[u] & \ker d_1 \arrow[r] \arrow[u] & E_\infty^{n,0} \arrow[r] & \mathrm{Ext}^n(BP_*)
\end{tikzcd}$$

which we denote by

$$\kappa\colon \mathrm{Ext}^0(N^n) \to \mathrm{Ext}^n(BP_*).$$

κ and α differ by sign, i.e.,

5.1.18. PROPOSITION. $\kappa = (-1)^{[(n+1)/2]}\alpha$, *where* $[x]$ *is the largest integer not exceeding* x.

PROOF. The image y_0 of $i(v_n^t)$ in M^n is an element in the chromatic complex (5.1.10) cohomologous to some class in the cobar complex $C(BP_*)$. Inductively we can find $x_s \in C^s(M^{n-s-1})$, and $y_s \in C^s(M^{n-s})$ such that $d_e(x_s) = y_s$ and $d_i(x_s) = y_{s+1}$. Moreover $y_n \in C^n(M^0)$ is the image of some $x_n \in C^n(BP_*)$. It follows from the definition of the connecting homomorphism that x_n is a cocycle representing $\alpha(i(v_n^t)) = \alpha_t^{(n)}$. On the other hand, y_s is cohomologous to

$(-1)^{n-s}y_{s+1}$ in $CC(BP_*)$ by 5.1.10 and $\prod_{s=0}^{n-1}(-1)^{n-s} = (-1)^{[n+1/2]}$ so x_n represents $(-1)^{[n+1/2]}\kappa(i(v_n^t))$. $\square$

5.1.19. Definition. *If $x \in \mathrm{Ext}^0(M^n)$ is in the image of $\mathrm{Ext}^0(N^n)$ (and hence gives a permanent cycle in the chromatic spectral sequence) and has the form*

$$\frac{v_n^t}{p^{i_0}v_1^{i_1}\cdots v_{n-1}^{i_{n-1}}} \bmod I_n$$

(i.e., x is the indicated fraction plus terms with larger annihilator ideals) then we denote $\alpha(x)$ by $\alpha^{(n)}_{t/i_{n-1},\ldots,i_0}$; if for some $m < n$, $i_k = 1$ for $k \le m$ then we abbreviate $\alpha(x)$ by $\alpha^{(n)}_{t/i_{n-1},\ldots,i_{m+1}}$. $\square$

5.1.20. Examples and Remarks. We will compute the image of β_t in $\mathrm{Ext}^2(BP_*/I_2)$ for $p > 2$ in two ways.

(a) We regard β_t as an element in $\mathrm{Ext}^0(M^2)$ and compute its image under connecting homomorphisms δ_0 to $\mathrm{Ext}^1(M_1^1)$ and then δ_1 to $\mathrm{Ext}^2(M_2^0)$, which is $E_1^{0,2}$ in the chromatic spectral sequence for $\mathrm{Ext}(BP_*/I_2)$. To compute δ_0, we pick an element in $x \in M^2$ such that $px = \beta_t$, and compute its coboundary in the cobar complex $C(M^2)$. The result is necessarily a cocycle of order p, so it can be pulled back to $\mathrm{Ext}^1(M_1^1)$. To compute δ_1 on this element we take a representative in $C^1(M_1^1)$, divide it by v_1, and compute its coboundary.

Specifically β_t is $\frac{v_2^t}{pv_1} \in M^2$, so we need to compute the coboundary of $x = \frac{v_2^t}{p^2v_1}$. It is convenient to write x as $\frac{v_1^{p-1}v_2^t}{p^2v_1^p}$, then the denominator is the product of elements generating an invariant regular ideal, which means that we need to compute η_R on the numerator only. We have

$$\eta_R(v_1^{p-1}) \equiv v_1^{p-1} - pv_1^{p-2}t_1 \quad \bmod (p^2)$$

and

$$\eta_R(v_2^t) \equiv v_2^t + tv_2^{t-1}(v_1t_1^p + pt_2) \quad \bmod (p^2, pv_1, v_1^2).$$

These give

$$d\left(\frac{v_t^{p-1}v_2^t}{p^2v_1^p}\right) = \frac{-v_2^tt_1}{pv_1^2} + \frac{tv_2^{t-1}}{pv_1}(t_2 - t_1^{1+p}).$$

This is an element of order p in $C^1(M^2)$, so it is in the image of $C^1(M_1^1)$. In this group the p in the denominator is superfluous, since everything has order p, so we omit it. To compute δ_1 we divide by v_1 and compute the coboundary; i.e., we need to find

$$d\left(\frac{-v_2^tt_1}{v_1^3} + \frac{tv_2^{t-1}(t_2 - t_1^{1+p})}{v_1^2}\right).$$

Recall (4.3.15)

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 + v_1b_{10}$$

where

$$b_{10} = -\sum_{0<i<p}\frac{1}{p}\binom{p}{i}t_1^i \otimes t_1^{p-i}$$

as in 4.3.14. From this we get

$$\begin{aligned} d\left(\frac{-v_2^t t_1}{v_1^3}+\frac{tv_2^{t-1}}{v_1^2}(t_2-t_1^{1+p})\right) =& \frac{-tv_2^{t-1}t_1^p|t_1}{v_1^2}-\binom{t}{2}\frac{v_2^{t-2}}{v_1}t_1^{2p}|t_1 \\ &+t(t-1)\frac{v_2^{t-2}}{t_1^p}|(t_2-t_1^{1+p}) \\ &+t\frac{v_2^{t-1}}{v_1^2}(-v_1b_{10}+t_1^p|t_1) \\ =&\binom{t}{2}\frac{v_2^{t-2}}{v_1}(2t_1^p|t_2-2t_1^p|t_1^{1+p}-t_1^{2p}|t_1) \\ &-t\frac{v_2^{t-1}}{v_1}b_{10}. \end{aligned}$$

We will see below that $\mathrm{Ext}^2(M_2^0)$ has generators k_0 represented by $2t_1^p|t_2-2t_1^p|t_1^{1+p}-t_1^{2p}|t_1$ and b_{10}. Hence the mod I_2 reduction of $-\beta_t$ is

$$\binom{t}{2}v_2^{t-2}k_0+tv_2^{t-1}b_{1,0}.$$

(b) In the chromatic complex $CC(BP_*)$ (5.1.10), $\beta_t\in M^2$ is cohomologous to elements in $C^1(M^1)$ and $C^2(M^0)$. These three elements pull back to N^2, $C^1(N^1)$, and $C^2(N^0)$, respectively. In theory we could compute the element in $C^2(N^0)=C^2(BP_*)$ and reduce mod I_2, but this would be very laborious. Most of the terms of the element in $C^0(BP_*)$ are trivial mod I_2, so we want to avoid computing them in the first place. The passage from $C^0(N^2)$ to $C^2(BP_*)$ is based on the four-term exact sequence

$$0\to BP_*\to M^0\to M^1\to N^2\to 0.$$

Since $\frac{v_2^t}{pv_1}\in N^2$ is in the image of $\Sigma^{-q}BP_*/I_2$, we can replace this sequence with

$$0\to BP_*\xrightarrow{p}BP_*\xrightarrow{v_1}\Sigma^{-q}BP_*/I_1\to\Sigma^{-q}BP_*/I_2\to 0.$$

We are going to map the first BP_* to BP_*/I_2; we can extend this to a map of sequences to

$$0\to BP_*/I_2\xrightarrow{p}BP_*/(p^2,pv_1,v_1^2)\xrightarrow{v_1}\Sigma^{-q}BP_*/(p,v_1^3)\to\Sigma^{-q}BP_*/I_2\to 0,$$

which is the identity on the last comodule. [The reader may be tempted to replace the middle map by

$$BP_*/(p^2,v_1)\xrightarrow{v_1}\Sigma^{-q}BP_*/(p,v_1^2)$$

but $BP_*/(p^2,v_1)$ is not a comodule.] This sequence tells us which terms we can ignore when computing in the chromatic complex, as we will see below.

Specifically we find (ignoring signs) that $\frac{v_2^t}{pv_1}\in M^2$ is cohomologous to

$$\frac{tv_2^{t-1}t_1^p}{p}+\binom{t}{2}\frac{v_1v_2^{t-2}}{p}t_1^{2p}+\text{higher terms}.$$

Note that the first two terms are divisible by v_1 and v_1^2 respectively in the image of $C^1(\Sigma^{-q}BP_*/(p))$ in $C^1(M^1)$. The higher terms are divisible by v_1^3 and can therefore be ignored.

In the next step we will need to work mod I_2^2 in the image of $C^2(BP_*)$ in $C^2(M^0)$ via multiplication by p. From the first term above we get

$$t(t-1)v_2^{t-2}t_2|t_1^p + tv_2^{t-1}b_{10},$$

while the second term gives

$$\binom{t}{2}v_2^{t-2}t_1|t_1^{2p}$$

and their sum represents the same element obtained in (a).

Our next result is

5.1.21. PROPOSITION. *For $n \geq 3$,*

$$\alpha_1^{(n)} = (-1)^n \alpha_1 \alpha_{p-1}^{(n-1)}. \qquad \square$$

For $n = 3$ this gives $\gamma_1 = -\alpha_1\beta_{p-1}$. In the controversy over the nontriviality of γ_1 (cf. the paragraph following 1.3.18) the relevant stem was known to be generated by $\alpha_1\beta_{p-1}$, so what follows is an easy way (given all of our machinery) to show $\gamma_1 \neq 0$.

PROOF OF 5.1.2. α_1 is easily seen to be represented by t_1 in $C(BP_*)$, while $\alpha_1^{(n)}$ and $\alpha_{p-1}^{(n-1)}$ are represented by

$$(-1)^{[n+1/2]}\frac{v_n}{pv_1\cdots v_{n-1}} \in M^n \quad \text{and} \quad (-1)^{[n/2]}\frac{v_{n-1}^{p-1}}{pv_1\cdots v_{n-2}} \in M^{n-1},$$

respectively. Hence $(-1)^n\alpha_1\alpha_{p-1}^{(n-1)} = -\alpha_{p-1}^{(n-1)}\alpha_1$ is represented by

$$(-1)^{[n/2]}\frac{v_{n-1}^{p-1}t_1}{pv_1\cdots v_{n-2}} \in C^1(M^{n-1}) \subset CC^n(BP_*)$$

and it suffices to show that this element is cohomologous to $\frac{(-1)^{[n+1/2]}v_n}{(pv_1\cdots v_{n-1})}$ in $CC(BP_*)$.

Now consider

$$x = \frac{v_{n-1}^{-1}v_n}{pv_1\cdots v_{n-2}} - \frac{v_{n-1}^p}{pv_1\cdots v_{n-3}v_{n-2}^{1+p}} \in M^{n-1}.$$

Clearly

$$d_e(x) = \frac{v_n}{pv_1\cdots v_{n-1}}.$$

To compute $d_i(x)$ we need to know $\eta_R(v_{n-1}^{-1}v_n)$ mod I_{n-1} and $\eta_R(v_{n-1}^p)$ mod $(p, v_1, \ldots, v_{n-3}, v_{n-2}^{1+p})$ since $d_i(x) = \eta_R(x) - x$. We know

$$\eta_R(v_n) \equiv v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \mod I_{n-1}$$

by 4.3.21, so

$$\eta_R(v_{n-1}^p) \equiv v_{n-1}^p + v_{n-2}^p t_1^{p^{n-1}} - v_{n-2}^{p^2}t_1^p \mod I_{n-2}.$$

Hence

$$\eta_R(v_{n-1}^{-1}v_n) - v_{n-1}^{-1}v_n \equiv t_1^{p^{n-1}} - v_{n-1}^{p-1}t_1 \mod I_{n-1}$$

and

$$\eta_R(v_{n-1}^p) - v_{n-1}^p \equiv v_{n-2}^p t_1^{p^{n-1}} \mod (p, v_1, \ldots, v_{n-3}, v_{n-2}^{1+p}).$$

It follows that

$$d_i(x) = \frac{-v_{n-1}^{p-1}t_1}{pv_1 \cdots v_{n-2}}$$

so

$$d(x) = \frac{v_n}{pv_1 \cdots v_{n-1}} + (-1)^n \frac{v_{n-1}^{p-1}t_1}{pv_1 \cdots v_{n-2}}$$

and a simple sign calculation gives the result. □

For $p = 2$ 5.1.21 says $\alpha_1^{(n)} = \alpha_1^{n-2}\alpha_1^{(2)}$ for $n \geq 2$. We will show that each of these elements vanishes and that they are killed by higher differentials (d_{n-1}) in the chromatic spectral sequence. We do not know if there are nontrivial d_r's for all $r \geq 2$ for odd primes.

5.1.22. THEOREM. *In the chromatic spectral sequence for $p = 2$ there are elements $x_n \in E_{n-1}^{1,n-2}$ for $n \geq 2$ such that*

$$d_{n-1}(x_n) = \frac{v_n}{2v_1 \cdots v_{n-1}} \in E_{n-1}^{n,0}.$$

PROOF. Fortunately we need not worry about signs this time. Equation 4.3.1 gives $\eta_R(v_1) = v_1 - 2t_1$ and $\eta_R(v_2) \equiv v_2 + v_1 t_1^2 + v_1^2 t_1 \mod (2)$. We find then that

$$x_2 = \frac{v_1^2 + 4v_1^{-1}v_2}{8}$$

has the desired property. For $n > 2$ we represented x_n by

$$\frac{[(t_2 - t_1^3 + v_1^{-1}v_2 t_1)|t_1| \cdots |t_1|]}{2} \in C^{n-2}(M^1)$$

with $n-3$ t_1's. To compute $d_{n-1}(x_n)$ let

$$\tilde{x}_n = x_n + \sum_{i=1}^{n-2} \frac{(v_{i+1}^2 - v_i^2 v_{i+1}^{-1} v_{i+2})t_1| \cdots |t_1}{2v_1 \cdots v_{i-1}v_i^3} \in CC(BP_*),$$

where the ith term has $(n-2-i)$ t_1's. Then one computes

$$d(\tilde{x}_n) = \frac{v_n}{2v_1 \cdots v_{n-1}},$$

so

$$d_{n-1}(x_n) = \frac{v_n}{2v_1 \cdots v_{n-1}}$$

unless this element is killed by an earlier differential, in which case x_n, would represent a nontrivial element in $\mathrm{Ext}^{n-1,2n}(BP_*)$, which is trivial by 5.1.23 below. □

5.1.23. EDGE THEOREM.
(a) For all primes p $\mathrm{Ext}^{s,t}(BP_*) = 0$ for $t < 2s$,
(b) for $p = 2$ $\mathrm{Ext}^{s,2s}(BP_*) = \mathbf{Z}/(2)$ for $s \geq 1$, and
(c) for $p = 2$ $\mathrm{Ext}^{s,2s+2}(BP_*) = 0$ for $s \geq 2$.

PROOF. We use the cobar complex $C(BP_*)$ of A1.2.11. Part (a) follows from the fact that $C^{s,t}$ for $t < 2s$. $C^{s,2s}$ is spanned by $t_1| \cdots |t_1$ while $C^{s,2s+2}$ is spanned by $v_1 t_1| \cdots |t_1$ and $e_j = t_1| \cdots t_1|t_1^2|t_1 \cdots t_1$ with t_1^2 in the jth position, $1 \leq j \leq s$. Since $d(t_1^3) = -3t_1|t_1^2 - 3t_1^2|t_1$, the e_j's differ by a coboundary up to sign. Part (b) follows from

$$d(e_1) = 2t_1| \cdots |t_1 = -d(v_1 t_1| \cdots |t_1)$$

and (c) follows from

$$d(t_2|t_1|\cdots|t_1) = -v_1 t_1|\cdots|t_1 - e_1.$$

□

We conclude this section by tying up some loose ends in Section 4.4. For $p > 2$ we need

5.1.24. LEMMA. *For odd primes, $\alpha_1\beta_p$ is divisible by p but not by p^2. (This gives the first element of order p^2 in* $\mathrm{Ext}^s(BP_*)$ *for $s \geq 2$.)*

PROOF. Up to sign $\alpha_1\beta_p$ is represented by $\frac{v_2^p t_1}{pv_1}$. Now $\frac{v_2^p t_1}{p^2 v_1}$ is not a cocycle, but if we can get a cocycle by adding a term of order p then we will have the desired divisibility. It is more convenient to write this element as $\frac{v_1^{p-1} v_2^p t_1}{p^2 v_1^p}$; then the factors of the denominator form an invariant sequence [i.e., $\eta_R(v_1^p) \equiv v_1^p \mod (p^2)$], so to compute the coboundary it suffices to compute $\eta_R(v_1^{p-1} v_2) \mod (p^2, v_1^p)$. We find

$$d\left(\frac{v_1^{p-1} v_2^p}{p^2 v_1^p}\, t_1\right) = \frac{-v_2^p t_1|t_1}{p v_1^2} = \frac{1}{2}\, d\left(\frac{v_2^p t_1^2}{p v_1^2}\right)$$

so the desired cocycle is

$$\frac{v_1^{p-1} v_2^p t_1}{p^2 v_1^p} - \frac{1}{2}\,\frac{v_2^p t_1^2}{p v_1^2}.$$

This divisibility will be generalized in (5.6.2).

To show that $\alpha_1\beta_p$ is not divisible by p^2 we compute the mod (p) reduction of our cocycle. More precisely we compute its image under the connecting homomorphism associated with

$$0 \to M_1^1 \to M_0^2 \xrightarrow{p} M_0^2 \to 0$$

(see 5.1.16). To do this we divide by p and compute the coboundary. Our divided (by p) cocycle is

$$\frac{v_1^{p^2-1} v_2^p t_1}{p^3 v_1^{p^2}} - \frac{1}{2}\,\frac{v_1^{p-2} v_2^p t_1^2}{p^2 v_1^p}$$

and its coboundary is

$$\frac{v_2^p (t_1^2|t_1 + t_1|t_1^2)}{p v_1^3} + \frac{v_2^{p-1} t_2|t_1}{p v_1} - \frac{1}{2}\,\frac{v_2^{p-1} t_1^p|t_1^2}{p v_1} - \frac{v_2^{p-1} t_1^{1+p}|t_1}{p v_1}$$

We can eliminate the first term by adding $\frac{1}{3}\frac{v_2^p t_1^3}{p v_i^3}$ (even if $p = 3$). For $p > 3$ the resulting element in $C^2(M_1^1)$ is

$$\frac{v_2^{p-1}(t_2|t_1 - t_1^{1+p}|t_1 - t_1^p|t_2)}{v_1}.$$

Reducing this mod I_2 in a similar fashion gives a unit multiple of $\bar{\phi}$ in 4.1.14. For $p = 3$ we add $\frac{-v_2 t_1^6}{3 v_1^2}$ to the divided cocycle and get

$$\frac{v_2^{p-1}(t_2|t_1 - \cdots)}{v_1} + \frac{v_2}{v_1}(t_1^3|t_1^6 + t_1^6|t_1^3),$$

which still gives a nonzero element in $\mathrm{Ext}^2(M_1^1)$. □

For $p = 2$ we need to prove 4.4.38 and 4.4.40, i.e., 5.1.25. LEMMA. *In the notation of* 4.4.32 *for* $p = 2$ (a) $\delta_0(\beta_3) \equiv \beta_{2/2}^2 + \eta_1 \mod (2)$,

(b) $\delta_0(\eta_2) \equiv c_0 \mod (2)$.

PROOF. For (a) we have

$$d\left(\frac{v_1 v_2^3}{4v_1^2} + \frac{v_2 v_3}{2v_1^2}\right) = \frac{v_2^2 t_1^4}{2v_1^2} + \frac{v_3 t_1^2 + v_2 t_2^2 + v_2^2 t_2 + v_2 t_1^6}{2v_1},$$

which gives the result.

For (b) we use Massey products. We have $\eta_2 \langle \eta_1, v_1, \beta_1 \rangle$ so by A1.4.11 we have $\delta_0(\eta_2) \equiv \langle \eta_1, h_{10}, \beta_1 \rangle \mod (2)$. Hence we have to equate this product with c_0, which by 4.4.31 is represented by $\frac{x_{22}}{v_1}$, where x_{22} is defined by 4.4.25. To expedite this calculation we will use a generalization of Massey products not given in A1.4 but fully described by May [**?**]. We regard η_1 as an element in $\mathrm{Ext}^1(M_1^1)$, and h_{10}, and β_1 as elements in $\mathrm{Ext}^1(BP_*/I_1)$ and use the pairing $M_1^1 \otimes BP_*/I_1 \to M_1^1$ to define the product. Hence the cocycles representing η_1, h_{10} and β_1 are

$$\frac{v_3 t_1^2 + v_2(t_2^2 + t_1^6) + v_2^2 t_2}{v_1}, \quad t_1, \quad \text{and} \quad t_1^2 + v_1 t_1,$$

respectively. The cochains whose coboundaries are the two successive products are

$$\frac{v_3(t_2 + t_1^3) + v_2(t_3 + t_1 t_2^2 + t_1^4 t_2 + t_1^7) + v_2^2(t_1^4 + t_1 t_2)}{v_1} \quad \text{and} \quad t_2.$$

If we alter the resulting cochain representative of the Massey product by the coboundary of

$$\frac{v_3 t_1^2 t_2 + v_2(t_2^3 + t_2 t_1^6 + t_1^9) + v_2^2(t_1^6 + t_2^2)}{v_1} + \frac{v_2^4(t_2 + t_1^3)}{v_1^4} + \frac{v_2^5 t_1}{v_1^5}$$

we get the desired result. □

2. $\mathrm{Ext}^1(BP_*/I_n)$ and Hopf Invariant One

In this section we compute $\mathrm{Ext}^1(BP_*/I_n)$ for all n. For $n > 0$ our main results are 5.2.14 and 5.2.17. For $n = 0$ this group is $E_2^{1,*}$ in the Adams–Novikov spectral sequence and is given in 5.2.6. In 5.2.8 we will compute its image in the classical Adams spectral sequence, thereby obtaining proofs of the essential content of the Hopf invariant one theorems 1.2.12 and 1.2.14. More precisely, we will prove that the specified h_i's are not permanent cycles, but we will not compute $d_2(h_i)$. The computation of $\mathrm{Ext}^1(BP_*/I_n)$ is originally due to Novikov [**?**] for $n = 0$ and to Miller and Wilson [**?**] for $n > 0$ (except for $n = 1$ and $p \geq 2$).

To compute $\mathrm{Ext}^1(BP_*)$ with the chromatic spectral sequence we need to know $\mathrm{Ext}^1(M^0)$ and $\mathrm{Ext}^0(M^1)$. For the former we have

5.2.1. THEOREM. (a) $\mathrm{Ext}^{s,t}(M^0) = \begin{cases} \mathbf{Q} & \text{if } s = t = 0 \\ 0 & \text{otherwise} \end{cases}$

(b) $\mathrm{Ext}^{0,t}(BP_*) = \begin{cases} \mathbf{Z}_{(p)} & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$.

PROOF. (a) Since $M^0 = \mathbf{Q} \otimes BP_*$, we have $\mathrm{Ext}(M^0) = \mathrm{Ext}_\Gamma(A, A)$ where $A = M^0$ and $\Gamma = \mathbf{Q} \otimes BP_*(BP)$. Since t_n is a rational multiple of $\eta_R(v_n) - v_n$ modulo decomposables, Γ is generated by the image of η_R and η_L and is therefore a unicursal Hopf algebroid (A1.1.11). Let $\bar{v}_n = \eta_R(v_n)$, so $\Gamma = A[\bar{v}_1, \bar{v}_2, \dots]$. The coproduct in Γ is given by $\Delta(v_n) = v_n \otimes 1$ and $\Delta(\bar{v}_n) = 1 \otimes \bar{v}_n$. The map $\eta_R\colon A \to \Gamma = A \otimes_A \Gamma$ makes A a right Γ-comodule. Let R be the complex $\Gamma \otimes E(y_1, y_2, \dots)$ where $E(y_1, y_2, \dots)$ is an exterior algebra on generators y_i of degree 1 and dimension $2(p^i - 1)$. Let the coboundary d be a derivation with $d(y_n) = d(\bar{v}_n) = 0$ and $d(v_n) = y_n$. Then R is easily seen to be acyclic with $H^0(R) = A$. Hence R is a suitable resolution for computing $\mathrm{Ext}_\Gamma(A, A)$ (A1.2.4). We have $\mathrm{Hom}_\Gamma(A, R) = A \otimes E(y_1, \dots)$ and this complex is easily seen to be acyclic and gives the indicated Ext groups for M^0.

For (b) $\mathrm{Ext}^0 BP_* = \ker d_e \subset \mathrm{Ext}^0(M_0^0)$ and $d_e(x) \neq 0$ if x is a unit multiple of a negative power of p. $\square$

To get at $\mathrm{Ext}(M^1)$ we start with

5.2.2. THEOREM.

(a) *For $p > 2$, $\mathrm{Ext}(M_1^0) = K(1)_* \otimes E(h_0)$ where $h_0 \in \mathrm{Ext}^{1,q}$ is represented by t_1 in $C^1(M_1^0)$ (see 5.1.12) and $q = 2p - 2$ as usual.*

(b) *For $p = 2$, $\mathrm{Ext}(M_1^0) = K(1)_* \otimes P(h_0) \otimes E(\rho_1)$, where h_0 is as above and $\rho_1 \in \mathrm{Ext}^{1,0}$ is represented by $v_1^{-3}(t_2 - t_1^3) + v_1^{-4} v_2 t_1$.* $\square$

This will be proved below as 6.3.21.

Now we use the method of 5.1.17 to find $\mathrm{Ext}^0(M^1)$; in the next section we will compute all of $\mathrm{Ext}(M^1)$ in this way. From 4.3.3 we have $\eta_R(v_1^{p^i}) \equiv v_1^{p^i} \mod (p^{i+1})$, so $\frac{v_1^{sp^i}}{p^{i+1}} \in \mathrm{Ext}^0(M^1)$. For p odd we have

$$\eta_R(v_1^{sp^i}) \equiv v_1^{sp^i} + sp^{i+1} v_1^{sp^i - 1} t_1 \mod (p^{i+2}) \tag{5.2.3}$$

so in 5.1.17 we have

$$\delta\left(\frac{v_1^{sp^i}}{p^{i+1}}\right) = s v_1^{sp^i - 1} h_0 \in \mathrm{Ext}^1(M_1^0)$$

for $p \nmid s$, and we can read off the structure of $\mathrm{Ext}^0(M_1^0)$ below.

For $p = 2$, 5.2.3 fails for $i > 0$, e.g.,

$$\eta_R(v_1^2) = v_1^2 + 4v_1 t_1 + 4t_1^2 \mod (8).$$

The element $t_1^2 + v_1 t_1 \in C^1(M_1^0)$ is the coboundary of $v_1^{-1} v_2$, so

$$\alpha_{2/3} = \frac{(v_1^2 + 4v_1^{-1} v_2)}{8} \in \mathrm{Ext}^0(M^1);$$

i.e., we can divide by at least one more power of p than in the odd primary case. In order to show that further division by 2 is not possible we need to show that $\alpha_{2/3}$ has a nontrivial image under δ (5.1.17). This in turn requires a formula for $\eta_R(v_2)$ mod (4). From 4.3.1 we get

$$\eta_R(v_2) = v_2 + 13 v_1 t_1^2 - 3v_1^2 t_1 - 14 t_2 - 4t_1^3. \tag{5.2.4}$$

[This formula, as well as $\eta_R(v_1) = v_1 - 2t_1$, are in terms of the v_i defined by Araki's formula A2.2.2. Using Hazewinkel's generators defined by A2.2.1 gives $\eta_R(v_1) = v_1 + 2t_1$ and $\eta_R(v_2) = v_2 - 5v_1 t_1^2 - 3v_1^2 t_1 + 2t_2 - 4t_1^3$.]

Let $x_{1,1} = v_1^2 + 4v_1^{-1} v_2$. Then 5.2.4 gives

$$\eta_R(x_{1,1}) \equiv x_{1,1} + 8(v_1^{-1} t_2 + v_1^{-1} t_1^3 + v_1^{-2} v_2 t_1) \mod (16) \tag{5.2.5}$$

so $\delta(\alpha_{2/3}) = v_1^2 \rho_1 \neq 0 \in \mathrm{Ext}^1(M_1^0)$.

5.2.6. THEOREM.
(a) *For p odd*

$$\mathrm{Ext}^{0,t}(M^1) = \begin{cases} 0 & \text{if } q \nmid t \text{ where } q = 2p-2 \\ \mathbf{Q}/\mathbf{Z}_{(p)} & \text{if } t = 0 \\ \mathbf{Z}/(p^{i+1}) & \text{if } t = sp^i \text{ and } p \nmid s \end{cases}$$

These groups are generated by

$$\frac{v_1^{sp^i}}{p^{i+1}} \in M^1.$$

(b) *For p odd*

$$\mathrm{Ext}^{1,t}(BP_*) = \begin{cases} \mathrm{Ext}^{0,t}(M^1) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

(c) *For $p = 2$*

$$\mathrm{Ext}^{0,t}(M^1) = \begin{cases} 0 & \text{if } t \text{ is odd} \\ \mathbf{Q}/\mathbf{Z}_{(2)} & \text{if } t = 0 \\ \mathbf{Z}/(2) & \text{if } t \equiv 2 \mod 4 \\ \mathbf{Z}/(2^{i+3}) & \text{if } t = 2^{i+2}s \text{ for odd } s \end{cases}$$

These groups are generated by $\frac{v_1^s}{2}$ *and* $\frac{x_{1,1}^{2^i s}}{2^{i+3}} \in M^1$ *where* $x_{1,1}$ *is as in* 5.2.5.
(d) *For $p = 2$*

$$\mathrm{Ext}^{1,t}(BP_*) = \begin{cases} 0 & \text{if } t \le 0 \\ \mathrm{Ext}^{0,t}(M^1) & \text{if } t > 0 \text{ and } t \ne 4 \\ \mathbf{Z}/(4) & \text{if } t = 4 \end{cases}$$

and $\mathrm{Ext}^{1,4}(BP_*)$ *is generated by* $\alpha_{2,2} = \pm\frac{v_1^2}{4}$.

We will see in the next section (5.3.7) that in the Adams–Novikov spectral sequence for $p > 2$, each element of $\mathrm{Ext}^1(BP_*)$ is a permanent cycle detecting an element in the image of the J-homomorphism (1.1.13). For $p = 2$ the generators of $\mathrm{Ext}^{1,2t}$ are permanent cycles for $t \equiv 0$ and $1 \mod (4)$ while for $t \equiv 2$ and 3 the generators support nontrivial d_3's (except when $t = 2$) and the elements of order 4 in $\mathrm{Ext}^{1,8t+4}$ are permanent cycles. The generators of $E_4^{1,4t} = E_\infty^{1,4t}$ detect elements in $\mathrm{im}\, J$ for all $t > 0$.

PROOF OF 5.2.6. Part (a) was sketched above. We get $\mathbf{Q}/\mathbf{Z}_{(p)}$ in dimension zero because $1/p^i$ is a cocycle for all $i > 0$. For (b) the chromatic spectral sequence gives an short exact sequence

$$0 \to E_\infty^{1,0} \to \mathrm{Ext}^1(BP_*) \to E_\infty^{0,1} \to 0$$

and $E_\infty^{0,1}$ by 5.2.1. $E_\infty^{1,0} = E_2^{1,0} = \ker d_e / \mathrm{im}\, d_e$. An element in $E_1^{1,0} = \mathrm{Ext}^0(M^1)$ has a nontrivial image under d_e iff it has terms involving negative powers of v_1, so $\ker d_e \subset E_1^{1,0}$ is the subgroup of elements in nonnegative dimensions. The zero-dimensional summand $\mathbf{Q}/\mathbf{Z}_{(p)}$ is the image of d_e, so $E_2^{1,0} = \mathrm{Ext}^1(BP_*)$ is as stated.

For (c) the computation of $\mathrm{Ext}^0(M_1^0)$ is more complicated for $p=2$ since 5.2.3 no longer holds. From 5.2.5 we get

$$\text{(5.2.7)} \qquad \eta_R(x_{1,1}^{2^i s}) \equiv x_{1,1}^{2^i s} + 2^{i+3} x_{1,1}^{2^i s - 1}(v_1^{-1}t_2 + v_1^{-1}t_1^3 + v_1^{-2}v_2t_1) \qquad \mathrm{mod}\ (2^{i+4})$$

for odd s, from which we deduce that $\frac{x_{1,1}^{2^i s}}{2^{i+3}}$ is a cocycle whose image under δ (see 5.1.17) is $v_1^{2^{i+1}s}\rho_1$. Equation 5.2.3 does hold for $p=2$ when $i=0$, so $\mathrm{Ext}^{0,2s}(M_1^0)$ is generated by $\frac{v_1^s}{2}$ for odd s. This completes the proof of (c).

For (d) we proceed as in (b) and the situation in nonpositive dimensions is the same. We need to compute $d_e\left(\frac{x_{1,1}^{2^i s}}{2^{i+3}}\right)$. Since $x_{1,1} = v_1^2 + 4v_1^{-1}v_2$, we have

$$\frac{x_{1,1}^{2^i s}}{2^{i+3}} = \frac{v_1^{2^i s} + s - 2^{i+2}v_1^{2^i s - 3}v_2}{2^{i+3}}.$$

For $2^i s = 1$ (but for no $2^i s > 1$) this expression has a negative power of v_1 and we get

$$d_e\left(\frac{x_{1,1}}{8}\right) = \frac{v_2}{2v_1} \in M^2.$$

This gives a chromatic d_1 (compare 5.1.21) and accounts for the discrepancy between $\mathrm{Ext}^{0,4}(M^1)$ and $\mathrm{Ext}^{1,4}(BP_*)$. $\square$

Now we turn to the Hopf invariant one problem. Theorems 1.2.12 and 1.2.14 say which elements of filtration 1 in the classical Adams spectral sequence are permanent cycles. We can derive these results from our computation of $\mathrm{Ext}^1(BP_*)$ as follows. The map $BP \to H/(p)$ induces a map Φ from the Adams–Novikov spectral sequence to the Adams spectral sequence. Since both spectral sequences converge to the same thing there is essentially a one-to-one correspondence between their E_∞-terms. A nontrivial permanent cycle in the Adams spectral sequence of filtration s corresponds to one in the Adams–Novikov spectral sequence of filtration $\leq s$.

To see this consider BP_* and mod (p) Adams resolutions (2.2.1 and 2.1.3)

$$\begin{tikzcd}
S^0 \arrow[r, equal] & X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
S^0 \arrow[r, equal] & Y_0 & Y_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

where the vertical maps are the ones inducing Φ. An element $x \in \pi_*(S^0)$ has Adams filtration s if it is in $\mathrm{im}\,\pi_*(Y_s)$ but not in $\mathrm{im}\,\pi_*(Y_{s+1})$. Hence it is not in $\mathrm{im}\,\pi_*(X_{s+1})$ and its Novikov filtration is at most s.

We are concerned with permanent cycles with Adams filtration 1 and hence of Novikov filtration 0 or 1. Since $\mathrm{Ext}^0(BP_*)$ is trivial in positive dimensions [5.2.1(b)] it suffices to prove

5.2.8. THEOREM. *The image of*

$$\Phi\colon\ \mathrm{Ext}^1(BP_*) \to \mathrm{Ext}^1_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

is generated by h_1, h_2, *and* h_3, *for* $p=2$ *and by* $h_0 \in \mathrm{Ext}^{1,q}$ *for* $p>2$. (*These elements are permanent cycles*; *cf.* 1.2.11 *and* 1.2.13.)

PROOF. Recall that $A_* = \mathbf{Z}/(p)[t_1, t_2, \dots] \otimes E(e_0, e_1, \dots)$ with

$$\Delta(t_n) = \sum_{0 \le i \le n} t_i \otimes t_{n-i}^{p^i} \qquad \text{and} \qquad \Delta(e_n) = 1 \otimes e_n + \sum_{1 \le i \le n} e_i \otimes t_{n-i}^{p^i}$$

where $t_0 = 1$. Here t_n and e_n are the conjugates of Milnor's ξ_n and τ_n (3.1.1). The map $BP_*(BP) \to A_*$ sends $t_n \in BP_*(BP)$ to $t_n \in A_*$.

Now recall the I-adic filtration of 4.4.4. We can extend it to the comodules M^n and N^n by saying that a monomial fraction $\frac{v^I}{v^J}$ is in F^k iff the sum of the exponents in the numerator exceeds that for the denominator by at least k. (This k may be negative and there is no k such that $F^k M^n = M^n$ or $F^k N^n = N^n$. However, there is such a k for any finitely generated subcomodule of M^n or N^n.) For each $k \in \mathbf{Z}$ the sequence

$$0 \to F^k N^n \to F^k M^n \to F^k N^{n+1} \to 0$$

is exact. It follows that $\alpha\colon \operatorname{Ext}^s(N^n) \to \operatorname{Ext}^{s+n}(BP_*)$ (5.1.18) preserves the I-adic filtration and that if $x \in F^1 \operatorname{Ext}^0(N^1)$ then $\Phi\alpha(x) = 0$.

Easy inspection of 5.2.6 shows that the only elements in $\operatorname{Ext}^0(M^1)$ not in F^1 are α_1 and, for $p = 2$, $\alpha_{2/2}$, and $\alpha_{4/4}$, and the result follows. □

Now we turn to the computation of $\operatorname{Ext}^1(BP_*/I_n)$ for $n > 0$; it is a module over $\operatorname{Ext}^0(BP_*/I_n)$ which is $\mathbf{Z}/(p)[v_n]$ by 4.3.2. We denote this ring by $k(n)_*$. It is a principal ideal domain and $\operatorname{Ext}^1(BP_*/I_n)$ has finite type so the latter is a direct sum of cyclic modules, i.e., of free modules and modules of the form $k(n)_*/(v_n^i)$ for various $i > 0$. We call these the v_n-torsion free and v_n-torsion summands, respectively. The rank of the former is obtained by inverting v_n, i.e., by computing $\operatorname{Ext}^1(M_n^0)$. The submodule of the v_n-torsion which is annihilated by v_n is precisely the image of $\operatorname{Ext}^0(BP_*/I_{n+1}) = k(n+1)_*$ under the connecting homomorphism for the short exact sequence

$$0 \to \Sigma^{\dim v_n} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0. \tag{5.2.9}$$

We could take these elements in $\operatorname{Ext}^1(BP_*/I_n)$ and see how far they can be divided by v_n by analyzing the long exact sequence for 5.2.9, assuming we know enough about $\operatorname{Ext}^1(BP_*/I_{n+1})$ to recognize nontrivial images of elements of $\operatorname{Ext}^1(BP_*/I_n)$ when we see them. This approach was taken by Miller and Wilson [**?**].

The chromatic spectral sequence approach is superficially different but one ends up having to make the same calculation either way. From the chromatic spectral sequence for $\operatorname{Ext}(BP_*/I_n)$ (5.1.11) we get an short exact sequence

$$0 \to E_\infty^{1,0} \to \operatorname{Ext}^1(BP_*/I_n) \to E_\infty^{0,1} \to 0, \tag{5.2.10}$$

where $E_\infty^{1,0} = E_2^{1,0}$ is a subquotient of $\operatorname{Ext}^0(M_{n+1}^1)$ and is the v_n-torsion summand, while $E_\infty^{0,1} = E_3^{0,1} \operatorname{Ext}^1(M_n^0)$ is the v_n-torsion free quotient. To get at $\operatorname{Ext}^0(M_{n+1}^1)$ we study the long exact sequence for the short exact sequence

$$0 \to M_{n+1}^0 \xrightarrow{j} \Sigma^{\dim v_n} M_n^1 \xrightarrow{v_n} M_n^1 \to 0$$

as in 5.1.17; this requires knowledge of $\operatorname{Ext}^0(M_{n+1}^0)$ and $\operatorname{Ext}^1(M_{n+1}^0)$. To determine the subgroup $E_\infty^{0,1}$ of $\operatorname{Ext}^1(M_n^0)$ we need the explicit representatives of generators of the latter constructed by Moreira [**?, ?**].

The following result (to be proved later as 6.3.12) then is relevant to both $E_\infty^{0,1}$ and $E_\infty^{1,0}$ in 5.2.10.

5.2.11. THEOREM. *$\mathrm{Ext}^1(M_n^0)$ for $n > 0$ is the $K(n)_*$-vector space generated by $h_i \in \mathrm{Ext}^{1,p^iq}$ for $0 \le i \le n-1$ represented by $t_1^{p^i}$, $\zeta_n \in \mathrm{Ext}^{1,0}$ (for $n \ge 2$)represented for $n = 2$ by $v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}) - v_2^{-1-p}v_3t_1^p$, and (if $p = 2$ and $n \ge 1$) $\rho_n \in \mathrm{Ext}^{1,0}$. ($\zeta_n$ and ρ_n will be defined in 6.3.11).* □

5.2.12. REMARK. For $i \ge n$, h_i does not appear in this list because the equation

$$\eta_R(v_{n+1}) \equiv v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \mod I_n$$

leads to a cohomology between h_{n+i} and $v_n^{(p-1)p^i}h_i$.

Now we will describe $\mathrm{Ext}^0(M_n^1)$ and $E_\infty^{1,0}$. The groups are v_n-torsion modules. The submodule of the former annihilated by v_n is generated by $\left\{\frac{v_{n+1}^t}{v_n} : t \in \mathbf{Z}\right\}$. Only those elements with $t > 0$ will appear in $E_\infty^{1,0}$; if $t = 0$ the element is in $\operatorname{im} d_1$, and $\ker d_1$ is generated by those elements with $t \ge 0$. We need to see how many times we can divide by v_n and (still have a cocycle). An easy calculation shows that if $t = sp^i$ with $p \nmid s$, then $\frac{v_{n+1}^t}{v_n^{p^i}}$ is a cocycle whose image in $\mathrm{Ext}^1(M_{n+1}^0)$ is $sv_{n+1}^{(s-1)p^i}h_{n+i}$, but by 5.2.12 these are not linearly independent, so this is not the best possible divisibility result. For example, for $n = 1$ we find that

$$\frac{v_2^{p^2}}{v_1^{1+p^2}} - \frac{v_2^{p^2-p+1}}{v_1^2} - \frac{v_2^{-p}v_3^p}{v_1}$$

is a cocycle.

The general result is this.

5.2.13. THEOREM. *As a $k(n)_*$-module, $\mathrm{Ext}^0(M_n^1)$ is the direct sum of*
(i) *the cyclic submodules generated by $\frac{x_{n+1,i}^s}{v_n^{a_{n+1,i}}}$ for $i \ge 0$, $p \nmid s$; and*
(ii) *$K(n)_*/k(n)_*$, generated by $\frac{1}{v_n^j}$ for $j \ge 1$.*
The $x_{n,i}$ are defined as follows.

$$\begin{aligned}
x_{1,0} &= v_1,\\
x_{1,1} &= v_1^p \quad \text{if } p > 2 \quad \text{and} \quad v_1^2 + 4v_1^{-1}v_2 \quad \text{if } p = 2,\\
x_{1,i} &= x_{1,i-1}^p \qquad \text{for} \quad i \ge 2,\\
x_{2,0} &= v_2,\\
x_{2,1} &= v_2^p - v_1^p v_2^{-1} v_3,\\
x_{2,2} &= x_{2,1}^p - v_1^{p^2-1}v_2^{p^2-p+1} - v_1^{p^2+p-1}v_2^{p^2-2p}v_3,\\
x_{2,i} &= x_{2,i-1}^2 \qquad \text{for } i \ge 3 \quad \text{if } p = 2
\end{aligned}$$

and

$$x_{2,i-1}^p - 2v_1^{b_{2,i}}v_2^{(p-1)p^{i-1}+1} \qquad \text{for } i \ge 3 \quad \text{if } p > 2,$$

where

$$\begin{aligned}
b_{2,i} &= (p+1)(p^{i-1}-1), \\
x_{n,0} &= v_n && \textit{for} \quad n>2, \\
x_{n,1} &= v_n^p - v_{n-1}^p v_n^{-1} v_{n+1}, \\
x_{n,i} &= x_{n,i-1}^p && \textit{for} \quad i>1 \quad \textit{and} \quad i \not\equiv 1 \mod (n-1), \\
x_{n,i} &= x_{n,i-1}^p - v_{n-1}^{b_{n,i}} v_n^{p^i - p^{i-1}+1} && \textit{for} \quad i>1, \quad \textit{and} \quad i \equiv 1 \mod (n-1)
\end{aligned}$$

where

$$b_{n,i} = \frac{(p^{i-1}-1)(p^n-1)}{p^{n-1}-1} \quad \textit{for} \quad i \equiv 1 \mod (n-1).$$

The $a_{n,i}$ are defined by

$$\begin{aligned}
a_{1,0} &= 1 \\
a_{1,i} &= i+2 && \textit{for } p=2 \quad \textit{and} \quad i \geq 1, \\
a_{i,1} &= i+1 && \textit{for } p>2 \quad \textit{and} \quad i \geq 1, \\
a_{2,0} &= 1, \\
a_{2,i} &= p^i + p^{i-1} - 1 && \textit{for } p>2 \quad \textit{and} \quad i \geq 1 \quad \textit{or } p=2 \quad \textit{and} \quad i=1, \\
a_{2,i} &= 3 \cdot 2^{i-1} && \textit{for } p=2 \textit{ and } i>1, \\
a_{n,0} &= 1 && \textit{for } n>2, \\
a_{n,1} &= p, \\
a_{n,i} &= pa_{n,i-1} && \textit{for } i>1 \quad \textit{and} \quad i \not\equiv 1 \mod (n-1),
\end{aligned}$$

and

$$a_{n,i} = pa_{n,i} + p - 1 \qquad \textit{for } i>1 \quad \textit{and} \quad i \equiv 1 \mod (n-1).$$

□

This is Theorem 5.10 of Miller, Ravenel, and Wilson [**?**], to which we refer the reader for the proof.

Now we need to compute the subquotient $E_2^{1,0}$ of $\mathrm{Ext}^0(M_n^1)$. It is clear that the summand of (ii) above is in the image of d_1 and that ker d_1 is generated by elements of the form $\frac{x_{n+1,i}^s}{v_1^j}$ for $s \geq 0$. Certain of these elements for $s>0$ are not in $\ker d_1$; e.g., we saw in 5.2.6 that $d_1\left(\frac{x_{1,1}}{8}\right) \neq 0$. More generally we find $d_1\left(\frac{x_{n+1,i_s}}{v_n^j}\right) \neq 0$ iff $s=1$ and $p^i < j \leq a_{n+1,i}$ (see Miller and Wilson [**?**]), so we have

5.2.14. Corollary. *The v_n-torsion summand of* $\mathrm{Ext}^1(BP_*/I_n)$ *is generated by the elements listed in* 5.2.13(i) *for $s>0$ with* ($when$ $s=1$) $\frac{x_{n+1,i}}{v_n^{a_{n+1,i}}}$ *replaced by* $\frac{x_{n+1,i}}{v_n^{p^i}}$. □

Now we consider the $k(n)_*$-free summand $E_\infty^{0,1} \subset \mathrm{Ext}^1(M_n^0)$. We assume $n>1$ ($n=1$ is the subject of 5.2.2); 5.2.11 tells us that $E_\infty^{0,1}$ has rank $n+1$ for $p>2$ and $n+2$ for $p=2$. We need to determine the image of $\mathrm{Ext}^1(BP_*/I_n)$ in $\mathrm{Ext}^1(M_n^0)$. To show that an element in the former is not divisible by v_n we must show that it has a nontrivial image in $\mathrm{Ext}^1(BP_*/I_{n+1})$. The elements $h_i \in \mathrm{Ext}^1(M_n^0)$ clearly are in the image of $\mathrm{Ext}^1(BP_*/I_n)$ and have nontrivial images in $\mathrm{Ext}^1(BP_*/I_{n+1})$. The elements ζ_n and ρ_n are more complicated. The formula given in 5.2.11 for ζ_2 shows that $v_2^{1+p}\zeta_2$ pulls back to $\mathrm{Ext}^1(BP_*/I_2)$ and projects to $v_3h_1 \in \mathrm{Ext}^1(BP_*/I_3)$. This element figures in the proof of 5.2.13 and in the computation of $\mathrm{Ext}^2(BP_*)$ to be described in Section 4.

The formula of Moreira [**?**] for a representative of ζ_n is

$$T_n = \sum_{1\le i\le j\le k\le n} u_{2n-k}^{p^{k-i}} t_j^{p^{n-i}} c(t_{k-j})^{p^{n-i+j}} \tag{5.2.15}$$

where the $u_{n+i} \in M_n^0$ are defined by

$$u_n = v_n^{-1} \quad \text{and} \quad \sum_{0\le i\le k} u_{n+i} v_{n+k-i}^{p^i} = 0 \quad \text{for } k>0. \tag{5.2.16}$$

One sees from 5.2.16 that $u_{n+i-1}v_n^{(p^i-1)/(p-1)} \in BP_*/I_n$ so $\widehat{T}_n = v_n^{(p^n-1)/(p-1)}T_n \in BP_*(BP)/I_n$. In 5.2.15 the largest power of v_n^{-1} occurs in the term with $i=j=k=1$; in T_n this term is $v_n^{(p^n-1)/(p-1)}u_{2n-1}t_1^{p^{n-1}}$ and its image in $\mathrm{Ext}^1(BP_*/I_{n+1})$ is $(-1)^{n+1}v_{n+1}^{(p^{n-1}-1)/(p-1)}h_{n-1}$.

The formula of Moreira [**?**] for a representative U_n of ρ_n is very complicated and we will not reproduce it. From it one sees that $v_n^{2^{2n}-1+2^{n-1}}U_n \in BP_*(BP)/I_n$ reduces to $v_{n+1}^{2^{2n-1}-1}t_1^{2^{n-1}} \in BP_*(BP)/I_{n+1}$.

Combining these results gives

5.2.17. THEOREM. *The $k(n)_*$-free quotient $E_\infty^{0,1}$ of $\mathrm{Ext}^1(BP_*/I_n)$ for $n\ge 1$ is generated by $h_i \in \mathrm{Ext}^{1,p^iq}$ for $0\le i\le n-1$, $\hat\zeta = v_n^{(p^n-1)/(p-1)}\zeta_n$, and (for $p=2$) $\hat\rho_n = v_n^{2^{2n}+2^{n-1}-1}\rho_n$. The imagas of $\hat\zeta_n$ and $\hat\rho_n$ in $\mathrm{Ext}^1(BP_*/I_{n+1})$ are $(-1)^{n+1}v_{n+1}^{(p^{n-1}-1)/(p-1)}h_{n-1}$ and $v_{n+1}^{2^{2n-1}-1}h_{n-1}$, respectively.* □

3. Ext(M^1) and the J-Homomorphism

In this section we complete the calculation of $\mathrm{Ext}(M^1)$ begun with 5.2.6 and describe the behavior of the resulting elements in the chromatic spectral sequence and then in the Adams–Novikov spectral sequence. Then we will show that the elements in $\mathrm{Ext}^1(BP_*)$ (and, for $p=2$, Ext^2 and Ext^3) detect the image of the homomorphism $J\colon \pi_*(SO)\to\pi_*^S$ (1.1.12). This proof will include a discussion of Bernoulli numbers. Then we will compare these elements in the Adams–Novikov spectral sequence with corresponding elements in the Adams spectral sequence.

We use the method of 5.1.17 to compute $\mathrm{Ext}(M^1)$; i.e., we study the long exact sequence of Ext groups for

$$0 \to M_1^0 \xrightarrow{j} M^1 \xrightarrow{p} M^1 \to 0. \tag{5.3.1}$$

$\mathrm{Ext}(M_1^0)$ is described in 5.2.6 and the computation of $\mathrm{Ext}^0(M^1)$ is given in 5.2.6 Let δ be the connecting homomorphism for 5.3.1. Then from the proof of 5.2.6 we have

5.3.2. COROLLARY. *The image of δ in $\mathrm{Ext}^1(M_1^0)$ is generated by* (a) *$v_1^th_0$ for all $t\ne 1$ when p is odd and*

(b) *$v_1^th_0$ for all even t and $v_1^t\rho_1$ for all $t\ne 0$ when $p=2$.* □

For odd primes this result alone determines all of $\mathrm{Ext}(M^1)$. $\mathrm{Ext}^s(M_1^0)=0$ for $s>1$ and there is only one basis element of $\mathrm{Ext}^1(M_1^0)$ not in $\mathrm{im}\,\delta$, namely

$v_1^{-1}h_0$. Its image under j is represented by $\frac{v_1^{-1}t_1}{p}$. Since $\mathrm{Ext}^2(M_1^0) = 0$, there is no obstruction to dividing $j(v_1^{-1}h_0)$ by any power of p, so we have

$$\mathrm{Ext}^{1,t}(M^1) = \begin{cases} \mathbf{Q}/\mathbf{Z}_{(p)} & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases} \tag{5.3.3}$$

for any odd prime p. We can construct a representative of an element of order p^k in $\mathrm{Ext}^{1,0}(M^1)$ as follows. From 4.3.1 we have $\eta_R(v_1) = v_1 = put_1$ where $u = 1 - p^{p-1}$. Then a simple calculation shows that

$$y_k = -\sum_{1=i}(-1)^i \frac{v_1^{-i}u^i t_1^i}{ip^{k+1-i}} \tag{5.3.4}$$

is the desired cocycle. (This sum is finite although the ith term for some $i > k$ could be nonzero if $p \mid i$.) The group $\mathrm{Ext}^{1,0}(M^1) + E_1^{1,1,0}$ cannot survive in the chromatic spectral sequence because it would give a nontrivial $\mathrm{Ext}^{2,0}(BP_*)$ contradicting the edge theorem, 5.1.23. It can be shown (lemma 8.10 of Miller, Ravenel, and Wilson [**?**]) that this group in fact supports a d_1 with trivial kernel. Hence we have

5.3.5. Theorem.
(a) *for $p > 2$ the group* $\mathrm{Ext}^{s,t}(M^1)$ *is*

$\mathbf{Q}/\mathbf{Z}_{(p)}$ *generated by* $\frac{1}{p^k}$ *for* $(s,t) = (0,0)$.

$\mathbf{Z}/(p^{i+1})$ *generated by* $\frac{v_1^{rp^i}}{p^{i+1}}$ *for* $p \nmid r$ *and* $(s,t) = (0, rp^i q)$,

$\mathbf{Q}/\mathbf{Z}_{(p)}$ *generated by* y_k (5.3.4) *for* $(s,t) = (1,0)$ *and*

0 *otherwise.*

(b) *In the chromatic spectral sequence, where* $\mathrm{Ext}^{s,t}(M^1) = E_1^{1,s,t} E_1^{1,0,0} \subset \operatorname{im} d_1$ *and* $\ker d_1 \bigoplus_{t \geq 0} E_1^{1,0,t}$, *so* $E_\infty^{1,*} = \mathrm{Ext}^1(BP_*)$ *and* $\ker d_1 = \bigoplus_{t\geq 0} E_1^{1,0,t}$, *so* $E_\infty^{1,*} = \mathrm{Ext}^1(BP_*)$ *is generated by the groups* $\mathrm{Ext}^{0,t}(M^1)$ *for* $t > 0$. □

We will see below that each generator of $\mathrm{Ext}^1(BP_*)$ for $p > 2$ is a permanent cycle in the Adams–Novikov spectral sequence detecting an element in the image of J (1.1.12).

The situation for $p = 2$ is more complicated because $\mathrm{Ext}(M_1^0)$ has a polynomial factor not present for odd primes. We use 5.3.2 and 5.2.2 to compute $\mathrm{Ext}^s(M^1)$ for $s > 1$. The elements of order 2 in $\mathrm{Ext}^{1,0}(M^1)$ are the images under j (5.3.1) of $v_1^t h_0$ for t odd and $v_1^t \rho_1$ for t odd and $t = 0$.

We claim $j(\rho_1)$ is divisible by any power of 2, so $\mathrm{Ext}^{1,0}(M^1)$ contains a summand isomorphic to $\mathbf{Q}/\mathbf{Z}_{(2)}$ as in the odd primary case. To see this use 5.2.4 to compute

$$\eta_R\Big(\frac{v_1^{-3}v_2}{4}\Big) = \frac{v_1^{-3}(-v_1t_1^2 + v_1^2t_1 + v_2)}{4} + \frac{v_1^{-4}}{2}(v_2t_1 + v_1t_1^3 + v_1t_2),$$

showing that y_2 (5.3.4) represents $j(\rho_1)$; the same calculation shows that $y_1 = \frac{v_1^{-1}t_1 + v_1^{-2}t_1^2}{2}$ is a coboundary. Hence the y_k for $k \geq 2$ give us the cocycles we need.

Next we have to deal with $j(v_1^t h_0)$ and $j(v_1^t\rho_1)$ for odd t. These are not divisible by 2 since an easy calculation gives $\delta_j(v_1^t x) = v_1^{t-1}h_0 x$ for t odd and $x = h_0^{i+1}$ or $h_0^i \rho_1$ for any $i \geq 0$. Indeed this takes care of all the remaining elements in the short exact sequence for 5.3.1 and we get

5.3.6. THEOREM.
(a) *For $p = 2$,* $\mathrm{Ext}^{s,t}(M^1)$ *is*

$$\begin{cases} \mathbf{Q}/\mathbf{Z}_{(2)} \text{ generated by } \frac{1}{2^j} \\ \qquad\qquad\qquad \text{for } (s,t) = (0,0), \\ \mathbf{Z}/(2) \text{ generated by } \frac{v_1^r}{2} \\ \qquad\qquad\qquad \text{for } (s,t) = (0,2r) \text{ and } r \text{ odd}, \\ \mathbf{Z}/(2^{i+3}) \text{ generated by } \frac{x_{1,1}^{r2^i}}{2^{i+3}} \\ \qquad\qquad\qquad \text{for } (s,t) = (0, r2^{i+2}) \text{ and } r \text{ odd}, \\ \mathbf{Q}/\mathbf{Z}_{(2)} \oplus \mathbf{Z}/(2) \text{ generated by } y_k \ (k \geq 2) \text{ and } \frac{v_1^{-1}t_1}{2} \\ \qquad\qquad\qquad \text{for } (s,t) = (1,0), \\ \mathbf{Z}/(2) \text{ generated by } j(v_1^r h_0^s) \\ \qquad\qquad\qquad \text{for } s > 0,\ t = 2(r+s),\ r \text{ odd, and } (s,t) \neq (1,0) \\ \mathbf{Z}/(2) \text{ generated by } j(v_1^r \rho_1 h_0^{s-1}) \\ \qquad\qquad\qquad \text{for } s > 0,\ t + 2(r+s-1), \text{ and } r \text{ odd}, \\ \qquad\qquad\qquad\qquad \text{and} \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

(b) *In the chromatic spectral sequence for $p = 2$, $E_\infty^{1,2,t}$ is*

$$\begin{cases} \mathrm{Ext}^{s,t}(M^1) & \text{for } t = 2s + 2r \text{ and } r \geq 1,\ r \neq 2, \\ \mathbf{Z}/(4) \text{ generated by } \frac{v_1^2}{4} & \text{for } (s,t) = (0,4), \text{ and } \mu_{2t-1} \in \pi_{8t+1}^S, \\ & \qquad \text{and} \\ 0 & \text{otherwise} \end{cases}$$

(*See* 5.1.22 *for a description of differentials originating in* $E_r^{1,s,2s+4}$.) *In other words the subquotient of* $\mathrm{Ext}(BP_*)$ *corresponding to* $E_\infty^{1,*}$ *is generated by* $\mathrm{Ext}^1(BP_*)$ (5.2.6) *and products of its generators* (*excluding* $\alpha_{2/2} \in \mathrm{Ext}^{1,4}$) *with all positive powers of* $\alpha_1 \in \mathrm{Ext}^{1,2}$.

PROOF. Part (a) was proved above. For (b) the elements said to survive, i.e., those in $E_1^{1,0}$ and $j(v_1^r\rho_1 h_0^{s-1})$ for $s > 0$ with odd $r \geq 5$ and $j(v_1^r h_0^s)$ for $s > 0$ with odd $r \geq 1$, are readily seen to be permanent cycles. The other elements in $E_1^{1,s}$ for $s > 0$ have to support nontrivial differentials by the edge theorem, 5.1.23. □

Now we describe the behavior of the elements of 5.3.5(b) and 5.3.6(b) in the Adams–Novikov spectral sequence. The result is

5.3.7. THEOREM.
(a) *For $p > 2$, each element in* $\mathrm{Ext}^1(BP_*)$ *is a permanent cycle in the Adams–Novikov spectral sequence represented by an element of* $\mathrm{im}\, J$ (1.1.13) *having the same order.*

(b) *For $p = 2$ the behavior of* $\mathrm{Ext}^{1,2t}(BP_*)$ *in the Adams–Novikov spectral sequence depends on the residue of t* mod (4) *as follows. If $t \equiv 1 \mod 4$ the generator α_t is a permanent cycle represented by the element $\mu_{2t-1} \in \pi_{8t-1}^S$ of order 2 constructed by Adams* [**?**]. *In particular α_1 is represented by η* (1.1.13). *$\alpha_1\alpha_t$ is represented by $\mu_{2t} = \eta\mu_{2t-1}$ and $\alpha_1^2\alpha_t$ is represented by an element of order* 2 *in* $\mathrm{im}\, J \subset \pi_{2t+1}^S$ (*the order of this group is an odd multiple of* 8). *$\alpha^{s+3}\alpha_t = d_3(\alpha_1^s\alpha_{t+2})$ for all $s \geq 0$.*

If $t \equiv 0 \mod (4)$ then the generator $\bar{\alpha}_t$ of $\mathrm{Ext}^{1,2t}(BP_*)$ is a permanent cycle represented by an element of $\operatorname{im} J$ *having the same order, as are $\alpha_1\bar{\alpha}_t$, and $\alpha_1^2\bar{\alpha}_t$. $\alpha_1^{s+3}\bar{\alpha}_t = d_3(\alpha_1^s\alpha_{t+2/3})$ for $s \geq 0$. In particular y_4 is represented by $\sigma \in \pi_7^S$ (1.1.13).*

If $t \equiv 2 \mod (4)$, $\alpha_{t/2}$ (twice the generator except when $t = 2$) is a permanent cycle represented by an element in $\operatorname{im} J$ *of order 8. ($\alpha_{t/2}$ has order 4 and 4 times the generator of* $\operatorname{im} J$ *represents $\alpha_1^2\alpha_{t-2}$ as remarked above). In particular $\alpha_{2/2}$ is represented by $\nu \in \pi_3^S$ (1.1.13).* □

This result says that the following pattern occurs for $p = 2$ in the Adams–Novikov spectral sequence E_∞-term as a direct summand for all $k > 0$:

(5.3.8)

Where all elements have order 2 except $\alpha_{4k+2/2}$, which has order 4, and $\bar{\alpha}_{4k}$, whose order is the largest power of 2 dividing $16k$; the broken vertical line indicates a nontrivial group extension. The image of J represents all elements shown except α_{4k+1} and $\alpha_1\alpha_{4k+1}$.

Our proof of 5.3.7 will be incomplete in that we will not prove that $\operatorname{im} J$ actually has the indicated order. This is done up to a factor of 2 by [**?**] Adams [**?**], where it is shown that the ambiguity can be removed by proving the Adams conjecture, which was settled by Quillen [**?**] and Sullivan [**?**].

We will actually use the complex J-homomorphism $J\colon \pi_*(U) \to \pi_*^S$, where U is the unitary group. Its image is known to coincide up to a factor of 2 with that of the real J-homomorphism. We will comment more precisely on the difference between them in due course.

An element $x \in \pi_{2t-1}(U)$ corresponds to a stable complex vector bundle ξ over S^{2t}. Its Thom spectrum $T(\xi)$ is a 2-cell CW-spectrum $S^0 \cup e^{2t}$ with attaching map $J(x)$ and there is a canonical map $T(\xi) \to MU$. We compose it with the standard map $MU \to BP$ and get a commutative diagram

(5.3.9)
$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d] & T(\xi) \arrow[r] \arrow[d] & S^{2t} \arrow[d] \\
S^0_{(p)} \arrow[r] & BP \arrow[r] & \overline{BP} \arrow[d] \\
 & & BP \wedge \overline{BP}
\end{tikzcd}$$

where the two rows are cofibre sequences. The map $S^{2t} \to \overline{BP}$ is not unique but we do get a unique element $e(x) \in \pi_{2t}(BP \wedge \overline{BP})/\operatorname{im}\pi_{2t}(BP)$. Now $E_2^{1,2t}$ of the Adams–Novikov spectral sequence is by definition a certain subgroup of this quotient containing $e(x)$, so we regard the latter as an element in $\mathrm{Ext}^{1,2t}(BP_*)$. Alternatively, the top row in 5.3.9 gives an short exact sequence of comodules which is the extension corresponding to $e(x)$. We need to show that if x generates $\pi_{2t-1}(U)$ then $e(x)$ generates $\mathrm{Ext}^1(BP_*)$ up to a factor of 2.

For a generator x_t of $\pi_{2t-1}(U)$ we obtain a lower bound on the order of $e(x)$ as follows. If $je(x_t) = 0$ for some integer j then for the bundle given by $x = jx_t \in \pi_{2t-1}(U)$ the map $S^{2t} \to \overline{BP}$ in 5.3.9 lifts to BP, so we get an element in $\pi_{2t}(BP)$.

Now consider the following diagram

$$
\begin{tikzcd}
\pi_*(BU) \arrow[d] & \pi_*(MU) \arrow[r, "\theta"] \arrow[d] & \mathbf{Z} \arrow[d] \\
H_*(BU) \arrow[r, phantom, "\cong"] & H_*(MU) \arrow[r] & \mathbf{Q}
\end{tikzcd}
\tag{5.3.10}
$$

where the two left-hand vertical maps are the Hurewicz homomorphisms and θ is some ring homomorphism; it extends as indicated since $\pi_*(MU)\otimes\mathbf{Q} \cong H_*(MU)\otimes\mathbf{Q}$ by 3.1.5. Let ϕ be the composite map (not a ring homomorphism) from $\pi_*(BU)$ to $\mathbf{Q}$. If $\phi(x_t)$ has denominator j_t, then j_t divides the order of $e(x_t)$.

According to Bott [**?**] the image of x_t in $H_{2t}(BU)$ is $(t-1)!\, s_t$ where s_t is a primitive generator of $H_{2t}(BU)$. By Newton's formula

$$s(z) = \frac{z}{b(z)}\frac{db(z)}{dz},$$

where $s(z) = \sum_{t\geq 0} s_t z^t$ and $b(z) = \sum_{t\geq 0} b_t z^t$, the b_t being the multiplicative generators of $H_*(BU) \cong H_*(MU)$ (3.1.4).

Now by Quillen's theorem, 4.1.6, θ defines a formal group law over $\mathbf{Z}$ (see Appendix 2), and by 4.1.11

$$\theta(b(z)) = \frac{\exp(z)}{z}$$

so

$$\theta(s(z)) = \frac{z}{\exp(z)}\frac{d\exp(z)}{dz} - 1,$$

where $\exp(z)$ is the exponential series for the formal group law defined by θ, i.e., the functional inverse of the logarithm (A2.1.5).

The θ we want is the one defining the multiplicative formal group law (A2.1.4) $x+y+xy$. An easy calculation shows $\exp(z) = e^z - 1$ so

$$\theta(s(z)) = \frac{ze^z}{e^z-1} - 1.$$

This power series is essentially the one used to define Bernoulli numbers (see appendix B of Milnor and Stasheff [**?**]), i.e., we have

$$\theta(s(z)) = \frac{z}{2} + \sum_{k\geq 1}(-1)^{k+1}\frac{B_k z^{2k}}{(2k)!}$$

where B_k is the kth Bernoulli number. Combining this with the above formula of Bott we get

5.3.11. Theorem. *The image of a generator x_t of $\pi_{2t-1}(U) = \pi_{2t}(BU)$ under the map $\phi\colon \pi_*BU \to \mathbf{Q}$ of* 5.3.10 *is $\frac{1}{2}$ if $t=1$, 0 for odd $t>1$, and $\pm B_k/2k$ for $t=2k$. Hence the order of x_t in $\mathrm{Ext}^1(BP_*)$ is divisible by 2 for $t=1$, 1 for $t>1$, and the denominator j_{2k} of $B_k/2k$ for $t=2k$.* □

This denominator j_{2k} is computable by a theorem of von Staudt proved in 1845; references are given in Milnor and Stasheff [**?**]. The result is that $p \mid j_{2k}$ iff $(p-1) \mid 2k$ and that if p^i is the highest power of such a prime which divides $2k$ then p^{i+1} is the highest power of p dividing j_{2k}. Comparison with 5.2.6 shows that $\mathrm{Ext}^{1,4k}(BP_*)$ also has order p^{i+1} except when $p=2$ and $k>1$, in which case it has order 2^{i+2}. This gives

5.3.12. COROLLARY. *The subgroup of* $\mathrm{Ext}^{1,2t}(BP_*)$ *generated by* $e(x_t)$ (5.3.9), *i.e., by the image of the complex J-homomorphism, has index* 1 *for* $t = 1$ *and* 2. *and* 1 *or* 2 *for* $t \geq 3$. *Moreover each element in this subgroup is a permanent cycle in the Adams–Novikov spectral sequence.* □

This completes our discussion of im J for odd primes. We will see that the above index is actually 2 for all $t \geq 3$, although the method of proof depends on the congruence class of $t \mod (4)$. We use the fact that the complex J-homomorphism factors through the real one. Hence for $t \equiv 3 \mod (4)$, $e(x_t) = 0$ because $\pi_{2t-1}(SO) = 0$.

For $t \equiv 0$ the map $\pi_{2t-1}(U) \to \pi_{2t-1}(SO)$ has degree 2 in Bott [**?**] (and for $t \equiv 2$ it has degree 1) so $e(x_t)$ is divisible by 2 and the generator y of $\mathrm{Ext}^1(BP_*)$ is as claimed in 5.3.7. This also shows that ηy_t, and $\eta^2 y_t$, detect elements in im J. Furthermore η^3 kills the generator of $\pi_{2t-1}(SO)$ by 3.1.26, so $\alpha_1^3 y_t$ must die in the Adams–Novikov spectral sequence. It is nonzero at E_2, so it must be killed by a higher differential and the only possibility is $d_3(\alpha_{t+2/3}) = \alpha_1^3 y_t$ [here we still have $t \equiv 0 \mod (4)$].

For $t \equiv 1$ the generator of $\pi_{2t-1}(SO) = \mathbf{Z}/(2)$ is detected by $\eta^2 y_{t-1}$ as observed above, so $e(x_t) = 0$. For $t \equiv 2$ we just saw that the generator $\alpha_{t/3}$ of $\mathrm{Ext}^{1,2t}$ supports a nontrivial d_3 for $t > 2$, so we must have $e(x_t) = \alpha_{t/2}$.

To complete the proof of 5.3.7 we still need to show three things: for $t \equiv 1 \mod (4)$, α_t is a permanent cycle, for $t \equiv 3$, $d_3(\alpha_t) = \alpha_t^3 \alpha_{t-2}$, and for $t \equiv 2\mathrm{m}$ α_t is represented by an element of order 4 whose double is detected by $\alpha_1^2 \alpha_{t-1}$. To do this we must study the Adams–Novikov spectral sequence for the mod (2) Moore spectrum $M(2)$. Since $BP_*(M(2)) = BP_*/(2)$ is a comodule algebra, the Adams–Novikov E_2-term for $M(2)$, $\mathrm{Ext}(BP_*/(2))$, is a ring (A1.2.14). However, since $M(2)$ is not a ring spectrum, the Adams–Novikov spectral sequence differentials need not respect this ring structure. The result we need is

5.3.13. THEOREM. (a) $\mathrm{Ext}(BP_*/(2))$ *contains* $\mathbf{Z}/(2)[v_1, h_0] \otimes \{1, u\}$ *as a direct summand where* $v_1 \in \mathrm{Ext}^{0,2}$, $h_0 \in \mathrm{Ext}^{1,2}$, *and* $u \in \mathrm{Ext}^{1,8}$ *are represented by* v_1, t_1, *and* $t_1^4 + v_1 t_1^3, v_1^2 t_1^2 + v_1 t_2 + v_2 t_1$ *respectively. This summand maps isomorphically to* $E_\infty^{0,*}$ *in the chromatic spectral sequence for* $\mathrm{Ext}(BP_*/(2))$ (5.1.11).

(b) *In the Adams–Novikov spectral sequence for* $M(2)$, $v_1^t h_0^s u^e$ *is a permanent cycle for* $s \geq 0$, $e = 0, 1$, *and* $t \equiv 0$ *or* $1 \mod (4)$. *If* $t \equiv 2$ *or* 3 *then* $d_3(v_1^t h_0^s u^e) = v_1^{t-2} h_0^{s+3} u^e$. *For* $t \equiv 3$, $v_1^t u^e$ *is represented by an element of order* 4 *in* $\pi_{2t+7e}(M(2))$ *whose double is detected by* $h_0^2 v_1^{t-1} u^e$.

(c) *Under the reduction map* $BP_* \to BP_*/(2)$ *induced by* $S^0 \to M(2)$, *if* t *is odd then the generator* α_t *of* $\mathrm{Ext}^{1,2t}(BP_*)$ *maps to* $v_1^{t-1} h_0$. *If* t *is even and at least* 4 *then the generator* y_t *of* $\mathrm{Ext}^{1,2t}(BP_*)$ *maps to* $v_1^{t-4} u$.

(d) *Under the connecting homomorphism* $\delta\colon \mathrm{Ext}^s(BP_*/(2)) \to \mathrm{Ext}^{s+1}(BP_*)$ *induced by* $M(2) \to S^1$ (2.3.4), v_1^t *maps to* $\alpha_t \in \mathrm{Ext}^{1,2t}(BP_*)$ *for all* $t > 0$; uv_1 *maps to* $\alpha_1 y_{t+3}$ *if* t *is odd and to* 0 *if* t *is even.* □

In other words, the Adams–Novikov E_∞-term for $M(2)$ has the following pattern as a summand in low dimensions:

(5.3.14)

where the broken vertical line represents a nontrivial group extension. [Compare this with 3.1.28(a) and 5.3.8.] The summand of (a) also contains the products of these elements with $v_1^{4t} u^e$ for $t \geq 0$ and $e = 0, 1$. The only other generators of

$\mathrm{Ext}^{s,t}(BP_*/(2))$ for $t-s\le 13$ are $\beta_1\in\mathrm{Ext}^{1,4}$, $\beta_1^2\in\mathrm{Ext}^{2,8}$, $h_0^s\beta_{2/2}\in\mathrm{Ext}^{1+s,8+2s}$ for $s=0,1,2$ (where $h_0^2\beta_{2/2}=\beta_1^3$), and $h_0^s\beta_2\in\mathrm{Ext}^{1+s,10+2s}$ for $s=0,1$.

Before proving this we show how it implies the remaining assertions of 5.3.7 listed above. For $t\equiv 1 \mod (4)$, $\alpha_t=\delta(v_1^t)$ by (d) and is therefore a permanent cycle by (b). For $t\equiv 3$, $\alpha_1=\delta(v_1^t)$ and δ commutes with differentials by 2.3.4, so

$$\begin{aligned} d_3(\alpha_t) &= \delta d_3(v_1^t)=\delta(h_0^3v_1^{t-2})\\ &=\alpha_1^3\alpha_{t-2}. \end{aligned}$$

For the nontrivial group extension note that for $t\equiv 1$ $\alpha_1^2\alpha_t$ maps to an element killed by a differential so it is represented in $\pi_*(S^0)$ by an element divisible by 2. Alternatively, α_{t+1} is not the image under δ of a permanent cycle so it is not represented by an element of order 2.

Proof of 5.3.13. Recall that in the chromatic spectral sequence converging to $\mathrm{Ext}(BP_*/(2))$, $\mathrm{Ext}_1^{0,*}=\mathrm{Ext}(M_1^0)$, which is described in 5.2.2. Once we have determined the subgroup $E_\infty^{1,*}\subset E_1^{1,*}$ then (c) and (d) are routine calculations, which we will leave to the reader. Our strategy for proving (b) is to make low-dimensional computations by brute force (more precisely by comparison with the Adams spectral sequence) and then transport this information to higher dimensions by means of a map $\alpha\colon\Sigma^8M(2)\to M(2)$ which induces multiplication by v_1^4 in BP-homology. [For an odd prime p there is a map $\alpha\colon\Sigma^qM(p)\to M(p)$ inducing multiplication by v_1. v_1^4 is the smallest power of v_1 for which such a map exists at $p=2$.]

To prove (a), recall (5.2.2) that $\mathrm{Ext}(v_1^{-1}BP_*/(2))=K(1)_*[h_0,\rho_1]/(\rho_1^2)$ with $h_0\in\mathrm{Ext}^{1,2}$ and $\rho_1\in\mathrm{Ext}^{1,0}$. We will determine the image of $\mathrm{Ext}(BP_*/(2))$ in this group. The element u maps to $v_1^4\rho_1$. [Our representative of u differs from that of $v_1^4\rho_1$ given in 5.2.2 by an element in the kernel of this map. We choose this u because it is the mod (2) reduction of $y_4\in\mathrm{Ext}^{1,8}(BP_*)$.] It is clear that the image contains the summand described in (a). If the image contains $v_1^{-1}h_0^s$ or $v_1^{4-t}h_0^2\rho_1$ for any $t>0$, then it also contains that element times any positive power of h_0. One can show then that such a family of elements in $\mathrm{Ext}(BP_*/(2))$ would contradict the edge theorem, 5.1.23.

To prove (b) we need some simple facts about $\pi_*(S^0)$ in dimensions ≤ 8 which can be read off the Adams spectral sequence (3.2.11). First we have $\eta^3=4\nu$ in $\pi_3(S^0)$. This means h_0^3x must be killed by a differential in the Adams–Novikov spectral sequence for $M(2)$ for any permanent cycle x. Hence we get $d_3(v_1^2)=h_0^3$ and $d_3(v_1^3)=v_1h_0^3$. Next, if we did not have $\pi_2(M(2))=\mathbf{Z}/(4)$ then $v_1\in\pi_1(M(2))$ would extend to a map $\Sigma^2(M(2))\to M(2)$ and by iterating it we could show that all powers of v_1 are permanent cycles, contradicting the above.

Now suppose we can show that v_1^4 and u are permanent cycles representing elements of order 2 in $\pi_*(M(2))$, i.e., maps $S^n\to M(2)$ which extend to self-maps $\Sigma^nM(2)\to M(2)$. Then we can iterate the resulting $\alpha\colon\Sigma^8M(2)\to M(2)$ and compare with the map extending u to generalize the low-results above to all of (b).

A simple calculation with the Adams spectral sequence shows that $\pi_7(M(2))$ and $\pi_8(M(2))$ both have exponent 2 and contain elements representing u and v_1^4, respectively, so we have both the desired self-maps. $\square$

4. Ext^2 and the Thom Reduction

In this section we will describe $\mathrm{Ext}^2(BP_*)$ and what is known about its behavior in the Adams–Novikov spectral sequence. We will not give all the details of the calculation; they can be found in Miller, Ravenel, and Wilson [**?**] for odd primes and in Shimomura [**?**] for $p = 2$. The main problem is to compute $\mathrm{Ext}^0(M^2)$ and the map d_e^* from it to $\mathrm{Ext}^0(M^3)$. From this will follow (5.4.4) that the $\gamma_t \in \mathrm{Ext}^3(BP_*)$ are nontrivial for all $t > 0$ if p is odd. (We are using the notation of 5.1.19.) They are known to be permanent cycles for $p \geq 7$ (1.3.18).

We will also study the map Φ from Ext^2 to $E_2^{2,*}$ of the Adams spectral sequence as in 5.2.8 to show that most of the elements in the latter group, since they are not $\mathrm{im}\,\Phi$, cannot be permanent cycles (5.4.7). The result is that $\mathrm{im}\,\Phi$ is generated by

$$\{\Phi(\beta_{p^n/p^n-1}), \Phi(\beta_{p^n/p^n})\colon n \geq 1\}$$

and a certain finite number of other generators. It is known that for $p = 2$ the $\Phi(\beta_{p^n/p^n-1})$ are permenet cycles. They are the $\eta_{n+2} \in \Pi^S_{2^{n+2}}$ constructed by Mahowald [**?**] using Brown–Gitler spectra. For odd primes it follows that some element closely resembling β_{p^n/p^n-i} for $1 \leq i \leq p^n - 1$ is a nontrivial permanent cycle (5.4.9) and there is a similar more complicated result for $p = 2$ (5.4.10).

For $p = 2$, $\Phi(\beta_{2^n/2^n}) = b_{n+1}^2$ is known to be a permanent cycle iff there is a framed $(2^{n+2}-2)$-manifold with Kervaire invariant one (Browder [**?**]), and such are known to exist for $0 \leq n \leq 4$ (Barratt *et al.* [**?**]). The resulting element in $\pi_{2^{j+1}-2}$ is known as θ_j and its existence is perhaps the greatest outstanding problem in homotopy theory. It is known to have certain ramifications in the EHP sequence (1.5.29).

For odd primes the situation with $\Phi(\beta_{p^n/p^n})$ is quite different. We showed in Ravenel [**?**] that this element is *not* a permanent cycle for $p \geq 5$ and $n \geq 1$, and that β_{p^n/p^n} itself is *not* a permanent cycle in the Adams–Novikov spectral sequence for $p \geq 3$ and $n \geq 1$; see 6.4.1.

To compute Ext^2 with the chromatic spectral sequence we need to know $E_\infty^{0,2}$, $E_\infty^{1,1}$, and E_∞^{0}. The first vanishes by 5.2.1; the second is given by 5.3.5 for $p > 2$ and 5.3.6 for $p = 2$. For odd primes $\mathrm{Ext}^1(M^1) = E_1^{1,1}$ vanishes in positive dimensions; for $p = 2$ it gives elements in $\mathrm{Ext}^2(BP_*)$ which are products of α_1 with generators in $\mathrm{Ext}^2(BP_*)$. The main problem then is to compute $E_1^{0,2} = \mathrm{Ext}^0(M^2)$. We use the short exact sequence

$$0 \to M_1^1 \to M^2 \xrightarrow{p} M^2 \to 0$$

and our knowledge of $\mathrm{Ext}^0(M_1^1)$ (5.2.13). The method of 5.1.17 requires us to recognize nontrivial elements in $\mathrm{Ext}^1(M_1^1)$. This group is not completely known but we have enough information about it to compute $\mathrm{Ext}^0(M^2)$. We know $\mathrm{Ext}^1(M_2^0)$ by 5.2.11, and in proving 5.2.13 one determines the image of $\mathrm{Ext}^0(M_1^1)$ in it. Hence we know all the elements in $\mathrm{Ext}^1(M_1^1)$ which are annihilated by v_1, so any other element whose product with some v_1^i is one of these must be nontrivial.

To describe $\mathrm{Ext}^0(M^2)$ we need some notation from 5.2.13. We treat the odd primary case first. There we have

$$\begin{aligned} x_{2,0} &= v_2, \\ x_{2,1} &= v_2^p - v_1^p v_2^{-1} v_3, \\ x_{2,2} &= x_{2,1}^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \quad \text{and} \\ x_{2,i} &= x_{2,i-1}^p - 2v_1^{b_{2,i}} v_2^{(p-1)p^{i-1}+1} \quad \text{for } i \geq 3, \end{aligned}$$

where $b_{2,i} = (p+1)(p^{i-1}-1)$. Also $a_{2,0} = 1$ and $a_{2,i} = p^i + p^{i-1} - 1$ for $i \geq 1$. Then

5.4.1. THEOREM (Miller, Ravenel, and Wilson [?]). *For odd primes p, $\mathrm{Ext}^0(M^2)$ is the direct sum of cyclic p-groups generated by*

(i) $\frac{x_{2,i}^s}{p^{k+1}v_1^j}$ *with $p \nmid s$, $j \geq 1$, $k \geq 0$ such that $p^k \mid j$ and $j \leq a_{2,i-k}$ and either $p^{k+1} \nmid j$ or $a_{i-k-1} < j$; and*

(ii) $\frac{1}{p^{k+1}v_1^j}$ *for $k \geq 0$, $p^k \mid j$, and $j \geq 1$.* □

Note that s may be negative.

For $p = 2$ we define $x_{2,i}$ as above for $0 \leq i \leq 2$, $x_{2,i} = x_{2,i-1}^2$ for $i \geq 3$, $a_{2,0} = 1$, $a_{2,1} = 2$, and $a_{2,i} = 3 \cdot 2^{i-1}$ for $i \geq 2$. We also need $x_{1,0} = v_1$, $x_{1,1} = v_1^2 + 4v_1^{-1}v_2$, and $x_{1,i} = x_{1,i-1}^2$ for $i \geq 2$. In the following theorem we will describe elements in $\mathrm{Ext}^0(M^2)$ as fractions with denominators involving $x_{1,i}$, i.e., with denominators which are not monomials. These expressions are to be read as shorthand for sums of fractions with monomial denominators. For example, in $\frac{1}{8x_{1,1}}$ we multiply numerator and denominator by $x_{1,1}$ to get $\frac{x_{1,1}}{8x_{1,1}^2}$. Now $x_{1,1}^2 \equiv v_1^4 \mod (8)$ so we have

$$\frac{1}{8x_{1,1}} = \frac{v_1^2 + 4v_1^{-1}v_2}{8v_1^4} = \frac{1}{8v_1^2} + \frac{v_2}{2v_1^5}.$$

5.4.2. THEOREM (Shimomura [?]). *For $p = 2$, $\mathrm{Ext}^0(M^2)$ is the direct sum of cyclic 2-groups generated by*

(i) $\frac{v_2^s}{2v_1}$, $\frac{x_{2,1}^s}{2v_1^j}$, $\frac{x_{2,2}^s}{2v_1^k}$, *and* $\frac{x_{2,2}^s}{8x_{11}}$ *for s odd, $j = 1$ or 2 and $k = 1, 3, 4, 5,$ or 6 ($k = 2$ is excluded because $\beta_{4s/2}$ is divisible by 2);*

(ii) $\frac{x_{2,i}^s}{2v_1^j}$ *for s odd, $i \geq 3$, $j \leq a_{2,i}$, and either j is odd or $a_{2,i-1} < j$;*

(iii) $\frac{x_{2,j}^s}{2^{k+1}v_1^{j2^k}}$ *for s odd, $j, k \geq 1$, $i \geq 3$, and $a_{2,i-k-1} < j2^k \leq a_{2,i-k}$;*

(iv) $\frac{x_{2,i}^s}{2^{k+2}x_{1,k}^j}$ *for s odd $i \geq 3$, $k \geq 1$, j odd and ≥ 1, and $2^k j \leq a_{2,i-k-1}$; and*

(v) $\frac{1}{2v_1^j}$, $\frac{1}{2^{k+2}x_{1,k}^j}$ *for j odd and ≥ 1 and $k \geq 1$.* □

This result and the subsequent calculation of $\mathrm{Ext}^2(BP_*)$ for $p = 2$ were obtained independently by S. A. Mitchell.

These two results give us $E_1^{2,0}$ in the chromatic spectral sequence. The image of d_1 is the summand of 5.4.1(ii) and 5.4.2(v) and, for $p = 2$, the summand generated by β_1; this is the same d_1 that we needed to find $\mathrm{Ext}^1(BP_*)$ (5.2.6). We know that $\operatorname{im} d_2 = 0$ since its source, $E_2^{0,1}$, is trivial by 5.2.1. The problem then is to compute $d_1\colon E_1^{2,0} \to E_1^{3,0}$. Clearly it is nontrivial on all the generators with negative exponent s. The following result is proved for $p > 2$ as lemma 7.2 in Miller, Ravenel, and Wilson [?] and for $p = 2$ in section 4 of Shimomura [?].

5.4.3. LEMMA. *In the chromatic spectral sequence, $d_1\colon E_1^{2,0} \to E_1^{3,0}$ is trivial on all of the generators listed in* 5.4.1 *and* 5.4.2 *except the following*:

(i) *all generators with $s < 0$;*

(ii) $\frac{x_{2,i}}{pv_1^j}$ *with $p^i < j \le a_{2,i}$, and $i \ge 2$, the image of this generator being* $\frac{-v_3^{p^{i-1}}}{pv_1^{j-p^i}v_2^{p^{i-1}}}$; *and*

(iii) (*for $p = 2$ only*) $\frac{x_{2,2}}{8x_{1,1}}$, *whose image is* $\frac{v_3^2}{2v_1v_2}$. □

From this one easily read off both the structure of $\mathrm{Ext}^2(BP_*)$ and the kernel of $\alpha\colon \mathrm{Ext}^0(N^3) \to \mathrm{Ext}^3(BP_*)$, i.e., which Greek letter elements of the γ-family are trivial. We treat the latter case first. The kernel of α consists of $\operatorname{im} d_1 \oplus \operatorname{im} d_2 \oplus \operatorname{im} d_3$. For $p = 2$ we know that $\gamma_1 \in \operatorname{im} d_2$ by 5.1.22. d_2 for $p > 2$ and d_3 for all primes are trivial because $E_2^{1,1}$ (in positive dimensions) and $E_3^{0,2}$ are trivial by 5.3.5 and 5.2.1, respectively.

5.4.4. COROLLARY. *The kernel of $\alpha\colon \mathrm{Ext}^0(N^3) \to \mathrm{Ext}^3(BP_*)$* (5.1.18) *is generated by $\gamma_{p^i/p^i,j}$ for $i \ge 1$ with $1 \ge j \ge p^i - 1$ for $p > 2$ and $1 \le j \le p^i$ for $p = 2$; and (for $p = 2$ only) γ_1 and γ_2. In particular $0 \ne \gamma_t \in \mathrm{Ext}^3(BP_*)$ for all $t > 0$ if $p > 2$ and for all $t > 2$ if $p = 2$.*

5.4.5. COROLLARY.

(a) *For p odd, $\mathrm{Ext}^2(BP_*)$ is the direct sum of cyclic p-groups generated by $\beta_{sp^i/j,1+\phi(i,j)}$ for $s \ge 1$, $p \nmid s$, $j \ge 1$, $i \ge 0$, and $\phi(i,j) \ge 0$ where $\phi(i,j)$ is the largest integer k such that $p^k \mid j$ and*

$$j \le \begin{cases} a_{2,i-k} & \text{if } s > 1 \text{ or } k > 0 \\ p^i & \text{if } s = 1 \text{ and } k = 0 \end{cases}$$

This generator has order $p^{1+\phi(i,j)}$ and internal dimension $2(p^2-1)sp^i - 2(p-1)j$. It is the image under α (5.1.18) *of the element* $\frac{x_{2,i}^s}{p^{1+\phi(i,j)}v_1^j}$ *of* 5.4.1.

(b) *For $p = 2$, $\mathrm{Ext}^2(BP_*)$ is the direct sum of cyclic 2-groups generated by $\alpha_1\bar{\alpha}_t$, where $\bar{\alpha}_t$ generates $\mathrm{Ext}^{1,2t}(BP_*)$ for $t \ge 1$ and $t \ne 2$ (see* 5.2.6*), and by $\beta_{s2^i/j,1+\phi(i,j)}$ for $s \ge 1$, s odd, $j \ge 1$, $i \ge 0$, and $\phi(i,j) \ge 0$ where*

$$\phi(i,j) = \begin{cases} 0 & \text{if } 2 \mid j \text{ and } a_{2,i-1} < j \le a_{2,i}, \\ 0 & \text{if } j \text{ is odd and } j \le a_{2,i}, \\ 2 & \text{if } j = 2 \text{ and } i = 2, \\ k \ge 2 & \text{if } j \equiv 2^{k-1} \mod (2^k),\ j \le a_{2,i-k}, \text{ and } i \ge 3, \\ k \ge 1 & \text{if } 2^k \mid j,\ a_{2,i-k-1} < j \le a_{2,i-k}, \text{ and } i \ge 3, \\ -1 & \text{otherwise} \end{cases}$$

unless $s = 1$, in which case $a_{2,i}$ is replaced by 2^i in cases above where $\phi(i,j) = 0$, $\phi(2,2) = 1$, and β_1 is omitted. The order, internal dimension, and definition of this generator are as in (a). □

For example when $p=2$, $i=3$ and s is odd with $s>1$, we have generators

$$\begin{cases} \beta_{8s/j,2} & = & \dfrac{x_{2,2}^{2s}}{4v_1^j} \quad \text{for } j=2,4,6 \\ \beta_{8s/j} & = & \dfrac{x_{2,2}^{2s}}{2v_1^j} \quad \text{for } 1\le j\le 12 \text{ and } j\ne 2,4,6, \end{cases}$$

but $\beta_{8/j}$ is not defined for $9\le j\le 12$. Similarly when $p>2$, $i=4$ and s is prime to p with $s>1$, we have generators

$$\begin{cases} \beta_{p^4s/p^2,3} & = & \dfrac{x_{2,4}^{s}}{p^3v_1^{p^2}} \\ \beta_{p^4s/j,2} & = & \dfrac{x_{2,4}^{s}}{p^2v_1^j} \quad \text{for } p|j,\ j\ne p^2 \text{ and } j\le p^3+p^2-1 \\ \beta_{p^4s/j} & = & \dfrac{x_{2,4}^{s}}{pv_1^j} \quad \text{for other } j\le p^4+p^3-1, \end{cases}$$

but $\beta_{p^4/j}$ is not defined for $p^4<j\le p^4+p^3-1$.

Next we study the Thom reduction map Φ from $\mathrm{Ext}^2(BP_*)$ to $E_2^{2,*}$ in the classical Adams spectral sequence. This map on Ext^1 was discussed in 5.2.8. The group $E_2^{2,*}$ was given in 3.4.1 and 3.4.2. The result is

5.4.6. THEOREM. *The generators of* $\mathrm{Ext}^2(BP_*)$ *listed in* 5.4.5 *map to zero under the Thom reduction map* $\Phi\colon \mathrm{Ext}(BP_*)\to \mathrm{Ext}_{A_*}(\mathbf{Z}/(p),\mathbf{Z}/(p))$ *with the following exceptions.*

(a) (S. A. Mitchell). *For* $p=2$

$$\Phi(\alpha_1^2)=h_1^2,\qquad \Phi(\alpha_1\alpha_{4/4})=h_1h_3,$$
$$\Phi(\beta_{2^j/2^j})=h_{j+1}^2\qquad \textit{for } j\ge 1,$$
$$\Phi(\beta_{2^j/2^j-1})=h_1h_{j+2}\qquad \textit{for } j\ge 2,$$
$$\Phi(\beta_{4/2,2})=h_2h_4\quad \textit{and}\quad \Phi(\beta_{8/6,2})=h_2h_5.$$

(b) (Miller, Ravenel, and Wilson [?]). *For* $p>2$ $\Phi(\beta_{p^j/p^j})=-b_j$ *for* $j\ge 0$*;* $\Phi(\beta_{p^j/p^j-1})=h_0h_{j+1}$ *for* $j\ge 1$*, and* $\Phi(\beta_2)=\pm k_0$*.*

PROOF. We use the method of 5.2.8. For (a) we have to consider elements of $\mathrm{Ext}^1(N^1)$ as well as $\mathrm{Ext}^0(N^2)$. Recall (5.3.6) that the former is spanned by $\frac{v_1^s\rho_1}{2}$ for odd $s\ge 5$ and $\frac{v_1^st_1}{2}$ for odd $s\ge 1$. We are looking for elements with I-adic filtration ≥ 0, and the filtrations of t_1 and ρ_1 are 0 and -4, respectively. Hence we need to consider only $\frac{v_1^5\rho_1}{2}$ and $\frac{v_1t_1}{2}$, which give the first two cases of (a).

The remaining cases come from $\mathrm{Ext}^0(N^2)$. The filtration of $x_{2,i}$ is p^i so $\beta_{i/j,k}$ has filtration $i-j-k$, and this number is positive in all cases except those indicated above. We will compute $\Phi(\beta_{2/2})$ and $\Phi(\beta_{4/2,2})$, leaving the other cases of (a) and (b) to the reader. [The computation of $\Phi(\beta_1)$ and $\Phi(\beta_2)$ for $p>2$ were essentially done in 5.1.20.] Using the method of 5.1.20(a), we find that $\beta_{2/2}$ reduces to $\frac{v_2t_1^2}{v_1} \mod (2)$, which in turn reduces to $t_1^2|t_1^2 \mod I_2$, which maps under Φ to h_2^2. Similarly, $\beta_{4/2,2}$ reduces to $\frac{v_2^3t_1^2}{v_1}+\frac{v_2^4(t_1^2+v_1t_1)}{v_1^4} \mod (2)$ and to $v_2^2t_1^2|t_1^2+t_1^8|t_1^2 \mod I_2$, which maps under Φ to h_2h_4. $\square$

This result limits the number of elements in $\mathrm{Ext}^2_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ which can be permanent cycles. As remarked above (5.2.8), any such element must correspond to one having Novikov filtration ≤ 2. Theorem 5.4.6 tells us which elements in $\mathrm{Ext}(BP_*)^2$ map nontrivially to the Adams spectral sequence. Now we need to see which elements in $\mathrm{Ext}^1(BP_*)$ correspond to elements of Adams filtration 2. This amounts to looking for elements in $\mathrm{Ext}^0(N^1)$ with I-adic filtration 1. From 5.2.8 we see that $\alpha_{2/2}$ and $\alpha_{4/4}$ for $p = 2$ have I-adic filtration 0, so α_2 and $\alpha_{4/3}$ have filtration 1 and correspond to h_0h_2 and h_0h_3, respectively. More generally, α_t for all primes has filtration $t-1$ and therefore corresponds to an element with Adams filtration $\geq t$. Hence we get

5.4.7. COROLLARY. *Of the generators of* $\mathrm{Ext}^2_{A_*}(\mathbf{Z}/(p), \mathbf{Z}(p))$ *listed in* 3.4.1 *and* 3.4.2*, the only ones which can be permanent cycles in the Adams spectral sequence are*

(a) *for* $p = 2$, h_0^2, h_0h_2, h_0h_3, h_j^2 *for* $j \geq 1$, h_1h_j *for* $j \geq 3$, h_2h_4, *and* h_2h_5; *and*

(b) *for* $p > 2$, a_0^2, b_j *for* $j \geq 0$, a_1, a_0h_1 *for* $p = 3$, h_0h_j *for* $j \geq 2$, *and* k_0. $\square$

Part (a) was essentially proved by Mahowald and Tangora [**?**], although their list included h_3h_6. In Barratt, Mahowald, and Tangora [**?**] it was shown that h_2h_5 is not a permanent cycle. It can be shown that $d_3(\beta_{8/6,2}) \neq 0$, while $\beta_{4/2,2}$ is a permanent cycle. The elements h_0^2, h_0h_2, h_0h_3 for $p = 2$ and a_0^2, a_1, a_0h_1 $(p = 3)$ for odd primes are easily seen to be permanent cycles detecting elements in $\mathrm{im}\, J$.

This leaves two infinite families to be considered: the b_j (or h_{j+1}^2 for $p = 2$) for $j \geq 0$ and the h_0h_j (or h_1h_{j+1} for $p = 2$) for $j \geq 1$. These are dealt with in 3.4.4 and 4.4.22. In Section 6.4 we will generalize the latter to

5.4.8. THEOREM. (a) *In the Adams–Novikov spectral sequence for* $p \geq 3$,

$$d_{2p-1}(\beta_{p^j/p^j}) \equiv \alpha_1\beta^p_{p^{j-1}/p^{j-1}} \neq 0$$

modulo a certain indeterminacy for $j \geq 1$.

(b) *In the Adams spectral sequence for* $p \geq 5$, b_j *is not a permanent cycle for* $j \geq 1$. $\square$

The restriction on p in 5.4.8(b) is essential; we will see (6.4.11) that b_2 is a permanent cycle for $p = 3$.

The proof of 3.4.4(b) does not reveal which element in $\mathrm{Ext}^2(BP_*)$ detects the constructed homotopy element. 5.4.5 implies that $\mathrm{Ext}^{2,(1+p^j)q}$ is a $\mathbf{Z}/(p)$ vector space of rank $[j/2]$; i.e., it is spanned by elements of the form $\delta_0(x)$ for $x \in \mathrm{Ext}^1(BP_*/(p))$. (This group is described in 5.2.14 and 5.2.17.) The x that we want must satisfy $v_1^{p^{j-1}-2}x = \delta_1(v_2^{p^j})$. ($\delta_0$ and δ_1 are defined in 5.1.2.) The fact that the homotopy class has order p, along with 2.3.4, means that x itself [as well as $\delta_0(x)$] is a permanent cycle, i.e., that the map $f\colon S^m \to S^0$ for $m = q(1+p^j)-3$ given by 3.3.4(d) fits into the diagram

$$\begin{tikzcd} S^m \arrow[r, "f"] \arrow[d] & S^0 \\ \Sigma^m M(p) \arrow[r, "\tilde{f}"] & \Sigma^{-1} M(p) \arrow[u] \end{tikzcd}$$

where $M(p)$ denotes the mod (p) Moore spectrum and the vertical maps are inclusion of the bottom cell and projection onto the top cell. Now $\tilde{f}$ can be composed with any iterate of the map $\alpha\colon \Sigma^q M(p) \to M(p)$ inducing multiplication by v_1 in BP-homology, and the result is a map $S^{m+qt} \to S^0$ detected by $\delta_0(v_1^t x)$. This gives

5.4.9. THEOREM. (R. Cohen [?]) *Let $\zeta_{j-1} \in \pi_{m-1}^S$ be the element given by* 3.4.4(d), *where $m = (1+p^j)q-2$. It is detected by an element $y_{j-1} \in \mathrm{Ext}^{3,2+m}(BP_*)$ congruent to $\alpha_1\beta_{p^{j-1}/p^{j-1}}$ modulo elements of higher I-adic filtration (i.e., modulo* $\ker\Phi$). *Moreover for $j \geq 3$ and $0 < i < p^{j-1} - p^{j-2} - p^{j-3}\zeta_{j-1,i} \in \langle \zeta_{j-1}, p, \alpha_1\rangle \subset \pi_{m-1+qi}^S$, obtained as above, is nontrivial and detected by an element in* $\mathrm{Ext}^{3,2+m+qi}(BP_*)$ *congruent to $\alpha_1\beta_{p^{j-1}/p^{j-1}-i}$.* □

The range of i in 5.4.9(b) is smaller than in (a) because $\alpha_1\beta_{p^j/p^{j-1}+p^{j-2}} = 0$ for $j \geq 2$. To see this compute the coboundary of $\frac{v_1 v_2^{p^j}}{p^2 v_1^{(p+1)p^{j-2}}}$.

The analogous results for $p = 2$ are more complicated. $\eta_j \in \pi_{2^j}^S$ is *not* known to have order 2, so we cannot extend it to a map $\Sigma^{2^j} M(2) \to S^0$ and compose with elements in $\pi_*(M(2))$ as we did in the odd primary case above. In fact, there is reason to believe the order of η_j is 4 rather than 2. To illustrate the results one might expect, suppose $\beta_{2^j/2^j}$ is a permanent cycle represented by an element of order 2. (This would imply that the Kervaire invariant element θ_{j+1} exists; see 1.5.29.) Then we get a map $f\colon \Sigma^{2^{j+2}-2} M(2) \to S^0$ which we can compose with the elements of $\pi_*(M(2))$ given by 5.3.13. In particular, fv_1^{4k} would represent $\beta_{2^j/2^j-4k}$, which is nontrivial for $k < 2^{j-2}$, fv_1 would represent $\beta_{2^j/2^j-1}$ (i.e., would be closely related to η_{j+2}), and $2fv_1$ would represent $\alpha_1^2\beta_{2^j/2^j}$, leading us to expect η_{j+2} to have order 4. Since the elements of 5.3.13 have filtration ≤ 3, the composites with f would have filtration ≤ 5. Hence their nontriviality in $\mathrm{Ext}(BP_*)$ is not obvious.

Now 5.3.13 describes 12 families of elements in $\mathrm{Ext}(BP_*/(2))$ (each family has the form $\{v_1^{4k}x\colon k \geq 0\}$) which are nontrivial permanent cycles: the six shown in 5.3.14 and their products with u. Since we do not know θ_{j+1} exists we cannot show that these are permanent cycles directly. However, five of them ($v_1\alpha_1$, $v_1\alpha_1^2$, uv_1 $uv_1\alpha_1$, and $uv_1\alpha_1^2$) can be obtained by composing v_1 with mod (2) reductions of permanent cycles in $\mathrm{Ext}(BP_*)$, and hence correspond to compositions of η_{j+1} with elements in π_*^S. Four of these five families have been shown to be nontrivial by Mahowald [?] without use of the Adams–Novikov spectral sequence.

5.4.10. THEOREM (Mahowald [?]). *Let $\mu_{8k+1} \in \pi_{8k-1}^S$ be the generator constructed by Adams* [?] *and detected by $\alpha_{4k+1} \in \mathrm{Ext}^{1,8k+2}(BP_*)$, and let $\rho_k \in \pi_{8k-1}^S$ be a generator of* $\mathrm{im}\, J$ *detected by a generator y_{4k} of* $\mathrm{Ext}^{1,8k}(BP_*)$. *Then for $0 < k < 2^{j-4}$ the compositions $\mu_{8k+1}\eta_j$, $\eta\mu_{8k+1}\eta_j$, $\rho_k\eta_j$, and $\eta\rho_k\eta_j$ are essential. They are detected in the Adams spectral sequence respectively by $P^k h_1^2 h_j$, $P^k h_1^3 h_j$, $P^{k-1}c_0 h_j$, and $P^{k-1}c_0 h_1 h_j$.* □

This result provides a strong counterexample to the "doomsday conjecture", which says that for each $s \geq 0$, only finitely many elements of $E_2^{s,*}$ are permanent cycles (e.g., 1.5.29 is false). This is true for $s = 0$ and 1 by the Hopf invariant one theorem, 1.2.12, but 5.4.10 shows it is false for each $s \geq 2$.

5. Periodic Families in Ext^2

This section is a survey of results of other authors concerning which elements in $\mathrm{Ext}^2(BP_*)$ are nontrivial permanent cycles. These theorems constitute nearly all of what is known about systematic phenomena in the stable homotopy groups of spheres.

First we will consider elements various types of β's. The main result is 5.5.5. Proofs in this area tend to break down at the primes 2 and 3. These difficulties can sometimes be sidestepped by replacing the sphere with a suitable torsion-free finite complex. This is the subject of 5.5.6 ($p = 3$) and 5.5.7 ($p = 2$).

In 5.5.8 we will treat decomposable elements in Ext^2.

The proof of Smith [**?**] that β_t is a permanent cycle for $p \geq 5$ is a model for all results of this type, the idea being to show that the algebraic construction of β_t can be realized geometrically. There are two steps here. First, show that the first two short exact sequences of 5.1.2 can be realized by cofiber sequences, so there is a spectrum $M(p, v_1)$ with $BP_*(M(p, v_1)) = BP_*/I_2$, denoted elsewhere by $V(1)$. [Generally if $I = (q_0, q_1, \ldots, q_{n-1}) \in BP_*$ is an invariant regular ideal and there is a finite spectrum X with $BP_*(X) = BP_*/I$ then we will denote X by $M(q_0, \ldots, q_{n-1})$.] This step is quite easy for any odd prime and we leave the details to the reader. It cannot be done for $p = 2$. Easy calculations (e.g., 5.3.13) show that the map $S^2 \to M(2)$ realizing v_1 does not have order 2 and hence does not extend to the required map $\Sigma^2 M(2) \to M(2)$. Alternatively, one could show that $H^*(M(2, v_1); \mathbf{Z}/(2))$, if it existed, would contradict the Adem relation $Sq^2Sq^2 = Sq^3Sq^1$.

The second step, which fails for $p = 3$, is to show that for all $t > 0$, $v_1^t \in \mathrm{Ext}^0(BP_*/I_2)$ is a permanent cycle in the Adams–Novikov spectral sequence for $M(p, v_1)$. Then 2.3.4 tells us that $\beta_t = \delta_0\delta_1(v_2^t)$ detects the composite

$$S^{2t(p^2-1)} \xrightarrow{v_2^t} M(p, v_1) \to \Sigma^{q+1} M(p) \to S^{q+2},$$

where $q = 2p - 2$ as usual. One way to do this is to show that the third short exact sequence of 5.1.2 can be realized, i.e., that there is a map $\beta\colon \Sigma^{2(p^2-1)} M(p, v_1) \to M(p, v_1)$ realizing multiplication by v_2. This self-map can be iterated t times and composed with inclusion of the bottom cell to realize v_2^t. To construct β one must first show that v_2 is a permanent cycle in the Adams–Novikov spectral sequence for $M(p, v_1)$. One could then show that the resulting map $S^{2(p^2-1)} \to M(p, v_1)$ extends cell by cell to all of $\Sigma^{2(p^2-1)} M(p, v_1)$ by obstruction theory. However, this would be unnecessary if one knew that $M(p, v_1)$ were a ring spectrum, which it is for $p \geq 5$ but not for $p = 3$. Then one could smash v_2 with the identity on $M(p, v_1)$ and compose with the multiplication, giving

$$\Sigma^{2(p^2-1)} M(p, v_1) \to M(p, v_1) \wedge M(p, v_1) \to M(p, v_1),$$

which is the desired map β.

Showing that $M(p, v_1)$ is a ring spectrum, i.e., constructing the multiplication map, also involves obstruction theory, but in lower dimensions than above.

We will now describe this calculation in detail and say what goes wrong for $p = 3$. We need to know $\mathrm{Ext}^{s,t}(BP_*/I_2)$ for $t - s \leq 2(p^2 - 1)$. This deviates from $\mathrm{Ext}(BP_*/I) = \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ only by the class $v_2 \in \mathrm{Ext}^{0,2(p^2-1)}$. It follows from 4.4.8 that there are five generators in lower dimensions, namely $1 \in \mathrm{Ext}^{0,0}$,

$h_0 \in \mathrm{Ext}^{1,q}$, $b_0 \in \mathrm{Ext}^{2,pq}$, $h_0b_0 \in \mathrm{Ext}^{3(p+1)q}$, $h_1 \in \mathrm{Ext}^{1,pq}$, and $\mathrm{Ext}^{s,t} = 0$ for $t-s = 2(p^2-1)-1$ so v_2 is a permanent cycle for any odd prime.

To show $M(p, v_1)$ is a ring spectrum we need to extend the inclusion $S^0 \to M(p, v_1)$ to a suitable map from $X = M(p, v_1) \wedge M(p, v_1)$. We now assume $p = 5$ for simplicity. Then X has cells in dimensions 0, 1, 2, 9, 10, 11, 18, 19, and 20, so obstructions occur in $\mathrm{Ext}^{s,t}$ for $t-s$ one less than any of these numbers. The only one of these groups which is nontrivial is $\mathrm{Ext}^{0,0} = \mathbf{Z}/(p)$. In this case the obstruction is p times the generator (since the 1-cells in X are attached by maps of degree p), which is clearly zero. Hence for $p \geq 5$ $M(p, v_1)$ is a ring spectrum and we have the desired self-map β needed to construct the β_t's.

However, for $p = 3$ obstructions occur in dimensions 10 and 11, where the Ext groups are nonzero. There is no direct method known for calculating an obstruction of this type when it lies in a nontrivial group. In Toda [**?**] it is shown that the nontriviality of one of these obstructions follows from the nonassociativity of the multiplication of $M(3)$.

We will sketch another proof now. If $M(3, v_1)$ is a ring spectrum then each β_t is a permanent cycle, but we will show that β_4 is not. In $\mathrm{Ext}^{6,84}(BP_*)$ one has $\beta_1^2\beta_4$ and $\beta_1\beta_{3/3}^2$. These elements are actually linearly independent, but we do not need this fact now. It follows from 4.4.22 that $d_5(\beta_1\beta_{3/3}^2) = \pm\alpha_1\beta_1^4\beta_{3/3} \neq 0$. The nontriviality of this element can be shown by computing the cohomology of P_* in this range.

Now $\beta_2^3 \in \mathrm{Ext}^{6,84}(BP_*)$ is a permanent cycle since β_2 is. If we can show

$$\beta_2^3 = \pm\beta_1\beta_{3/3}^2 \pm \beta_1^2\beta_4 \tag{5.5.1}$$

then $\beta_1^2\beta_4$ and hence β_4 will have to support a nontrivial d_5. We can prove 5.5.1 by reducing to $\mathrm{Ext}(BP_*/I_2)$. By 5.1.20 the images of β_1, β_2, and β_4 in this group are $\pm b_{10}$, $\pm v_2b_{10} \pm k_0$, and $\pm v_2^3b_{10}$, respectively, and the image of $\beta_{3/3}$ is easily seen to be $\pm b_{11}$. Hence the images of $\beta_1^2\beta_4$, $\beta_1\beta_{3/3}^2$, and β_2^3 are $\pm v_2^3b_{10}^3$, $\pm b_{10}b_{11}^2$ and $\pm v_2^3b_{10}^3 \pm k_0^3$ respectively. Thus 5.5.1 will follow if we can show $k_0^3 = \pm b_{10}b_{11}^2$. (At any larger prime p we would have $k_0^p = 0$.) k_0 is the Massey product $\pm\langle h_0, h_1, h_1\rangle$. Using A1.4.6 we have up to sign

$$\begin{aligned}
k_0^2 &= \langle h_0, h_1, h_1\rangle\langle h_0, h_1, h_1\rangle \\
&= \langle h_0\langle h_0, h_1, h_1\rangle, h_1, h_1\rangle \\
&= \langle\langle h_0, h_0, h_1\rangle h_1, h_1, h_1\rangle \\
&= \langle h_0, h_0, h_1\rangle\langle h_1, h_1, h_1\rangle \\
&= g_0b_{11}
\end{aligned}$$

and

$$\begin{aligned}
k_0^3 &= \langle h_0, h_1, h_1\rangle\langle h_0, h_0, h_1\rangle b_{11} \\
&= \langle h_0\langle h_0, h_0, h_1\rangle, h_1, h_1\rangle b_{11} \\
&= \langle\langle h_0, h_0, h_0\rangle h_1, h_1, h_1\rangle b_{11} \\
&= \langle h_0, h_0, h_0\rangle\langle h_1, h_1, h_1\rangle b_{11} \\
&= b_{10}b_{11}^2 \quad \text{as claimed.}
\end{aligned}$$

5.5.2. THEOREM (Smith [**?**]). *Let $p \geq 5$*

(a) *$\beta_t \in \mathrm{Ext}^{2,q((p+1)t-1)}$ is a nontrivial permanent cycle in the Adams–Novikov spectral sequence for all $t > 0$.*

(b) *There is a map $\beta\colon \Sigma^{2(p^2-1)}M(p,v_1) \to M(p,v_1)$ inducing multiplication by v_2 in BP-homology. β_t detects the composite*

$$S^{2t(p^2-1)} \to \Sigma^{2t(p^2-1)}M(p,v_1) \xrightarrow{\beta^t} M(p,v_1) \to S^{2p}.$$

(c) *$M(p,v_1)$ is a ring spectrum.* □

5.5.3. Theorem (Behrens and Pemmaraju [?]). (a) *For $p = 3$ the complex $V(1)$ admits a self-map realizing multiplication by v_2^9 in BP-homology.*

(b) *The element $\beta_t \in \mathrm{Ext}^{2,q((p+1)t-1)}$ is a nontrivial permanent cycle in the Adams–Novikov spectral sequence for t congruent to 0, 1,2, 5, or 6 modlulo 9.*

To realize more general elements in $\mathrm{Ext}^2(BP_*)$ one must replace I_2 in the above construction by an invariant regular ideal. For example a self-map β of $M(p^2, v_1^p)$ inducing multiplication by $v_2^{p^2}$ (such a map does not exist) would show that $\beta_{tp^2/p,2}$ is a permanent cycle for each $t > 0$. Moreover we could compose β^t on the left with maps other than the inclusion of the bottom cell to get more permanent cycles. $\mathrm{Ext}^0(BP_*/(p^2, v_1^p))$ contains pv_1^i for $0 \le i < p$, and each of these is a permanent cycle and using it we could show that $\beta_{tp^2/p-i}$ is a permanent cycle.

It is easy to construct $M(p^{i+1}, v_1^{sp^i})$ for $s > 0$ and p odd. Showing that it is a ring spectrum and constructing the appropriate self-map is much harder. The following result is a useful step.

5.5.4. Theorem. (a) (Mahowald [?]). *$M(4, v_1^{4t})$ is a ring spectrum for $t > 0$.*

(b) (Oka [?]). *$M(2^{i+2}, v_1^{2^i t} + 2^{i+1}tv_1^{2^i t-3}v_2)$ is a ring spectrum for $i \ge 2$ and $t \ge 2$.*

(c) (Oka [?]). *For $p > 2$, $M(p^{i+1}v_1^{p^i t})$ is a ring spectrum for $i \ge 0$ and $t \ge 2$ [Recall $M(p,v_1)$ is a ring spectrum for $p \ge 5$ by* 5.5.2(c).] □

Note that $M(p^i, v_1^j)$ is not unique; the theorem means that there is a finite ring spectrum with the indicated BP-homology.

Hence we have a large number of four-cell ring spectra available, but it is still hard to show that the relevant power of v_2 is a permanent cycle in Ext^0.

5.5.5. Theorem.

(a) (Davis and Mahowald [?], Theorem 1.3). *For $p = 2$, there is a map $\Sigma^{48}M(2, v_1^4) \to M(2, v_1^4)$ inducing multiplication by v_2^8, so $\beta_{8t/4}$ and $\beta_{8t/3}$ are permanent cycles for all $t > 0$.*

(b) *For $p \ge 5$ the following spectra exist*: $M(p, v_1^{p-1}, v_2^p)$ (Oka [?, ?], Smith [?], Zahler [?]); *$M(p, v_2^p, v_2^{tp})$ for $t \ge 2$* (Oka [?]); $M(p, v_1^{2p-2}, v_2^{p^2})$ (Oka [?]); *$M(p, v_2^{2p}, v_2^{tp^2})$ for $t \ge 2$* (Oka [?]); *$M(p^2, v_1^p, v_2^{tp^2})$ for $t \ge 2$* (Oka [?]); *and consequently the following elements in* $\mathrm{Ext}^2(BP_*)$ *are nontrivial permanent cycles*: *$\beta_{tp/i}$ for $t > 0$, $1 \le i \le p-1$; $\beta_{tp/p}$ for $t \ge 2$; $\beta_{tp^2/i}$ for $t > 0$, $1 \le i \le 2p-2$; $\beta_{tp^2/2p}$ and $\beta_{tp^2/2p-1}$ for $t \ge 2$; and $\beta_{tp^2/p,2}$ for $t \ge 2$.*

(c) (Oka [?]). *For $p \ge 5$ the spectra $M(p, v_1^{2n-2p}, v_2^{p^n t})$ for $t \ge 2$ and $n \ge 3$, and $M(p, v_1^{2^{n-3}p}, v_2^{p^n})$ for $n \ge 3$ exist. Consequently the following elements are nontrivial permanent cycles: $\beta_{p^n t/s}$ for $t \ge 2$, $n \ge 3$, and $1 \le s \le 2^{n-2}p$; and $\beta_{p^n t/s}$ for $t \ge 1$, $n \ge 3$, and $1 \le s \le 2^{n-3}p$. In particular the p-rank of π_k^S can be arbitrarily large.* □

Note that in (a) $M(2,v_1^4)$ is not a ring spectrum since $M(2)$ is not, so the proof involves more than just showing that $v_2^8 \in \mathrm{Ext}^0(BP_*/(2,v_1^4))$ is a permanent cycle.

When a spectrum $M(p^i,v_1^j,v_2^k)$ for an invariant ideal $(p^i,v_1^j,v_2^k) \subset BP_*$ does not exist one can look for the following sort of substitute for it. Take a finite spectrum X with torsion-free homology and look for a finite spectrum $XM(p^i,v_1^j,v_2^k)$ whose BP homology is $BP_*(X) \otimes_{BP_*} BP_*/(p^i,v_1^j,v_2^k)$. Then the methods above show that the image $\bar\beta_{k/j,i}$ of $\beta_{k/j,i}$ induced by the inclusion $S^0 \to X$ [assuming X is (-1)-connected with a single 0-cell] is a permanent cycle. The resulting homotopy class must "appear" on some cell of X, giving us an element in π_*^S which is related to $\beta_{k/j,i}$. The first example of such a result is

5.5.6. THEOREM (Oka and Toda [**?**]). *Let $p=3$ and $X = S^0 \cup_{\beta_1} e^{11}$, the mapping cone of β_1.*

(a) *The spectrum $XM(3,v_1,v_2)$ exists so $\bar\beta_t \in \mathrm{Ext}^2(BP_*(X))$ is a permanent cycle for each $t>0$.*

(b) *The spectrum $XM(3,v_1^2,v_2^3)$ exists so $\bar\beta_{3t/2} \in \mathrm{Ext}^2(BP_*(X))$ is a permanent cycle for each $t>0$.*

Let $p=5$ and $X = S^0 \cup_{\beta_1} e^{39}$.

(c) *The spectrum $XM(5,v_1,v_2,v_3)$ exists so $\bar\gamma_t \in \mathrm{Ext}^3(BP_*(X))$ is a permanent cycle for all $t>0$.* □

Hence $\bar\beta_t$ detects an element in $\pi_{16t-6}(X)$ which we also denote by $\bar\beta_t$. The cofibration defining X gives an long exact sequence

$$\cdots \to \pi_n(S^0) \xrightarrow{i} \pi_n(X) \xrightarrow{j} \pi_{n-11}(S^0) \xrightarrow{\beta_1} \pi_{n-1}(S^0) \to \cdots$$

where the last map is multiplication by $\beta_1 \in \pi_{10}(S^0)$. If $\bar\beta_t \notin \mathrm{im}\, i$ then $j(\beta_t) \neq 0$, so for each $t>0$ we get an element in either π_{16t-6}^S or π_{16t-17}^S. For example, in the Adams–Novikov spectral sequence for the sphere one has $d_5(\beta_4) = \alpha_1\beta_1^2\beta_{3/3}$ so $\beta_4 \notin \mathrm{im}\, i$ and $j(\beta_4) \in \pi_{47}^S$ is detected by $\alpha_1\beta_1\beta_{3/3}$, i.e., $j(\beta_4) = \beta_1\varepsilon'$ (see 5.1.1). We can regard $j(\beta_t)$ as a substitute for β_t when the latter is not a permanent cycle.

In the above example we had $BP_*(X) = BP_* \oplus \Sigma^i BP_*$ as a comodule, so $\mathrm{Ext}(BP_*)$ is a summand of $\mathrm{Ext}(BP_*(X))$. In the examples below this is not the case, so it is not obvious that $\beta_{k/j,i} \neq 0$.

5.5.7. THEOREM (Davis and Mahowald [**?**] and Mahowald [**?**]). *Let $p=2$, $X = S^0 \cup_\eta e^2$, $W = S^0 \cup_\nu e^4$, and $Y = X \wedge W$. Part* (a) *below is essentially theorem* 1.4 *of Davis and Mahowald* [**?**]*, while the numbers in succeeding statements refer to theorems in Mahowald* [**?**]*. Their Y and A_1 are $XM(2)$ and $XM(2,v_1)$ in our notation.*

(a) *$XM(2,v_1,v_2^8)$ exists and $\bar\beta_{8t} \in \mathrm{Ext}^2(BP_*(X))$ is a nontrivial permanent cycle.*

(b) (1.4) *In the Adams–Novikov spectral sequence for S^0, β_{8t} is not a permanent cycle and $\bar\beta_{8t} \in \pi_{4qt-4}(X)$ projects under the pinching map $X \to S^2$ to an element detected by $\alpha_1^2\beta_{8t/3}$ if this element is nontrivial.*

(c) (1.5) *$\bar v_2^{8t+1} \in \mathrm{Ext}^0(BP_*(X)/I_2)$ and $\bar\beta_{8t+1} \in (BP_*(X))$ are nontrivial permanent cycles. $\beta_{8t+1} \in \mathrm{Ext}^2(BP_*)$ is not a permanent cycle and $\beta_{8t+1} \in \pi_{48t+2}(X)$ projects to an element detected by $\alpha_1\alpha_{4/4}\beta_{8t/3} \in \mathrm{Ext}^4(BP_*)$ if this element is nontrivial.*

PROOF. (a) Davis and Mahowald [?] showed that $XM(2, v_1)$ admits a self-map realizing v_2^8. This gives the spectrum and the permanent cycles. To show $\bar{\beta}_{8t} \neq 0$ it suffices to observe that $\beta_{8t} \in \operatorname{Ext}^2(BP_*)$ is not divisible by α_1.

(b) Mahowald [?] shows that $\bar{\beta}_{8t} \in \pi_{48t-4}(X)$ projects nontrivially to π_{48t-6}^S. By duality there is a map $f\colon \Sigma^{48t-4}(X) \to S^0$ which is nontrivial on the bottom cell. From 5.3.13 one can construct a map $\Sigma^{48t-4}X \to \Sigma^{48t-10}M(2)$ which is $v_1\eta^2$ on the bottom cell and such that the top cell is detected by $v_1^3 \in \operatorname{Ext}^0(BP_*/(2))$. Now compose this with the extension of $\beta_{8t/4}\Sigma^{48t-10}M(2) \to S^0$ given by 5.5.4(a). The resulting map $g\colon \Sigma^{48t-4}X \to S^0$ is $\alpha_1^2\beta_{8t/3}$ on the bottom cell and the top cell is detected by β_{8t}. Hence this map agrees with f modulo higher Novikov filtration. If $\alpha_1^2\beta_{8t/3} \neq 0 \in \operatorname{Ext}^4(BP_*)$ it follows that the bottom cell on f is detected by that element. [It is likely that $\alpha_1^3\beta_{8t/3} = 0$ (this is true for $t = 1$), so the differential on β_{8t} is not a d_3.]

(c) As in (b) Mahowald [?] shows the projection of β_{8t+1} in π_{48t}^S is nontrivial. To show that $\alpha_1\alpha_{4/4}\beta_{8t/3}$ detects our element if it is nontrivial we need to make a low-dimensional computation in the Adams–Novikov spectral sequence for $M(2, v_1^4)$ where we find that $v_1^3v_2 \in \operatorname{Ext}^{0,12}(BP_*/(2, v_1^4))$ supports a differential hitting $v_1\alpha_{4/4}\alpha_1^2 \in \operatorname{Ext}^{3,14}$. It follows that ${}_1\sigma\eta \in \pi_{11}(M(2, v_1^4))$ extends to a map $\Sigma^{10}X \to M(2, v_1^4)$ with the top cell detected by $v_2v_1^3$. Suspending $48t - 10$ times and composing with the extension of $\beta_{8t/4}$ to $\Sigma^{48t-10}M(2, v_1^4)$ gives the result. $\square$

Now we consider products of elements in Ext^1.

5.5.8. THEOREM. *Let $\bar{\alpha}_t$ be a generator of* $\operatorname{Ext}^{1,qt}(BP_*)$ (*see* 5.2.6).
(a) (Miller, Ravenel, and Wilson [?]). *For $p > 2$, $\bar{\alpha}_s\bar{\alpha}_t = 0$ for all $s, t > 0$.*
(b) *For $p = 2$*

(i) *If s or t is odd and neither is* 2 *then $\bar{\alpha}_1\bar{\alpha}_t = \alpha_1\bar{\alpha}_{s+t-1} \neq 0$.*
(ii) $\bar{\alpha}_2^2 = \beta_{2/2}$.
(iii) $\bar{\alpha}_4^2 = \beta_{4/4}$.

(*Presumably, all other products of this sort vanish.*)

PROOF. Part (a) is given in Miller, Ravenel, and Wilson [?] as theorem 8.18. The method used is similar to the proof of (b) below.

For (b)(i) assume first that s and t are both odd. Then $\bar{\alpha}_s = \frac{v_1^s}{2}$ and the mod (2) reduction of $\bar{\alpha}_t$ is $v_1^{t-1}t_1$. Hence $\bar{\alpha}_s\bar{\alpha}_t = \frac{v_1^{s+t-1}}{2}t_1 = \bar{\alpha}_{s+t-1}\bar{\alpha}_1$.

For s odd and $t = 2$ we have

$$\alpha_s\bar{\alpha}_2 = \frac{v_1^s}{2}(t_1^2 + vt_1) = d\left(\frac{v_1^{s-t}v_2}{2}\right) \quad \text{so} \quad \bar{\alpha}_s\bar{\alpha}_2 = 0.$$

For t even and $t > 2$, recall that

$$\bar{\alpha}_t = \frac{x^{t/2}}{4t} \quad \text{where} \quad x = v_1^2 - 4v_1^{-1}v_2$$

and

$$d(x) = 8\rho$$

where

$$\rho \equiv v_1^{-2}v_2t_1 - v_1^{-1}(t_2 + t_1^3) + 2(v_1t_1 + v_1^{-2}t_1^4 + v_1^{-2}t_1t_2 + v_1^{-3}v_2t_1^3) \mod (4).$$

Hence for even $t > 2$ the mod (2) reduction of $\bar{\alpha}_t$ is $v_1^{t-2}\rho$ and for odd s

$$\bar{\alpha}_s\bar{\alpha}_t = \frac{v_1^{s+t-2}}{2}\rho = \frac{v_1 x^{(s+t-1)/2}}{2}\rho.$$

Since

$$d\left(\frac{v_1 x^{(s+t+1)/2}}{2^3(s+t+1)}\right) = \frac{v_1}{2}x^{(s+t-1)/2}\rho + \frac{x^{(s+t+1)/2}t_1}{2^2(s+t+1)}$$

so $\bar{\alpha}_s\bar{\alpha}_t = \bar{\alpha}_1\bar{\alpha}_{s+t-1}$ as claimed.

For (ii) we have $\bar{\alpha}_2^2 = \frac{v_1^2(t_1^2+v_1t_1)}{4}$. The coboundary of $\frac{v_1^4}{2^5} + \frac{v_1^{-2}v_2^2}{2}$ shows this is cohomologous to $\beta_{2/2}$.

For (iii) we have $\alpha_{4/4}^2 \in \mathrm{Ext}^{2,16}$ which is $(\mathbf{Z}/(2))^3$ generated by $\alpha_1\alpha_7$, β_3, and $\beta_{4/4}$. $\alpha_1\alpha_7$ is not a permanent cycle (5.3.7) so $\alpha_{4/4}^2$ must be a linear combination of $\beta_{4/4}$ and β_3. Their reductions mod I_2, $t_1^4|t_1^4$ and $v_2t_1^4|t_1$, are linearly independent so it suffices to compute $\alpha_{4/4}^2 \mod I_2$. The mod I_2 reduction of $\alpha_{4/4}$ is t_1^4, so the result follows. □

6. Elements in Ext^3 and Beyond

We begin by considering products of elements in Ext^2 with those in Ext^1 and Ext^2. If x and y are two such elements known to be permanent cycles, then the nontriviality of xy in Ext implies that the corresponding product in homotopy is nontrivial, but if $xy = 0$ then the homotopy product could still be nontrivial and represent an element in a higher Ext group. The same is true of relations among and divisibility of products of permanent cycles; they suggest but do not imply (without further argument) similar results in homotopy.

Ideally one should have a description of the subalgebra of $\mathrm{Ext}(BP_*)$ generated by Ext^1 and Ext^2 for all primes p. Our results are limited to odd primes and fall into three types (see also 5.5.8). First we describe the subgroup of Ext^3 generated by products of elements in Ext^1 with elements of order p in Ext^2 (5.6.1). Second we note that certain of these products are divisible by nontrivial powers of p (5.6.2). These two results are due to Miller, Ravenel, and Wilson [**?**], to which we refer for most of the proofs.

Our third result is due to Oka and Shimomura [**?**] and concerns products of certain elements in Ext^2 (5.4.4–5.4.7). They show further that in certain cases when a product of permanent cycles is trivial in Ext^4, then the corresponding product in homotopy is also trivial.

This brings us to γ's and δ's. Toda [**?**] showed that γ_t is a permanent cycle for $p \geq 7$ (1.3.18), but left open the case $p = 5$. In Section 7.5 we will make calculations to show that γ_3 does not exist. We sketch the argument here. As remarked in Section 4.4, 4.4.22 implies that $d_{33}(\alpha_1\beta_{5/5}^4) = \beta_1^{21}$ (up to a nonzero scalar). Calculations show that $\alpha_1\beta_{5/5}^4$ is a linear combination of $\beta_1^3\gamma_3$ and $\beta_1\langle\alpha_1\beta_3, \beta_4, \gamma_2\rangle$. Hence if the latter can be shown to be a permanent cycle then we must have $d_{33}(\gamma_3) = \beta_1^{18}$. Each of the factors in the above Massey product is a permanent cycle, so it suffices to show that the products $\alpha_1\beta_3\beta_4 \in \pi_{323}(S^0)$ and $\beta_4\gamma_2 \in \pi_{619}(S^0)$ both vanish. Our calculation shows that both of these stems have trivial 5-torsion.

To construct δ_t one could proceed as in the proof of 5.5.2. For $p \geq 7$ there is a finite complex $V(3)$ with $BP_*(V(3)) = BP_*/I_4$. According to Toda [**?**] it is a ring spectrum for $p \geq 11$. Hence there is a self-map realizing multiplication by

v_4 iff there is a corresponding element in $\pi_*(V(3))$. We will show (5.6.13) that the group $\mathrm{Ext}^{2p-1,2(p^4+p-2)}(BP_*/I_4)$ is nonzero for all $p \geq 3$, so it is possible that $d_{2p-1}(v_4) \neq 0$.

The following result was proved in Miller, Ravenel, and Wilson [?] as theorem 8.6.

5.6.1. THEOREM. *Let $m \geq 0$, $p \nmid s$, $s \geq 1$, $1 \leq j \leq a_{2,m}$ (where $a_{2,m}$ is as in* 5.4.1) *for $s > 1$ and $1 \leq j \leq p^m$ for $s = 1$. Then $\alpha_1\beta_{sp^m/j} \neq 0$ in $\mathrm{Ext}^3(BP_*)$ iff one of the following conditions holds*

(i) *$j = 1$ and either $s \not\equiv -1 \mod (p)$ or $s \equiv -1 \mod (p^{m+2})$.*

(ii) *$j = 1$ and $s = p - 1$.*

(iii) *$j > 1 + a_{2,m-\nu(j-1)-1}$.*

In case (ii), *we have $\alpha_1\beta_{p-1} = -\gamma_1$ and for $m \geq 1$, $2\alpha_1\beta_{(p-1)p^m} = -\gamma_{p^m/p^m,p^m}$. The onfy linear relations among these classes are*

$$\alpha_1\beta_{sp^2/p+2} = s\alpha_1\beta_{sp^2-1},$$

and

$$\alpha_1\beta_{sp^{2m+2}/2+a_{2,m+1}} = 2s\alpha_1\beta_{sp^{2m+2}-p^m} \quad \text{for} \quad m \geq 1. \qquad \square$$

This result implies that some of these products vanish and therefore certain Massey products (A1.4.1) are defined. For example, $\alpha_1\beta_{(tp-1)p^m} = 0$ if $t > 1$ and $p^{m+2} \nmid t$ so we have Massey products such as $\langle\beta_{2p-1}, \alpha_1, \alpha_1\rangle$ represented up to nonzero scalar multiplication by

$$\frac{v_2^{2p}t_1}{pv_1^{1+p}} + \frac{v_2^{2p-1}t_1^2 - 2v_2^{p-1}v_3t_1}{pv_1}.$$

This product has order p^2 but many others do not. For example, $\alpha_1\beta_{p/2} = 0$ and $\langle\beta_{p/2}, \alpha_1, \alpha_1\rangle$ is represented by

$$\frac{2v_1^{p-1}v_2^pt_1}{p^2v_1^p} - \frac{v_2^pt_1^2}{pv_1^2}$$

which has order p^2 and $p\langle\beta_{p/2}, \alpha_1, \alpha_1\rangle = \alpha_1\beta_p$ up to nonzero scalar multiplication. Similarly, one can show

$$\alpha_1\beta_{p^2} = p\langle\beta_{p^2/2}, \alpha_1, \alpha_1\rangle = p^2\langle\beta_{p^2/3}, \alpha_1, \alpha_1, \alpha_1\rangle.$$

The following results were 2.8(c) and 8.17 in Miller, Ravenel, and Wilson [?].

5.6.2. THEOREM. *With notation as in* 5.6.1, *if $\alpha_1\beta_{sp^m/j} \neq 0$ in $\mathrm{Ext}^3(BP_*)$, then it is divisible by at least p^i whenever $0 < i \leq m$ and $j \geq a_{2,m-i}$.* $\square$

5.6.3. THEOREM. *With notation as above and with t prime to p, $\alpha_{sp^k/k+1}\beta_{tp^m/j} = s\alpha_1\beta_{tp^m/j-sp^k+1}$ in $\mathrm{Ext}^3(BP_*)$.* $\square$

Now we consider products of elements in Ext^2, which are studied in Oka and Shimomura [?].

5.6.4. THEOREM. *For $p \geq 3$ we have $ij\beta_s\beta_t = st\beta_i\beta_j$ in Ext^4 for $i + j = s + t$.*

PROOF. To compute $\beta_s\beta_t$ we need the mod I_2 reduction of β_t, which was computed in 5.1.20. Hence we find $\beta_s\beta_t$ is represented by

$$\frac{-tv_2^{s+t-1}b_{10} + \binom{t}{2}v_2^{s+t-2}k_0}{pv_1}.$$

Now let

$$u_m = \frac{v_2^m v_1^{p-1} t_1^p}{p^2 v_1^p} - \frac{v_2^m t_2}{p v_1^2} + \frac{k v_2^{k-1}}{p v_1}(t_1^p t_2 - t_1^{2p+1}).$$

A routine computation gives

$$d\Big(\frac{t}{2}\, u_{s+t-1}\Big) = \frac{s v_2^{s+t-1} b_{10}}{p v_1} - \frac{s}{2}(s+t-1) v_2^{s+t-2} k_0$$

and hence $\beta_s\beta_t$ is represented by $-\frac{st}{2}\frac{v_2^{s+t-2}k_0}{pv_1}$ and the result follows. $\square$

The analogous result in homotopy for $p \geq 5$ was first proved by Toda [**?**]. The next three results are 6.1, A, and B of Oka and Shimomura [**?**].

5.6.5. THEOREM. *For $p \geq 3$ the following relations hold in* Ext^4 *for $s, t > 0$.*
(i) $\beta_s\beta_{tp^k/r} = 0$ *for $k \geq 1$, $t \geq 2$ and $r < a_{2,k}$.*
(ii) $\beta_s\beta_{tp^2/p,2} = \beta_{s+t(p^2-p)}\beta_{tp/p}$.
(iii) *For $t, k \geq 2$,*

$$\begin{aligned} \beta_2\beta_{tp^k/a_{2,k}} &= \beta_{s+(tp-1)(p^{k-1}-p)}\beta_{tp^2/a_{2,2}} \\ &= (t/2)\beta_{s+(tp-1)p^{k-1}-(2p-1)p}\beta_{2p^2/a_{2,2}}. \quad \square \end{aligned}$$

5.6.6. THEOREM. *For $p \geq 5$, $0 < r \leq p$, with $r \leq p-1$ if $t = 1$, the element $\beta_s\beta_{tp/r}$ is trivial in $\pi_*(S^0)$ if one of the following conditions holds.*
(i) $r \leq p-2$.
(ii) $r = p-1$ *and* $s \not\equiv -1 \mod (p)$.
(iii) $r = p-1$ *or* p *and* $t \equiv 0 \mod (p)$. $\square$

5.6.7. THEOREM. *For $p \geq 5$, $s \not\equiv 0$ or 1, $t \not\equiv 0 \mod (p)$, and $t \geq 2$, the elements $\beta_s\beta_{tp/p}$ and $\beta_s\beta_{tp^2/p,2}$ are nontrivial.* $\square$

Now we will display the obstruction to the existence of $V(4)$, i.e., a nontrivial element in $\mathrm{Ext}^{2p-1,2(p^4+p-2)}(BP_*/I_4)$. This group is isomorphic to the corresponding Ext group for $P_* = P[t_1, t_2, \dots]$, the dual to the algebra of Steenrod reduced powers. To compute this Ext we use a method described in Section 3.5. Let $P(1)_* = P/(t_1^{p^2}, t_2^p, t_3, \dots)$, the dual to the algebra generated by P^1 and P^p. We will give P_* a decreasing filtration so that $P(1)_*$ is a subalgebra of E_0P_*. We let $t_1, t_2 \in F_0$, and $t_i^{p^2}, t_{i+1}^p, t_{i+2} \in F^{(p^i-1)/(p-1)}$ for $i \geq 1$. Then as an algebra we have

$$E_0P_* = P(1)_* \otimes T(t_{i+2,0}, t_{i+1,1}) \otimes P(t_{i,2}), \tag{5.6.8}$$

where $i \geq 1$, $t_{i,j}$ corresponds to $t_i^{p^j}$, and T denotes the truncated polynomial algebra of height p. Let R denote the tensor product of the second two factors in 5.6.8. Then

$$P(1)_* \to E_0P_* \to R \tag{5.6.9}$$

is an extension of Hopf algebras (A1.1.5) for which there is a Cartan–Eilenberg spectral sequence (A1.3.14) converging to

$$\mathrm{Ext}_{E_0P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

with

$$E_2 = \mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p), \mathrm{Ext}_R(\mathbf{Z}/(p), \mathbf{Z}/(p))). \tag{5.6.10}$$

The filtration of P_* gives a spectral sequence (A1.3.9) converging to

$$\mathrm{Ext}_{P^*}(Z/(p), Z/(p))$$

with

$$E_2 = \mathrm{Ext}_{E_0 P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)). \tag{5.6.11}$$

In the range of dimensions we need to consider, i.e., for $t - s \leq 2(p^4 - 1)$ Ext_R is easy to compute. We leave it to the reader to show that it is the cohomology of the differential $P(1)_*$-comodule algebra

$$E(h_{12}, h_{21}, h_{30}, h_{13}, h_{22}, h_{31}, h_{40}) \otimes P(b_{12}, b_{21}, b_{30})$$

with $d(h_{22}) = h_{12}h_{13}$, $d(h_{31}) = h_{21}h_{13}$, and $d(h_{40}) = h_{30}h_{13}$. In our range this cohomology is

$$E(h_{12}, h_{21}, h_{30}, h_{13})/h_{13}(h_{12}, h_{21}, h_{30}) \otimes P(b_{12}, b_{21}, b_{30}), \tag{5.6.12}$$

where the nontrivial action of $P(1)$ is given by

$$P^1 h_{30} = h_{21}, \quad P^p h_{21} = h_{12}, \quad \text{and} \quad P^p b_{30} = b_{21}.$$

We will not give all of the details of the calculations since our aim is merely to find a generator of $\mathrm{Ext}_*^{2p-1, 2(p^4+p-2)}$. The element in question is

$$b_{20}^{p-3} h_{11} h_{20} h_{12} h_{21} h_{30}. \tag{5.6.13}$$

We leave it to the interested reader to decipher this notation and verify that it is a nontrivial cocycle.